LECTURE 8: GEOMETRIC FLAVOR AND SUBWORD PROPERTY OF BRUHAT ORDER

WANG, QIANG

1. Geometric flavor

This section is a brief introduction on the role of Bruhat order in the study of Flag and Schubert varieties.

Let G be an algebraic group, in our following discussion we concentrate on $G = GL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup of G, which in the case of $G = GL_n$ is the set of all upper triangular matrices. Then G/B has the structure of smooth projective variety.

Let V be an n-dimensional complex vector space. A **flag** is a sequence

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

of subspaces of V. If we denote by $d_i = \dim(V_i)$, then $(d_0, \cdots, d_k)$ is a strictly increasing sequence with $d_0 = 0$ and $d_k = n$, which is called the **signature** of the flag. We say that a flag is **complete** if $d_i = i$ for all $i = 0, \ldots, n$.

Fix an ordered basis $\mathcal{B} = (v_1, \cdots, v_n)$ of V. The **standard flag** of V is given by setting $V_i = \text{span}\{v_1, \cdots, v_i\}$. It is clear that each V_i is invariant under B.

The group $G = GL_n$ acts transitively on the set of all complete flags and B is the stabilizer. Thus the set of complete flags can be thought as the smooth projective variety G/B. In the case of partial flags one obtains G/P where P is a parabolic subgroup. A (partial) flag variety of signature $(d_0 = 0, d_1, d_2 = n)$ is just a **Grassmannian** of all d_1-dimensional subspaces of V.

It is known that G can be decomposed in terms of the **Bruhat decomposition**

$$G = BWB,$$

where W is a Weyl subgroup of G, and in the case of $G = GL_n$, W is the subgroup of all permutation matrices ($\cong S_n$). Then $G/B = \bigcup_{w \in W} BwB/B$ is the disjoint union of **Schubert cells** $C_w := BwB/B$ indexed by $w \in W$.

Let $X_w = \overline{C_w}$ be the topological closure of C_w. X_w is the **Schubert variety** in flag manifold $F = F(V)$ of all complete flags in V. The following theorem connects Bruhat order to the study of flag varieties.

Theorem 1.1. *$X_v \subseteq X_w$ if and only if $v \leq w$ in Bruhat order.*

Let $H^*(F; \mathbb{Z})$ be the cohomology ring associated with F. Each closed subvariety X of F determines an element $[X] \in H^*(F, \mathbb{Z})$. Recall the Schubert polynomials σ_w from Lecture 1. The next theorem relates Schubert classes with Schubert polynomials.

Date: January 23, 2009.

Theorem 1.2. *There is a surjective ring homomorphism*

$$\varphi : \mathbb{Z}[x_1, \cdots, x_n] \to H^*(F;\mathbb{Z})$$
$$\sigma_w \mapsto [X_w].$$

2. Subword property of Bruhat order

In this section we continue the discussion of Bruhat order in Lecture 7.

Let (W, S) be a Coxeter system.

Definition 2.1. *A subword of a word $s_1 s_2 \cdots s_q$ is a word of the form $s_{i_1} \cdots s_{i_k}$ where $1 \le i_1 < \cdots < i_k \le q$. Write $s_{i_1} \cdots s_{i_k} \prec s_1 s_2 \cdots s_q$.*

Lemma 2.2. *Let $u, w \in W$ and $u \neq w$. Suppose w has a reduced expression $s_1 s_2 \cdots s_q$ and u has a reduced expression which is a subword of $s_1 s_2 \cdots s_q$. Then there exists $v \in W$ such that*

(1) $u < v$
(2) $\ell(v) = \ell(u) + 1$
(3) *v has a reduced expression which is a subword of $s_1 s_2 \cdots s_q$.*

Proof. Let $u = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_k} \cdots s_q$ be the reduced word of u such that i_k is minimal among all possible choices.

Let $t = t_{i_k} \in \hat{T}(s_q s_{q-1} \cdots s_1)$. Then $ut = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots s_{i_k} \cdots s_q$ (adding s_{i_k} back). At the least, we know $\ell(ut) \le \ell(u) + 1$. We claim that $ut > u$. Assuming this claim, we can let $v = ut$, all conditions are easily checked.

