THE ROOK MONOID IS LEXICOGRAPHICALLY SHELLABLE

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1. Introduction.

Let n be a positive integer and let $[n] = \{1, ..., n\}$. Let P be a finite graded poset of rank n with minimum and maximum elements denoted by $\hat{0}$ and $\hat{1}$, respectively. Let $C(P)$ be the set of pairs $(x, y) \in P \times P$ such that y covers x. P is called *lexicographically shellable* (see Section 2 for a slightly more general definition), if there exists a map $f : C(P) \to [n]$ such that

(1) in every interval $[x, y]$ of P there is a unique unrefinable chain $\mathfrak{c}$: $x = x_0 < x_1 < \cdots < x_{k+1} = y$ such that $f(x_i, x_{i+1}) < f(x_{i+1}, x_{i+2})$ for all $i = 0, ..., k-1$,
(2) the sequence $f(\mathfrak{c}) := (f(x, x_1), ..., f(x_k, y))$ of the unique chain $\mathfrak{c}$ from (1) is lexicographically first in $\{f(\mathfrak{d}) |\ \mathfrak{d}$ is an unrefinable chain in $[x, y]\}$.

The concept of lexicographically shellability is introduced by Björner in [3] and is shown to imply the weaker property of shellability of $\Delta(P)$, the simplicial complex of all chains of P. Shellability of a simplicial complex is a combinatorial property with important topological and algebraic consequences. For more see [5], [17], or [19].

In this paper we are concerned with the question of shellability for R_n, the *rook monoid* of $0/1$ matrices of size n with at most one 1 in each row and each column. One can view elements of R_n as non-attacking rook placements on an $n \times n$ chess board. This explains the nomenclature. The partial ordering on R_n that we are interested in comes from the topology of the "matrix Schubert varieties." Let us explain.

Let K be an algebraically closed field, and let $G = GL_n$ be the general linear group over K. Denote by $B \subset G$ the subgroup of invertible upper triangular matrices. Let M_n be the set of all $n \times n$ matrices over K. Note that M_n is a monoid under matrix multiplication. The Zariski closure in M_n of an orbit of the action

$$(x, y) \cdot g = xgy^{-1},\ g \in G,\ x, y \in B \tag{1.1}$$

of $B \times B$ on G is called a matrix Schubert variety. It is well known that the matrix Schubert varieties are parametrized by the symmetric group S_n, and G has the "Bruhat-Chevalley

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decomposition"

$$G = \bigsqcup_{w \in S_n} BwB. \tag{1.2}$$

Clearly, the action (1.1) on G extends to an action on M_n. In [13], Renner shows that the orbits of the extended action are parametrized by the rook monoid R_n, furthermore, the analogue of (1.2) holds:

$$M_n = \bigsqcup_{r \in R_n} BrB. \tag{1.3}$$

The *Bruhat-Chevalley-Renner ordering* on R_n is defined by

$$r \leq t \iff BrB \subseteq \overline{BtB}.$$

Here, the bar on the orbit BtB denotes the Zariski closure in M_n.

The main result of this paper is that the rook monoid R_n with respect to Bruhat-Chevalley-Renner ordering is a lexicographically shellable poset. Consequently, we know that for any interval I in R_n, the simplicial complex $\Delta(I)$ has the homotopy type of a wedge of spheres or balls.

Reductive monoids. The monoid of $n \times n$ matrices is an important member of the family of varieties called algebraic monoids. To place our work appropriately in this general setting and to help the reader unfamiliar with the theory of algebraic monoids let us briefly recall the definitions and relevant combinatorial results without detail. See one of [14], [11] or [16] for more.

Let G be a reductive group. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. The *Weyl group* W associated with (G, T) is defined to be the quotient group $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. In the case of $G = GL_n$ the Weyl group is isomorphic to the symmetric group S_n. The Bruhat-Chevalley order on the Weyl group W is defined by $w \leq v \iff BwB \subseteq \overline{BvB}$. It is shown by different authors that the Bruhat-Chevalley orders are lexicographically shellable (see [10], [4], and in the special case of the symmetric group, see [7]).

The generalization of the Bruhat-Chevalley ordering in the realm of algebraic monoids is due to Renner, [13]. An algebraic monoid is an algebraic variety M together with an associative binary operation $m : M \times M \to M$ which is a morphism of varieties.

An interesting class of algebraic monoids can be described as follows. Let $\rho : G_0 \to GL(V)$ be a rational representation of a semisimple algebraic group G_0. By abuse of notation, let K^* denote the scalar matrices in the (affine) space $End(V)$ of linear transformations on V. Then, the Zariski closure $M = \overline{K^* \cdot \rho(G_0)}$ in $End(V)$ is a *reductive monoid.*

Let G be the (reductive) group of invertible elements of a reductive monoid M, and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. It is shown in [13] that reductive monoids have decompositions into double cosets of B

$$M = \bigsqcup_{r\in R} B\dot{r}B,\ \dot{r} \in \overline{N_G(T)}/T,$$

indexed by a finite monoid R, now called the *Renner monoid* of M. Here $\overline{N_G(T)}$ is the Zariski closure in M of the normalizer in G of T. The Bruhat-Renner ordering on R is defined as before. In the special case of the defining representation $\rho : G_0 \to GL(K^n)$ of $G_0 = SL_n$, the Renner monoid R is isomorphic to the rook monoid R_n. The Weyl group W of (G,T) forms the group of invertible elements in the Renner monoid R, and the Bruhat-Chevalley ordering on W extends to the Bruhat-Renner ordering on R.

There is a cross section lattice $\Lambda \subset R$ of idempotents, parametrizing the $G\times G-$orbits in M

$$M = \bigsqcup_{e\in\Lambda} GeG.$$

Furthermore,

$$R = \bigsqcup_{e\in\Lambda} WeW.$$

Let $e \in \Lambda$. In [12], Putcha shows that the subposets $WeW \subseteq R$ of $W \times W$-orbits in R are lexicographically shellable posets. It is also known that the cross section lattice $\Lambda \subseteq R$ is an (upper) semimodular lattice, hence shellable. However, showing that a Renner monoid is shellable seems to be a difficult problem.

2. Background.

2.0.1. *Lexicographic shellability.* Let P be a finite poset with a maximum and a minimum element, denoted by $\hat{1}$ and $\hat{0}$ respectively. We assume that P is *graded* of *rank* n. In other words, all maximal chains of P have equal length n. Denote by $C(P)$ the set of covering relations

$$C(P) = \{(x,y) \in P \times P :\ y \text{ covers } x\}.$$

An *edge-labeling* on P is a map of the form $f = f_{P,\Gamma} : C(P) \to \Gamma$ for some poset Γ. The *Jordan-Hölder sequence* (with respect to f) of a maximal chain $\mathfrak{c} : x_0 < x_1 < \cdots < x_{n-1} < x_n$ of P is the n-tuple

$$f(\mathfrak{c}) = (f((x_0,x_1)), f((x_1,x_2)), \dots, f((x_{n-1},x_n))) \in \Gamma^n.$$

Fix an edge labeling f, and a maximal chain $\mathfrak{c} :\ x_0 < x_1 < \cdots < x_n$. We call both of the maximal chain $\mathfrak{c}$ and its image $f(\mathfrak{c})$ *increasing*, if

$$f((x_0,x_1)) \leq f((x_1,x_2)) \leq \cdots \leq f((x_{n-1},x_n))$$

holds in Γ.

