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Ten Fantastic Facts on Bruhat Order

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Bruhat Order on Coxeter Groups

Coxeter Groups. generators : $s_1, s_2, \ldots s_n$
relations : $s_i^2 = 1$ and $(s_i s_j)^{m(i,j)} = 1$

Coxeter Graph. $V = \{1, \ldots, n\}$, $E = \{(i,j) : m(i,j) \geq 3\}$.

Define. If $w \in W$= Coxeter Group,

- $w = s_{i_1} s_{i_2} \ldots s_{i_p}$ is a *reduced expression* if p is minimal.
- $l(w)$ = *length* of w =p.

Example. S_n = Permutations generated by $s_i = (i \leftrightarrow i+1)$, $i < n$, with relations

$$s_i s_i = 1$$
$$(s_i s_j)^2 = 1 \text{ if } |i - j| > 1$$
$$(s_i s_{i+1})^3 = 1$$

$w = 4213 = s_1 s_3 s_2 s_1$ and $l(w) = 4$

Other Examples. Weyl groups and dihedral groups.

Bruhat Order on Coxeter Groups

Natural Partial Order on W.

$v \leq w$ if *any* reduced expression for w contains a subexpression which is a reduced expression for v.

Example. $s_1 s_3 s_2 s_1 > s_3 s_1 > s_1$

Chevalley-Bruhat Order on Coxeter Groups

Natural Partial Order on W.

$v \leq w$ if *any* reduced expression for w contains a subexpression which is a reduced expression for v.

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Ehresmann-Chevalley-Bruhat Order on Coxeter Groups

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Bruhat-et.al Order on Coxeter Groups

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Bruhat-et.al Order on Coxeter Groups

- $v \leq w$ if *any* reduced expression for w contains a subexpression which is a reduced expression for v.

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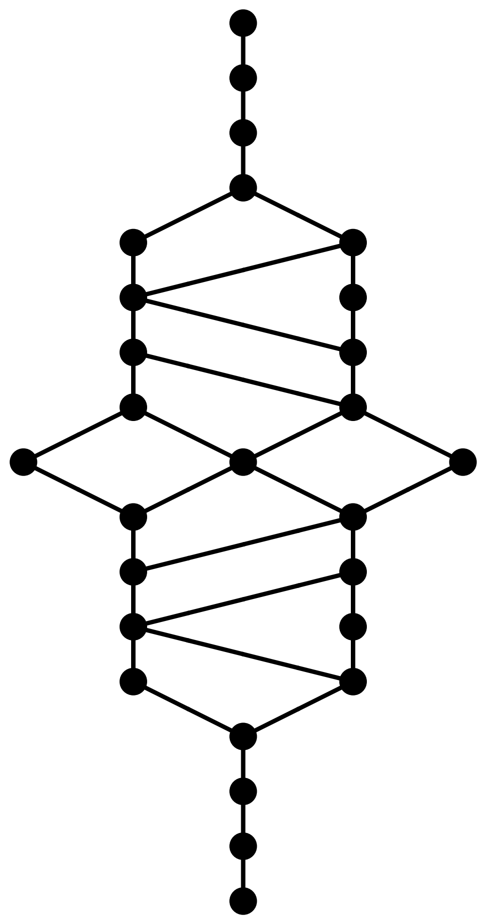
- Covering relations: w covers $v \iff w = s_{i_1} s_{i_2} \ldots s_{i_p}$ (reduced) and there exists j such that $v = s_{i_1} \ldots \widehat{s_{i_j}} \ldots s_{i_p}$ (reduced).

- Covering relations: w covers $v \iff w = vt$ and $l(w) = l(v) + 1$ where $t \in \{u s_i u^{-1} : u \in W\}$= *Reflections in W*.



Quotient E_6 modulo S_6

Fact 1: Bruhat Order Characterizes Inclusions of Schubert Varieties

- *Bruhat Decomposition*: $G = GL_n = \bigcup_{w \in S_n} BwB$
- *Flag Manifold*: G/B a complex projective smooth variety for any semisimple or Kac-Moody group G and Borel subgroup B
- *Schubert Cells*: BwB/B
- *Schubert Varieties*: $\overline{BwB/B} = X(w)$

Chevalley. (ca. 1958) $X(v) \subset X(w)$ if and only if $v \leq w$ *i. e.*

$$\overline{BwB/B} = \bigcup_{v \leq w} BvB/B$$

$\Longrightarrow$ The *Poincaré polynomial* for $H^*(X(w))$ is $P_w(t^2) = \sum_{v \leq w} t^{2l(v)}$

Fact 2: Contains Young's Lattice

- *Grassmannian Manifold*: $\{k$-dimensional subspaces of $\mathbb{C}^n\} = GL_n/P$ for P=maximal parabolic subgroup.
- *Schubert Cells*: BwB/P indexed by elements of
$$W^J = W/\langle s_i : i \in J\rangle$$
- *Schubert Varieties*: $X(w) = \overline{BwB/P} = \bigcup_{w \geq v \in W^J} BvB/P$.
- Elements of W^J can be identified with partitions inside a box, and the induced order is equivalent to containment of partitions.