So we need to prove the claim. First note that by definition of Bruhat order, ut is always comparable with u. Suppose $ut < u$, then $\ell(ut) < \ell(u)$. By the corollary of S.E.P (Strong Exchange Property) we know $t = t_p \in \hat{T}(s_q \cdots \hat{s}_{i_k} \cdots \hat{s}_{i_{k-1}} \cdots \hat{s}_{i_1} \cdots s_1)$. Either $p < q + 1 - i_k$ or not. If $p < q + 1 - i_k$, then t is of then form

$$t = s_q s_{q-1} \cdots s_{p+1} s_p s_{p-1} \cdots s_q$$

otherwise

$$t = s_q \cdots \hat{s}_{i_k} \cdots \hat{s}_{i_d} \cdots s_r \cdots \hat{s}_{i_d} \cdots \hat{s}_{i_k} \cdots s_q$$

for some $r < i_k$ and $r \neq i_j$ for any $j \in [k]$.

In the first case, consider

$$w = wtt = (s_1 \cdots s_q)(s_q \cdots s_{i_k} \cdots s_q)(s_q s_{q-1} \cdots s_{p+1} s_p s_{p-1} \cdots s_q)$$
$$= s_1 \cdots \hat{s}_{i_k} \cdots \hat{s}_p \cdots s_q.$$

But this contradicts to our assumption that $\ell(w) = q$.

In the second case, consider

$$u = utt$$
$$= (s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots s_q)(s_q \cdots \hat{s}_{i_k} \cdots \hat{s}_{i_d} \cdots s_r \cdots \hat{s}_{i_d} \cdots \hat{s}_{i_k} \cdots s_q)(s_q \cdots s_{i_k} \cdots s_q)$$
$$= s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_r \cdots s_{i_k} \cdots s_q.$$

But this contradicts to the minimality of i_k. □

Theorem 2.3 (Subword Property; S.P.)**.** *Let $s_1 s_2 \cdots s_q$ be a reduced expression of w, then $u \le w$ if and only if u has a reduced expression that is a subword of w.*

Proof. ⇒:
Assume $u \leq w$, that means we have the following sequence:

$$u = u_0 \xrightarrow{t_1} u_1 \cdots \xrightarrow{t_m} u_m = w$$

Then $u_{m-1} = wt_m = s_1 \cdots \hat{s_i} \cdots s_q$ for some i by the S.E.P (Strong Exchange Property). Repeat this argument to $u_{m-2}, \cdots, u_0$, we get an expression of u that is a subword of w. This subword may not be reduced yet, but D.P. (Deletion Property) promise us that it contains as a subword a reduced expression of u.
⇐:
If u has a reduced expression that is a subword of $s_1 s_2 \cdots s_q$, then the above lemma allows us to construct a sequence $u < v_1 < \cdots < v_s$ such that their *length* are strictly increasing by one but each has a reduced word that is a subword of $s_1 s_2 \cdots s_q$. Then it is clear that $v_s = w$. □

Corollary 2.4. *For any $u, w \in W$ the following are equivalent:*

(1) *$u \leq w$.*
(2) *Every reduced expression of w has a subword that is a reduced expression of u.*
(3) *Some reduced expression of w has a subword that is a reduced expression of u.*

Proof. This follows from a pure logical consideration, formally: If A, P are first order formulas, and A does not involve x (P may or may not involve x), then

$$\forall_x (A \rightarrow P(x)) \Leftrightarrow A \rightarrow \forall_x (P(x))$$

and

$$\forall_x (P(x) \rightarrow A) \Leftrightarrow \exists_x (P(x)) \rightarrow A$$

Here A is the statement that "$u < v$". $P(x)$ is the statement that "x is a reduced expression of w, and it has a subword that is a reduced expression of u". Then S.P. is the formula $\forall_x (A \leftrightarrow P(x)$. □

Corollary 2.5. *For any $u, w \in W$ the interval $[u, w] := \{x \in W \mid u \leq x \leq w\}$ is always finite.*

Proof. We argue a stronger statement that indeed $[e, w]$ is finite where e is the identity element of W (the least element of the Bruhat order). Pick a reduced expression $s_1 s_2 \cdots s_q$ of w, then any $x \in [e, w]$, by above corollary, can be written as a subword of $s_1 s_2 \cdots s_q$, there are only at most 2^q of them. □