Let $k > 0$ be a positive integer. We consider the lexicographic (total) ordering on the $k-$fold cartesian product $\Gamma^k = \Gamma \times \cdots \times \Gamma$. An edge labeling $f : C(P) \to \Gamma$ is called an $EL-$*labeling*, if

(1) in every interval $[x, y] \subseteq P$ of rank $k > 0$ there exists a unique maximal chain $\mathfrak{c}$ such that $f(\mathfrak{c}) \in \Gamma^k$ is increasing,
(2) the Jordan-Hölder sequence $f(\mathfrak{c}) \in \Gamma^k$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among the Jordan-Hölder sequences of maximal chains $x = x_0 < x_1 < \cdots < x_k = y$.

A poset P is called *EL-shellable*, if it has an $EL-$labeling.

Remark 2.1. There are various lexicographic shellability conditions in the literature and the $EL-$shellability defined here is among the stronger ones. A deep relationship between $EL-$shellability of a Coxeter group W and the Kazhdan-Lusztig theory of the Hecke algebra associated with W is found by Dyer in [6].

2.0.2. *The symmetric group.* S_n is the set of all permutations of $[n]$. Let us represent the elements of S_n in one line notation $w = (w_1, ..., w_n) \in S_n$ so that $w(i) = w_i$. It is well known that the S_n is a graded poset with respect to Bruhat-Chevalley ordering. Let B be the invertible upper triangular matrices in SL_n. Grading on S_n is given by the length function

$$\ell(w) = \dim(BwB) - \dim(B) = inv(w), \tag{2.1}$$

where

$$inv(w) = |\{(i, j) : \ 1 \leq i < j \leq n, \ w_i > w_j\}|. \tag{2.2}$$

Note that $\dim B = \binom{n+1}{2}$.

The Bruhat-Chevalley ordering on S_n is the smallest partial order generated by the transitive closure of the following (covering) relations. The permutation $x = (a_1, ..., a_n)$ is covered by the permutation $y = (b_1, ..., b_n)$, if $\ell(y) = \ell(x) + 1$ and

(1) $a_k = b_k$ for $k \in \{1, ..., \widehat{i}, ..., \widehat{j}, ..., n\}$ (hat means omit those numbers),
(2) $a_i = b_j$, $a_j = b_i$, and $a_i < a_j$.

An $EL-$labeling for S_n is constructed by Edelman [7] as follows. Let $\Gamma = [n] \times [n]$ be the poset of pairs, ordered lexicographically: $(i, j) \leq (r, s)$ if $i < r$, or $i = r$ and $j < s$. Define $f((x, y)) = (a_i, a_j)$, if $y = (b_1, ..., b_n)$ covers $x = (a_1, ..., a_n)$ such that

(1) $a_k = b_k$ for $k \in \{1, ..., \widehat{i}, ..., \widehat{j}, ..., n\}$,
(2) $a_i = b_j$, $a_j = b_i$, and $a_i < a_j$.

For $n = 3$, the $EL-$labeling of S_3 is as depicted in the Figure 1.

Theorem 2.2. *([7]) The symmetric group S_n with Bruhat-Chevalley ordering is lexicographically shellable.*

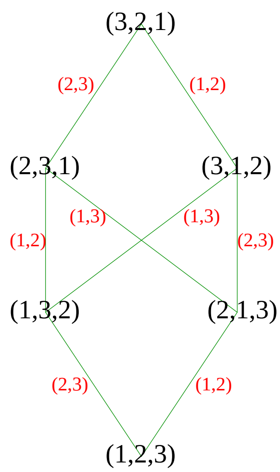

FIGURE 1. $EL-$labeling of S_3

2.0.3. *The rook monoid.* Recall from [13] that the rank function on R_n is given by

$$\ell(x) = \dim(BxB),\ x \in R_n. \tag{2.3}$$

There is a combinatorial formula for $\ell(x)$, $x \in R_n$ similar to (2.1). To explain let us represent the elements of R_n by n-tuples, as we did implicitly for S_n in the previous subsection. Let $x = (x_{ij}) \in R_n$ and define the sequence $(a_1, ..., a_n)$ by

$$a_j = \begin{cases} 0 & \text{if the } j\text{'th column consists of zeros,} \\ i & \text{if } x_{ij} = 1. \end{cases} \tag{2.4}$$

By abuse of notation, we denote both the matrix and the sequence $(a_1, ..., a_n)$ by x. For example, the associated sequence of the partial permutation matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is $x = (3, 0, 4, 0)$.

Let $x = (a_1, ..., a_n) \in R_n$. A pair (i, j) of indices $1 \leq i < j \leq n$ is called a *coinversion pair* for x, if $0 < a_i < a_j$. By abuse of notation, we use *coinv* for both the set of coinversion pairs of x, as well as its cardinality.

Example 2.3. Let $x = (4, 0, 2, 3)$. Then, the only coinversion pair for x is $(3, 4)$. Therefore, $coinv(x) = 1$.

In [2], we show that the dimension, $\ell(x) = \dim(BxB)$ of the orbit BxB, $x \in R_n$ is given by

$$\ell(x) = (\sum_{i=1}^{n} a_i^*) - coinv(x), \text{ where } a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0 \\ 0, & \text{if } a_i = 0 \end{cases} \tag{2.5}$$

If $x = (a_1, ..., a_n) \in S_n$ be a permutation. Then

$$\begin{aligned} \ell(x) &= (\sum_{i=1}^{n} a_i + n - i) - coinv(x) \\ &= \binom{n+1}{2} + \binom{n}{2} - coinv(x) \\ &= \binom{n+1}{2} + inv(x), \end{aligned}$$

which agrees with the formula (2.1). In fact, using (2.5) it is easy to see that if $x \in R_n$, then

$$\ell(x) = \sum a_i + inv(x), \tag{2.6}$$

where

$$inv(x) = |\{(i,j) :\ 1 \leq i < j \leq n,\ a_i > a_j\}|.$$

In [9], a characterization of the Bruhat-Chevalley ordering on the rook monoid R_n is given.

Theorem 2.4. [9] *Let $x = (a_1, ..., a_n)$, $y = (b_1, ..., b_n) \in R_n$. The Bruhat-Chevalley order on R_n is the smallest partial order on R_n generated by declaring $x \leq y$ if either*

1. *there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_j = a_j$ for all $j \neq i$, or*
2. *there exist $1 \leq i < j \leq n$ such that $b_i = a_j,\ b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i, j\}$, $b_k = a_k$.*

The following two Lemmas proved in [2] are critical for deciding whether $x \leq y$ is a covering relation or not.

Lemma 2.5. *Let $x = (a_1, ..., a_n)$ and $y = (b_1, ..., b_n)$ be elements of R_n. Suppose that $a_k = b_k$ for all $k = \{1, ..., \widehat{i}, ..., n\}$ and $a_i < b_i$. Then, $\ell(y) = \ell(x) + 1$ if and only if either*

1. *$b_i = a_i + 1$, or*
2. *there exists a sequence of indices $1 \leq j_1 < \cdots < j_s < i$ such that the set $\{a_{j_1}, ..., a_{j_s}\}$ is equal to $\{a_i + 1, ..., a_i + s\}$, and $b_i = a_i + s + 1$.*

Example 2.6. Let $x = (4, 0, 5, 0, 3, 1)$, and let $y = (4, 0, 5, 0, 6, 1)$. Then $\ell(x) = 21$, and $\ell(y) = 22$. Let $z = (6, 0, 5, 0, 3, 1)$. Then $\ell(z) = 23$.

