A "LOCAL MEAN OSCILLATION" DECOMPOSITION AND SOME ITS APPLICATIONS

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1. Introduction

A recent result by the author [39] establishes a pointwise control of an arbitrary measurable function in terms of its local mean oscillations. Soon after that, in a surprising work [12], D. Cruz-Uribe, J. Martell and C. Pérez showed that this result can be effectively applied in a variety of questions, including sharp weighted inequalities for classical singular integrals and the dyadic square function. In turn, based on [12] and on a recent concept of the intrinsic square function by M. Wilson [54], the author [40] obtained sharp weighted estimates for essentially any Littlewood-Paley operator.

The aim of these notes is to present a unified, extended and almost self-contained exposition of the above-mentioned works [39, 12, 40].

2. The space BMO

2.1. **The classical approach to BMO.** The mean oscillation of a locally integrable function f over a cube $Q \subset \mathbb{R}^n$ is defined by

$$\Omega(f;Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx,$$

where $f_Q = \frac{1}{|Q|}\int_Q f$. It is easy to see that

$$\inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - c|dx \le \Omega(f;Q) \le 2\inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - c|dx.$$

The space of functions with bounded mean oscillation, $BMO(\mathbb{R}^n)$, consists of all $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO} \equiv \sup_{Q\subset\mathbb{R}^n} \Omega(f;Q) < \infty.$$

This space was introduced by F. John and L. Nirenberg in [30]. In the same work the following fundamental property of BMO-functions was

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established: for any $f \in BMO$, any cube $Q \subset \mathbb{R}^n$, and for all $\alpha > 0$,

$$|\{x \in Q : |f(x) - f_Q| > \alpha\}| \le 2|Q| \exp\Big(- \frac{\alpha}{c_n \|f\|_{BMO}} \Big). \tag{2.1}$$

For example, it follows from this inequality that

$$\|f\|_{BMO} \asymp \sup_Q \Big(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \Big)^{1/p} \tag{2.2}$$

for any $p \ge 1$.

The sharp maximal function is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \Omega(f; Q),$$

where the supremum is taken over all cubes Q containing the point x. This operator was introduced by C. Fefferman and E.M. Stein [20].

Recall that the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Since $f^{\#}(x) \le 2Mf(x)$, we have that $f^{\#}$ is bounded on L^p for $p > 1$. The basic property of $f^{\#}$ proved in [20] says that the converse inequality is also true, namely if $f \in L^{p_0}$, then

$$\|f\|_{L^p} \le c\|f^{\#}\|_{L^p} \quad (1 < p < \infty). \tag{2.3}$$

As we shall see below, the assumption that $f \in L^{p_0}$ can be relaxed until $f^*(+\infty) = 0$.

Originally, inequality (2.3) was applied to describing the intermediate spaces between BMO and L^p. A number of other applications of (2.3) concerns L^p-norm inequalities involving various operators in harmonic analysis (see D. Kurtz [31]). Typically, one can obtain a pointwise estimate $(T_1 f)^{\#}(x) \le cT_2 f(x)$, where T_1 is a certain singular-type operator, and T_2 is a maximal-type operator. Combining this with (2.3) yields a norm estimate of T_1 by T_2. Perhaps the first such application was found by A. Cordoba and C. Fefferman in [10] where it was shown that for a singular integral T,

$$(Tf)^{\#}(x) \le cM(|f|^r)(x)^{1/r} \quad (r > 1). \tag{2.4}$$

2.2. **Median values and local mean oscillations.** Given a measurable function f, its non-increasing rearrangement is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\} \quad (0 < t < \infty).$$

Observe that we have defined the rearrangement as the left-continuous function. Usually in the literature (see, e.g., [1, p. 39]) one can find the right-continuous rearrangement defined by

$$\inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \le t\}.$$

The left-continuous rearrangement can be also defined in the following very convenient form (cf. [5]):

$$f^*(t) = \sup_{|E|=t} \inf_{x \in E} |f(x)|.$$

Definition 2.1. By a median value of f over Q we mean a possibly nonunique, real number $m_f(Q)$ such that

$$|\{x \in Q : f(x) > m_f(Q)\}| \le |Q|/2$$

and

$$|\{x \in Q : f(x) < m_f(Q)\}| \le |Q|/2.$$

If follows from this definition that

$$|\{x \in Q : |f(x)| \ge |m_f(Q)|\}| \ge |Q|/2,$$

which implies

$$|m_f(Q)| \le (f\chi_Q)^*(|Q|/2).$$

Also, if $m_f(Q)$ is a median value of f, then for any constant c, $m_f(Q)-c$ is a median value of $f-c$. Hence, applying the previous inequality, we get

$$|m_f(Q) - c| \le \big((f-c)\chi_Q\big)^*\big(|Q|/2\big). \tag{2.5}$$

Definition 2.2. Given a measurable function f on $\mathbb{R}^n$ and a cube Q, the local mean oscillation of f over Q is defined by

$$\omega_\lambda(f;Q) = \big((f - m_f(Q))\chi_Q\big)^*\big(\lambda|Q|\big) \quad (0 < \lambda < 1).$$

It follows from (2.5) that for $0 < \lambda \le 1/2$,

$$\inf_{c \in \mathbb{R}} \big((f-c)\chi_Q\big)^*\big(\lambda|Q|\big) \le \omega_\lambda(f;Q) \le 2 \inf_{c \in \mathbb{R}} \big((f-c)\chi_Q\big)^*\big(\lambda|Q|\big).$$

Therefore, the median value $m_f(Q)$ plays the same role for the local mean oscillation (in the case $0 < \lambda \le 1/2$) as the usual integral mean value f_Q does for the integral mean oscillation.

2.3. **A "local mean oscillation" approach to BMO.** By Chebyshev's inequality, $f^*(t) \le \frac{1}{t}\|f\|_{L^1}$, and hence,

$$\omega_\lambda(f;Q) \le \frac{2}{\lambda}\Omega(f;Q),$$

which shows that

$$\sup_Q \omega_\lambda(f;Q) \le \frac{2}{\lambda}\|f\|_{BMO}.$$

The main result about local mean oscillations obtained by F. John [29] says that for $\lambda < 1/2$ the converse inequality holds:

$$\|f\|_{BMO} \le c\sup_Q \omega_\lambda(f;Q). \tag{2.6}$$

This result was rediscovered by J.-O. Strömberg [51] who showed also that (2.6) remains true for $\lambda = 1/2$. Note that this exponent is sharp. Indeed, if $\lambda > 1/2$, then $\omega_\lambda(f;Q) = 0$ for any function f taking only two values, and therefore (2.6) fails.

Observe that (2.6) allows to "recover" BMO under the minimal assumptions on f. In particular, in the definition of $\omega_\lambda(f;Q)$ we suppose only that f is merely measurable. Note also that (2.6) shows that (2.2) holds for any $p > 0$.

Denote

$$\|f\|_{BMO_\lambda} = \sup_Q \omega_\lambda(f;Q).$$

By the above discussion we have the John-Strömberg equivalence

$$\|f\|_{BMO} \asymp \|f\|_{BMO_\lambda} \quad (0 < \lambda \le 1/2).$$

The proof of this result is based on the following version of the John-Nirenberg inequality (2.1):

$$|\{x \in Q : |f(x) - m_f(Q)| > \alpha\}| \le 2|Q| \exp\Big(-\frac{\alpha}{c_n\|f\|_{BMO_\lambda}}\Big). \tag{2.7}$$

It is easy to see that actually (2.7) implies the John-Nirenberg inequality. Indeed, by (2.5) and by Chebyshev's inequality,

$$|m_f(Q) - f_Q| \le 2\Omega(f;Q) \le 2\|f\|_{BMO}.$$

Hence, if $\alpha > 4\|f\|_{BMO}$, we get from (2.7) that

$$\begin{aligned} |\{x \in Q : |f(x) - f_Q| > \alpha\}| &\le |\{x \in Q : |f(x) - m_f(Q)| > \alpha/2\}| \\ &\le 2|Q| \exp\Big(-\frac{\alpha}{c_n'\|f\|_{BMO}}\Big). \end{aligned}$$

On the other hand, for $\alpha \le 4\|f\|_{BMO}$ this inequality holds trivially with suitable c_n'.

An interesting analysis of the John-Nirenberg and John-Strömberg inequalities (and in particular, an attempt to obtain (2.7) with dimension free constants) can be found in a recent work [14] by M. Cwikel, Y. Sagher and P. Shvartsman.

Similarly to the Fefferman-Stein sharp function, one can define the local sharp maximal function by

$$M_\lambda^\# f(x) = \sup_{Q \ni x} \omega_\lambda(f;Q).$$

It was shown by J.-O. Strömberg [51] that for any $p > 0$,

$$\|f\|_{L^p} \le c\|M_\lambda^\# f\|_{L^p} \quad (0 < \lambda \le 1/2). \tag{2.8}$$

As we shall see below (Corollary 4.2), it is enough to assume here that $f^*(+\infty) = 0$.

On one hand it seems that (2.8) is an improvement of the Fefferman-Stein inequality (2.3) since we easily have that $M_\lambda^\# f(x) \le \frac{2}{\lambda} f^\#(x)$. On the other hand, B. Jawerth and A. Torchinsky [28] showed that $f^\#$ is essentially the Hardy-Littlewood maximal function of $M_\lambda^\# f$: for any $x \in \mathbb{R}^n$,

$$c_1 M M_\lambda^\# f(x) \le f^\#(x) \le c_2 M M_\lambda^\# f(x) \tag{2.9}$$

(this result with $\lambda = 1/2$ one can find in [34]). It follows from this that inequalities (2.3) and (2.8) are equivalent for $p > 1$.

Inequalities (2.9) show that the local sharp maximal function $M_\lambda^\# f$ is much smaller than $f^\#$. This advantage is invisible in the unweighted L^p spaces. However, as soon as the weighted inequalities are concerned, the use of $M_\lambda^\# f$ usually leads to much more precise results. As a typical example, consider Coifman's inequality [6] saying that if $w \in A_\infty$, then a singular integral operator T satisfies

$$\|Tf\|_{L^p(w)} \le c\|Mf\|_{L^p(w)} \quad (0 < p < \infty). \tag{2.10}$$

It is well known that both inequalities (2.3) and (2.8) hold if we replace the L^p-norm by the $L^p(w)$-norm for $w \in A_\infty$ (we shall prove this fact in Theorem 4.6 below). Further, it was shown in [28] that

$$M_\lambda^\#(Tf)(x) \le cMf(x). \tag{2.11}$$

Combining this with the weighted Strömberg's inequality (2.8), we immediately obtain (2.10). Next, observe that (2.11) along with (2.9) yields an improvement of (2.4):

$$(Tf)^\#(x) \le cMMf(x).$$

But even this improved version combined with the weighted Fefferman-Stein inequality (2.3) gives only (2.10) with MMf instead of Mf on the right-hand side.

3. The main decomposition result

3.1. On theorems of C. Fefferman and L. Carleson. Denote $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. Given a cube $Q \subset \mathbb{R}^n$, set

$$\widetilde{Q} = \{(y,t) \in \mathbb{R}^{n+1}_+ : y \in Q, 0 < t < \ell(Q)\},$$

where $\ell(Q)$ denotes the side length of Q.

