The scheme of connected components of the Néron–Raynaud model of an algebraic torus

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Given a local field K with ring of integers $\mathcal{O}$ and perfect residual field k, and an algebraic torus T over K, one can construct a model of T over $\mathcal{O}$ satisfying all the properties of Néron models except of being of finite type ([1], Chapter 10). We call such a model $\mathcal{T}$ the Néron–Raynaud model of T. It is a smooth group scheme over $\mathcal{O}$ locally of finite type. (One can mention other approaches to constructing integer models of algebraic tori [2], [3]; their interrelations with Néron–Raynaud models will be described elsewhere).

Let $\mathcal{T}^\circ$ denote the identity component of $\mathcal{T}$. Denote by $i\colon \mathrm{Spec}\ k \to \mathrm{Spec}\ \mathcal{O}$ the natural closed immersion, and let $\phi = i^*(\mathcal{T}/\mathcal{T}^\circ)$ be the scheme of connected components of the reduction of $\mathcal{T}$. This scheme was studied by Xarles [4]. Let $G_K = \mathrm{Gal}\,(\bar{K}/K)$ and $G_k = \mathrm{Gal}\,(\bar{k}/k)$ be the Galois groups of K and k, respectively, and let $X = \mathrm{Hom}(T_{\bar{K}}, \mathbf{G}_{m,\bar{K}})$ be the G_K-module of characters of T. Xarles [4] described the G_k-module ϕ in terms of the module X. To be more precise, it is shown ([4], Corollary 2.18) that ϕ fits into an exact sequence of G_k-modules

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(H^1(I,X), \mathbf{Q}/\mathbf{Z}) \to \phi \to \mathrm{Hom}_{\mathbf{Z}}(X^I, \mathbf{Z}) \to 0, \qquad (1)$$

where I denotes the inertia subgroup of G_K and X^I denotes the module of I-invariants. Moreover, an important observation of [4] is that the sequence (1) determines ϕ in the case when $\mathrm{dim} T \leq 2$, but there exist 3-dimensional tori T_1 and T_2 having isomorphic G_k-modules $H^i(I,X_1) \cong H^i(I,X_2) (i=0,1)$, with non-isomorphic G_k-modules ϕ_1 and ϕ_2. Unfortunately, the example given in [4], §4, to confirm this statement, is wrong. In fact, with the notation of [4], we have $H^1(I,X_1) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ (and not $\mathbf{Z}/2\mathbf{Z}$, as stated in [4]), while $H^1(I,X_2) = \mathbf{Z}/2\mathbf{Z}$, so 1-cohomology groups do not coincide.

Below we give a correct example.

Let K be a local field, and let L/K be a finite Galois extension with Galois group $G \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and the inertia subgroup $I \cong \mathbf{Z}/2\mathbf{Z}$. We consider two 3-dimensional tori T_1 and T_2 defined over K and splitting by L whose character modules X_1 and X_2 are defined by the Galois representations

$$\rho_i : G \to GL(3, \mathbf{Z}), \qquad i = 1, 2,$$

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$$\rho_i(\sigma)=\begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, \qquad i=1,2,$$

$$\rho_1(\tau)=\begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, \qquad \rho_2(\tau)=\begin{pmatrix} -1 & -1 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

where σ and τ stand for the generators of G, and σ generates I.

Proposition. *Let T_1 and T_2 be as above. Then we have $\phi_i \cong \mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}(i=1,2)$ with the splitting action of $G_k =< \bar\tau >$ on ϕ_1 : $\bar\tau(a,\bar b)=(-a,\bar b)$ and the non-splitting action on ϕ_2 : $\bar\tau(a,\bar b)=(-a,\overline{a+b})$.*

Proof. As in [4], we consider the exact sequences

$$0\to X_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} N_i \to 0, \qquad i=1,2, \tag{2}$$

where $M_i=\mathrm{Ind}^1_{\rho_i(G)}(X_i)$ is the induced module and α_i is the natural inclusion. The module M_i is cohomologically trivial. We have $\phi_i\cong \mathrm{Coker}((N_i^I)^\vee \to (M_i^I)^\vee)$, where $^\vee$ denotes $\mathrm{Hom}(\cdot,\mathbf{Z})$ [4].

As a basis of M_i, choose

$$E_{j,g}=e_j\otimes g, \tag{3}$$

where $e_j(j=1,2,3)$ denotes the basis of X_i corresponding to the representation ρ_i, and g runs over G; we have

$$\alpha_i(e_j)=\sum_{g\in G} E_{j,g}. \tag{4}$$

Take $F_{j,g}=\beta_i(E_{j,g})(g\neq\sigma\tau)$ for a basis of N_i; one deduces from (3), (4) that G acts on N_i as follows:

$$\sigma:\ \begin{array}{lcl} F_{1,1} & \to & -F_{1,\sigma}\\ F_{1,\sigma} & \to & -F_{1,1}\\ F_{1,\tau} & \to & F_{1,1}+F_{1,\sigma}+F_{1,\tau}\\ F_{2,1} & \to & -F_{3,\sigma}\\ F_{2,\sigma} & \to & -F_{3,1}\\ F_{2,\tau} & \to & F_{3,1}+F_{3,\sigma}+F_{3,\tau}\\ F_{3,1} & \to & -F_{2,\sigma}\\ F_{3,\sigma} & \to & -F_{2,1}\\ F_{3,\tau} & \to & F_{2,1}+F_{2,\sigma}+F_{2,\tau}\end{array} \qquad (i=1,2); \tag{5}$$

For $i=1$:

$$\tau: \begin{array}{lcl} F_{1,1} & \to & F_{1,\tau} \\ F_{1,\sigma} & \to & -F_{1,1}-F_{1,\sigma}-F_{1,\tau} \\ F_{1,\tau} & \to & F_{1,1} \\ F_{j,1} & \to & -F_{j,\tau} \\ F_{j,\sigma} & \to & F_{j,1}+F_{j,\sigma}+F_{j,\tau} \\ F_{j,\tau} & \to & -F_{j,1} \end{array} \qquad (j=2,3); \tag{6}$$

For $i=2$:

$$\tau: \begin{array}{lcl} F_{1,1} & \to & -F_{1,\tau} \\ F_{1,\sigma} & \to & F_{1,1}+F_{1,\sigma}+F_{1,\tau} \\ F_{1,\tau} & \to & -F_{1,1} \\ F_{2,1} & \to & -F_{1,\tau}+F_{3,\tau} \\ F_{2,\sigma} & \to & F_{1,1}+F_{1,\sigma}+F_{1,\tau}-F_{3,1}-F_{3,\sigma}-F_{3,\tau} \\ F_{2,\tau} & \to & -F_{1,1}+F_{3,1} \\ F_{3,1} & \to & F_{1,\tau}+F_{2,\tau} \\ F_{3,\sigma} & \to & F_{1,1}+F_{1,\sigma}+F_{1,\tau}-F_{2,1}-F_{2,\sigma}-F_{2,\tau} \\ F_{3,\tau} & \to & -F_{1,1}+F_{2,1}. \end{array} \tag{7}$$

It follows from (3), (4) that for the modules of I-invariants we have

$$\begin{array}{rcl} M_i^I & = & < E_{1,1}-E_{1,\sigma}, E_{1,\tau}-E_{1,\sigma\tau}, E_{2,1}-E_{3,\sigma}, E_{2,\sigma}-E_{3,1}, E_{2,\tau}-E_{3,\sigma\tau}, E_{2,\sigma\tau}-E_{3,\tau} > \\ & = & < A_1, \ldots, A_6 >, \\ N_i^I & = & < F_{1,\sigma}+F_{1,\tau}, F_{1,1}+F_{1,\tau}, F_{2,1}-F_{3,\sigma}, F_{2,\sigma}-F_{3,1}, F_{2,\sigma}+F_{2,\tau}+F_{3,\sigma}+F_{3,\tau} > \\ & = & < B_1, \ldots, B_5 > . \end{array} \tag{8}$$

Taking into account (3), (6), (7), we obtain that $G_k =< \bar{\tau} >$ acts on the modules of I-invariants as follows:

For $i=1$:

$$\bar{\tau}: \begin{array}{lcl} A_1 & \to & A_2 \\ A_2 & \to & A_1 \\ A_3 & \to & A_5 \\ A_4 & \to & A_6 \\ A_5 & \to & A_3 \\ A_6 & \to & A_4 \end{array} \qquad \begin{array}{lcl} B_1 & \to & -B_1 \\ B_2 & \to & B_2 \\ B_3 & \to & B_4-B_5 \\ B_4 & \to & B_3+B_5 \\ B_5 & \to & B_5 \end{array} \tag{9}$$

For $i=2$:

$$\bar{\tau}: \begin{array}{lcl} A_1 & \to & -A_2 \\ A_2 & \to & -A_1 \\ A_3 & \to & -A_2-A_6 \\ A_4 & \to & A_2-A_5 \\ A_5 & \to & -A_1-A_4 \\ A_6 & \to & A_1-A_3 \end{array} \qquad \begin{array}{lcl} B_1 & \to & B_1 \\ B_2 & \to & -B_2 \\ B_3 & \to & -B_1-B_2+B_3+B_5 \\ B_4 & \to & B_1+B_2+B_4-B_5 \\ B_5 & \to & 2B_1-B_5 \end{array} \tag{10}$$

Taking I-cohomology of (2), we obtain an exact sequence

$$0 \to \mathbf{Z} \to M_i^I \to N_i^I \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

According to (8), the map of dual modules $\gamma_i : (N_i^I)^\vee \to (M_i^I)^\vee$ is given by

$$\gamma_i(x_1, \ldots, x_5) = (-x_1 + x_2, x_1 + x_2, x_3, x_4, -x_4 + x_5, -x_3 + x_5). \tag{11}$$

Denote by $(\varepsilon_i, \delta_i)$ the basis of $\phi_i \cong \mathrm{Coker}\gamma_i \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Take for ε_i the class of $E_6^\vee$ (generating the first direct summand $\mathbf{Z}$), and for δ_i the class of $E_1^\vee$ (generating the second direct summand $\mathbf{Z}/2\mathbf{Z}$), where $^\vee$ denotes the dual basis. One can calculate the action of G_k on ϕ_i from (9), (10), (11) :

$$\bar{\tau} : \begin{array}{lll} \varepsilon_1 & \to & -\varepsilon_1 \\ \delta_1 & \to & \delta_1 \end{array} \qquad\qquad \begin{array}{lll} \varepsilon_2 & \to & -\varepsilon_2 + \delta_2 \\ \delta_2 & \to & \delta_2 \end{array}$$

Hence ϕ_1 and ϕ_2 are not isomorphic as G_k-modules. This proves the proposition.

Q.E.D.

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