Lemma 2.7. *Let $x = (a_1, ..., a_n)$ and $y = (b_1, ..., b_n)$ be two elements of R_n. Suppose that $a_j = b_i$, $a_i = b_j$ and $b_j < b_i$ where $i < j$. Furthermore, suppose that for all $k \in \{1, ...\widehat{i}, ..., \widehat{j}, ..., n\}$, $a_k = b_k$. Then, $\ell(y) = \ell(x) + 1$ if and only if for $s = i+1, ..., j-1$, either $a_j < a_s$, or $a_s < a_i$.*

Example 2.8. Let $x = (2, 6, 5, 0, 4, 1, 7)$, and let $y = (4, 6, 5, 0, 2, 1, 7)$. Then $\ell(x) = 35$, and $\ell(y) = 36$. Let $z = (7, 6, 5, 0, 4, 1, 2)$. Then $\ell(z) = 42$.

3. An $EL-$labeling of R_n.

Recall that covering relations of the Bruhat-Renner ordering on R_n are characterized by the Lemma 2.5, and 2.7. For simplicity, a covering relation is called *type 1* if it is as in Lemma 2.5, and it is called *type 2* if it is as in Lemma 2.7.

Using these two lemmas, we define an $EL-$labeling on R_n

$$F : C(R_n) \longrightarrow \Gamma,$$

where Γ is the poset $\Gamma = \{0, 1, ..., n\} \times \{0, 1, ..., n\}$ with respect to lexicographic ordering.

Let $(x, y) \in C(R_n)$. We define

$$F((x, y)) = \begin{cases} (a_i, b_i), & \text{if } y \text{ covers } x \text{ by type } 1 \\ (a_i, a_j), & \text{if } y \text{ covers } x \text{ by type } 2. \end{cases} \tag{3.1}$$

For $n = 3$, the $EL-$labeling is as depicted in the Figure 2 below.

Remark 3.1. (1) Let $F((x, y)) = (a, b)$ for some $(x, y) \in C(R_n)$. Then, b is never 0.

(2) If y covers x by type 2, then the set of nonzero entries of y is the same as the set of nonzero entries of x. If y covers x by type 1, then the symmetric difference of the set of nonzero entries of y and the set of nonzero entries of x has at most 2, and at least 1 elements.

Theorem 3.2. *Let $\Gamma = \{0, 1, ..., n\} \times \{0, 1, ..., n\}$, and let $F : C(R_n) \longrightarrow \Gamma$ be the edge-labeling, defined as in (3.1). Then F is an $EL-$labeling for R_n.*

We prove this theorem in the next Section. The complete labeling of R_3 is shown in Figure 2.

4. Proofs.

Let R_n be the rook monoid. Let $\Gamma = \{0, ..., n\} \times \{0, ..., n\}$. Then, for any $k > 0$, $\Gamma^k = \Gamma \times \cdots \times \Gamma$ is totally ordered with respect to the lexicographic ordering. Let F be the labeling on R_n, as defined in (3.1). Let $[x, y] \subseteq R_n$ be an interval, and $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ be a maximal chain in $[x, y]$. Let $F(\mathfrak{c})$ the Jordan-Hölder sequence of labels of $\mathfrak{c}$:

$$F(\mathfrak{c}) = (F((x_0, x_1)), \ldots, F((x_{k-1}, x_k))) \in \Gamma^k. \tag{4.1}$$

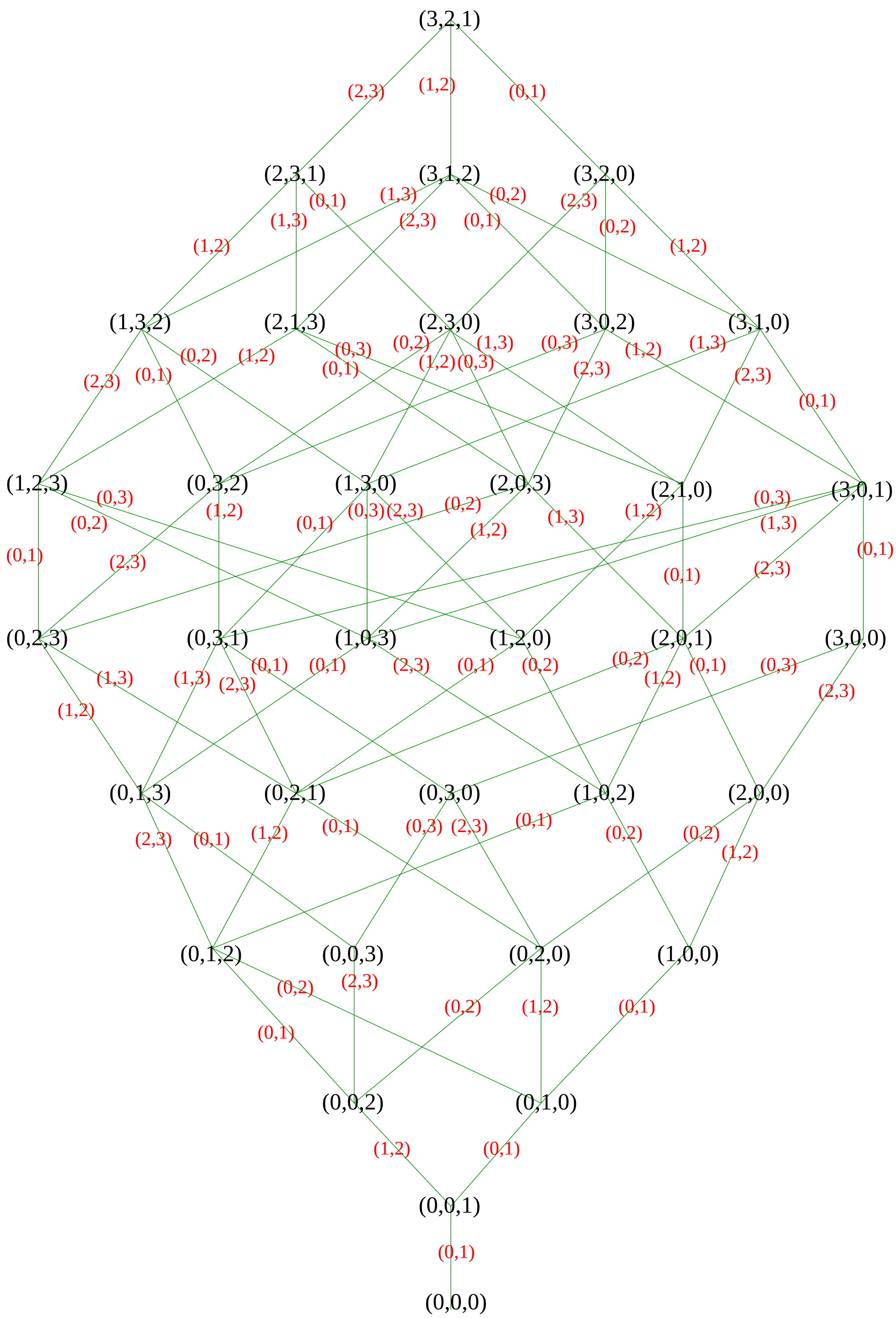

FIGURE 2. $EL-$labeling of the rook monoid R_3

Proposition 4.1. *Let* $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ *be a maximal chain in* $[x, y]$ *such that its Jordan-Hölder sequence* $F(\mathfrak{c})$ *is lexicographically smallest among all Jordan-Hölder sequences (of chains from* $[x, y]$*) in* Γ^k*. Then,*

$$F((x_0, x_1)) \leq F((x_1, x_2)) \leq \cdots \leq F((x_{k-1}, x_k)). \tag{4.2}$$

Before we start our proof, let us give an example in the case of $n = 3$.