We say that σ is a Carleson measure on $\mathbb{R}^{n+1}_+$ if

$$C(\sigma) \equiv \sup_{Q \subset \mathbb{R}^n} \frac{|\sigma|(\widetilde{Q})}{|Q|} < \infty,$$

where $|\sigma|$ is the total variation of σ.

Let K be a positive function with $\int_{\mathbb{R}^n} K = 1$ and such that

$$K(x) \le \frac{c}{1 + |x|^{n+1}}.$$

Write $K_t(x) = t^{-n}K(x/t), t > 0$.

The famous duality theorem by C. Fefferman [18, 20] says that $BMO(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$. This result has several equivalent formulations. One of them is the following (see [4] and [22, p. 272]): if $f \in BMO(\mathbb{R}^n)$ with compact support, then there is $g \in L^\infty$ and there is a Carleson measure σ such that

$$f(x) = g(x) + \int_{\mathbb{R}^{n+1}_+} K_t(x-y)d\sigma(y,t), \tag{3.1}$$

where

$$\|g\|_{L^\infty} \le c\|f\|_{BMO} \quad \text{and} \quad C(\sigma) \le c\|f\|_{BMO}.$$

L. Carleson [4] gave a direct and constructive proof of (3.1) providing a new proof of the $H^1 - BMO$ duality.

3.2. On theorems of J. Garnett and P. Jones, and N. Fujii. We say that $I \subset \mathbb{R}$ is a dyadic interval if I is of the form $(\frac{j}{2^k}, \frac{j+1}{2^k})$ for some integers j and k. A dyadic cube $Q \subset \mathbb{R}^n$ is a Cartesian product of n dyadic intervals of equal lengths. Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$.

Given a cube Q_0, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0 (that is, they are formed by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes).

J. Garnett and P. Jones [23] obtained the following dyadic version of Carleson's theorem: given $f \in BMO$ and a cube Q_0, there is $g \in L^\infty$

with $\|g\|_{L^\infty} \le 2\|f\|_{BMO}$ and there exists a sequence $\{Q_k\} \in \mathcal{D}(Q_0)$, and a sequence a_k of real numbers such that for a.e. $x \in Q_0$,

$$f(x) - f_{Q_0} = g(x) + \sum_k a_k \chi_{Q_k}(x)$$

and for each $Q \in \mathcal{D}(Q_0)$,

$$\sum_{Q_k \subset Q} |a_k||Q_k| \le c\|f\|_{BMO}|Q|.$$

If we replace $\mathbb{R}^{n+1}_+$ by its discrete subset $\{p_Q = (c_Q, \ell(Q)), Q \in \mathcal{D}\}$, where c_Q is the center of Q, and consider on this subset the measure σ having mass $a_k|Q_k|$ at p_{Q_k}, then we get a correspondence between the Garnett-Jones and Carleson theorems.

The above-mentioned results deal with decompositions of BMO-functions. A remarkable observation of N. Fujii [21] is that almost the same proof as in [23] yields actually a decomposition of an arbitrary locally integrable function: given a cube Q,

$$f(x) - f_Q = g(x) + \sum_{j=1}^{\infty}\sum_{\nu} a_\nu^{(j)} \chi_{Q_\nu^j}(x) \quad \text{for a.e. } x \in Q. \tag{3.2}$$

Here $|g(x)| \le c f_Q^{\#}(x)$, where $f_Q^{\#}(x)$ is the Fefferman-Stein sharp function relative to Q, and

$$|a_\nu^{(j)}| \le c \sup_{Q \supset Q' \supset Q_j^k} \Omega(f; Q'). \tag{3.3}$$

3.3. **A decomposition in terms of local mean oscillations.** It was mentioned by N. Fujii [21] without the proof that replacing f_Q in (3.2) by a median value $m_f(Q)$, one can obtain a similar decomposition but with the local sharp maximal function instead of $f_Q^{\#}(x)$. Indeed, following [21], we get a variant of (3.2) with a control of g by $M_\lambda^{\#} f$ and with (3.3) replaced by

$$|a_\nu^{(j)}| \le c \sup_{Q \supset Q' \supset Q_j^k} \omega_\lambda(f; Q').$$

Our key result stated below says that such a variant can be further improved with a control of $a_\nu^{(j)}$ it terms of single local mean oscillations. As we shall see below, this point is crucial for many important applications.

If $Q \in \mathcal{D}(Q_0)$ and $Q \ne Q_0$, we denote by $\widehat{Q}$ its dyadic parent, that is, the unique cube from $\mathcal{D}(Q_0)$ containing Q and such that $|\widehat{Q}| = 2^n|Q|$.

For $x \in Q_0$ set

$$M^{\#,d}_{\lambda;Q_0}f(x) = \sup_{x\in Q\in \mathcal{D}(Q_0)} \omega_\lambda(f;Q),$$

where the supremum is taken over all dyadic cubes with respect to Q_0 containing the point x.

The following theorem was proved in [39].

Theorem 3.1. *Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) collection of cubes $Q_j^k \in \mathcal{D}(Q_0)$ such that*

(i) *for a.e. $x \in Q_0$,*

$$|f(x) - m_f(Q_0)| \le 2M^{\#,d}_{1/4;Q_0}f(x) + 2\sum_{k=1}^{\infty}\sum_j \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}_j^k)\chi_{Q_j^k}(x);$$

(ii) *for each fixed k the cubes Q_j^k are pairwise disjoint;*

(iii) *if $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$;*

(iv) $|\Omega_{k+1} \cap Q_j^k| \le \frac{1}{2}|Q_j^k|$.

Observe that properties (ii)-(iv) here are the same as those of cubes obtained from the classical Calderón-Zygmund decomposition (and indeed, the cubes Q_j^k in Theorem 3.1 are also obtained from a similar stopping-time process). In particular, we shall use below the following standard trick. Let $E_j^k = Q_j^k \setminus \Omega_{k+1}$. It follows from the properties (ii)-(iv) of Theorem 3.1 that $|E_j^k| \ge |Q_j^k|/2$ and the sets E_j^k are pairwise disjoint.

Proof of Theorem 3.1. We divide the proof into several parts.

The 1st part. We claim that there exists a (possibly empty) collection of pairwise disjoint cubes $\{Q_j^1\} \in \mathcal{D}(Q_0)$ such that

$$\sum_j |Q_j^1| \le \frac{1}{2}|Q_0| \tag{3.4}$$

and

$$f - m_f(Q_0) = g_1 + \sum_j \alpha_{j,1}\chi_{Q_j^1} + \sum_j (f - m_f(Q_j^1))\chi_{Q_j^1}, \tag{3.5}$$

where

$$|g_1(x)| \le M^{\#,d}_{1/4;Q_0}f(x) \quad \text{for a.e. } x \in Q_0 \setminus \cup_j Q_j^1 \tag{3.6}$$

and the numbers $\alpha_{j,1}$ satisfy

$$|a_{j,1}| \le \omega_{1/2^{n+1}}(f;\widehat{Q}_j^1) + \omega_{1/4}(f;Q_0). \tag{3.7}$$

Set $f_1(x) = f(x) - m_f(Q_0)$ and

$$E_1 = \{x \in Q_0 : |f_1(x)| > \omega_{1/4}(f; Q_0)\}.$$

We may assume that $|E_1| > 0$ since otherwise we trivially have

$$|f(x) - m_f(Q_0)| \le \omega_{1/4}(f; Q_0) \le M^{\#,d}_{1/4;Q_0} f(x) \quad \text{for a.e. } x \in Q_0$$

(and so the collection $\{Q^1_j\}$ is empty).

Let

$$m^{\Delta}_{Q_0} f_1(x) = \sup_{Q \in \mathcal{D}(Q_0), x \in Q} |m_{f_1}(Q)|,$$

and consider

$$\Omega_1 = \{x \in Q_0 : m^{\Delta}_{Q_0} f_1(x) > \omega_{1/4}(f; Q_0)\}.$$

Observe that by (2.5) and by Chebyshev's inequality

$$|m_f(Q) - f(x)| \le \frac{2}{|Q|} \int_Q |f(y) - f(x)| dy.$$

From this, by Lebesgue's differentiation theorem we get that for a.e. $x \in \mathbb{R}^n$,

$$\lim_{|Q| \to 0, Q \ni x} m_f(Q) = f(x).$$

Hence, $m^{\Delta}_{Q_0} f_1(x) \ge |f_1(x)|$ for a.e. x, and therefore $|\Omega_1| \ge |E_1| > 0$.

For each $x \in \Omega_1$ there exists a cube $Q_x \in \mathcal{D}(Q_0)$ containing x and such that $|m_{f_1}(Q_x)| > \omega_{1/4}(f; Q_0)$. Note that $Q_x \ne Q_0$ since $m_{f_1}(Q_0) = 0$. Therefore, choosing the maximal dyadic cubes (exactly as in the classical Calderón-Zygmund decomposition), we get that $\Omega_1 = \cup_j Q^1_j$, where Q^1_j are pairwise disjoint cubes from $\mathcal{D}(Q_0)$ such that

$$|m_{f_1}(Q^1_j)| > \omega_{1/4}(f; Q_0) \tag{3.8}$$

and

$$|m_{f_1}(\widehat{Q}^1_j)| \le \omega_{1/4}(f; Q_0).$$

It follows from (3.8) that

$$(f_1 \chi_{Q_0})^*(|Q_0|/4) < (f_1 \chi_{Q^1_j})^*(|Q^1_j|/2).$$

Hence,

$$|\{x \in Q^1_j : |f_1(x)| > (f_1 \chi_{Q_0})^*(|Q_0|/4)\}| \ge |Q^1_j|/2,$$

and thus

$$\begin{aligned} \frac{1}{2} \sum_j |Q^1_j| &\le \sum_j |\{x \in Q^1_j : |f_1(x)| > (f_1 \chi_{Q_0})^*(|Q_0|/4)\}| \\ &\le |\{x \in Q_0 : |f_1(x)| > (f_1 \chi_{Q_0})^*(|Q_0|/4)\}| \le |Q_0|/4, \end{aligned}$$

which proves (3.4) (here we have used the well known property of the rearrangement saying that $\mu_f(f^*(t)) \le t$, where μ_f is the distribution function of f).

Further, observe that $m_{f_1}(Q_j^1) = m_f(Q_j^1) - m_f(Q_0)$. Hence we can write

$$f - m_f(Q_0) = f_1\chi_{Q_0\setminus\Omega_1} + \sum_j m_{f_1}(Q_j^1)\chi_{Q_j^1} + \sum_j (f - m_f(Q_j^1))\chi_{Q_j^1},$$

which yields (3.5) with $g_1 = f_1\chi_{Q_0\setminus\Omega_1}$ and $\alpha_{j,1} = m_{f_1}(Q_j^1)$. Since $m^{\Delta}_{Q_0}f_1(x) \ge |f_1(x)|$ a.e., we obviously have (3.6). Next,

$$\begin{aligned} |\alpha_{j,1}| &\le |m_{f_1}(Q_j^1) - m_{f_1}(\widehat{Q}_j^1)| + |m_{f_1}(\widehat{Q}_j^1)| \\ &\le \big((f - m_f(\widehat{Q}_j^1))\chi_{Q_j^1}\big)^*(|Q_j^1|/2) + \omega_{1/4}(f;Q_0) \\ &\le \omega_{1/2^{n+1}}(f;\widehat{Q}_j^1) + \omega_{1/4}(f;Q_0), \end{aligned}$$

which gives (3.7), and hence our claim is proved.

