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MATHEMATICS

Word Maps and Word Maps with Constants of Simple Algebraic Groups[1]

N. L. Gordeev[a,b,*], B. E. Kunyavskii[c,], and E. B. Plotkin[c]**

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Abstract—In the present paper, we consider word maps w: $G^m \to G$ and word maps with constants w_Σ: $G^m \to G$ of a simple algebraic group G, where w is a nontrivial word in the free group F_m of rank m, $w_\Sigma = w_1\sigma_1 w_2 \cdots w_r\sigma_r w_{r+1}$, $w_1, \ldots, w_{r+1} \in F_m$, $w_2, \ldots, w_r \neq 1$, $\Sigma = \{\sigma_1, \ldots, \sigma_r \mid \sigma_i \in G\backslash Z(G)\}$. We present results on the images of such maps, in particular, we prove a theorem on the dominance of "general" word maps with constants, which can be viewed as an analogue of a well-known theorem of Borel on the dominance of genuine word maps. Besides, we establish a relationship between the existence of unipotents in the image of a word map and the structure of the representation variety $R(\Gamma_w, G)$ of the group $\Gamma_w = F_m/\langle w\rangle$.

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1. WORD MAPS WITH CONSTANTS

For any group G and any non-empty word w in the free group F_m of rank m one can define the *word map* w: $G^m \to G$ by the formula $w((g_1, \ldots, g_m)) = w(g_1, \ldots, g_m)$ (we substitute the elements g_i instead of the variables x_i). Recently one could observe growing interest to the study of word maps of simple algebraic groups (see [1–3, 5, 8, 9]). For such groups we also consider here word maps with constants. Namely, let G be a simple algebraic group defined over an algebraically closed field K (we identify the group G with the group of points $G(K)$), and let $w_1,\ldots,w_{r+1} \in F_m$, where $w_2,\ldots,w_r \neq 1$, $\Sigma = \{\sigma_1,\ldots,\sigma_r \mid \sigma_i \in G\backslash Z(G)\}$ (we allow $\sigma_i = \sigma_j$ for $i \neq j$). The expression $w_\Sigma = w_1\sigma_1 w_2 \cdots w_r\sigma_r w_{r+1}$ is called a word with constants (we regard usual words as words with constants by setting $\Sigma = \phi$, $w = w_1$). The behaviour of words with constants on simple algebraic groups was studied, in particular, in [6, 7, 15]. A word with constants also gives rise to a natural word map with constants w_Σ: $G^m \to G$. In [6] such maps were used for studying products of conjugacy classes, and in [8] they served as a method for studying genuine word maps.

One of the main questions of the theory of word maps concerns their surjectivity (the answer is unknown even for the group $G = \mathrm{SL}_2(C)$, see [9]). According to a theorem of Borel [4], the word maps of the simple algebraic groups are dominant, i.e., the image Imw of such a map contains a dense open subset of G. A word map with constants is not necessarily dominant (for example, for $w_\Sigma = x\sigma x^{-1}$). However, for a "general" word with constants such a map turns out to be dominant.

Theorem 1. *Let* $\Omega_r = (w_1, \ldots, w_{r+1})$ *be a sequence of words from* F_m *where* $w_2, \ldots, w_r \neq 1$. *Suppose that* $\prod_{i=1}^{r+1} w_i \notin [F_m, F_m]$. *Then there is a non-empty Zariski open subset* $U(\Omega_r) \subset G^r$ *such that for every sequence* $\Sigma = (\sigma_1, \ldots, \sigma_r) \in U(\Omega_r)$ *the map* w_Σ: $G^m \to G$ *is dominant.*

We also consider the case of word maps with constants for which we have $\prod_{i=1}^{r+1} w_i = 1$. Namely, let $w(x, y) \in F_2$ and $\sigma \in G$. Then the map w_σ: $G \to G$ defined by the formula $w_\sigma(x) = w(x, \sigma)$ is a word map with constants (here the constants are powers of σ). We have the following theorem, which can serve as a tool in studying word maps in two variables (see [8]).

[1] The article was translated by the authors.