Example 4.2. Let $x = (0, 1, 0)$ and $y = (3, 1, 2)$ in R_3. It is easy to check from Figure 2 that in $[x, y]$ the maximal chain

$$\mathfrak{c} : \ x < (1, 0, 0) < (1, 0, 2) < (1, 2, 0) < (1, 2, 3) < (2, 1, 3) < y$$

has the (lexicographically) smallest Jordan-Hölder sequence. Obviously,

$$F(\mathfrak{c}) = ((0, 1), (0, 2), (0, 2), (0, 3), (1, 2), (2, 3))$$

is a non-decreasing sequence.

Proof. Assume that (4.2) is not true. Then, there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathfrak{c}$, such that

$$F((x_{t-1}, x_t)) > F((x_t, x_{t+1})). \tag{4.3}$$

Obviously, we have the following 4 cases to consider.

Case 1: $type(x_t, x_{t+1}) = 1$, and $type(x_{t-1}, x_t) = 1$.
Case 2: $type(x_t, x_{t+1}) = 1$, and $type(x_{t-1}, x_t) = 2$.
Case 3: $type(x_t, x_{t+1}) = 2$, and $type(x_{t-1}, x_t) = 1$.
Case 4: $type(x_t, x_{t+1}) = 2$, and $type(x_{t-1}, x_t) = 2$.

In each of these cases we construct an element $z \in [x, y]$ which covers x_{t-1}, and such that $F((x_{t-1}, z)) < F((x_{t-1}, x_t))$. Since we assume that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence, this provides us with the contradictions we seek. To this end, let $x_{t-1} = (a_1, ..., a_n)$, $x_t = (b_1, ..., b_n)$ and $x_{t+1} = (c_1, ..., c_n)$.

Case 1: Since $type(x_{t-1}, x_t) = 1$, there exists an index $1 \leq r \leq n$ such that $b_k = a_k$ for all $k \neq r$ and $a_r < b_r$. Likewise, there exists $1 \leq s \leq n$ such that $c_k = b_k$ for all $k \neq s$, and $b_s < c_s$. Therefore, $F((x_{t-1}, x_t)) = (a_r, b_r)$ and $F((x_t, x_{t+1})) = (b_s, c_s)$. Furthermore, by the assumption, $(a_r, b_r) > (b_s, c_s)$. Since $a_r < b_r$, r cannot be equal to s. Otherwise, $F((x_{t-1}, x_t)) = (a_r, b_r) > F((x_t, x_{t+1})) = (b_r, c_r)$, which is absurd. Therefore, either $r > s$, or $r < s$. Hence, $b_s = a_s$.

Suppose first that $r > s$. Define $z = (d_1, ..., d_n) \in R_n$ by $d_k = a_k$ for $k \neq s$, and $d_s = c_s$. It is easy to check that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_s, c_s)$. Since $F((x_{t-1}, x_t)) = (a_r, b_r) > (b_s, c_s) = (a_s, c_s) = F((x_{t-1}, z))$, we find a contradiction.

Next, suppose that $r < s$. Observe that $a_r = a_s = 0$ is not possible (because, $type(x_{t-1}, x_t) = 1$). Similar to the previous case, define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \neq s$, and $d_s = b_r$. It is easy to check that z covers x_{t-1} and that $F((x_{t-1}, z)) = (a_s, b_r)$ is less than $F((x_{t-1}, x_t)) = (a_r, b_r)$. This, too, contradicts the hypotheses (on $F(\mathfrak{c})$). Therefore, Case 1 is finished.

Case 2: Since $type(x_t, x_{t+1}) = 1$, there exists $r \in [n]$ such that $b_k = c_k$ for $k \neq r$, and $b_k < c_k$, and since $type(x_{t-1}, x_t) = 2$ there exist $i < j$ such that $b_k = a_k$ for $k \notin \{i, j\}$, and $b_i = a_j,\ b_j = a_i$ with $a_i < a_j$. Then $(a_i, a_j) > (b_r, c_r)$ by (4.3).

Suppose first that either $r < i$ or $r > j$ is true. Let $d_k = a_k$ for $k \neq r$ and let $d_r = c_r$. Put $z = (d_1, ..., d_n) \in R_n$. Then, z covers x_{t-1} and $F((x_{t-1}, z)) = (a_r, c_r) < F((x_{t-1}, x_t)) = (a_i, a_j)$. This is a contradiction, as before.

Next, suppose that $i \leq r \leq j$. Then, the first case is $i < r < j$. Since $type(x_{t-1}, x_t) = 2$, either $a_r > a_j$, or $a_r < a_i$. If $a_r > a_j$, then $a_r > a_i$. This contradicts $F((x_t, x_{t+1})) = (b_r, c_r) = (a_r, c_r) < F((x_{t-1}, x_t)) = (a_i, a_j)$. Therefore, $a_r \leq a_i$. If $a_r = a_i$, then it is easy to see that $a_r = a_i = 0$. But $r > i$, and $type(x_{t-1}, x_t) = 2$. Therefore, $a_r = a_i = 0$ is not possible. So, we conclude that $a_i > a_r$. Since $type(x_t, x_{t+1}) = 2$, $c_r < a_i$. Note that $a_i < a_j$. Finally, define $z = (d_1, ..., d_n) \in R_n$ by letting $d_k = a_k$ for $k \neq r$, and $d_r = c_r$. It is easy to see that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_r, c_r) < F((x_{t-1}, x_t)) = (a_i, a_j)$. This is a contradiction, as before.

The remaining cases are $r = i$ and $r = j$. If $r = j$, then $F((x_{t-1}, x_t)) = (a_i, a_j)$, and $F((x_t, x_{t+1})) = (a_i, c_j)$. Therefore $c_j < a_j$. Define $z = (d_1, ..., d_n) \in R_n$ by $d_k = a_k$ for $k \neq i$, and $d_i = c_j$. It is easy to see that z covers x_{t-1}, and that $F((x_{t-1}, z)) < F((x_{t-1}, x_t))$. This is a contradiction. Finally, if $r = i$, then $F((x_{t-1}, x_t)) = (a_i, a_j) < F((x_t, x_{t+1})) = (a_j, c_i)$, contradicting (4.3). This finishes Case 2.

The Case 3 is similar to the Case 2, so we omit the proof.

Case 4: Since $type(x_{t-1}, x_t) = 2$, there exist $1 \leq i < j \leq n$ such that $a_i < a_j$, $b_i = a_j$, $b_j = a_i$, and since $type(x_t, x_{t+1}) = 2$, there exist $1 \leq r < s \leq n$ such that $b_r < b_s$, $c_r = b_s$ and $c_s = b_r$.