The 2nd part. Observe that each function $f - m_f(Q_j^1)$ has the same behavior on Q_j^1 that $f - m_f(Q_0)$ has on Q_0. Therefore we can repeat the process for any Q_j^1, and continue by induction.

Denote by Q_j^k the cubes obtained at k-th stage. Let $\Omega_k = \cup_j Q_j^k$ and $f_{i,k}(x) = f(x) - m_f(Q_i^{k-1})$. Denote

$$\mathcal{I}_{1,k} = \{i : \Omega_k \cap Q_i^{k-1} = \emptyset\} \quad \text{and} \quad \mathcal{I}_{2,k} = \{i : \Omega_k \cap Q_i^{k-1} \ne \emptyset\}.$$

Assume that $i \in \mathcal{I}_{2,k}$. Setting

$$\mathcal{J}_{i,k} = \{j : Q_j^k \subset Q_i^{k-1}\},$$

we have

$$\Omega_{i,k} = \{x \in Q_i^{k-1} : m^{\Delta}_{Q_i^{k-1}}(f_{i,k})(x) > \omega_{1/4}(f;Q_i^{k-1})\} = \cup_{j\in\mathcal{J}_{i,k}} Q_j^k,$$

and the numbers $\alpha_{j,k}^{(i)} = m_{f_{i,k}}(Q_j^k)$, similarly to (3.7), satisfy

$$|\alpha_{j,k}^{(i)}| \le \omega_{1/2^{n+1}}(f;\widehat{Q}_j^k) + \omega_{1/4}(f;Q_i^{k-1}) \quad (j \in \mathcal{J}_{i,k}). \tag{3.9}$$

Further, similarly to (3.4),

$$|\Omega_k \cap Q_i^{k-1}| = \sum_{j\in\mathcal{J}_{i,k}} |Q_j^k| \le \frac{1}{2}|Q_i^{k-1}|. \tag{3.10}$$

We write below how (3.5) will transformed after the first repetition of the above described process. Set $\psi_1 = \sum_i (f - m_f(Q_i^1))\chi_{Q_i^1}$. We have

$$\psi_1 = \sum_{i\in\mathcal{I}_{1,2}} (f - m_f(Q_i^1))\chi_{Q_i^1} + \sum_{i\in\mathcal{I}_{2,2}} (f - m_f(Q_i^1))\chi_{Q_i^1}.$$

Next,

$$\sum_{i\in\mathcal{I}_{2,2}} = \sum_{i\in\mathcal{I}_{2,2}}(f-m_f(Q_i^1))\chi_{Q_i^1\setminus\Omega_{i,2}} + \sum_{i\in\mathcal{I}_{2,2}}(f-m_f(Q_i^1))\chi_{\Omega_i^2},$$

and (we use that $\alpha_{j,2}^{(i)} = m_f(Q_j^2) - m_f(Q_i^1)$)

$$\sum_{i\in\mathcal{I}_{2,2}}(f-m_f(Q_i^1))\chi_{\Omega_i^2} = \sum_{i\in\mathcal{I}_{2,2}}\sum_{j\in\mathcal{J}_{i,2}}(f-m_f(Q_i^1))\chi_{Q_j^2}$$

$$= \sum_{i\in\mathcal{I}_{2,2}}\sum_{j\in\mathcal{J}_{i,2}}\alpha_{j,2}^{(i)}\chi_{Q_j^2} + \sum_{i\in\mathcal{I}_{2,2}}\sum_{j\in\mathcal{J}_{i,2}}(f-m_f(Q_j^2))\chi_{Q_j^2}$$

Combining the previous equations along with (3.5), we get

$$f-m_f(Q_0) = g_1+g_2+\sum_j \alpha_{j,1}\chi_{Q_j^1} + \sum_{i\in\mathcal{I}_{2,2}}\sum_{j\in\mathcal{J}_{i,2}}\alpha_{j,2}^{(i)}\chi_{Q_j^2} + \psi_2,$$

where

$$g_2 = \sum_{i\in\mathcal{I}_{1,2}}(f-m_f(Q_i^1))\chi_{Q_i^1} + \sum_{i\in\mathcal{I}_{2,2}}(f-m_f(Q_i^1))\chi_{Q_i^1\setminus\Omega_{i,2}}$$

and

$$\psi_2 = \sum_{i\in\mathcal{I}_{2,2}}\sum_{j\in\mathcal{J}_{i,2}}(f-m_f(Q_j^2))\chi_{Q_j^2}.$$

Similarly, after $(k-1)$-th repetition of the process we obtain that

$$f(x)-m_f(Q_0) = \sum_{\nu=1}^{k} g_\nu + \sum_{\nu=1}^{k}\sum_{i\in\mathcal{I}_{2,\nu}}\sum_{j\in\mathcal{J}_{i,\nu}}\alpha_{j,\nu}^{(i)}\chi_{Q_j^\nu}(x) + \psi_k(x),$$

where

$$g_k = \sum_{i\in\mathcal{I}_{1,k}} f_{i,k}\chi_{Q_i^{k-1}}(x) + \sum_{i\in\mathcal{I}_{2,k}} f_{i,k}\chi_{Q_i^{k-1}\setminus\Omega_{i,k}}(x)$$

and

$$\psi_k(x) = \sum_{i\in\mathcal{I}_{2,k}}\sum_{j\in\mathcal{J}_{i,k}}\big(f-m_f(Q_j^k)\big)\chi_{Q_j^k}(x)$$

(so, by our notation, $\sum_{i\in\mathcal{I}_{2,1}}\sum_{j\in\mathcal{J}_{i,1}}\alpha_{j,1}^{(i)}\chi_{Q_j^1}(x) \equiv \sum_j \alpha_{j,1}\chi_{Q_j^1}(x)$).

By (3.10), $|\Omega_k| \le |\Omega_{k-1}|/2$, and hence $|\Omega_k| \le |Q_0|/2^k$. Since the support of ψ_k is Ω_k we obtain that $\psi_k \to 0$ a.e. as $k\to\infty$. Therefore, for a.e. $x\in Q_0$,

$$\begin{aligned} f(x)-m_f(Q_0) &= \sum_{\nu=1}^{\infty} g_\nu + \sum_{\nu=1}^{\infty}\sum_{i\in\mathcal{I}_{2,\nu}}\sum_{j\in\mathcal{J}_{i,\nu}}\alpha_{j,\nu}^{(i)}\chi_{Q_j^\nu}(x) \\ &\equiv S_1(x)+S_2(x). \end{aligned}$$

The 3rd part. It is easy to see that the supports of g_ν are pairwise disjoint and for any ν and for a.e. $x \in Q_0$,

$$|g_\nu(x)| \le M^{\#,d}_{1/4;Q_0} f(x).$$

Therefore,

$$|S_1(x)| \le M^{\#,d}_{1/4;Q_0} f(x) \quad \text{a.e. in } Q_0.$$

Next, we write

$$S_2(x) = \sum_j \alpha_{j,1}\chi_{Q_j^1}(x) + \sum_{\nu=2}^{\infty}\sum_{i\in\mathcal{I}_{2,\nu}}\sum_{j\in\mathcal{J}_{i,\nu}} \alpha^{(i)}_{j,\nu}\chi_{Q_j^\nu}(x). \tag{3.11}$$

By (3.7),

$$\sum_j |\alpha_{j,1}|\chi_{Q_j^1}(x) \le \sum_j \omega_{1/2^{n+1}}(f;\widehat{Q}_j^1)\chi_{Q_j^1}(x) + \omega_{1/4}(f;Q_0).$$

Applying (3.9), we get that the second term on the right-hand side of (3.11) is bounded by

$$\begin{aligned}
&\sum_{\nu=2}^{\infty}\sum_{i\in\mathcal{I}_{2,\nu}}\sum_{j\in\mathcal{J}_{i,\nu}} \Big(\omega_{1/2^{n+1}}(f;\widehat{Q}_j^\nu) + \omega_{1/4}(f;Q_i^{\nu-1})\Big)\chi_{Q_j^\nu}(x)\\
&\le \sum_{\nu=2}^{\infty}\sum_j \omega_{1/2^{n+1}}(f;\widehat{Q}_j^\nu)\chi_{Q_j^\nu}(x) + \sum_{\nu=2}^{\infty}\sum_i \omega_{1/4}(f;Q_i^{\nu-1})\chi_{Q_i^{\nu-1}}(x).
\end{aligned}$$

Combining this with the previous estimate yields

$$\begin{aligned}
|S_2(x)| &\le \sum_{\nu=1}^{\infty}\sum_j \big(\omega_{1/2^{n+1}}(f;\widehat{Q}_j^\nu) + \omega_{1/4}(f;Q_j^\nu)\big)\chi_{Q_j^\nu}(x) + \omega_{1/4}(f;Q_0)\\
&\le 2\sum_{\nu=1}^{\infty}\sum_j \omega_{1/2^{n+2}}(f;\widehat{Q}_j^\nu)\chi_{Q_j^\nu}(x) + M^{\#}_{1/4;Q_0}f(x),
\end{aligned}$$

which along with the estimate for S_1 completes the proof. □

4. Applications of Theorem 3.1 to $M^{\#}_{\lambda} f$ and BMO

By a weight we mean a non-negative locally integrable function. We start with the following result.

Theorem 4.1. *For any weight w, cube Q, and a measurable function f,*

$$\int_Q |f - m_f(Q)| w\,dx \le 6\int_Q M^{\#}_{\lambda;Q} f(x)^{\delta} M_Q\big(M^{\#}_{\lambda;Q} f(x)^{1-\delta} w\big)(x)\,dx, \tag{4.1}$$

where a constant λ depends only on n, and $0 < \delta \le 1$.

Proof. Applying Theorem 3.1 with $Q = Q_0$ yields

$$\begin{aligned}\int_{Q_0} |f - m_f(Q_0)| w\,dx \;\; &\le \;\; 2\int_{Q_0} M^{\#}_{1/4;Q_0} f(x) w\,dx \\ &+ \;\; 2\sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\int_{Q^k_j} w.\end{aligned}$$

Since

$$\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j) \le \inf_{x\in Q^k_j} M^{\#}_{\lambda_n;Q_0} f(x) \quad (\lambda_n = 1/2^{n+2}),$$

we have

$$\begin{aligned}&\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\int_{Q^k_j} w \le \sum_{k,j}\big(\inf_{Q^k_j} M^{\#}_{\lambda_n;Q_0} f\big)\int_{Q^k_j} w \\ &\le 2\sum_{k,j}\Big(\int_{E^k_j}\big(M^{\#}_{\lambda_n;Q_0} f\big)^{\delta}\Big)\frac{1}{|Q^k_j|}\int_{Q^k_j}\big(M^{\#}_{\lambda_n;Q_0} f\big)^{1-\delta} w \\ &\le 2\sum_{k,j}\int_{E^k_j}\big(M^{\#}_{\lambda_n;Q_0} f\big)^{\delta} M_{Q_0}\big((M^{\#}_{\lambda_n;Q_0} f)^{1-\delta} w\big)dx \\ &\le 2\int_{Q_0}\big(M^{\#}_{\lambda_n;Q_0} f\big)^{\delta} M_{Q_0}\big((M^{\#}_{\lambda_n;Q_0} f)^{1-\delta} w\big)dx.\end{aligned}$$

Also,

$$\int_{Q_0} M^{\#}_{1/4;Q_0} f(x) w\,dx \le \int_{Q_0} M^{\#}_{1/4;Q_0} f(x)^{\delta} M_{Q_0}\big(M^{\#}_{1/4;Q_0} f(x)^{1-\delta} w\big)(x)dx.$$

Combining the obtained estimates completes the proof. □

Theorem 4.1 implies easily both the John-Nirenberg-John-Strömberg and the Fefferman-Stein-Strömberg inequalities, namely, we have the following.