[a] Herzen State Pedagogical University of Russia, St. Petersburg
[b] St. Petersburg State University
[c] Bar-Ilan University, Ramat Gan, Israel
*e-mail: nickgordeev@mail.ru
**e-mail: kunyav@gmail.com

Theorem 2. *Let $w \in [F_2, F_2]$. Then there is a nonempty Zariski open subset $U(w) \subset G$ such that for every $\sigma \in U(w)$ the set $\{g(\mathrm{Im}\, w_\sigma)g^{-1} | g \in G\}$ is Zariski dense in G.*

2. SEMISIMPLE ELEMENTS IN Im w

In the paper [3] it was proven that for $G = \mathrm{SL}_2(K)$ the image of the word map w: $\mathrm{SL}_2(K)^m \to \mathrm{SL}_2(K)$ contains all semisimple elements of $\mathrm{SL}_2(K)$ (here K is an algebraically closed field) except possibly $-\mathbf{1}$ ($\mathbf{1}$ denotes the identity matrix). Using the fact that for all simple algebraic groups except those of types A_r, D_{2r+1}, E_6, the corresponding root system contains a subsystem of the same rank which consists of the union of disjoint subsystems of type A_1, one can get the following assertion.

Theorem 3. *Let G be a simple algebraic group, and let w: $G^m \to G$ be a word map. Suppose that G is not of type A_r, $r > 1$, D_{2r+1}, or E_6. Then every regular semisimple element of G is contained in* Im *w. Moreover, for every semisimple element $g \in G$ there exists an element g_0 of order two such that $gg_0 \in$* Im*w.*

3. UNIPOTENT ELEMENTS IN Im w AND THE REPRESENTATION VARIETY OF A FINITELY GENERATED GROUP

Let T and W denote, respectively, a fixed maximal torus and the Weyl group of G, and let π: $G \to T/W$ be the quotient morphism (see [14]). For a word map w: $G^m \to G$ define $Y_w = w^{-1}(\mathbf{1})$, $\Xi_w = (\pi \cdot w)^{-1}(\mathbf{1})$ (here $\mathbf{1}$ denotes the identity element of G and also the image of the identity element of T in T/W). Then $Y_w \subset \Xi_w$ are affine subvarieties of G^m, $\Xi_w = \{(g_1, \ldots, g_m) \in G \,|\, w(g_1, \ldots, g_m)$ is a unipotent element$\}$, $Y_w = R(\Gamma_w, G)$ is the variety of representations of the one-relator group $\Gamma_w = F_m/\langle w \rangle$ in the group G. Thus, the existence of nontrivial unipotent elements in Im w is equivalent to the inequality $Y_w \neq \Xi_w$. For example, in the simplest case $G = \mathrm{SL}_2(K)$ and $m = 2$ all irreducible components of the variety Ξ_w are of dimension 5, and thus the existence of nontrivial unipotent elements in Im w follows from the existence of irreducible components of Y_w of dimension ≤4.

The representation variety of a group is an important object which can be regarded from various points of view (see, e.g., [10–13]) and may be crucial for answering the question on the existence of unipotent elements in Im w.

The existence of unipotent elements in Im w is an open question even in the case $G = \mathrm{SL}_2(K)$. In [3], Bandman and Zarhin proved that for $G = \mathrm{SL}_2(K)$ (ch $K = 0$) and $w \notin [[F_m, F_m], [F_m, F_m]]$, the set Im w contains all unipotent elements. Besides, they gave an example of a computer-aided calculation for a word $w \in [[F_m, F_m], [F_m, F_m]]$ such that Im w also contains all unipotent elements. We consider a similar example for which we calculate (without using computer) the varieties Y_w, Ξ_w. Let K be an algebraically closed field (ch $K = 0$), and let w: $\mathrm{SL}_2(K)^2 \to \mathrm{SL}_2(K)$ be the word map induced by the word $w(x, y) = [[x, y], x[x, y]x^{-1}]$. Let B and T denote, respectively, the upper triangular and diagonal matrices in $\mathrm{SL}_2(K)$, let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let C_ω be the conjugacy class of ω in $\mathrm{SL}_2(K)$. Then we have the following fact.

Theorem 4. *The variety Ξ_w has exactly three irreducible components:*

$$\Xi_w^0 = \overline{\{g(B \times B)g^{-1} | g \in G\}},$$
$$\Xi_w^1 = \overline{\{g(T \times \omega B)g^{-1} | g \in G\}},$$
$$\Xi_w^2 = C_\omega \times G,$$

and the variety Y_w also has exactly three irreducible components:

$$Y_w^0 = \Xi_w^0, Y_w^1 = \overline{\{g(T \times \omega T)g^{-1} | g \in G\}} \subset \Xi_w^1,$$
$$\dim Y_w^1 = 4, \quad Y_w^2 = \Xi_w^2$$

(*here bar stands for the Zariski closure*).

Thus the existence of the component Y_w^1 of dimension 4 guarantees that all unipotents lie in Im w. Although a full proof of this theorem requires significant technical arguments, the mere fact that all unipotents belong to Im w is proved by an elementary calculation of the value of $w(s, \omega b)$ for $s \in T$, $s^4 \neq \mathbf{1}$, $b \in B$, $b \notin T$.

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