If $j < r$ or $s < i$, define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \neq \{r, s\}$, and $d_s = a_r$, $d_r = a_s$. It is easy to check that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_r, a_s) < F((x_{t-1}, x_t)) = (a_i, a_j)$. This contradicts the hypotheses. Therefore, one of the following holds:

(a) $i \leq r \leq j \leq s$, or
(b) $r \leq i \leq s \leq j$

We proceed with (a). If $i < r < j < s$, we see that $a_i > a_r$. Since $type(x_t, x_{t+1}) = 2$, we see further that $a_j > a_i > a_s$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{r, s\}$ and $d_r = a_s$, $d_s = a_r$. It is easy to see that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_r, a_s) < (a_i, a_j) = F((x_{t-1}, x_t)))$. A contradiction, as before. If $i = r < j < s$, then $F((x_{t-1}, x_t)) = (a_i, a_j) < F((x_t, x_{t+1})) = (a_j, a_s)$. This contradicts (4.3). The case $i < r < j = s$ is similar, so, we omit the proof. If $i < r = j < s$, then $F((x_{t-1}, x_t)) = (a_i, a_r) > F((x_t, x_{t+1})) = (a_i, a_s)$. Therefore $a_r > a_s$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{i, s\}$, and $d_i = a_s,\ d_s = a_i$. We claim that z covers x_{t-1} by the type 2. It is enough to show that for $i < l < s$, either $d_l < a_i$, or $d_l > a_s$. Note that, if $i < l < r$, then $d_l = a_l$. Since $type(x_{t-1}, x_t) = 2$, either $d_l < a_i$, or $d_l > a_s = a_j$. Similarly, if $r < l < s$, then, $d_l = a_l = b_l$. Since $type(x_t, x_{t+1}) = 2$, either $a_l > b_s = a_s$, or $a_l < b_i = a_i$. Therefore, either $d_l < a_i$, or $a_l > a_s$. In other words, z covers x_{t-1}. Clearly, $F((x_{t-1}, z)) = (a_i, a_s) < F((x_{t-1}, x_t)) = (a_i, a_r)$. This is a contradiction as before.

We proceed with (b): $r \leq i \leq s \leq j$.

Suppose first that $r < i < s < j$. If $a_i > a_r$, then $a_s < a_i$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{r, s\}$ and $d_r = a_s$, $d_s = a_r$. It is easy to check that z covers x_{t-1}, and $F((x_{t-1}, z)) = (a_r, a_s) < F((x_{t-1}, x_t)) = (a_i, a_j)$. Contradiction, as before. Therefore, by (4.3) we have to have $a_i = a_r = 0$. This forces $a_j > a_s$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{i, s\}$ and $d_i = a_s$, $d_s = a_i$. It is easy to check that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_i, a_s) < F((x_{t-1}, x_t)) = (a_i, a_j)$. Contradiction.

The remaining possibilities are $r = i < s < j$, $r < i = s < j$, or $r < i < s = j$.

If $r = i$, it is easy to see that $F((x_{t-1}, x_t)) = (a_i, a_j) < F((x_t, x_{t+1})) = (a_j, a_s)$, which contradicts (4.3).

If $i = s$, $(a_i, a_j) > (a_r, a_j)$ by (4.3). Therefore, $a_r \leq a_i$. Obviously, if $a_r = a_i = 0$, then $(a_i, a_j) > (a_r, a_j)$ is not possible. Thus, we have $a_r < a_i$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{r, i\}$ and $d_r = a_i$, $d_i = a_r$. It is easy to check that z covers x_{t-1}, and $F((x_{t-1}, z)) = (a_r, a_i) < F((x_{t-1}, x_t)) = (a_i, a_j)$. Contradiction.

Finally, if $r < i < s = j$, then $a_r < a_i$. Define $z = (d_1, ..., d_n)$ by $d_k = a_k$ for $k \notin \{r, i\}$, and $d_r = a_i$, $d_i = a_r$. It is easy to check that z covers x_{t-1}, and that $F((x_{t-1}, z)) = (a_r, a_i) < F((x_{t-1}, x_t)) = (a_i, a_j)$. Contradiction.

The proof is complete and the chain which is lexicographically first is increasing. □

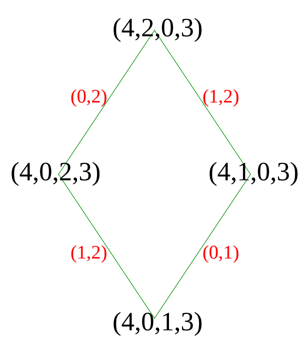

FIGURE 3. Two opposite edges have identical labels.

Proposition 4.3. *Let $[x,y]$ be an interval of length 2 in R_n, and let F be as in (3.1). Let*

$$\mathfrak{c}:\ x = x_0 < x_1 < x_2 = y$$

be the chain such that $F(\mathfrak{c}) = (F((x,x_1)), F((x_1,y)))$ is the lexicographically smallest Jordan-Hölder sequence. Let $x_0 < x_1' < x_2$ be any other chain between x and y. Then either $F((x_0,x_1')) = F((x_1,x_2))$, or $F((x_1',x_2)) = F((x_0,x_1))$.

Proof. There are two cases to consider:

Case A: $type(x_0,x_1) = 1$,
Case B: $type(x_0,x_1) = 2$.

Let $x_0 = (a_1,...,a_n)$, $x_1 = (b_1,...,b_n)$, $x_2 = (c_1,...,c_n)$ and let $x_1' = (d_1,...,d_n)$.

Case A: Since $type(x_0,x_1) = 1$, there exists $i \in [n]$ such that $a_i < b_i$, and $a_k = b_k$ for $k \neq i$.

Suppose that $type(x_1,x_2) = 1$. Then, there exists $j \in [n]$ such that $b_j < c_j$, and $b_k = c_k$ for $k \neq j$. We may assume that $i \neq j$, otherwise, $x_1' = x_1$ is forced and in this case there is nothing to prove. Therefore, we assume that $x_1' \neq x_1$. Thus,

$$\{c_1,...,c_n\} \setminus \{a_1,...,a_n\} = \{c_i,c_j\}. \tag{4.4}$$

Therefore, by Remark (3.1) neither $type(x_0,x_1') = 2$ nor $type(x_1',x_2) = 2$ is possible. The remaining possibility is $type(x_0,x_1') = type(x_1',x_2) = 1$. Let $r \in [n]$ be such that $a_r < d_r$, and $a_k = d_k$ for $k \neq r$. Let $s \in [n]$ be such that $d_s < c_s$, and $d_k = c_k$ for $k \neq s$. Observe that $r = s$ is not possible. Observe also that $\{c_s,c_r\} = \{c_i,c_j\}$.

Since $c_i,c_j,c_r,c_s \neq 0$, unless the subscripts are the same, they can not be equal. Therefore, either $r = i$ and $s = j$, or $r = j$ and $s = i$. The former implies the contradiction that $x_1 = x_1'$, and the latter implies $F((x_1',x_2)) = (a_i,c_i) = F((x_0,x_1))$. Therefore, if $type(x_1,x_2) = 1$, then we are done.

Next, suppose that $type(x_1,x_2) = 2$. Clearly, both $type(x_0,x_1')$ and $type(x_1',x_2)$ can not be equal to 1 at the same time. Let $F((x_0,x_1')) = (\alpha,\beta)$, for some $\alpha \in \{a_1,...,a_n\},\ \beta \in \{d_1,...,d_n\}$. Since $F(\mathfrak{c})$ is lexicographically smallest, $F((x_0,x_1)) = (a_i,b_i) \leq F((x_0,x_1'))$. Hence, either $a_i < \alpha$, or $a_i = \alpha$. We proceed with the former. We are going to show that $type(x_0,x_1') = 2$. This, in turn, implies that $F((x_0,x_1)) = F((x_1',x_2)) = (a_i,b_i)$.