Corollary 4.2. *Let* $0 < \lambda \le \lambda_n$.

(i) *For any* $f \in BMO$ *and any cube* Q,

$$|\{x \in Q : |f(x) - m_f(Q)| > \alpha\}| \le 2|Q|\exp\Big(-\frac{\alpha}{c_n\|f\|_{BMO_\lambda}}\Big); \tag{4.2}$$

(ii) *For any measurable* f *with* $f^*(+\infty) = 0$ *and for all* $p > 0$,

$$\|f\|_{L^p} \le c\|M^{\#}_{\lambda} f\|_{L^p}. \tag{4.3}$$

Proof. Theorem 4.1 with $\delta = 1$ along with Stein's $L\log L$ characterization [50] yields

$$\int_Q |f - m_f(Q)| w dx \le 6\|f\|_{BMO_\lambda}\int_Q M_Q w \le c\|f\|_{BMO_\lambda}\|w\|_{L\log L(Q)}.$$

From this, by the $\exp(L)$-$L\log L$ duality [1, p. 243],

$$\|f - m_f(Q)\|_{expL(Q)} \le c\|f\|_{BMO_\lambda},$$

which is equivalent to (4.2).

Further, if $f^*(+\infty) = 0$, then $|m_f(Q)| \to 0$ when Q tends to $\mathbb{R}^n$. Hence, letting $Q \to \mathbb{R}^n$ in Theorem 4.1 with $\delta = 1$, we get

$$\int_{\mathbb{R}^n} |f(x)|w dx \le 6 \int_{\mathbb{R}^n} M_\lambda^\# f(x) Mw dx \tag{4.4}$$

(observe that this inequality was proved by means of different ideas in [35]). From this, using the L^p boundedness of M and the $L^p - L^{p'}$ duality, we get (4.3) for $p > 1$.

The case $p = 1$ follows immediately if we take $w \equiv 1$. Further, if $0 < r < 1$, we have

$$\omega_\lambda(|f|^r; Q) \le 2((|f|^r - |m_f(Q)|^r)\chi_Q)^*(\lambda|Q|) \le 2\omega_\lambda(f; Q)^r,$$

which implies

$$M_\lambda^\#(|f|^r)(x) \le 2M_\lambda^\# f(x)^r.$$

Combining this estimate with the proved case $p = 1$, we obtain (4.3) for $0 < p < 1$. □

Remark 4.3. Observe that our approach does not allow to obtain the sharp John-Strömberg exponent $\lambda = 1/2$ in Corollary 4.2. On the other hand, from point of view of most applications, it is enough to have (4.2) and (4.3) for some λ depending on n.

Remark 4.4. It follows from the definition of the rearrangement that $f^*(+\infty) = 0$ if and only if

$$|\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < \infty$$

for any $\alpha > 0$ (cf., [36, Prop. 2.1]).

Remark 4.5. Theorem 4.1 contains also a part of Jawerth-Torchinsky inequality (2.9). Indeed, by (4.1) with $\delta = 1$ and $w \equiv 1$,

$$\int_Q |f(x) - f_Q| dx \le 2\int_Q |f(x) - m_f(Q)| dx \le 12 \int_Q M_\lambda^\# f(x) dx,$$

which yields the right-hand side of (2.9).

We now extend (4.3) to the $L^p(w)$-norms with $w \in A_\infty$. Recall first several definitions.

We say that a weight w satisfies the $A_p, 1 < p < \infty$, condition if

$$\|w\|_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w\, dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\, dx\right)^{p-1},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

In the case $p = 1$ we say that w is an A_1 weight if there is a finite constant c such that $Mw(x) \le c\,w(x)$ a.e.

Denote $A_\infty = \bigcup_{1\le p<\infty} A_p$. There are several equivalent characterizations of A_∞ (see [7]). In particular, $w \in A_\infty$ if and only if there exist constants $c, \varepsilon > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and for any measurable subset $E \subset Q$,

$$\frac{w(E)}{w(Q)} \le c\left(\frac{|E|}{|Q|}\right)^{\varepsilon} \tag{4.5}$$

(we use a notation $w(E) = \int_E w(x)dx$). Also, the Lebesgue measure and the w-measure here can be reversed, namely, $w \in A_\infty$ if and only if there exist constants $c, \delta > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and for any measurable subset $E \subset Q$,

$$\frac{|E|}{|Q|} \le c\left(\frac{w(E)}{w(Q)}\right)^{\delta}. \tag{4.6}$$

Theorem 4.6. *Let $w \in A_\infty$. Then for any measurable f satisfying $f^*(+\infty) = 0$ and for all $p > 0$,*

$$\|f\|_{L^p(w)} \le c\|M^{\#}_{\lambda} f\|_{L^p(w)} \quad (0 < \lambda \le \lambda_n).$$

Proof. As in the previous proof, it is enough to prove the theorem for $p \ge 1$. Applying Theorem 3.1, we have

$$\begin{aligned}\|f - m_f(Q_0)\|_{L^p(Q_0,w)} &\le 2\|M^{\#}_{1/4} f\|_{L^p(w)} \\ &+ 2\|\sum_{k=1}^{\infty}\sum_j \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\chi_{Q^k_j}\|_{L^p(w)}.\end{aligned}$$

It suffices to show that the second term here is bounded by $\|M^{\#}_{\lambda} f\|_{L^p(w)}$. Then letting $Q_0 \to \mathbb{R}^n$ will complete the proof.

Set $E_k = \Omega_k \setminus \Omega_{k+1}$ (we use the notation from Theorem 3.1). Then

$$\sum_{k=1}^{\infty}\sum_j \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\chi_{Q^k_j} = \sum_{l=0}^{\infty}\sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\chi_{E_{k+l}\cap Q^k_j}$$

Therefore

$$\begin{aligned}&\|\sum_{k=1}^{\infty}\sum_j \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\chi_{Q^k_j}\|_{L^p(w)} \\ &\le \sum_{l=0}^{\infty}\|\sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}^k_j)\chi_{E_{k+l}\cap Q^k_j}\|_{L^p(w)}.\end{aligned}$$

Further,

$$\|\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}_j^k)\chi_{E_{k+l}\cap Q_j^k}\|_{L^p(w)}^p=\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}_j^k)^p w(E_{k+l}\cap Q_j^k)$$

By properties (ii)-(iv) of Theorem 3.1 we have

$$|E_{k+l}\cap Q_j^k|\le|\Omega_{k+l}\cap Q_j^k|\le 2^{-l}|Q_j^k|.$$

From this and from (4.5) and (4.6),

$$w(E_{k+l}\cap Q_j^k)\le c2^{-l\varepsilon}w(Q_j^k)\le c2^{-l\varepsilon}w(E_j^k).$$

Therefore,

$$\begin{aligned}\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}_j^k)^p w(E_{k+l}\cap Q_j^k) &\le c2^{-l\varepsilon}\sum_{k,j}\inf_{Q_j^k}(M^{\#}_{\lambda_n}f)^p w(E_j^k)\\ &\le c2^{-l\varepsilon}\|M^{\#}_{\lambda_n}f\|_{L^p(w)}^p.\end{aligned}$$

Combining the obtained estimates, we get

$$\begin{aligned}\sum_{l=0}^{\infty}\|\sum_{k,j}\omega_{\frac{1}{2^{n+2}}}(f;\widehat{Q}_j^k)\chi_{E_{k+l}\cap Q_j^k}\|_{L^p(w)} &\le c\sum_{l=0}^{\infty}2^{-l\varepsilon/p}\|M^{\#}_{\lambda_n}f\|_{L^p(w)}\\ &\le c\|M^{\#}_{\lambda_n}f\|_{L^p(w)},\end{aligned}$$

which completes the proof. □

A different proof of Theorem 4.6 can be found in [33].

5. Applications of Theorem 3.1 to sharp weighted inequalities

It is well known that most of the classical operators in harmonic analysis are bounded on $L^p(w)$ for $w\in A_p$. The question about the sharp $L^p(w)$ operator norm of a given operator in terms of $\|w\|_{A_p}$ has been a subject of intense research during the last decade.

The first sharp result in this direction was obtained by S. Buckley [3] who showed that the Hardy-Littlewood maximal operator M satisfies

$$\|M\|_{L^p(w)}\le c(n,p)\|w\|_{A_p}^{\frac{1}{p-1}}\quad(1<p<\infty),\tag{5.1}$$

and the exponent $\frac{1}{p-1}$ is sharp for any $p>1$.

It turned out that for singular integrals the question is much more complicated. Very recently this problem has been solved due to efforts of many mathematicians.

To be more precise, we have that any Calderón-Zygmund operator T satisfies

$$\|T\|_{L^p(w)}\le c(T,n,p)\|w\|_{A_p}^{\max(1,\frac{1}{p-1})}\quad(1<p<\infty).\tag{5.2}$$

Further, for a large class of Littlewood-Paley operators S,

$$\|S\|_{L^p(w)} \le c(T,n,p)\|w\|_{A_p}^{\max(\frac{1}{2},\frac{1}{p-1})} \quad (1<p<\infty). \tag{5.3}$$

The exponents in (5.2) in (5.3) are best possible for any $p > 1$.

Inequality (5.2) was first proved by S. Petermichl [45, 46] for the Hilbert and Riesz transforms and by S. Petermichl and S. Volberg [48] for the Ahlfors-Beurling operator. The proofs in [45, 46, 48] are based on the so-called Haar shift operators combined with the Bellman function technique.

A unified approach to the above results was given by M. Lacey, S. Petermichl and M. Reguera [32], who proved (5.2) for a general class of "dyadic shifts"; their proof employed a two-weight "Tb theorem" for such shifts due to F. Nazarov, S. Treil and S. Volberg [41]. The proof in [32] was essentially simplified by D. Cruz-Uribe, J. Martell and C. Pérez [12]; as the main tool they used Theorem 3.1.

Very soon after that, C. Pérez, S. Treil and A. Volberg [43] showed that for general Calderón-Zygmund operators the problem is reduced to proving the corresponding weak-type estimate. Based on this work, T. Hytönen et al. [26] solved (5.2) for Calderón-Zygmund operators with sufficiently smooth kernels. Inequality (5.2) in full generality was proved by T. Hytönen [25] (see also the subsequent work [27]).

The history of (5.3) in brief is the following. First it was proved for $1 < p \le 2$ for the dyadic square function by S. Hukovic, S. Treil and A. Volberg [24], and, independently by J. Wittwer [56]. Also, (5.3) in the case $1 < p \le 2$ was proved by J. Wittwer [57] for the continuous square function.

In [12], D. Cruz-Uribe, J. Martell and C. Pérez proved (5.3) for the dyadic square function for any $p > 1$; the key tool was again Theorem 3.1. Soon after that, the author [39] showed that (5.3) holds for the intrinsic square function, establishing by this (5.3) for essentially any square function.

Our goal is to describe below the key points from [12] and [39]. But first we mention the following very useful tool commonly used in these questions.

5.1. **Extrapolation.** The famous extrapolation theorem of J. Rubio de Francia [49] says that if a sublinear operator T is bounded on $L^{p_0}(w)$ for any $w \in A_{p_0}$, then it is bounded on $L^p(w)$ for any $p > 1$ for all $w \in A_p$. In [15], O. Dragičević et al. found the sharp dependence of the corresponding norms on $\|w\|_{A_p}$ in this theorem. Very recently, a different and much simplified proof of this result was given by J. Duoandikoetxea [17]. We shall use the following version from [17].

Theorem 5.1. *Assume that for some family of pairs of nonnegative functions* (f, g)*, for some* $p_0 \in [1, \infty)$*, and for all* $w \in A_{p_0}$ *we have*

$$\|g\|_{L^{p_0}(w)} \le cN(\|w\|_{A_{p_0}})\|f\|_{L^{p_0}(w)},$$

where N *is an increasing function and the constant* c *does not depend on* w*. Then for all* $1 < p < \infty$ *and all* $w \in A_p$ *we have*

$$\|g\|_{L^p(w)} \le c_1 N(c_2\|w\|_{A_p}^{\max(1,\frac{p_0-1}{p-1})})\|f\|_{L^p(w)}.$$

Observe that by extrapolation, it is enough to prove (5.2) for $p = 2$; similarly, it is enough to prove (5.3) for $p = 3$.

5.2. **The key result.** The main idea found in [12] can be described as follows.

Theorem 5.2. *Assume that for some family of pairs of nonnegative functions* (f, g) *with* $g^*(+\infty) = 0$ *we have*

$$\omega_\lambda(|g|^r; Q) \le c\Big(\frac{1}{|\gamma Q|}\int_{\gamma Q}|f|dx\Big)^r \tag{5.4}$$

for any dyadic cube $Q \subset \mathbb{R}^n$*, where* $r, \gamma \ge 1$*, and the constant* c *does not depend on* Q*. Then for any* $1 < p < \infty$*,*

$$\|g\|_{L^p(w)} \le c\|w\|_{A_p}^{\max(\frac{1}{r},\frac{1}{p-1})}\|f\|_{L^p(w)},$$

where the constant c *does not depend on* w*.*

Proof. Let Q_0 be a dyadic cube with respect to $\mathbb{R}^n$. Then any cube dyadic with respect to Q_0 will be also dyadic with respect to $\mathbb{R}^n$. Applying condition (5.4) along with Theorem 3.1, we get that for a.e. $x \in Q_0$,

$$||g(x)|^r - m_{|g|^r}(Q_0)|^{1/r} \le c\big(Mf(x) + \mathcal{A}_{3\gamma,r}f(x)\big),$$

where

$$\mathcal{A}_{\gamma,r}f(x) = \left(\sum_{j,k}\Big(\frac{1}{|\gamma Q_j^k|}\int_{\gamma Q_j^k}|f|dx\Big)^r\chi_{Q_j^k}(x)\right)^{1/r}$$

(we have used that $\gamma\widehat{Q} \subset 3\gamma Q$).

Therefore, the question is reduced to showing the corresponding bounds for Mf and $\mathcal{A}_{\gamma,r}f$. For M this is Buckley's estimate (5.1). For the sake of completeness we give here a short proof of (5.1) found recently in [38]. It is interesting to note that very similar ideas will be used in order to bound the operator $\mathcal{A}_{\gamma,r}$.

Denote by M_w^c the weighted centered maximal operator, that is,

$$M_w^c f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)| w(y) dy,$$

where the supremum is taken over all cubes Q centered at x. If $w \equiv 1$, we drop the subscript w.

Denote $A_p(Q) = w(Q)\sigma(3Q)^{p-1}/|Q|^p$, where $\sigma = w^{-\frac{1}{p-1}}$. Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q |f| &= A_p(Q)^{\frac{1}{p-1}} \left\{ \frac{|Q|}{w(Q)} \Big(\frac{1}{\sigma(3Q)} \int_Q |f| \Big)^{p-1} \right\}^{\frac{1}{p-1}} \\ &\leq 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} \left\{ \frac{1}{w(Q)} \int_Q M_\sigma^c (f\sigma^{-1})^{p-1} dx \right\}^{\frac{1}{p-1}}. \end{aligned}$$

From this and from the fact that $Mf(x) \leq 2^n M^c f(x)$ we get

$$Mf(x) \leq 2^n 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} M_w^c \big(M_\sigma^c (f\sigma^{-1})^{p-1} w^{-1} \big)(x)^{\frac{1}{p-1}}.$$

It is well known that by the Besicovitch covering theorem, $\|M_w^c\|_{L_w^{p'}}$ and $\|M_\sigma^c\|_{L_\sigma^p}$ are bounded uniformly in w. Therefore, from the previous estimate we get

$$\begin{aligned} \|Mf\|_{L^p(w)} &\leq 2^n 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} \|M_w\big(M_\sigma(f\sigma^{-1})^{p-1} w^{-1}\big)\|_{L^{p'}(w)}^{\frac{1}{p-1}} \\ &\leq c\|w\|_{A_p}^{\frac{1}{p-1}} \|M_\sigma(f\sigma^{-1})\|_{L^p(\sigma)} \leq c\|w\|_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}, \end{aligned}$$

which completes the proof of (5.1).

We turn now to showing that for any $1 < p < \infty$,

$$\|\mathcal{A}_{\gamma,r} f\|_{L^p(Q_0,w)} \leq c(p,\gamma,n) \|w\|_{A_p}^{\max(\frac{1}{r},\frac{1}{p-1})} \|f\|_{L^p(w)}.$$

By Theorem 5.1, it suffices to prove this estimate for $p = r+1$. By duality, this is equivalent to that for any $h \geq 0$ with $\|h\|_{L^{r+1}(w)} = 1$,

$$\begin{aligned} \int_{Q_0} (\mathcal{A}_{\gamma,r} f)^r hw\, dx &= \sum_{k,j} \Big(\frac{1}{|\gamma Q_j^k|} \int_{\gamma Q_j^k} |f| \Big)^r \int_{Q_j^k} hw \\ &\leq c\|w\|_{A_{r+1}} \|f\|_{L^{r+1}(w)}^r. \end{aligned}$$

Denote $A_{r+1}(Q)=\frac{w(Q)(w^{-1/r}(Q))^r}{|Q|^{r+1}}$. Then

$$\begin{aligned}&\Big(\frac{1}{|\gamma Q_j^k|}\int_{\gamma Q_j^k}|f|\Big)^r\int_{Q_j^k}hw\le cA_{r+1}(3\gamma Q_j^k)\\&\times\Big(\frac{1}{w^{-1/r}(3\gamma Q_j^k)}\int_{\gamma Q_j^k}|f|\Big)^r\Big(\frac{1}{w(3\gamma Q_j^k)}\int_{\gamma Q_j^k}hw\Big)|E_j^k|\\&\le c\|w\|_{A_{r+1}}\int_{E_j^k}M^c_{w^{-1/r}}(fw^{1/r})^rM^c_wh\,dx.\end{aligned}$$

Therefore (exactly as above we use the boundedness of the weighted centered maximal function),

$$\begin{aligned}&\sum_{k,j}\Big(\frac{1}{|\gamma Q_j^k|}\int_{\gamma Q_j^k}|f|\Big)^r\int_{Q_j^k}hw\\&\le c\|w\|_{A_{r+1}}\int_{\mathbb{R}^n}M^c_{w^{-1/r}}(fw^{1/r})^rM^c_wh\,dx\\&\le c\|w\|_{A_{r+1}}\|M^c_{w^{-1/r}}(fw^{1/r})\|^r_{L^{r+1}(w^{-1/r})}\|M^c_wh\|_{L^{r+1}(w)}\\&\le c\|w\|_{A_{r+1}}\|f\|^r_{L^{r+1}(w)},\end{aligned}$$

and we are done.

Combining the estimates for M and $\mathcal{A}_{\gamma,r}$, we get

$$\||g|^r-m_{|g|^r}(Q_0)|^{1/r}\|_{L^p(Q_0,w)}\le c\|w\|_{A_p}^{\max(\frac1r,\frac{1}{p-1})}\|f\|_{L^p(w)}. \tag{5.5}$$

Now, the finish of the proof is standard. Denote by $\mathbb{R}^n_i, 1\le i\le 2^n$, the n-dimensional quadrants in $\mathbb{R}^n$, that is, the sets $I^\pm\times I^\pm\times\cdots\times I^\pm$, where $I^+=[0,\infty)$ and $I^-=(-\infty,0)$. For each $i, 1\le i\le 2^n$, and for each $N>0$ let $Q_{N,i}$ be the dyadic cube adjacent to the origin of side length 2^N that is contained in $\mathbb{R}^n_i$. In (5.5) with $Q_0=Q_{N,i}$ we let $N\to\infty$. Observe that

$$m_{|g|^r}(Q_{N,i})\le g^*(|Q_{N,i}|/2)^r\to 0$$

as $N\to\infty$. Hence, by Fatou's lemma, we get

$$\|g\|_{L^p(\mathbb{R}^n_i,w)}\le c\|w\|_{A_p}^{\max(\frac1r,\frac{1}{p-1})}\|f\|_{L^p(w)}.$$

Summing over $1\le i\le 2^n$ completes the proof. □

Remark 5.3. The proof actually shows the following dependence of the constant on γ and r:

$$\|g\|_{L^p(w)}\le c\gamma^{\frac nr}\|w\|_{A_p}^{\max(\frac1r,\frac{1}{p-1})}\|f\|_{L^p(w)}.$$

Corollary 5.4. *Let T be a linear operator of weak type $(1,1)$ and such that $T(f\chi_{\mathbb{R}^n\setminus Q'})(x)$ is a constant on Q for any dyadic cube Q, where $Q'\subset\gamma Q$. Then for any $1<p<\infty$,*

$$\|Tf\|_{L^p(w)}\le c\|w\|_{A_p}^{\max(1,\frac{1}{p-1})}\|f\|_{L^p(w)},$$

where the constant c does not depend on w.

Proof. For any dyadic cube Q we have $Tf(x)=T(f\chi_{\gamma Q})(x)+c$. Therefore,

$$\omega_\lambda(Tf;Q)\le 2((Tf-c)\chi_Q)^*(\lambda|Q|)\le 2(T(f\chi_{Q'}))^*(\lambda|Q|)\le c|f|_{\gamma Q}.$$

It remains to apply Theorem 5.2 (we may assume that $f\in L^1$ and then $(Tf)^*(+\infty)=0$). □

Remark 5.5. In the next section we will define a general dyadic grid $\mathscr{D}$. It is easy to see that that Theorem 5.2 (and hence Corollary 5.4) remains true if we assume that the corresponding conditions hold for any $Q\in\mathscr{D}$ instead of $Q\in\mathcal{D}$.

In the next sections we consider applications of the obtained results to weighted estimated of various operators, and, in particular, to estimates (5.2) and (5.3).

6. Calderón-Zygmund and Haar shift operators

We start with a number of definitions.

Definition 6.1. A Calderón-Zygmund operator on $\mathbb{R}^n$ is an integral operator, bounded on $L^2(\mathbb{R}^n)$, and with kernel K satisfying the following conditions:

(i) $|K(x,y)|\le\frac{c}{|x-y|^n}$ for all $x\neq y$;
(ii) there exists $0<\alpha\le 1$ such that

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\le c\frac{|x-x'|^\alpha}{|x-y|^{n+\alpha}},$$

whenever $|x-x'|<|x-y|/2$.