Fact 3: Nicest Possible Möbius Function

Möbius Function on a Poset: unique function $\mu : \{x < y\} \to \mathbb{Z}$ such that

$$\sum_{x \leq y \leq z} \mu(x, y) = \begin{cases} 1 & x = z \\ 0 & x \neq z. \end{cases}$$

Theorem. (Verma, 1971) $\mu(x, y) = (-1)^{l(y)-l(x)}$ if $x \leq y$.

Theorem. (Deodhar, 1977) $\mu(x, y)^J = \begin{cases} (-1)^{l(y)-l(x)} & [x, y]^J = [x, y] \\ 0 & \text{otherwise} \end{cases}.$

Apply Möbius Inversion to

- Kazhdan-Lusztig polynomials.
- Kostant polynomials
- Any family of polynomials depending on Bruhat order.

Fact 4: Beautiful Rank Generating Functions

rank generating function: $W(t) = \sum_{u \in W} t^{l(u)} = \sum_{k \geq 0} a_k t^k$

Computing $W(t)$. for W = finite reflection group

- $W(t) = \prod (1 + t + t^2 + \cdots + t^{e_i})$ (Chevalley)
- $W(t) = \prod_{\alpha \in R^+} \frac{t^{\mathrm{ht}(\alpha)+1} - 1}{t^{\mathrm{ht}(\alpha)-1}}$ (Kostant '59, Macdonald '72)

Here, $e_i's$ = exponents of W, R^+=positive roots associated to W and $s_1, \ldots, s_n$, $\mathrm{ht}(\alpha) = k$ if $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$ (simple roots).

Fact 4: Beautiful Rank Generating Functions

- *Carrell-Peterson, 1994*: If $X(w)$ is smooth

$$P_{[\hat{0},w]}(t) = \sum_{v \leq w} t^{l(v)} = \prod_{\beta \in R_+ \sigma_\beta \leq w} \frac{t^{\mathrm{ht}(\beta)+1} - 1}{t^{\mathrm{ht}(\beta)} - 1}$$

- *Gasharov*: For $w \in S_n$, if $X(w)$ is rationally smooth

$$P_{[\hat{0},w]}(t) = \prod (1 + t + t^2 + \cdots + t^{d_i})$$

  for some set of d_i's.

- In 2001, Billey and Postnikov gave similar factorizations for all rationally smooth Schubert varieties of semisimple Lie groups.

Fact 5: Symmetric Interval $[\hat{0}, w] \implies$ $X(w)$ is Rationally Smooth

Definition. A variety X of dimension d is *rationally smooth* if for all $x \in X$,

$$H^i(X, X \setminus \{x\}, \mathrm{Q}) = \begin{cases} 0 & i \neq 2d \\ \mathbb{Q} & i = 2d. \end{cases}$$

Theorem. (Kazhdan-Lusztig '79) $X(w)$ is rationally smooth if and only if the Kazhdan-Lusztig polynomials $P_{v,w} = 1$ for all $v \leq w$.

Theorem. (Carrell-Peterson '94) X_w is rationally smooth if and only if $[\hat{0}, w]$ is rank symmetric.

Fact 5: Symmetric Interval $[\hat{0}, w] \implies$ $X(w)$ is Rationally Smooth

Fact 6: $[x, y]$ Determines the Composition Series for Verma Modules

- $\mathfrak{g}$ = complex semisimple Lie algebra
- $\mathfrak{h}$ = Cartan subalgebra
- λ = integral weight in $\mathfrak{h}^*$
- $M(\lambda)$ = Verma module with highest weight λ
- $L(\lambda)$ = unique irreducible quotient of $M(\lambda)$
- W = Weyl group corresponding to $\mathfrak{g}$ and $\mathfrak{h}$

Fact. $\{L(\lambda)\}_{\lambda \in \mathfrak{h}^*}$ = complete set of irreducible highest weight modules.