Assume to the contrary that $type(x_0,x_1') = 1$. Then, $\beta \notin \{a_1,...,a_n\}$. Since $\{b_1,...,b_n\} \setminus \{a_1,...,a_n\} = \{b_i\}$, if $\beta \in \{b_1,...,b_n\}$, then $\beta = b_i$, and hence $\alpha = a_i$; a contradiction. Therefore, β is not an entry of x_1. Since $type(x_1',x_2) \neq 1$, β has to be an entry of x_2. On the other hand, by Remark 3.1, β cannot be an entry of x_2. This contradiction shows that $type(x_0,x_1') = 2$, and consequently $type(x_1',x_2) = 1$.

Since b_i is not an entry of x_1', $F((x_1',x_2)) = (\gamma,b_i)$ for some $\gamma \in \{d_1,...,d_n\}$. Observe that, unless a_i is zero (and this is not a problem) a_i can not appear in x_2. However, since

$type(x_0, x_1') = 2$, a_i appears in x_1'. Therefore, $\gamma = a_i$. Then, $F((x_0, x_1)) = F((x_1', x_2)) = (a_i, b_i)$. We are done in this case.

We proceed with the case that $a_i = \alpha$. Suppose $\alpha = a_m$ for some $m \in [n]$. Suppose first that $m \neq i$. Then, $\alpha = a_m = a_i = 0$. Notice that β cannot be equal to b_i (otherwise, depending on the relative positions of i and m, we would have either $x_1 < x_1'$ or $x_1' < x_1$). We claim that $type(x_0, x_1') = 2$. If $type(x_0, x_1') = 1$, then β is not an entry of x_1, and since $type(x_1, x_2) = 2$, it is not an entry of x_2. On the other hand, if $type(x_0, x_1') = 1$, then $type(x_1', x_2) = 2$, hence β has to be an entry of x_2, a contradiction. Hence,$type(x_0, x_1') = 2$. Notice that $type(x_1', x_2) = 2$ is not possible (since $type(x_0, x_1) = 1$, and $type(x_1, x_2) = 2$). Therefore, $type(x_1', x_2) = 1$, and $F((x_1', x_2)) = (\gamma, b_i)$ for some $\gamma \in \{d_1, ..., d_n\}$. Since,

$$|\{c_k :\ c_k = 0\ k = 1, ..., n, \}| = |\{a_k :\ a_k = 0,\ k = 1, ..., n\}| - 1,$$

and since,

$$|\{d_k :\ d_k = 0\ k = 1, ..., n, \}| = |\{a_k :\ a_k = 0,\ k = 1, ..., n\}|,$$

γ has to be zero. Therefore, $F((x_1', x_2)) = F((x_0, x_1))$, and we are done in this case.

Next, suppose that $m = i$. Then, $type(x_0, x_1') = 2$. Similar to above, it follows that $type(x_1', x_2) = 1$, and that $F((x_1', x_2)) = (\gamma, b_i)$ for some $\gamma \in \{d_1, ..., d_n\}$. If $a_i = 0$, then the exact same argument as above shows that $F((x_1', x_2)) = F((x_0, x_1))$. If $a_i \neq 0$, then since a_i is not an entry of x_2, but it appears in x_1', we must have that $\gamma = a_i$. Hence we are done in Case A.

Case B: $type(x_0, x_1) = 2$. Since $F(\mathfrak{c})$ is lexicographically smallest, $type(x_0, x_1') = type(x_1', x_2) = 1$ is not possible. Let $F((x_0, x_1)) = (a_i, a_j)$ and let $F((x_0, x_1')) = (\alpha, \beta)$.

Suppose that $type(x_0, x_1') = 1$. Then, β is an entry of x_2, and α is not (unless $\alpha = 0$, but this is not a problem, because, in this case we argue about the number of 0 entries of x_0, x_1' and of x_2, as before). Therefore, $F((x_1, x_2)) = (\alpha, \beta)$. We are done in this case.

Next, suppose that $type(x_0, x_1') = 2$. Assume that $type(x_1, x_2) = 1$. Then, it is easy to see that $type(x_1', x_2) = 1$, and that $F((x_1, x_2)) = F((x_1', x_2))$. Then, $x_1 = x_1'$. Contradiction. Therefore, we may assume that $type(x_1, x_2) = 2$. It follows also that $type(x_1', x_2) = 2$.

Let $F((x_0, x_1)) = (a_i, a_j)$, and let $F((x_0, x_1')) = (a_l, a_m)$, for some entries $a_i < a_j,\ a_l < a_m$ of x_0. Since $F(\mathfrak{c})$ is lexicographically smallest, one of the following holds:

(a) $a_i < a_l$, or
(b) $a_i = a_l$.

We proceed with (a). Assume that $a_j \notin \{a_l, a_m\}$. Let $F((x_1, x_2)) = (b_{u_1}, b_{u_2})$ for some $1 \leq u_1 < u_2 \leq n$. If b_{u_1} or b_{u_2} is not equal to a_l, then it means that the position of a_l did not change in the chain in $x_0 < x_1' < x_2$, a contradiction. Therefore, either $b_{u_1} = a_l$, or $b_{u_2} = a_l$. Similarly, either b_{u_1} or b_{u_2} has to be equal to a_m. In other words, $\{b_{u_1}, b_{u_2}\} = \{a_l, a_m\}$.

Since $b_{u_1} < b_{u_2}$ and $a_l < a_m$, we have to have that $F((x_1, x_2)) = (b_{u_1}, b_{u_2}) = (a_l, a_m) = F((x_0, x_1'))$. Thus we are done under these assumptions.

Next, we assume that $a_j \in \{a_l, a_m\}$ with $a_j = a_m$. Then $a_i < a_l < a_j = a_m$. This contradicts $type(x_0, x_1) = 2$. Therefore, we may assume that $a_j = a_l$. If $a_m \notin \{b_{u_1}, b_{u_2}\}$, then $c_m = a_m$. Contradiction. Therefore, $a_m \in \{b_{u_1}, b_{u_2}\}$. If $a_j = a_l \in \{b_{u_1}, b_{u_2}\}$, then we are done. So, we assume that $a_j = a_l \notin \{b_{u_1}, b_{u_2}\}$. If $a_i \in \{b_{u_1}, b_{u_2}\}$, it follows that $F((x_1', x_2)) = (a_i, a_j) = F((x_0, x_1))$, we are done. So, we may assume that $a_i \notin \{b_{u_1}, b_{u_2}\}$. Then, there exists an entry a_p of x_0 such that $a_p \notin \{a_i, a_j = a_l, a_m\}$ and $\{a_p, a_m\} = \{b_{u_1}, b_{u_2}\}$. Thus, in x_2, $c_p = a_m$, and $c_m = a_p$. To get x_2 from x_1' by a type 2 covering relation, the entry d_l $(= a_m)$ is interchanged with $d_p = a_p$. Therefore, the index of a_p in x_2 is j, and this is a contradiction. Hence, $a_i \in \{b_{u_1}, b_{u_2}\}$, and hence, $F((x_1', x_2)) = (a_i, a_j) = F((x_0, x_1))$.