Definition 6.2. By a general dyadic grid $\mathscr{D}$ we mean a collection of cubes with the following properties:

(i) for any $Q\in\mathscr{D}$ its sidelength $\ell(Q)$ is of the form $2^k, k\in\mathbb{Z}$;
(ii) $Q\cap R\in\{Q,R,\emptyset\}$ for any $Q,R\in\mathscr{D}$;
(iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

Definition 6.3. We say that h_Q is a Haar function on a cube $Q\subset\mathbb{R}^n$ if

(i) h_Q is a function supported on Q, and is constant on the children of Q;
(ii) $\int h_Q = 0$;

We say that h_Q is a generalized Haar function if it is a linear combination of a Haar function on Q and χ_Q (in other words, only condition (i) above is satisfied).

Definition 6.4. Let $m, k \in \mathbb{N}$. We say that $\mathbb{S}$ is a (generalized) Haar shift operator with parameters m, k if

$$\mathbb{S}f(x) = \mathbb{S}^{m,k}_{\mathscr{D}} f(x) = \sum_{Q\in\mathscr{D}} \sum_{\substack{Q',Q''\in\mathscr{D}, Q',Q''\subset Q\\ \ell(Q')=2^{-m}\ell(Q),\ell(Q'')=2^{-k}\ell(Q)}} \frac{\langle f, h_{Q'}^{Q''}\rangle}{|Q|} h_{Q''}^{Q'}(x),$$

where $h_{Q'}^{Q''}$ is a (generalized) Haar function on Q', and $h_{Q''}^{Q'}$ is one on Q'' such that

$$\|h_{Q'}^{Q''}\|_{L^\infty}\|h_{Q''}^{Q'}\|_{L^\infty} \le 1.$$

The number $\max(m, k)$ is called the complexity of $\mathbb{S}$.

It can be easily verified that any Haar shift operator is bounded on $L^2(\mathbb{R}^n)$. In the case of a generalized Haar shift operator its L^2 boundedness is required additionally by the definition.

The importance of the Haar shift operators comes from the following result, see T. Hytönen [25] and T. Hytönen et al. [27].

Theorem 6.5. *Let T be a Calderón-Zygmund operator which satisfies the standard estimates with $\alpha \in (0, 1]$. Then for all bounded and compactly supported functions f and g,*

$$\langle g, Tf\rangle = C(T, n)\mathbb{E}_{\mathscr{D}} \sum_{k,m=0}^{\infty} 2^{-(m+k)\alpha/2}\langle g, \mathbb{S}^{m,k}_{\mathscr{D}} f\rangle, \tag{6.1}$$

where $\mathbb{E}_{\mathscr{D}}$ is the expectation with respect to a probability measure on the space of all general dyadic grids.

Observe that for several classical singular integrals of the form

$$Tf(x) = P.V. f * K(x)$$

a simpler representation was obtained earlier which involves only the Haar shifts of bounded complexity (and hence, the finite sum on the right-hand side of (6.1)). The known examples include:

(i) $K(x) = \frac{1}{x}$ (the Hilbert transform, see S. Petermichl [44]);
(ii) $K(x) = \frac{x_j}{|x|^{n+1}}, n \ge 2$ (the Riesz transform, see S. Petermichl, S. Treil and A. Volberg [47]);

(iii) $K(z) = \frac{1}{z^2}$ (the Beurling transform, see O. Dragičević and A. Volberg [16]);

(iv) $K(x)$ is any odd, one-dimensional C^2 kernel satisfying

$$|K^{(i)}(x)| \le c|x|^{-1-i} \quad (i = 0, 1, 2),$$

see A. Vagharshakyan [52].

By extrapolation and by Theorem 6.5 we have that (5.2) would follow from

$$\|\mathbb{S}_{\mathscr{D}}^{m,k} f\|_{L^2(w)} \le c(n)\varphi(\max(m,k))\|w\|_{A_2}\|f\|_{L^2(w)} \tag{6.2}$$

with sufficiently good dependence on the complexity. First, (6.2) was obtained in [32] with the exponential growth of φ. It is easy to see that such a dependence cannot be combined with (6.1). However, it is enough in order to handle the above mentioned classical singular integrals. In [12], a simpler proof was given based on Theorem 3.1; this proof also yields the exponential dependence on the complexity. After that a better estimate (which is enough for applying Theorem 6.5) was obtained in [25]; in [27] it was improved until $\varphi(t) = t^2$.

We give below the proof of (6.2) with the exponential growth of φ, which in turn yields (5.2) for the classical singular integrals mentioned above.

Given a cube $Q \in \mathscr{D}$ and $l \in \mathbb{N}$ denote by $Q^{(l)}$ its l-fold parent, that is, the unique cube from $\mathscr{D}$ such that $|Q^{(l)}| = 2^{nl}|Q|$.

Theorem 6.6. *For any $w \in A_2$,*

$$\|\mathbb{S}_{\mathscr{D}}^{m,k} f\|_{L^2(w)} \le c(n)\xi 4^{n\xi}\|w\|_{A_2}\|f\|_{L^2(w)} \quad (\xi = \max(m,k)). \tag{6.3}$$

Proof. First, the Haar shift operator is of weak type $(1,1)$. This was proved in [32] with the exponential dependence on the complexity, and it was further improved until the linear dependence in [25].

Next, we claim that $\mathbb{S}_{\mathscr{D}}^{m,k}(f\chi_{\mathbb{R}^n\setminus Q_0^{(\xi)}})(x)$ is a constant on Q_0. Indeed, take an arbitrary cube $Q \in \mathscr{D}$, and let us consider

$$\sum_{\substack{Q',Q''\in\mathscr{D},Q',Q''\subset Q\\ \ell(Q')=2^{-m}\ell(Q),\ell(Q'')=2^{-k}\ell(Q)}} \frac{\langle f\chi_{\mathbb{R}^n\setminus Q_0^{(\xi)}}, h_{Q'}^{Q''}\rangle}{|Q|} h_{Q''}^{Q'}(x).$$

Since $h_{Q''}$ is supported on Q'' and $x \in Q_0$, we may assume that $Q'' \cap Q_0 \ne \emptyset$. Further, $Q'' \subset Q$ implies $Q \cap Q_0^{(\xi)} \ne \emptyset$. Similarly we have that $Q' \cap \mathbb{R}^n \setminus Q_0^{(\xi)} \ne \emptyset$, and $Q' \subset Q$ implies $Q \cap \mathbb{R}^n \setminus Q_0^{(\xi)} \ne \emptyset$. Combining this with the previous fact, we have that $Q_0^{(\xi)} \subset Q$. Therefore, $|Q_0| < 2^{-kn}|Q| = |Q''|$, and hence $Q_0 \subset Q''$. From this, $h_{Q''}^{Q'}(x)$ is a constant on Q_0 (by the definition of the Haar function), which proves the claim.

It remains to apply Corollary 5.4. We only remark that after applying the weak type $(1,1)$ with the linear dependence on ξ we get

$$(\mathbb{S}^{m,k}_{\mathscr{D}}(f\chi_{Q_0^{(\xi)}}))^*(\lambda|Q_0|)\le \frac{c\xi}{\lambda|Q_0|}\int_{Q_0^{(\xi)}}|f|\le \frac{c\xi 2^{n\xi}}{\lambda}|f|_{Q_0^{(\xi)}}.$$

Since $Q^{(\xi)}\subset 3\cdot 2^{\xi}Q$, we use Remark 5.3 with $\gamma=3\cdot 2^{\xi}$, which yields (6.3). □

An interesting open question is whether it is possible to change somehow the machinery used in Theorems 3.1 and 5.2 in order to improve the dependence on the complexity in (6.3) We emphasize again that a much better dependence was recently obtained in [27] but by means of a different argument.

7. Some other applications

In this section we mention briefly some other applications found in [12] which are also based on Theorem 5.2.

7.1. **The dyadic paraproduct.** The dyadic paraproduct π_b is defined by

$$\pi_b f(x)=\sum_{I\in\mathcal{D}} f_I\langle b,h_I\rangle h_I(x).$$

Here the sum is taken over all dyadic intervals from $\mathbb{R}$, $b\in BMO(\mathbb{R})$, and h_I is the classical Haar function

$$h_I(x)=|I|^{-1/2}(\chi_{I_-}(x)-\chi_{I_+}(x)),$$

where I_- and I_+ are the left and right halves of I, respectively. In [2], O. Beznosova proved that

$$\|\pi_b f\|_{L^p(w)}\le c\|b\|_{BMO}\|w\|_{A_p}^{\max(1,\frac{1}{p-1})}\|f\|_{L^p(w)}\quad(1<p<\infty). \tag{7.1}$$

The proof in [2] was based on the Bellman function technique.

Now we observe that (7.1) follows from Corollary 5.4 since:

(i) $\pi_b(f\chi_{\mathbb{R}\setminus I_0})(x)$ is a constant on I_0 for any dyadic interval I_0 (this follows easily from the definition of π_b and from the basic property of dyadic intervals);
(ii) $\|\pi_b f\|_{L^{1,\infty}}\le c\|b\|_{BMO}\|f\|_{L^1}$ (the proof of this fact can be found in [42]).

7.2. **The dyadic square function.** The dyadic square function is defined by

$$S_d f(x) = \left(\sum_{Q\in\mathcal{D}} (f_Q - f_{\widehat{Q}})^2\chi_Q(x)\right)^{1/2}.$$

As we mentioned in Section 5, for this operator the sharp $L^2(w)$ bound

$$\|S_d f\|_{L^2(w)} \le c\|w\|_{A_2}\|f\|_{L^2(w)}$$

was obtained independently in [24] and [56]. By extrapolation, this yields also the sharp bounds for $1 < p \le 2$. It was conjectured in [37] for a general class of Littlewood-Paley operators S that they satisfy (5.3). In [12] this conjecture was proved for S_d for any $1 < p < \infty$. The proof is very similar to the ones given above.

Lemma 7.1. *For any dyadic cube Q_0,*

$$\omega_\lambda((S_d f)^2; Q_0) \le c(|f|_{Q_0})^2.$$

From this lemma and from Theorem 5.2 we get (5.3) for S_d. The proof of Lemma 7.1 is based on the same idea as the proof of Corollary 5.4. For $x \in Q_0$ we have

$$S_d f(x)^2 = \sum_{Q\in\mathcal{D}, Q\subsetneq Q_0} (f_Q - f_{\widehat{Q}})^2\chi_Q(x) + \sum_{Q\in\mathcal{D}, Q_0\subset Q} (f_Q - f_{\widehat{Q}})^2.$$

The second term is a constant, while the first term is controlled by $S_d(f\chi_{Q_0})(x)^2$. Hence, using the weak type $(1,1)$ of S_d (see, e.g.,[55]), we get

$$\begin{aligned}\omega_\lambda((S_d f)^2; Q_0) &\le 2\big((S_d(f\chi_{Q_0}))^2\big)^*\big(\lambda|Q_0|\big)\\ &= 2\big(S_d(f\chi_{Q_0})\big)^*\big(\lambda|Q_0|\big)^2 \le c(|f|_{Q_0})^2,\end{aligned}$$

and hence the proof is complete.

7.3. **The vector-valued maximal operator.** Given a vector-valued function $f = \{f_i\}$, and $q, 1 < q < 1$, the vector-valued maximal operator $\overline{M}_q$ is defined by

$$\overline{M}_q f(x) = \left(\sum_{i=1}^{\infty} Mf_i(x)^q\right)^{1/q}.$$

It was proved in [12] that for any $1 < p, q < \infty$,

$$\|\overline{M}_q f\|_{L^p(w)} \le c\|w\|_{A_p}^{\max(\frac{1}{q},\frac{1}{p-1})}\Big(\int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p w\,dx\Big)^{1/p}. \tag{7.2}$$

By the well known idea found by C. Fefferman and E.M. Stein [19], it suffices to prove the result for the dyadic vector-valued maximal function $\overline{M}_q^d f(x)$. Next, by Theorem 5.2, inequality (7.2) will follow from

$$\omega_\lambda((\overline{M}_q^d f)^q; Q_0) \le c\Big(\frac{1}{|Q_0|}\int_{Q_0}\|f(x)\|_{\ell^q}dx\Big)^q \quad (Q_0\in\mathcal{D}). \tag{7.3}$$

The proof of (7.3) follows similar lines as the previous proofs. Namely, setting

$$C^q=\sum_{i=1}^\infty\Big(\sup_{Q\in\mathcal{D},Q_0\subset Q}|f_i|_Q\Big)^q$$

for $x\in Q_0$ we get

$$0\le(\overline{M}_q^d f(x))^q - C^q \le (\overline{M}_q^d(f\chi_{Q_0})(x))^q.$$

It remains to use the weak type $(1,1)$ of $\overline{M}_q$ (this was proved in [19]).

8. Littlewood-Paley operators

In the previous section inequality (5.3) was proved for the dyadic square function S_d. The crucial point in the proof was the local nature of S_d, that is, the fact that for any dyadic cube Q, $S_d f(x)$ on Q is essentially $S_d(f\chi_Q)(x)$ plus a constant. More general square functions do not satisfy such a nice property. However, as it was shown in [40], a number of tricks (based mainly on the works of M. Wilson) allow to reduce the general problem to a local situation mentioned above. As a result, we obtain (5.3) for a large class of Littlewood-Paley operators.

8.1. **The classical S and g functions.** Let $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times\mathbb{R}_+$ and $\Gamma_\beta(x)=\{(y,t)\in\mathbb{R}^{n+1}_+ : |y-x|<\beta t\}$. Here and below we drop the subscript β if $\beta=1$. Set $\varphi_t(x)=t^{-n}\varphi(x/t)$.

If $u(x,t)=P_t*f(x)$ is the Poisson integral of f, the Lusin area integral S_β and the Littlewood-Paley g-function are defined respectively by

$$S_\beta(f)(x)=\left(\int_{\Gamma_\beta(x)}|\nabla u(y,t)|^2\frac{dydt}{t^{n-1}}\right)^{1/2}$$

and

$$g(f)(x)=\left(\int_0^\infty t|\nabla u(x,t)|^2dt\right)^{1/2}.$$

One can define similar operators by means of general but compactly supported kernel. Let $\psi\in C^\infty(\mathbb{R}^n)$ be radial, supported in $\{|x|\le 1\}$, and $\int \psi = 0$. The continuous square functions $S_{\psi,\beta}$ and g_ψ are defined by

$$S_{\psi,\beta}(f)(x) = \left(\int_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

8.2. **The intrinsic square function.** In [54] (see also [55, p. 103]), M. Wilson introduced a new square function called the intrinsic square function. It has a number of remarkable properties. This function is independent of the aperture and of any particular kernel ψ. On one hand, it dominates pointwise all the above defined square function. On the other hand, it has the same mapping properties. Finally, perhaps the most important property for us is that the intrinsic square function is "local" in a sense. This fact makes applicable the above described machinery to essentially any square function.

For $0 < \alpha \leq 1$, let $\mathcal{C}_\alpha$ be the family of functions supported in $\{x : |x| \leq 1\}$, satisfying $\int \psi = 0$, and such that for all x and x',

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha.$$

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+$, we define

$$A_\alpha(f)(y,t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.$$

The intrinsic square function is defined by

$$G_{\beta,\alpha}(f)(x) = \left(\int_{\Gamma_\beta(x)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

If $\beta = 1$, set $G_{1,\alpha}(f) = G_\alpha(f)$.

We mention several properties of $G_\alpha(f)$ (for the proofs we refer to [54] and [55, Ch. 6]). First of all, it is of weak type $(1,1)$:

$$|\{x \in \mathbb{R}^n : G_\alpha(f)(x) > \lambda\}| \leq \frac{c(n,\alpha)}{\lambda} \int_{\mathbb{R}^n} |f|\, dx. \tag{8.1}$$

As we have already seen, this fact is crucial for applications.

Second, if $\beta \geq 1$, then for all $x \in \mathbb{R}^n$,

$$G_{\beta,\alpha}(f)(x) \leq c(\alpha,\beta,n) G_\alpha(f)(x). \tag{8.2}$$

This is an interesting property since in general only an indirect control of the square function with bigger aperture by the one with smaller aperture is possible (see [8]).

Third, if S is anyone of the Littlewood-Paley operators defined above, then

$$S(f)(x) \le cG_\alpha(f)(x), \tag{8.3}$$

where the constant c is independent of f and x.

8.3. **A variant of the intrinsic square function.** In order to employ the dyadic analysis (crucial for all the above considerations), we consider the following operator equivalent to G_α.

Given a cube $Q \subset \mathbb{R}^n$, set

$$T(Q) = \{(y,t) \in \mathbb{R}^{n+1}_+ : y \in Q, \ell(Q)/2 \le t < \ell(Q)\}.$$

Denote $\gamma_Q(f)^2 = \int_{T(Q)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}}$ and let

$$\widetilde{G}_\alpha(f)(x)^2 = \sum_{Q\in\mathcal{D}} \gamma_Q(f)^2 \chi_{3Q}(x).$$

Lemma 8.1. *For any $x \in \mathbb{R}^n$,*

$$G_\alpha(f)(x) \le \widetilde{G}_\alpha(f)(x) \le c(\alpha,n)G_\alpha(f)(x). \tag{8.4}$$

Proof. For any $x \notin 3Q$ we have $\Gamma(x)\cap T(Q) = \emptyset$, and hence

$$\int_{\Gamma(x)\cap T(Q)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} \le \gamma_Q(f)^2\chi_{3Q}(x).$$

Therefore,

$$\begin{aligned} G_\alpha(f)(x)^2 &= \int_{\Gamma(x)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} \\ &= \sum_{Q\in\mathcal{D}} \int_{\Gamma(x)\cap T(Q)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} \le \widetilde{G}_\alpha(f)(x)^2. \end{aligned}$$

On the other hand, if $x \in 3Q$ and $(y,t) \in T(Q)$, then $|x-y| \le 2\sqrt{n}\ell(Q) \le 4\sqrt{n}t$. Thus,

$$\begin{aligned} &\widetilde{G}_\alpha(f)(x)^2 = \sum_{Q\in\mathcal{D}} \gamma_Q(f)^2\chi_{3Q}(x) \\ &\le \sum_{Q\in\mathcal{D}} \int_{T(Q)\cap\Gamma_{4\sqrt{n}}(x)} \big(A_\alpha(f)(y,t)\big)^2 \frac{dydt}{t^{n+1}} = G_{4\sqrt{n},\alpha}(f)(x)^2. \end{aligned}$$

Combining this with (8.2), we get the right-hand side of (8.4). $\square$

8.4. **Some tricks with dyadic cubes.** Our goal is to apply Theorem 5.2 to $\widetilde{G}_\alpha(f)$, that is, we are going to estimate the local mean oscillation of this function on any dyadic cube Q_0. Due to the definition of $\widetilde{G}_\alpha(f)$, instead of the family of dyadic cubes (as in the definition of the dyadic square function) we have to deal with the family $\{3Q : Q \in \mathcal{D}\}$. In order to understand what is the interaction of Q_0 with any cube from the latter family, we will need several tricks with dyadic cubes.

The following result can be found in [53, Lemma 2.1] or in [55, p. 91]. We give a slightly different proof here.

Lemma 8.2. *There exist disjoint families $\mathcal{D}_1, \dots \mathcal{D}_{3^n}$ of dyadic cubes such that $\mathcal{D} = \cup_{k=1}^{3^n} \mathcal{D}_k$, and, for every k, if Q_1, Q_2 are in $\mathcal{D}_k$, then $3Q_1$ and $3Q_2$ are either disjoint or one is contained in the other.*

Proof. It suffices to prove the lemma in the one-dimensional case. Indeed, if $\mathcal{I}$ is the set of all dyadic intervals in $\mathbb{R}$ and $\mathcal{I} = \cup_{j=1}^{3} \mathcal{I}_j$ is the representation from Lemma 8.2 in the case $n = 1$, then the required families in $\mathbb{R}^n$ are of the form

$$\mathcal{D}_k = \Big\{ \prod_{m=1}^{n} I_m : I_m \in \mathcal{I}_{\alpha_i}, \alpha_i \in \{1, 2, 3\} \Big\} \quad (k = 1, \dots, 3^n).$$

Suppose that $n = 1$. Denote by $\mathcal{D}^{(l)}$ the family of dyadic intervals with length $1/2^l, l \in \mathbb{Z}$. Fix $l_0 \in \mathbb{Z}$. We distribute the intervals from $\mathcal{D}^{(l_0)}$ into the families $\mathcal{I}_j, 1 \le j \le 3$, by the following way: for $i \in \mathbb{Z}$,

$$\Big(\frac{3i}{2^{l_0}}, \frac{3i+1}{2^{l_0}}\Big) \in \mathcal{I}_1, \quad \Big(\frac{3i+1}{2^{l_0}}, \frac{3i+2}{2^{l_0}}\Big) \in \mathcal{I}_2, \quad \Big(\frac{3i+2}{2^{l_0}}, \frac{3i+3}{2^{l_0}}\Big) \in \mathcal{I}_3.$$

The intervals from any $\mathcal{D}^{(l)}$ will be distributed in the same way. In order to do that we have to choose only one interval from $\mathcal{D}^{(l)}$ and to determine the correct family $\mathcal{I}_j$ for this interval; all other intervals from $\mathcal{D}^{(l)}$ will be distributed automatically.

Below we show how to choose the corresponding intervals from $\mathcal{D}^{(l_0-1)}$ and $\mathcal{D}^{(l_0+1)}$. Then by induction we obtain the distribution from any other family $\mathcal{D}^{(l)}$.

Take any interval from $\mathcal{D}^{(l_0-1)}$ such that its left half (which is from $\mathcal{D}^{(l_0)}$) belongs to $\mathcal{I}_1$; put such an interval to the family $\mathcal{I}_3$. Similarly, take any interval from $\mathcal{D}^{(l_0+1)}$ such that it is a left half of some interval from $\mathcal{I}_3$ and put it to the family $\mathcal{I}_1$.

Let I, J be two arbitrary intervals from $\mathcal{I}_i$. By our construction, it is easy to see that the statement of the lemma holds for them if one of them lies in $\mathcal{D}^{(l)}$ and another one is in one of the classes $\mathcal{D}^{(l)}$ or $\mathcal{D}^{(l\pm1)}$. But then by induction we get the same statement if one of them is in $\mathcal{D}^{(l)}$ and another one is in $\mathcal{D}^{(l\pm k)}$. ∎

Lemma 8.3. *For any cube $Q \in \mathcal{D}$ and for each $k = 1, \dots, 3^n$ there is a cube $Q_k \in \mathcal{D}_k$ such that $Q \subset 3Q_k \subset 5Q$.*

Proof. Let us consider first the one-dimensional case. Assume that $\mathcal{I} = \cup_{j=1}^3 \mathcal{I}_j$ is the representation from Lemma 8.2.

Take an arbitrary dyadic interval $J = (\frac{j}{2^k}, \frac{j+1}{2^k})$. Set $J_1 = J$. Consider the dyadic intervals $J_2 = (\frac{j-1}{2^k}, \frac{j}{2^k})$ and $J_3 = (\frac{j+1}{2^k}, \frac{j+2}{2^k})$. By the above construction, the intervals J_l lie in the different families $\mathcal{I}_j$. Also, $J \subset 3J_l \subset 5J$ for $l = 1, 2, 3$.

Consider now the multidimensional case. Take an arbitrary cube $Q \in \mathcal{D}$. Then $Q = \prod_{m=1}^n I_m$, where $I_m \in \mathcal{I}$ and $\ell_{I_m} = h$ for each m. Fix $\alpha_i \in \{1, 2, 3\}$. We have already proved that there exists $\widetilde{I}_m \in \mathcal{I}_{\alpha_i}$ such that $I_m \subset 3\widetilde{I}_m \subset 5I_m$. Observe also that, by the one-dimensional construction, $\ell_{\widetilde{I}_m} = \ell_{I_m} = h$. Therefore, setting $Q_k = \prod_{m=1}^n \widetilde{I}_m$, we obtain the required cube from $\mathcal{D}_k$. □

8.5. **A local mean oscillation estimate of $\widetilde{G}_\alpha$.** By Theorem 5.2, estimate (5.3) for $\widetilde{G}_\alpha$ (and hence, for any square function from Section 8.1) will follow from the following lemma.

Lemma 8.4. *For any cube $Q \in \mathcal{D}$,*

$$\omega_\lambda(\widetilde{G}_\alpha(f)^2; Q) \le c(n, \alpha, \lambda) \left(\frac{1}{|15Q|} \int_{15Q} |f| dx \right)^2.$$

Proof. Applying Lemma 8.2, we can write

$$\widetilde{G}_\alpha(f)(x)^2 = \sum_{k=1}^{3^n} \sum_{Q \in \mathcal{D}_k} \gamma_Q(f)^2 \chi_{3Q}(x) \equiv \sum_{k=1}^{3^n} \widetilde{G}_{\alpha,k}(f)(x)^2.$$

Hence,

$$\omega_\lambda(\widetilde{G}_\alpha(f)^2; Q) \le 2 \sum_{k=1}^{3^n} \omega_{\lambda/3^n}(\widetilde{G}_{\alpha,k}(f)^2; Q)$$

(we have used here the standard property of the rearrangement saying that $(f+g)^*(t_1 + t_2) \le f^*(t_1) + g^*(t_2)$).

By Lemma 8.3, for each $k = 1, \dots, 3^n$ there exists a cube $Q_k \in \mathcal{D}_k$ such that $Q \subset 3Q_k \subset 5Q$. Hence,

$$\begin{aligned} &\inf_{c \in \mathbb{R}} \Big(\big(\widetilde{G}_{\alpha,k}(f)^2 - c\big) \chi_Q \Big)^* \big(\lambda |Q| / 3^n\big) \\ &\le \inf_{c \in \mathbb{R}} \Big(\big(\widetilde{G}_{\alpha,k}(f)^2 - c\big) \chi_{3Q_k} \Big)^* \big(\lambda |Q| / 3^n\big). \end{aligned}$$

Using the main property of cubes from the family $\mathcal{D}_k$ (expressed in Lemma 8.2), for any $x \in 3Q_k$ we have

$$\widetilde{G}_{\alpha,k}(f)(x)^2 = \sum_{Q\in\mathcal{D}_k:3Q\subset 3Q_k} \gamma_Q(f)^2\chi_{3Q}(x) + \sum_{Q\in\mathcal{D}_k:3Q_k\subset 3Q} \gamma_Q(f)^2. \tag{8.5}$$

Arguing as in the proof of Lemma 8.1, we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_k:3Q\subset 3Q_k} \gamma_Q(f)^2\chi_{3Q}(x)\\
&\le \sum_{Q\in\mathcal{D}_k:3Q\subset 3Q_k} \int_{T(Q)\cap\Gamma_{4\sqrt{n}}(x)} \big(A_\alpha(f)(y,t)\big)^2\frac{dydt}{t^{n+1}}\\
&\le \int_{\widehat{T}(3Q_k)\cap\Gamma_{4\sqrt{n}}(x)} \big(A_\alpha(f)(y,t)\big)^2\frac{dydt}{t^{n+1}},
\end{aligned}$$

where $\widehat{T}(3Q_k) = \{(y,t) : y \in 3Q_k, 0 < t \le \ell(3Q_k)\}$. For any φ supported in $\{x : |x| \le 1\}$ and for $(y,t) \in \widehat{T}(3Q_k)$ we have

$$f * \varphi_t(y) = (f\chi_{9Q_k}) * \varphi_t(y).$$

Therefore,

$$\int_{\widehat{T}(3Q_k)\cap\Gamma_{4\sqrt{n}}(x)} \big(A_\alpha(f)(y,t)\big)^2\frac{dydt}{t^{n+1}} \le G_{4\sqrt{n},\alpha}(f\chi_{9Q_k})(x)^2.$$

Combining the latter estimates with (8.5) and setting

$$c = \sum_{Q\in\mathcal{D}_k:3Q_k\subset 3Q} \gamma_Q(f)^2,$$

we get

$$0 \le \widetilde{G}_{\alpha,k}(f)(x)^2 - c \le G_{4\sqrt{n},\alpha}(f\chi_{9Q_k})(x)^2 \quad (x \in 3Q_k).$$

From this, by (8.1) and (8.2) (we use also that $3Q_k \subset 5Q$ implies $9Q_k \subset 15Q$),

$$\begin{aligned}
&\inf_{c\in\mathbb{R}} \Big(\big(\widetilde{G}_{\alpha,k}(f)^2 - c\big)\chi_{3Q_k}\Big)^*(\lambda|Q|/3^n)\\
&\le c(n,\alpha)\big(G_\alpha(f\chi_{9Q_k})\big)^*(\lambda|Q|/3^n)^2\\
&\le c\left(\frac{3^n}{\lambda|Q|}\int_{9Q_k}|f|\right)^2 \le c\left(\frac{3^n}{\lambda|Q|}\int_{15Q}|f|\right)^2,
\end{aligned}$$

which completes the proof. □

The proof of the above lemma shows that the operator $\widetilde{G}_\alpha$ is "local" similarly to the dyadic square function. This follows from the fact that the intrinsic square function is defined by means of the uniformly compactly supported kernels. For example, the analogue of Lemma 8.4 is not true for the Lusin area integral. On the other hand, we have (5.3) for the Lusin area integral since, by (8.3), it is pointwise dominated by G_α. So, we have here a very interesting phenomenon that the "local" operator dominates the "non-local" one. An explanation of this phenomenon is in [55, pp. 114-118].

9. Applications to two-weight inequalities

In this section we mention briefly another application of Theorem 3.1, namely, the application to sharp two-weighted inequalities formulated in terms of Muckenhoupt-type conditions with the help of Orlicz bumps.

Suppose that T is a Calderón-Zygmund operator. A difficult and long-standing open problem in harmonic analysis is to characterize a pair of weights (u, v) yielding the two-weight inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p u\,dx \le c\int_{\mathbb{R}^n} |f(x)|^p v\,dx. \tag{9.1}$$

We consider sufficient Muckenhoupt-type conditions for (9.1). For a detailed history of such estimates we refer to [11, 12].

Given a Young function Φ, the mean Luxemburg norm of f on a cube Q is defined by

$$\|f\|_{\Phi,Q} = \inf\left\{\alpha > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f|}{\alpha}\right)dy \le 1\right\}.$$

If $\Phi(t) = t^p$, then we denote $\|f\|_{\Phi,Q} = \|f\|_{p,Q}$.

Given $p, 1 < p < \infty$, a Young function Φ satisfies the B_p condition if for some $c > 0$,

$$\int_c^\infty \frac{\Phi(t)}{t^p}\frac{dt}{t} < \infty.$$

Given a Young function Φ, there exists an associate Young function $\bar\Phi$ such that $t \le \Phi^{-1}(t)\bar\Phi^{-1}(t) \le 2t$.

In [13], D. Cruz-Uribe and C. Pérez conjectured the following.

Conjecture 9.1. Let A and B be two Young functions such that

$$\bar A \in B_{p'} \quad\text{and}\quad \bar B \in B_p. \tag{9.2}$$

If

$$\sup_Q \|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{B,Q} < \infty,$$

then (9.1) holds.

In the same work [13], Conjecture 9.1 was proved in the particular case when B is as in (9.2) and $A(t) = t^{pr}$ for some $r > 1$. In [11], this result was improved as follows: if $p > n$, B satisfies (9.2) and $A(t) = t^p[\log(e+t)]^{p-1+\delta}, \delta > 0$, then (9.1) holds. If $1 < p \le n$, an analogous result was proved in [11] but with larger Orlicz bump $A(t) = t^p[\log(e+t)]^{2p-1+\delta}$.

In [39] it was shown that Theorem 3.1 along with the local mean oscillation estimate

$$\omega_\lambda(Tf,Q) \le c\sum_{m=0}^{\infty}\frac{1}{2^m}\frac{1}{|2^mQ|}\int_{2^mQ}|f|$$

proves Conjecture 9.1 for $p > n$ (for example, for the Hilbert transform this shows that Conjecture 9.1 is true for all $p > 1$). In the case $n \ge 2$, the standard duality argument shows that Conjecture 9.1 holds in the case $p < n'$ as well. In the case when $n' \le p \le n$ the problem remains open, in general.

On the other hand, it was shown in [12] that the "Haar shift operator" approach combined with Theorem 3.1 proves Conjecture 9.1 for any classical singular integral mentioned after Theorem 6.5.

Consider a similar two-weighted problem for Littlewood-Paley operators:

$$\int_{\mathbb{R}^n}|Sf(x)|^p u\,dx \le c\int_{\mathbb{R}^n}|f(x)|^p v\,dx. \tag{9.3}$$

In [12], it was shown that for the dyadic square function, if $1 < p \le 2$ and $\bar{B} \in B_p$, then the condition

$$\sup_Q \|u^{1/p}\|_{p,Q}\|v^{-1/p}\|_{B,Q} < \infty$$

is sufficient for (9.3); if $2 < p < \infty$ and $\bar{A} \in B_{(p/2)'}, \bar{B} \in B_p$, then the condition

$$\sup_Q \|u^{2/p}\|_{A,Q}^{1/2}\|v^{-1/p}\|_{B,Q} < \infty$$

is sufficient for (9.3). The proof is based on Theorem 3.1 along with Lemma 7.1. By the arguments from Section 8, we have that the same conditions are sufficient for (9.3) with the intrinsic square function, and hence for S and g functions defined in Section 8.1.

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