Problem. Determine the formal character of $M(\lambda)$

$$\mathbf{ch}(M(\lambda)) = \sum_{\mu} [M(\lambda) : L(\mu)] \cdot \mathbf{ch}(L(\mu))$$

Fact 6: $[x, y]$ Determines the Composition Series for Verma Modules

Answer. Only depends on Bruhat order using the following reasoning:

- $[M(\lambda) : L(\mu)] \neq 0 \iff \begin{cases} \lambda = x \cdot \lambda_0 \\ \mu = y \cdot \lambda_0 \\ x < y \in W \end{cases}$

  (Verma, Bernstein-Gelfand-Gelfand, van den Hombergh)

- $[M(x \cdot \lambda_0) : L(y \cdot \lambda_0)] = m(x, y)$ independent of λ_0. (BGG '75)

- $m(x, y) = 1 \iff \#\{r \in \mathcal{R} : x < rx \leq z\} = l(z) - l(x)$ $\forall x \leq z \leq y.$ (Janzten '79)

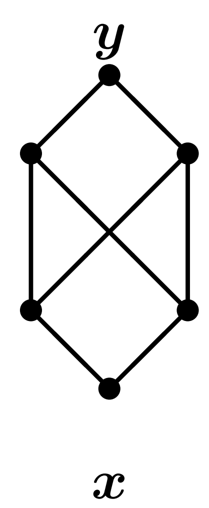
  $m(x, y) = P_{x,y}(1)$ = Kazhdan-Lusztig polynomial for $x < y$

  (Beilinson-Bernstein '81, Brylinski-Kashiwara '81)

Fact 6: $[x, y]$ Determines the Composition Series for Verma Modules

Conjecture. The Kazhdan-Lusztig polynomial $P_{x,y}(q)$ depends only on the interval $[x, y]$ (not on W or $\mathfrak{g}$ etc.)

Example.



$$\Longrightarrow \ m(x, y) = 1$$

Fact 7: Order Complex of (u, v) is Shellable

- *Order complex* $\Delta(u, v)$ has faces determined by the chains of the open interval (u, v), maximal chains determine the facets.

- Δ = pure d-dim complex is *shellable* if the maximal faces can be linearly ordered $C_1, C_2, \ldots$ such that for each $k \geq 1$, $(\overline{C_1} \cup \cdots \cup \overline{C_k}) \cap \overline{C_{k+1}}$ is pure $(d-1)$-dimensional.

Shellable Not Shellable

Fact 7: Order Complex of (u, v) is Shellable

Lexicographic Shelling of $[u, v]$: (Bjorner-Wachs '82, Proctor, Edelman)

- Each maximal chain $\longrightarrow$ label sequence

$$v = s_1 s_2 \dots s_p > s_1 \dots \widehat{s_j} \dots s_p > s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_p > \dots$$

maps to

$$(j, i, \dots)$$

- Order chains by lexicographically ordering label sequences.

Consequences:

1. $\Delta(u, w)^J$ is Cohen-Macaulay.
2. $\Delta(u, w)^J \equiv \begin{cases} \text{the sphere } S^{l(w)-l(u)-2} & (u, w)^J = (u, w) \\ \text{the ball } B^{l(w)-l(u)-2} & \text{otherwise} \end{cases}$

Fact 8: Rank Symmetric, Rank Unimodal and k-Sperner

1. P= ranked poset with maximum rank m
2. P is *rank symmetric* if the number of elements of rank i equals the number of elements of rank $m-i$.
3. P is *rank unimodal* if the number of elements on each rank forms a unimodal sequence.
4. P is *k-Sperner* if the largest subset containing no $(k+1)$-element chain has cardinality equal to the sum of the k middle ranks.

Theorem.(Stanley '80) For any subset $J \subset \{s_1, \dots s_n\}$, let W^J be the partially ordered set on the quotient W/W_J induced from Bruhat order. Then W^J is rank symmetric, rank unimodal, and k-Sperner.

(proof uses the Hard Lefschetz Theorem)

Fact 9: Efficient Methods for Comparison

Problem. Given two elements $u, v \in W$, what is the best way to test if $u < w$?
Don't use subsequences of reduced words if at all possible.

Tableaux Comparison in S_n. (Ehresmann)

- Take $u = 352641$ and $v = 652431.$
- Compare the sorted arrays of $\{u_1, \ldots u_i\} \leq \{v_1, \ldots, v_i\}$:

$$\begin{array}{cccccccccccccc} & & & & & 3 & \leq & 6 & & & & & \\ & & & & 3 & 5 & \leq & 5 & 6 & & & & \\ & & & 2 & 3 & 5 & \leq & 2 & 5 & 6 & & & \\ & & 2 & 3 & 5 & 6 & \leq & 2 & 4 & 5 & 6 & & \\ & 2 & 3 & 4 & 5 & 6 & \leq & 2 & 3 & 4 & 5 & 6 & \\ 1 & 2 & 3 & 4 & 5 & 6 & \leq & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Fact 9: Efficient Methods for Comparison

- Generalized to B_n and D_n and other quotients by Proctor (1982).
- *Open:* Find an efficient way to compare elements in $E_{6,7,8}$ in Bruhat order.