We proceed with (b); $a_i = a_l$. Then, $a_j < a_m$. Notice that, $m < j$. Let $F((x_1, x_2)) = (b_{u_1}, b_{u_2})$ for some $1 \leq u_1 < u_2 \leq n$. It is easy to see that $a_m \in \{b_{u_1}, b_{u_2}\}$. If $a_i \in \{b_{u_1}, b_{u_2}\}$, we are done. So, assume that $a_i \notin \{b_{u_1}, b_{u_2}\}$. It is easy to check that, $a_j \in \{b_{u_1}, b_{u_2}\}$ implies $F((x_1', x_2)) = F((x_0, x_1))$ (draw a picture). Assume that $a_j \notin \{b_{u_1}, b_{u_2}\}$. Then, there exists an entry a_p of x_0 such that $a_p \notin \{a_i, a_j = a_l, a_m\}$ and $\{a_p, a_m\} = \{b_{u_1}, b_{u_2}\}$. Thus, in x_2, $c_p = a_m$, and $c_m = a_p$. Since $type(x_1', x_2) = 2$, d_i, $(= a_m)$ is intrechanged with $d_p = a_p$. Then, the index of a_p in x_2 is i, and this is a contradiction, as before. Therefore, $a_j \in \{b_{u_1}, b_{u_2}\}$, and hence $F((x_1', x_2)) = (a_i, a_j) = F((x_0, x_1))$. The proof is complete.

□

As a corollary of the proof of the Proposition 4.3, it is easy to see that

Corollary 4.4. *Let $[x, y] \subseteq R_n$ be an interval of length 2. Then, $[x, y]$ is either a chain, or a diamond. In other words, either $[x, y] = \{x, x_1, x_1', y\}$ with $x < x_1 \neq x_1' < y$, or $[x, y] = \{x, x_1, y\}$ with $x < x_1 < y$.*

Lemma 4.5. *We use the notation of Proposition 4.3. Then, lexicographically smallest chain $\mathfrak{c}:\ x_0 < x_1 < x_2$ is the unique increasing maximal chain in $[x, y]$.*

Proof. Assume that there exists another increasing chain $x_0 < x_1' < x_2$ between $x = x_0$ and $y = x_2$. By the Lemma 4.3, either $F((x_0, x_1)) = F((x_1', x_2))$, or $F((x_1, x_2)) = F((x_0, x_1'))$. First assume that $F((x_0, x_1)) = F((x_1', x_2))$. Since $F((x_0, x_1))$ is lexicographically smallest in

$$\{F((x_0, z)) :\ (x_0, z) \in C(R_n)\},$$

we have $F((x_0, x_1)) = F((x_1', x_2)) < F((x_0, x_1'))$; a contradiction. So, we may assume that $F((x_1, x_2)) = F((x_0, x_1'))$, Let $F((x_1, x_2)) = (c, d)$, and let $F((x_0, x_1)) = (a, b)$, so that $(a, b) < (c, d)$. Let $F((x_1', x_2)) = (e, f)$. If $(e, f) > (c, d)$, then either $e > c$ or $f > max\{a, b, c, d\}$. Then, either e or f has to appear as an entry in x_2. This is impossible because the difference between the set of entries of x_2 and the set of entries of x_0 lies in the set $\{a, b, c, d\}$. This contradiction shows that $F((x'_1, x_2)) < F((x_0, x'_1))$. □

Proposition 4.6. *We use the notation of Proposition 4.1. There exists a unique maximal chain* $x = x_0 < \cdots < x_k = y$ *with* $F((x_0, x_1)) \leq \cdots \leq F((x_{k-1}, x_k))$.

Proof. We already know that the lexicographically first chain is increasing. Therefore, it is enough to show that there is no other increasing chain. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. If $\ell(y)-\ell(x) = 2$, then this is done by the Lemma 4.5. So, we assume that for any interval of length $k > 2$ there exists a unique increasing maximal chain.

Let $[x, y] \subseteq R_n$ be an interval of length $k + 1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

be the maximal chain such that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another increasing chain

$$\mathfrak{c}' : x = x_0 < x'_1 < \cdots < x'_k < x_{k+1} = y.$$

Since the length of the chain

$$x'_1 < \cdots < x'_k < x_{k+1} = y$$

is k, by the induction hypotheses, it is the lexicographically first chain between x'_1 and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $type(x_0, x_1) = 1$, and $type(x_0, x'_1) = 1$,
Case 2: $type(x_0, x_1) = 1$, and $type(x_0, x'_1) = 2$,
Case 3: $type(x_0, x_1) = 2$, and $type(x_0, x'_1) = 1$,
Case 4: $type(x_0, x_1) = 2$, and $type(x_0, x'_1) = 2$.

Let $x_0 = (a_1, ..., a_n)$, $x_1 = (b_1, ..., b_n)$, and $x'_1 = (c_1, ..., c_n)$.

Case 3: Suppose that x_1 covers x_0 by interchanging a_i and a_j (where $i < j$), and that x'_1 covers x_0 by the type 1; replacing a_r with c_r. Since $(a_i, a_j) = F((x_0, x_1)) \leq F((x_0, x'_1)) = (a_r, c_r)$, $a_i \leq a_r < c_r$. In fact, $(a_i, a_j) < (a_r, c_r)$.

Assume first that $r < i$. Define $z = (e_1, ..., e_n) \in R_n$ by $e_k = a_k$ for $k \notin \{r, i, j\}$ and $e_r = c_r$, $e_i = a_j$ and $e_j = a_i$. It is easy to check that z covers x'_1, and $F((x'_1, z)) = ((a_i, a_j))$. Since the Jordan-Hölder sequence of $x'_1 < \cdots < x'_n < x_{n+1} = y$ is lexicographically smallest in $[x'_1, y]$, and since $F(\mathfrak{c}')$ is increasing,

$$(a_r, c_r) = F((x, x'_1)) \leq F((x'_1, x'_2)) \leq F((x'_1, z)) = (a_i, a_j).$$

This contradicts $(a_i, a_j) < (a_r, c_r)$. Therefore, we may assume that $r \geq i$. A similar argument shows that we may assume $r \leq j$.

Next, assume that $r = i$. Since $type(x_0, x_1') = 1$, any number between a_i and c_i has to occur before the i'th position. This contradicts $(a_i, a_j) < (a_r, c_r) = (a_i, c_r)$.

Next, assume that $r = j$. Since $type(x_0, x_1') = 1$, any number between a_j and c_j has to occur before the j'th position. If all of them occur before i'th position, we define $z = (e_1, ..., e_n)$ by $e_k = a_k$ for $k \notin \{i, j\}$ and $e_i = c_j$, $e_j = a_i$. Then, z covers x_1' and $F((x_1', z)) = (a_i, c_j)$. This contradicts

$$(a_j, c_j) = F((x, x_1')) \leq F((x_1', x_2')) \leq F((x_1', z)) = (a_i, c_j).$$

If any of the numbers between a_j and c_j occur between the i'th and the j'th positions, define $z = (e_1, ..., e_n)$ as follows. Let $i < m < j$ be the smallest number such that $a_j < a_m < c_j$. Let $e_k = a_k$ for $k \notin \{i, m, j\}$, and let $e_i = a_m,\ e_m = a_i$, $e_j = c_j$. Then, z covers x_1', and $F((x_1', z)) = (a_i, a_m)$. Since $(a_j, c_j) = F((x, x_1'))$, we find a contradiction, as before.

Finally, assume that $i < r < j$. Define $z = (e_1, ..., e_n)$ by $e_k = a_k$ for $k \notin \{i, r, j\}$, and $e_i = a_j$, $e_j = a_i$, $e_r = c_r$. It is easy to check that z covers x_1', and that $F((x_1', z)) = (a_i, a_j)$. Since, $(a_j, c_j) = F((x_0, x_1'))$, we find a contradiction, as before. This finishes Case 3.

Case 4: Suppose that x_1 covers x_0 by interchanging a_i and a_j (where $i < j$), and that x_1' covers x_0 by interchanging a_r and a_s (where $r < s$). In the following situations "$r < s < i < j$, $i < j < r < s$, $r < i < j < s$, $i < r < s < j$, $i < r < j < s$, $r < i = s < j$, $r < i < s = j$," define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \notin \{i, j\}$, and $e_i = a_j = b_i$, $e_j = a_i = b_j$. Then, z covers x_1', and $F((x_1', z)) = (a_i, a_j)$. This contradicts

$$(a_r, a_s) = F((x, x_1')) \leq F((x_1', x_2')) \leq F((x_1', z)) = (a_i, a_j).$$

The remaining situations are "$r \leq i \leq s \leq j$," and "$i \leq r \leq j \leq s$."

If $r < i < s < j$, define $z = (e_1, ..., e_n)$ by $e_k = b_k$ for $k \notin \{i, s\}$, and $e_i = b_s = a_r$, $e_s = b_i = a_i$. Then, $F((x_1', z)) = (a_i, a_r)$. The contradiction is found as usual.

If $r = i < s < j$, define $z = (e_1, ..., e_n)$ by $e_k = b_k$ for $k \notin \{s, j\}$, and let $e_s = b_j = a_j$, $e_j = b_s = a_i$. Then, $F((x_1', z)) = (a_i, a_j)$. The contradiction is found as usual.

If $r = i < j < s$, since $type(x_0, x_1') = 2$, we see that $a_j > a_s$. This contradicts $F((x_0, x_1)) = (a_i, a_j) < (a_i, a_s) = F((x_0, x_1'))$.

Next, assume that $i < r = j < s$. Assume also that there exists an index $i < m < j$ such that $a_j < c_m = a_m < a_s$. Let m' be the smallest such index. Define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \notin \{i, m'\}$, and $e_i = c_{m'} = a_{m'}$, $e_{m'} = c_i = a_i$. Then, $F((x_1', z)) = (a_i, a_{m'}) < (a_j, a_s) = F((x_0, x_1'))$. This gives a contradiction as before. Therefore, we may assume that there does not exist any $i < m < j$ such that $a_j < a_m < a_s$. Then, for any $i < m < j$ we have either $c_m = a_m < c_i = a_i$, or $c_m = a_m > c_j = a_s$. In this case, define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \notin \{i, m'\}$, and $e_i = c_{m'} = a_{m'}$, $e_{m'} = c_i = a_i$. Then, $F((x_1', z)) = (a_i, a_{m'}) < (a_j, a_s) = F((x_0, x_1'))$ provides a contradiction, as before.

The final case is $i < r < j = s$. Observe that $a_r < a_i$ is forced. Thus, $F((x_0, x_1)) = (a_i, a_j) < (a_r, a_s) = F((x_0, x_1'))$ is a contradiction. Notice $a_i = a_r = 0$ is impossible, too. This finishes Case 4.

Case 1: There exists $1 \leq i \leq n$ such that $b_k = a_k$ for all $k \neq i$, and $b_i > a_i$, and there exists $1 \leq r \leq n$ such that $c_k = a_k$ for $k \neq r$, and $c_r > a_r$. Note that $r = i$ is impossible. Define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F((x_1', z)) = (a_i, b_i) < (a_r, c_r) = F((x_0, x_1'))$. The contradiction is found as usual.

Case 2: Suppose that x_1 covers x_0 by replacing a_i by b_i, and x_1' covers x_0 by interchanging a_r and a_s, where $r < s$.

If $i \leq r < s$, define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F((x_1', z)) = (a_i, b_i) < (a_r, a_s) = F((x_0, x_1'))$. The contradiction is found as usual.

Assume that $r < i < s$. Observe that a_r cannot be equal to a_i, otherwise, $a_r = a_i = 0$ forcing $type(x_0, x_1') \neq 2$. Therefore, we may assume that $a_i < a_r$. Then, either $b_i < a_r$, or $a_i < a_r < b_i$. If $a_r > b_i$, define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F((x_1', z)) = (a_i, b_i) < (a_r, a_s) = F((x_0, x_1'))$. This gives a contradiction as before. So, we assume that $a_i < a_r < b_i$.

Since $type(x_0, x_1) = 1$, any number between a_i and b_i (hence, any number between a_i and a_r) occur before i'th position. Since $type(x_0, x_1') = 2$, we know that $c_s = a_r$ and $c_i = a_i$, and furthermore if $i < k < s$, then either $c_k < a_r = c_s$, or $c_k > a_s = c_r$. Define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \notin \{i, s\}$, and $e_i = c_s = a_r$, $e_s = c_i = a_i$. Clearly $x_1' \leq z$. For $i < k < s$, either $e_k < a_i = c_i$, or $e_k > c_s = a_r$. Therefore, z covers x_1' and $F((x_1', z)) = (a_i, a_r) < (a_r, a_s) = F((x_0, x_1'))$ gives a contradiction, as before.

If $r < s < i$, define $z = (e_1, ..., e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F((x_1', z)) = (a_i, b_i) < (a_r, a_s) = F((x_0, x_1'))$ gives a contradiction, as before.

Finally, observe that $r < i = s$ is impossible. Otherwise a_r has to be less than a_i which contradicts the assumption that $F((x_0, x_1)) = (a_i, b_i) < (a_r, a_i) = F((x_0, x_1'))$. This finishes Case 2, and the proof is complete. □

Proof of Theorem 3.2. Let $\Gamma = \{0, 1, ..., n\} \times \{0, 1, ..., n\}$, and let $F : C(R_n) \longrightarrow \Gamma$ be the edge-labeling, as defined in (3.1). By Propositions 4.1 and 4.6, $F : C(R_n) \longrightarrow \Gamma$ is an $EL-$labeling .

5. FINAL REMARKS

Let P be a finite graded poset of rank n. Let $\hat{P}$ denote $P \cup \{\hat{0}, \hat{1}\}$. The Möbius function $\mu : I(P) \longrightarrow \mathbb{Z}$ is an integer valued function defined on the set of all intervals of $\hat{P}$, uniquely determined by the following conditions

$$\mu([x, y]) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu([x, z]) & \text{if } x < y. \end{cases}$$

It is well known that $\mu(\hat{P}) := \mu([\hat{0}, \hat{1}])$ is the reduced Euler characteristic of the simplicial complex $\Delta(P)$ of all chains in P.

Let $R_{n,k}$, $k = 0, \ldots, n$ denote the subposet of rank $k = 0, \ldots, n$ elements in R_n. In [1] it is shown that the Möbius function on $I(R_{n,k})$ takes values in $\{-1, 0, 1\}$. When $k = n$ $R_{n,k} = S_n$ is the symmetric group and the Möbius function on S_n is well known: [20], [18], [8]. It seems that, at the time of writing of this article the determination of the Möbius function on the whole R_n is still open. We wish to tackle this problem in a forthcoming article.

When $\hat{P}$ is an $EL-$shellable poset, $\Delta(P)$ has the homotopy type of a wedge of spheres or a ball. See Section 4.7 of [15]. As a corollary of our Theorem 3.2 we have

Corollary 5.1. *The order complex $\Delta([x, y])$ of an interval $[x, y] \subseteq R_n$ has the homotopy type of a wedge of spheres or a ball.*

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