Another criterion for Bruhat order on W.

$u \le v$ in $W \iff u \le v$ in W^J for each maximal proper $J \subset \{s_1, s_2, \ldots, s_n\}$.

Fact 10: Amenable to Pattern Avoidance

Patterns on Permutations. Small permutations serve as patterns in larger permutations.

Def. by Example. $w_1 w_2 \ldots w_n$ (one-line notation) *contains* the pattern $\mathbf{4231}$ if there exists $i < j < k < l$ such that

$$w_i = \text{4th}\{w_i, w_j, w_k, w_l\}$$
$$w_j = \text{2nd}\{w_i, w_j, w_k, w_l\}$$
$$w_k = \text{3rd}\{w_i, w_j, w_k, w_l\}$$
$$w_l = \text{1st}\{w_i, w_j, w_k, w_l\}$$

If w no such i, j, k, l exist, w *avoids* the pattern $\mathbf{4231}$.

Example: $w = \mathbf{625431}$ contains $\mathbf{6241} \sim \mathbf{4231}$
$w = \mathbf{612543}$ avoids $\mathbf{4231}$

Fact 10: Amenable to Pattern Avoidance

Or equivalently, w contains $\mathbf{4231}$ if matrix contains submatrix

$$\begin{bmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{1} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{0} & \ldots & \mathbf{1} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{1} & \ldots & \mathbf{0} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{1} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix}$$

Extending to other infinite families of Weyl groups: B_n and D_n: Use patterns on signed permutations.

Fact 10: Amenable to Pattern Avoidance

Applications of Pattern Avoidance.

1. (Knuth,Tarjan) Stack-sortable permutations are **231**-avoiding.

2. (Lascoux-Schützenberger) Vexillary permutations are 2143-avoiding. The number of reduced words for a vexillary permutation is equal to the number of standard tableau of some shape. Extended to types B,C, and D by Lam and Billey.

3. (Billey-Jockusch-Stanley) The reduced words of a 321-avoiding permutation all have the same content. Extended to fully commutative elements in other Weyl groups by Fan and Stembridge.

4. (Billey-Warrington) New formula for Kazhdan-Lusztig polynomial when second index is 321-hexagon-avoiding.

5. (Lakshmibai-Sandhya) For $w \in S_n$, X_w is smooth (equiv. rationally smooth) if and only if w *avoids* 4231 and 3412. Extended to types B, C, D to characterize all smooth and rationally smooth Schubert varieties by Billey.

Minimal List of Bad Patterns for Type B, C, D

Theorem. Let $w \in B_n$, the Schubert variety $X(w)$ is rationally smooth if and only if w avoids the following 26 patterns:

$$
\begin{array}{ccccccc}
\bar{1}2\bar{3} & 1\bar{2}\bar{3} & 12\bar{3} & 1\bar{3}\bar{2} & \bar{2}\bar{1}\bar{3} & \bar{2}1\bar{3} & 2\bar{1}\bar{3} \\
2\bar{3}\bar{1} & \bar{3}1\bar{2} & \bar{3}\bar{2}\bar{1} & \bar{3}\bar{2}1 & \bar{3}2\bar{1} & 3\bar{2}\bar{1} & 3\bar{2}1 \\
\bar{2}\bar{4}31 & 2\bar{4}31 & \bar{3}\bar{4}\bar{1}\bar{2} & \bar{3}4\bar{1}2 & \bar{3}412 & 34\bar{1}2 & 3412 \\
4\bar{1}3\bar{2} & 413\bar{2} & \bar{4}231 & 423\bar{1} & 4231 & &
\end{array}
$$

Theorem. Let $w \in D_n$, the Schubert variety $X(w)$ is rationally smooth if and only if w avoids the following 55 patterns:

Minimal List of Bad Patterns for Type B, C, D

Theorem. (Billey-Postnikov) Let W be the Weyl group of any semisimple Lie algebra. Let $w \in W$, the Schubert variety $X(w)$ is (rationally) smooth if and only if for every parabolic subgroup Y with a stellar Coxeter graph, the Schubert variety $X(f_Y(w)))$ is (rationally) smooth.

Summary of Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is Shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance