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On the quiver of the descent algebra ☆

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ABSTRACT

We study the quiver of the descent algebra of a finite Coxeter group W. The results include a derivation of the quiver of the descent algebra of types A and B. Our approach is to study the descent algebra as an algebra constructed from the reflection arrangement associated to W.

Contents

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1. Introduction

The descent algebra $\Sigma_k(W)$ of a finite Coxeter group W is a highly exceptional subalgebra of the group algebra of W. First introduced by Louis Solomon in 1976 [Sol76], it has enjoyed much attention because of several connections with various areas of mathematics, including the representation theory of Coxeter groups, free Lie algebras and higher Lie modules, Hochschild homology, and probability. These connections are described in a survey article by Manfred Schocker [Sch04].

In this article we study the quiver of the descent algebra. Our approach is to use a result of T.P. Bidigare that identities the descent algebra with the W-invariant subalgebra $(k\mathcal{F})^W$ of a semigroup algebra $k\mathcal{F}$ associated to the reflection arrangement of W [Bid97]. Then using results about $k\mathcal{F}$ we deduce some general properties about the quiver of the descent algebra and determine the quiver of the descent algebras of type A and B. The quiver of the descent algebra of type A has already been computed [Sch04], but the quiver of the descent algebra of type B was not previously known.

We briefly outline the contents and structure of the article. Section 2 defines finite Coxeter groups and reflection arrangements, and explains the connection between the descent algebra $\Sigma_k(W)$ and the W-invariant subalgebra $(k\mathcal{F})^W$. Section 3 recalls definitions and results about quivers of split basic algebras. In Section 4 we provide a proof that $k\mathcal{F}$ and $(k\mathcal{F})^W$ are split basic algebras, so there are (canonical) quivers associated to each. Section 5 constructs a complete system of primitive orthogonal idempotents for $k\mathcal{F}$ that leads to a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$. This allows us to define, in Section 6, a W-equivariant surjection $\varphi : k\mathcal{Q} \twoheadrightarrow k\mathcal{F}$, where $\mathcal{Q}$ is the quiver of $k\mathcal{F}$. We use this surjection in Section 7 to prove some general properties of the quiver of the descent algebra and in Sections 8 and 9 to determine the quiver of the descent algebras of types A and B, respectively. Finally, Section 10 discusses some future directions for this project.

The interested reader may also decide to consult recent work of Götz Pfeiffer who is taking a different approach to the problem of determining the quiver of the descent algebras [Pfe07].

2. The geometric approach to the descent algebra

For an introduction to the theory of Coxeter groups, see the books [Bro89,Hum90,Kan01,BB05]. The reader may wish to read Section 2.1 and Section 2.3 alongside Section 2.4 since the latter presents these ideas for the symmetric group $\mathcal{S}_n$. Also see Section 9.1, which describes some of these ideas for the hyperoctahedral group $\mathcal{B}_n$.

2.1. Coxeter systems and reflection arrangements

Let V be a finite-dimensional real vector space. A ***finite Coxeter group*** W is a finite group generated by a set of reflections of V. The ***reflection arrangement*** of W is the hyperplane arrangement $\mathcal{A}$ consisting of the hyperplanes of V fixed by some reflection in W. The connected components of the complement of $\bigcup_{H\in\mathcal{A}} H$ in V are called ***chambers***. A ***wall*** of a chamber c is a hyperplane $H \in \mathcal{A}$ such that $H \cap \bar{c}$ spans H, where $\bar{c}$ denotes the closure of the set c.

Fix a chamber c and let $S \subseteq W$ denote the set of reflections in the walls of c. Then S is a generating set of W [Bro89, §I.5A]. The pair (W, S) is called a ***Coxeter system***, and c is the ***fundamental chamber*** of (W, S).

2.2. The descent algebra

Fix a Coxeter system (W, S). For $J \subseteq S$, let $W_J = \langle J\rangle$ denote the subgroup of W generated by the elements in J. Each coset of W_J in W contains a unique element of minimal length, where the ***length*** $\ell(w)$ of an element w of W is the smallest number of generators $s_1, \dots, s_i \in S$ such that $w = s_1 \cdots s_i$ [Hum90, Proposition 1.10(c)].

For $J \subseteq S$, let X_J denote the set of ***minimal length coset representatives*** of W_J and let $x_J = \sum_{w\in X_J} w$ denote the sum of the elements of X_J in the group algebra kW of W with coefficients in a field k. Louis Solomon proved that the elements x_J form a k-vector space basis of a subalgebra of kW [Sol76, Theorem 1]. This subalgebra is denoted by $\Sigma_k(W)$ and is called the ***descent algebra*** of W. Throughout k will be a field of characteristic that does not divide the order of W.

2.3. The geometric approach to the descent algebra

Let (W, S) be a finite Coxeter system with fundamental chamber c, and let $\mathcal{A}$ be the reflection arrangement of W. The reader may want to read this section alongside Section 2.4.

2.3.1. Face semigroup algebra

For each hyperplane $H \in \mathcal{A}$, let H^+ and H^- denote the two open half spaces of V determined by H. The choice of labels H^+ and H^- is arbitrary, but fixed throughout. For convenience, let $H^0 = H$. A ***face*** of $\mathcal{A}$ is a nonempty intersection of the form $\bigcap_{H\in\mathcal{A}} H^{\sigma_H}$, where $\sigma_H \in \{+, 0, -\}$ for each hyperplane $H \in \mathcal{A}$. The sequence $(\sigma_H)_{H\in\mathcal{A}}$ is called the ***sign sequence*** of the face. We denote the sign sequence of a face x by $\sigma(x) = (\sigma_H(x))_{H\in\mathcal{A}}$.

Let $\mathcal{F}$ denote the set of all faces of $\mathcal{A}$. Define the product of two faces $x, y \in \mathcal{F}$ to be the face xy with sign sequence

$$\sigma_H(xy) = \begin{cases} \sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\ \sigma_H(y), & \text{if } \sigma_H(x) = 0, \end{cases}$$

where $\sigma(x)$ and $\sigma(y)$ are the sign sequences of x and y. This product has a geometric interpretation: xy is the face entered by moving a small positive distance along a straight line from a point in x towards a point in y. In the special case where y is a chamber, the product xy is the chamber that has x as a face and that is separated from y by the fewest number of hyperplanes in $\mathcal{A}$ [BD98, §2C]. It is straightforward to verify that this product gives $\mathcal{F}$ the structure of an associative semigroup with identity, and that $x^2 = x$ and $xyx = xy$ for all $x, y \in \mathcal{F}$. Semigroups satisfying these identities are called ***left regular bands***.

The semigroup algebra $k\mathcal{F}$ is called the ***face semigroup algebra*** of $\mathcal{A}$ over the field k. It consists of finite k-linear combinations of elements of $\mathcal{F}$ with multiplication extended k-linearly from the product of $\mathcal{F}$.

The semigroup $\mathcal{F}$ is also a partially ordered set with respect to the relation $x \leqslant y$ if and only if $xy = y$. Equivalently, $x \leqslant y$ if and only it $x \subseteq \bar{y}$, where $\bar{y}$ denotes the closure of the set y. Note that the chambers of the arrangement are precisely the faces that are maximal with respect to this partial order. If $x \leqslant y$, then we say that x is a face of y or that y contains x as a face.

2.3.2. Support map and intersection lattice

For each face $x \in \mathcal{F}$, the ***support*** of x, denoted by $\mathrm{supp}(x)$, is the intersection of all hyperplanes in $\mathcal{A}$ that contain x. Equivalently, $\mathrm{supp}(x)$ is the subspace of V spanned by the vectors in x. The ***dimension*** of x is the dimension of the subspace $\mathrm{supp}(x)$.

The ***intersection lattice*** $\mathcal{L}$ of $\mathcal{A}$ is the image of supp; that is, $\mathcal{L} = \mathrm{supp}(\mathcal{F})$. The elements of $\mathcal{L}$ are subspaces of V and are ordered by inclusion. (N.B. Some authors order $\mathcal{L}$ by reverse inclusion rather than inclusion.) With this partial order, $\mathcal{L}$ is a finite lattice, where the meet $(\vee)$ of two subspaces is their intersection, and the join $(\wedge)$ of two subspaces is the smallest subspace that contains both.

It is straightforward to show that $\mathrm{supp}(x) \leqslant \mathrm{supp}(y)$ for all $x, y \in \mathcal{F}$ with $x \leqslant y$. Therefore, supp is an order-preserving poset surjection. Moreover, $\mathrm{supp}(xy) = \mathrm{supp}(x) \vee \mathrm{supp}(y)$ for all $x, y \in \mathcal{F}$, so supp is also a semigroup homomorphism, where $\mathcal{L}$ is viewed as a semigroup with product $\vee$. The elements of $\mathcal{F}$ also satisfy $xy = x$ if $\mathrm{supp}(x) \geqslant \mathrm{supp}(y)$. Proofs of these statements can be found in [Bro00, Appendix A].

2.3.3. Invariant subalgebra

Since W is a group of orthogonal transformations of the vector space V, there is a natural action of W on V: the action of $w \in W$ on $\vec{v} \in V$ is the image of $\vec{v}$ under the transformation w. This action permutes the set $\mathcal{A}$ [Hum90, Proposition 1.2], so it induces an action of W on $\mathcal{L}$ and on $\mathcal{F}$. This induced action preserves the semigroup structure of $\mathcal{F}$ and $\mathcal{L}$, so it extends linearly to an action on $k\mathcal{F}$ and $k\mathcal{L}$.

Let $(k\mathcal{F})^W$ denote the subalgebra of $k\mathcal{F}$ consisting of the elements of $k\mathcal{F}$ fixed by all elements of W:

$$(k\mathcal{F})^W = \{a \in k\mathcal{F}\colon\ w(a) = a \text{ for all } w \in W\}.$$

The following was first proved by T.P. Bidigare [Bid97]. Another proof was given by K.S. Brown [Bro00, Theorem 7].

Theorem 2.1 *(T.P. Bidigare). Let W be a finite reflection group and let $k\mathcal{F}$ denote the face semigroup algebra of the reflection arrangement of W. The W-invariant subalgebra $(k\mathcal{F})^W$ is anti-isomorphic to the descent algebra $\Sigma_k(W)$ of W.*

We briefly describe an anti-isomorphism. The faces of the fundamental chamber c are parametrized by the subsets of S: if $J \subseteq S$, then there is a unique face c_J of c that is fixed by all elements in J [Bro89, §I.5F]. Furthermore, every face of $\mathcal{A}$ is in the W-orbit of a unique face of c [Bro89, §I.5F]. So if $\mathcal{O}_J$ denotes the W-orbit of c_J, then the elements $\boldsymbol{x}_J = \sum_{y \in \mathcal{O}_J} y$ form a basis of $(k\mathcal{F})^W$. The map defined by sending $\boldsymbol{x}_J$ to x_J is an anti-isomorphism from $(k\mathcal{F})^W$ onto $\Sigma_k(W)$.

2.4. The symmetric group

We describe the above ideas in combinatorial terms for the symmetric group $\mathcal{S}_n$. The results in this section are not crucial to what follows, and will only be used in the proof of Theorem 8.1 to give a combinatorial description of the quiver of the descent algebra $\Sigma_k(\mathcal{S}_n)$.

For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. A ***set partition*** of $[n]$ is a collection of nonempty subsets $B = \{B_1, \ldots, B_r\}$ of $[n]$ such that $\bigcup_i B_i = [n]$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. The sets B_i in B are called the ***blocks*** of B. A ***set composition*** of $[n]$ is an ordered set partition of $[n]$, which we denote by $(B_1, \ldots, B_r)$. An ***integer partition*** of $n \in \mathbb{N}$ is a collection of positive integers that sum to n.

2.4.1. Braid arrangement

Fix $n \in \mathbb{N}$. The ***braid arrangement*** is the hyperplane arrangement $\mathcal{A}$ in $V = \mathbb{R}^n$ consisting of the hyperplanes $H_{i,j} = \{\vec{v} \in V\colon\ v_i = v_j\}$ for $1 \leqslant i < j \leqslant n$. The group of transformations generated by the reflections in the hyperplanes in $\mathcal{A}$ is identified with the symmetric group acting on V by permuting coordinates: $\omega(v_1, \ldots, v_n) = (v_{\omega^{-1}(1)}, \ldots, v_{\omega^{-1}(n)})$ for $\omega \in \mathcal{S}_n$ and $\vec{v} \in V$. The reflections in the hyperplanes $H_{i,j}$ correspond to the transpositions $(i, j) \in \mathcal{S}_n$.

2.4.2. Faces

Let $\vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ be a vector in a chamber c of the braid arrangement $\mathcal{A}$. Then $\vec{v}$ is not on any of the hyperplanes $H_{i,j}$, so all the coordinates of $\vec{v}$ are distinct. Therefore, there exists $\omega \in \mathcal{S}_n$ such that $v_{\omega(1)} < \cdots < v_{\omega(n)}$. All vectors in c satisfy this identity, so c can be identified with the permutation ω. The faces of c are obtained by changing some of the inequalities to equalities, so the ***faces*** of $\mathcal{A}$ can be identified with set compositions of $[n]$. For example, the set composition $(\{5\}, \{1,3,4\}, \{2,6\})$ is identified with the face $\{\vec{v} \in \mathbb{R}^5\colon\ v_5 < v_1 = v_3 = v_4 < v_2 = v_6\} = H^-_{1,5} \cap H_{1,3} \cap H_{3,4} \cap H^-_{2,4} \cap H_{2,6}$, where $H^-_{i,j} = \{\vec{v}\colon\ v_i > v_j,\ i < j\}$.

The ***partial order*** is given by $(B_1, \ldots, B_m) \leqslant (C_1, \ldots, C_l)$ if and only if $(C_1, \ldots, C_l)$ consists of a set composition of B_1, followed by a set composition of B_2, and so forth. The ***action*** of $\mathcal{S}_n$ on set compositions is given by permuting the underlying set: $\omega(B_1, \ldots, B_r) = (\omega(B_1), \ldots, \omega(B_r))$. And if $(B_1, \ldots, B_l)$ and $(C_1, \ldots, C_m)$ are set compositions of $[n]$, their ***product*** is the set composition of $[n]$ given by the formula

$$(B_1, \ldots, B_l)(C_1, \ldots, C_m) = (B_1 \cap C_1, \ldots, B_1 \cap C_m, \ldots, B_l \cap C_1, \ldots, B_l \cap C_m)^{\wedge},$$

where $^\wedge$ means "delete empty intersections."

2.4.3. Intersection lattice

The elements of the intersection lattice $\mathcal{L}$ of $\mathcal{A}$ are identified with set partitions of $[n]$ via the following bijection,

$$\{B_1,\ldots,B_r\} \leftrightarrow \{\vec{v} \in V\colon\ v_i = v_j \text{ if } i,j \in B_h \text{ for some } h \in [r]\} = \bigcap_{h=1}^{r} \biggl(\bigcap_{i,j\in B_h} H_{ij} \biggr),$$

where $\{B_1,\ldots,B_r\}$ is a set partition of $[n]$.

Under this identification, if B and C are set partitions of $[n]$, then $B \lessdot C$ if and only if B is obtained from C by merging two blocks of C. The action of $\mathcal{S}_n$ on $\mathcal{L}$ is given by $\omega(\{B_1,\ldots,B_r\}) = \{\omega(B_1),\ldots,\omega(B_r)\}$. The ***support map*** sends a set composition $(B_1,\ldots,B_m)$ to the underlying set partition $\{B_1,\ldots,B_m\}$.

The $\mathcal{S}_n$-orbit of a set partition $\{B_1,\ldots,B_m\}$ of $[n]$ depends only on the sizes of the blocks B_i, so $\mathcal{L}/\mathcal{S}_n$ can be identified with the poset of integer partitions of n. Under this identification, for any two integer partitions p and q of n, we have $p \lessdot q$ if and only if p is obtained from q by adding two elements of q.

3. The quiver of a split basic algebra

This section recalls definitions and results from the theory of finite-dimensional algebras. Our main references are [ARS95,Ben98,ASS06].

Let k be a field and A a finite-dimensional k-algebra. An element $a \in A$ is an ***idempotent*** if $e^2 = e$. Two idempotents $e, f \in A$ are ***orthogonal*** if $ef = 0 = fe$. An idempotent $e \in A$ is ***primitive*** if it cannot be written as $e = f + g$ with f and g nonzero orthogonal idempotents of A. A ***complete system of primitive orthogonal idempotents*** of A is a set $\{e_1, e_2, \ldots, e_n\}$ of primitive idempotents of A that are pairwise orthogonal and that sum to $1 \in A$.

The ***Jacobson radical*** of A is the smallest ideal $\operatorname{rad}(A)$ of A such that $A/\operatorname{rad}(A)$ is semisimple. If $A/\operatorname{rad}(A)$ is isomorphic, as a k-algebra, to a direct product of copies of k, then A is said to be a ***split basic algebra***. Equivalently, A is a split basic algebra if and only if all the simple A-modules are one-dimensional.

The ***quiver*** of a split basic k-algebra A is the directed graph Q constructed as follows. Let $\{e_v\colon\ v \in \mathcal{V}\}$ be a complete system of primitive orthogonal idempotents of A, where $\mathcal{V}$ is some index set. There is one vertex v in Q for each idempotent e_v in $\{e_v\colon\ v \in \mathcal{V}\}$. If $x, y \in \mathcal{V}$, then the number of arrows in Q from x to y is $\dim_k e_y(\operatorname{rad}(A)/\operatorname{rad}^2(A))e_x$. This construction does not depend on the complete system of primitive orthogonal idempotents (see [Ben98, Definition 4.1.6] or [ASS06, Lemma II.3.2]).

If α is an arrow in a quiver (directed graph) beginning at a vertex x and ending at a vertex y, then we write $x \xrightarrow{\alpha} y$. If there is exactly one arrow from x to y, then we drop the label and write $x \to y$. The ***path algebra*** kQ of a quiver Q is the k-algebra with basis the set of paths in Q and with multiplication defined on paths by

$$\begin{aligned}
&(w_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_s} w_s) \cdot (v_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_r} v_r) \\
&\quad = \begin{cases} (v_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_r} v_r \xrightarrow{\alpha_1} w_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_s} w_s), & \text{if } w_0 = v_r, \\ 0, & \text{if } w_0 \neq v_r, \end{cases}
\end{aligned}$$

where $(w_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_s} w_s)$ and $(v_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_r} v_r)$ are paths in Q.

Let F denote the ideal in kQ generated by the arrows of Q. An ideal $I \subseteq kQ$ is said to be ***admissible*** if there exists an integer $m \geqslant 2$ such that $F^m \subseteq I \subseteq F^2$. This notion is useful for identifying the quiver of a split basic k-algebra as the following result demonstrates [ARS95, Theorem III.1.9(d)].

Theorem 3.1. *Q is the quiver of a finite-dimensional split basic k-algebra A if and only if $A \cong kQ/I$, where I is an admissible ideal of kQ. In particular, if $\varphi : kQ \twoheadrightarrow A$ is a surjection of k-algebras with an admissible kernel, then Q is the quiver of A.*

The following result will be helpful to define k-algebra morphisms. A proof can be found in [ASS06, Theorem II.1.8].

Theorem 3.2. *Let Q be a finite quiver and A a finite-dimensional k-algebra. If f is a function from the set of vertices and arrows of Q into A such that*

(1) $\sum_v f(v) = 1$, $f(v)^2 = f(v)$ *and* $f(u)f(v) = 0$ *for all vertices* u, v, *and*
(2) $f(u \to v) = f(v)f(u \to v)f(u)$ *for every arrow* $u \to v$,

then there exists a unique k-algebra homomorphism $\varphi : kQ \to A$ such that $\varphi(v) = f(v)$ and $\varphi(u \to v) = f(u \to v)$ for all vertices v and all arrows $u \to v$ of Q.

4. $k\mathcal{F}$ and $(k\mathcal{F})^W$ are split basic algebras

This section establishes that $k\mathcal{F}$ and $(k\mathcal{F})^W$ are split basic algebras. That $(k\mathcal{F})^W$ is a split basic algebra follows from various sources since the irreducible representations of the descent algebra are known to be one-dimensional (see, for example, [Sol76, Theorem 3]). We give a proof based on [Bid97, Bro00].

Proposition 4.1. *$k\mathcal{F}$ and $(k\mathcal{F})^W$ are split basic algebras.*

Proof. We begin by showing that $k\mathcal{F}$ is a split basic algebra. As mentioned in Section 2.3.2, the support map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ is a surjective semigroup homomorphism. Therefore, it extends linearly to a surjective k-algebra homomorphism $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$. The algebra $k\mathcal{L}$ is isomorphic to the k-algebra $\prod_{X \in \mathcal{L}} k$. Indeed, the elements defined recursively by the formula $E_X = X - \sum_{Y > X} E_Y$, one for each $X \in \mathcal{L}$, form a basis and a complete system of primitive orthogonal idempotents for $k\mathcal{L}$ [Sol67]. Since the kernel of supp is nilpotent, standard ring theory implies that $\ker(\operatorname{supp}) = \operatorname{rad}(k\mathcal{F})$. It follows that $k\mathcal{F}$ is a split basic algebra.

Since $\operatorname{supp} : k\mathcal{F} \twoheadrightarrow k\mathcal{L}$ is a surjective W-equivariant algebra homomorphism, it restricts to an algebra surjection $(k\mathcal{F})^W \twoheadrightarrow (k\mathcal{L})^W$, where $(k\mathcal{L})^W$ is the W-invariant subalgebra of $k\mathcal{L}$. Let E_X be the elements defined above. Since $w(E_X) = E_{w(X)}$ for all $w \in W$ and $X \in \mathcal{L}$, it follows that the elements $\sum_{X \in \mathcal{O}} E_X$, one for each W-orbit $\mathcal{O}$ of elements of $\mathcal{L}$, form a basis and a complete system of primitive orthogonal idempotents for $(k\mathcal{L})^W$. Thus, $(k\mathcal{L})^W \cong \prod_{\mathcal{O} \in \mathcal{L}/W} k$. Since $(k\mathcal{L})^W$ is semisimple and $\ker(\operatorname{supp}|_{(k\mathcal{F})^W})$ is nilpotent (because $\ker(\operatorname{supp})$ is), it follows that the radical of $(k\mathcal{F})^W$ is $\ker(\operatorname{supp}|_{(k\mathcal{F})^W})$. Thus, $(k\mathcal{F})^W$ is a split basic algebra. □

5. Complete systems of primitive orthogonal idempotents

In this section we construct a complete system of primitive orthogonal idempotents for $k\mathcal{F}$ that is permuted by the elements of W. This allows us to construct a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$. A complete system of primitive orthogonal idempotents for $\Sigma_k(W)$ was constructed previously [BBHT92], but the construction presented here is new and better suited to our needs because of the close relationship between the two systems.

For each $X \in \mathcal{L}$ let $\mathcal{O}_X = \{w(X)\colon\ w \in W\}$ denote the W-orbit of X. These orbits form a poset $\mathcal{L}/W = \{\mathcal{O}_X\colon\ X \in \mathcal{L}\}$ with partial order given by $\mathcal{O}_X \leqslant \mathcal{O}_Y$ if and only if there exists $w \in W$ with $w(X) \leqslant Y$.

Remark 5.1. The poset $\mathcal{L}/W$ is isomorphic to a poset of equivalence classes of subsets of S. Indeed, define a relation on subsets $J, K \subseteq S$ by setting $J \sim K$ if and only if $\operatorname{supp}(c_J)$ and $\operatorname{supp}(c_K)$ belong to the same W-orbit, where c_J and c_K are the largest faces of the fundamental chamber c that are fixed by J and K, respectively. Equivalently, $J \sim K$ if and only if W_J and W_K are conjugate subgroups of W. The poset $S/{\sim}$, with partial order induced by reverse inclusion of subsets of S, is isomorphic to $\mathcal{L}/W$.

Theorem 5.2. *For each $X \in \mathcal{L}$, fix a linear combination ℓ_X of faces of support X whose coefficients sum to* 1 *and suppose that they satisfy the identity*

$$w(\ell_X) = \ell_{w(X)} \quad \text{for all } w \in W,\ X \in \mathcal{L}. \tag{5.1}$$

Then the elements defined recursively using the equation

$$e_X = \ell_X - \sum_{Y > X} \ell_X e_Y \tag{5.2}$$

one for each $X \in \mathcal{L}$, form a complete system of primitive orthogonal idempotents for $k\mathcal{F}$, and they satisfy $w(e_X) = e_{w(X)}$ for every $w \in W$ and $X \in \mathcal{L}$. The elements

$$\varepsilon_{\mathcal{O}} = \sum_{X \in \mathcal{O}} e_X, \tag{5.3}$$

one for each $\mathcal{O} \in \mathcal{L}/W$, form a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$.

Examples of elements ℓ_X satisfying the above hypotheses will be presented below.

Proof. In [Sal07, Theorem 4.2] and [Sal08a, Theorem 5.2] it was shown that the elements e_X form a complete system of primitive orthogonal idempotents for $k\mathcal{F}$. (This is proved by first establishing Lemma 5.3 below and inducting on the codimension of X in V.) Induction on the codimension of $X \in \mathcal{L}$ establishes that $w(e_X) = e_{w(X)}$ for all $w \in W$ and all $X \in \mathcal{L}$. Therefore, the elements $\sum_{Y \in \mathcal{O}} e_Y$ are invariant under the action of W, so they belong to $(k\mathcal{F})^W$. They are orthogonal idempotents since sums of orthogonal idempotents are again orthogonal idempotents. They sum to 1 since $\sum_{X \in \mathcal{L}} e_X = 1$. Finally, they are primitive because there are enough of them: the number of elements in a complete system of primitive orthogonal idempotents for a split basic algebra A is the dimension of $A/\operatorname{rad}(A)$ (this follows from [Ben98, Corollary 1.7.4]), which in this case is $|\mathcal{L}/W|$ by the proof of Proposition 4.1. □

The idempotents e_X satisfy the following remarkable property that we will use on occasion. A proof can be found in [Sal07, Lemma 4.1] and [Sal08a, Lemma 5.1].

Lemma 5.3. *(See [Sal07,Sal08a].) Let $y \in \mathcal{F}$ and $X \in \mathcal{L}$. If $\operatorname{supp}(y) \not\leqslant X$, then $ye_X = 0$.*

Next we present some examples of elements ℓ_X satisfying the above hypotheses.

5.1. First complete system

For each $X \in \mathcal{L}$, let ℓ_X denote the normalized sum of all faces of support X:

$$\ell_X = \frac{1}{\#\{x \in \mathcal{F}\colon\ \operatorname{supp}(x) = X\}} \left(\sum_{\operatorname{supp}(x) = X} x \right).$$

Then $w(\ell_X) = \ell_{w(X)}$ for all $w \in W$ and $X \in \mathcal{L}$.

5.2. Second complete system

For every orbit $\mathcal{O} \in \mathcal{L}/W$, fix a face $f_{\mathcal{O}} \in \mathcal{F}$ such that $\operatorname{supp}(f_{\mathcal{O}}) \in \mathcal{O}$. For each $X \in \mathcal{L}$, let $f_X = f_{\mathcal{O}_X}$ and define

$$\ell_X = \frac{1}{L_X}\Bigg(\sum_{\substack{z \in \mathcal{O}_{f_X} \\ \operatorname{supp}(z)=X}} z\Bigg), \quad \text{where } L_X = \big|\big\{z \in \mathcal{O}_{f_X}\colon \operatorname{supp}(z) = X\big\}\big|. \tag{5.4}$$

Note that L_X is the index of the stabilizer subgroup W_x of x, where x is any face of support X, in the stabilizer subgroup W_X of X. It follows that every $w \in W$ induces a bijection between

$$T_X = \big\{z \in \mathcal{O}_{f_X}\colon \operatorname{supp}(z) = X\big\} \quad \text{and} \quad T_{w(X)} = \big\{z \in \mathcal{O}_{f_{w(X)}}\colon \operatorname{supp}(z) = w(X)\big\},$$

so $w(\ell_X) = \ell_{w(X)}$ for all $X \in \mathcal{L}$ and all $w \in W$.

5.3. Third complete system

If (W, S) is a Coxeter system with fundamental chamber c, then the faces of c are parametrized by the subsets of S: if $J \subseteq S$, then there is a unique largest face c_J of the fundamental chamber c that is fixed by all elements of J [Bro89, §I.5F]. For $J \subseteq S$, let $\boldsymbol{x}_J$ denote the sum of the faces in the W-orbit of c_J (see also Section 2.3.3). For each orbit $\mathcal{O} \in \mathcal{L}/W$, fix a subset $J_{\mathcal{O}} \subseteq S$ such that $\operatorname{supp}(c_{J_{\mathcal{O}}}) \in \mathcal{O}$ and define numbers

$$L_{\mathcal{O}} = \big|\big\{z \in \mathcal{O}_{x_{J_{\mathcal{O}}}}\colon \operatorname{supp}(z) = \operatorname{supp}(x_{J_{\mathcal{O}}})\big\}\big|.$$

Proposition 5.4. *The elements $\varepsilon_{\mathcal{O}}$, one for each $\mathcal{O} \in \mathcal{L}/W$, defined recursively by the formula*

$$\varepsilon_{\mathcal{O}} = \frac{1}{L_{\mathcal{O}}}\boldsymbol{x}_{J_{\mathcal{O}}} - \sum_{\mathcal{O}' > \mathcal{O}} \left(\frac{1}{L_{\mathcal{O}}}\boldsymbol{x}_{J_{\mathcal{O}}}\right)\varepsilon_{\mathcal{O}'},$$

form a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$.

Proof. For each $\mathcal{O} \in \mathcal{L}/W$, let $f_{\mathcal{O}} = x_{J_{\mathcal{O}}}$, and let $f_X = f_{\mathcal{O}_X}$ for each $X \in \mathcal{L}$. Define ℓ_X using Eq. (5.4). Then an induction on the corank of $\mathcal{O}$ establishes that the elements defined by Eq. (5.3) are equal to the elements $\varepsilon_{\mathcal{O}}$ defined above. □

Remark 5.5. Proposition 5.4 leads to a construction of a complete system of primitive orthogonal idempotents directly within the descent algebra $\Sigma_k(W)$. Let $S/{\sim}$ denote the poset defined in Remark 5.1. For each $\mathcal{O} \in S/{\sim}$, fix a subset $J_{\mathcal{O}} \subseteq S$ with $J_{\mathcal{O}} \in \mathcal{O}$ and define elements $\varepsilon_{\mathcal{O}}$, one for each $\mathcal{O} \in S/{\sim}$, recursively by the formula

$$\varepsilon_{\mathcal{O}} = \frac{1}{L_{\mathcal{O}}}x_{J_{\mathcal{O}}} - \sum_{\mathcal{O}' > \mathcal{O}} \varepsilon_{\mathcal{O}'}\left(\frac{1}{L_{\mathcal{O}}}x_{J_{\mathcal{O}}}\right),$$

where $x_{J_{\mathcal{O}}}$ are the basis elements of $\Sigma_k(W)$ as defined in Section 2.2 and where $L_{\mathcal{O}}$ is the index of W_J in the normalizer of W_J.

Remark 5.6. The construction of the idempotents $\varepsilon_{\mathcal{O}}$ in Remark 5.5 is very similar to the construction of the idempotents e_X in Theorem 5.2. In both cases we start with an algebra A and a function s from a basis of A to a poset P. Using s and P, a complete system of primitive orthogonal idempotents is constructed for A using Eq. (5.2). It would be interesting to determine conditions on A, P, and s to ensure that this construction provides a complete system of primitive orthogonal idempotents for A.

6. A *W*-equivariant surjection

In this section we define a quiver $\mathcal{Q}$ and a W-equivariant surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$ of k-algebras. We use this homomorphism in later sections to deduce properties of the quiver of $(k\mathcal{F})^W$. Recall that we write $Y \lessdot X$ if and only if $Y < X$ and there exists no Z such that $Y < Z < X$.

Definition 6.1. Let $\mathcal{Q}$ be the directed graph on the vertex set $\mathcal{L}$ and with exactly one arrow $X \to Y$ if and only if $Y \lessdot X$.

In [Sal08a, Corollary 8.4] it is shown that $\mathcal{Q}$ is the quiver of $k\mathcal{F}$, which will also follow from the theorem below (see Corollary 6.8). In that article it is also shown that $k\mathcal{F}$ is a Koszul algebra, but this fact will not be necessary here.

Theorem 6.2. *Let $\{e_X\}_{X \in \mathcal{L}}$ denote a complete system of primitive orthogonal idempotents for $k\mathcal{F}$ as defined in Theorem 5.2. Fix an orientation ϵ_X on each subspace $X \in \mathcal{L}$ and define numbers $[x : y]$ for pairs of faces satisfying $x \lessdot y$ by*

$$[x : y] = \epsilon_{\operatorname{supp}(x)}(\vec{x}_1, \dots, \vec{x}_d)\epsilon_{\operatorname{supp}(y)}(\vec{x}_1, \dots, \vec{x}_d, \vec{y}_1), \tag{6.1}$$

where $\vec{x}_1, \dots, \vec{x}_d$ is a basis of $\operatorname{supp}(x)$ *and $\vec{y}_1$ is a vector in y.*

Let φ be the function defined on the vertices and arrows of $\mathcal{Q}$ by

$$\varphi(X) = e_X \quad \textit{and} \quad \varphi(X \to Y) = \ell_Y\big([y : x]x + [y : x']x'\big)e_X,$$

where y is any face of support Y and where x and x' are the two faces of support X having y as a face. Then φ extends uniquely to a surjection of k-algebras $\varphi : k\mathcal{Q} \twoheadrightarrow k\mathcal{F}$, the kernel of φ is generated as an ideal by the sum of all the paths of length two in $\mathcal{Q}$, and φ is W-equivariant with respect to the following action of W on $k\mathcal{Q}$:

$$w(X_0 \to \cdots \to X_t) = \sigma_{X_0}(w)\sigma_{X_t}(w)\big(w(X_0) \to \cdots \to w(X_t)\big),$$

where $\sigma_X(w)$, for $X \in \mathcal{L}$ and $w \in W$, is defined by the equation

$$\sigma_X(w) = \epsilon_X(\vec{x}_1, \dots, \vec{x}_d)\epsilon_{w(X)}\big(w(\vec{x}_1), \dots, w(\vec{x}_d)\big), \tag{6.2}$$

where $\vec{x}_1, \dots, \vec{x}_d$ is a basis of X.

We will prove this by a sequence of lemmas. But before we do, let us record a few properties of the numbers defined in Eqs. (6.1) and (6.2).

It is straightforward to prove that the ***incidence numbers*** defined in Eq. (6.1) satisfy the identity

$$[x : y] = [x' : x'y], \quad \text{if } x, x' \in \mathcal{F} \text{ and } \operatorname{supp}(x') = \operatorname{supp}(x). \tag{6.3}$$

They also satisfy the following identity,

$$[z : y][y : x] + [z : u][u : x] = 0, \tag{6.4}$$

where y and u are the two faces in the interval $\{f \in \mathcal{F}\colon\ z \lessdot f \lessdot x\}$. A proof of this can be found in [BD98, Lemma 2 in §5C],

Remark 6.3. The incidence numbers were defined by Kenneth S. Brown and Persi Diaconis who used them to compute the multiplicities of the eigenvalues of random walks on the chambers of a hyperplane arrangement. The numbers get their name from the fact that they form a system of "incidence numbers" in the sense of homology theory of regular cell complexes. See [BD98, §5] for details.

The number $\sigma_X(w)$ defined in Eq. (6.2) measures whether w maps a positively oriented basis of X to a positively or negatively oriented basis of $w(X)$. Note that if $w(X) = X$, then $\sigma_X(w)$ is 1 if and only if the restriction of w to X is orientation-preserving, and is -1 otherwise. In particular, if $w(X) = X$, then the number $\sigma_X(w)$ does not depend on the choice of ϵ_X. And since w is an orthogonal transformation of V: if $w(X) = X$, $w(Y) = Y$ and $Y \lessdot X$, then $\sigma_X(w)\sigma_Y(w) = -1$ if and only if w interchanges the two halfspaces of X determined by Y.

6.1. Proof of Theorem 6.2

We begin by showing that φ is well defined.

Lemma 6.4. *$\varphi : k\mathcal{Q} \to k\mathcal{F}$ is a well-defined homomorphism of k-algebras.*

Proof. There are a three issues that need to be addressed with the definition of φ. First is the fact that there are exactly two faces of support X having y as a face, which is a well-known result [Bro89, §I.4E Proposition 3]. The second issue is the claim that $\varphi(X \to Y)$ does not depend on the choice of y. Indeed, since ℓ_Y is a linear combination of faces of support Y, we have $\ell_Y y' = \ell_Y$ for any face y' with $\operatorname{supp}(y') = Y$ (this is because $yy' = y$ if $\operatorname{supp}(y) \geqslant \operatorname{supp}(y')$; see Section 2.3.2). So if y' is another face with $\operatorname{supp}(y') = Y$, then

$$\begin{aligned}\ell_Y\big([y:x]x + [y:x']x'\big)e_X &= \ell_Y\big([y:x]y'x + [y:x']y'x'\big)e_X\\ &= \ell_Y\big([y':y'x]y'x + [y':y'x']y'x'\big)e_X,\end{aligned}$$

where we used Eq. (6.3) to obtain the last equality. Since $y'x$ and $y'x'$ are the two faces of support X having y' as a face, the claim follows.

The third issue is that φ extends uniquely to a homomorphism of k-algebras. Since the images of the vertices form a complete system of primitive orthogonal idempotents, it suffices to show that $\varphi(Y)\varphi(X \to Y)\varphi(X) = \varphi(X \to Y)$ for all arrows $X \to Y$ in $\mathcal{Q}$ (Theorem 3.2). Since $\varphi(X) = e_X$ is an idempotent for each vertex $X \in \mathcal{L}$, it follows immediately that $\varphi(X \to Y)\varphi(X) = \varphi(X \to Y)$. It remains to show that $\varphi(Y)\varphi(X \to Y) = \varphi(X \to Y)$. Using Eq. (5.2) we write,

$$\begin{aligned}\varphi(Y)\varphi(X \to Y) &= \Big(\ell_Y - \sum_{U>Y} \ell_Y e_U\Big)\ell_Y\big([y:x]x + [y:x']x'\big)e_X\\ &= \varphi(X \to Y) - \sum_{U>Y} \ell_Y e_U\big([y:x]x + [y:x']x'\big)e_X.\end{aligned}$$

We will show this is $\varphi(X \to Y)$ by showing each term in the summation is zero.

Suppose $U > Y$. Since $Y \lessdot X$, either $U = X$ or $U \not\leqslant X$. If $U = X$, then let u be a face of support U and note that $ux = u = ux'$ because $\operatorname{supp}(u) \geqslant \operatorname{supp}(x)$ (Section 2.3.2). Since e_U is a linear combination of elements of support at least $U = X$, we have $e_U u = e_U$. Thus,

$$e_U\big([y:x]x + [y:x']x'\big) = e_U\big([y:x]u + [y:x']u\big) = 0$$

since $[y:x] = -[y:x']$. If $U \not\leqslant X$, then the fact that $(e_U a)u = e_U a$ for any $a \in k\mathcal{F}$ and any $u \in \mathcal{F}$ with $\operatorname{supp}(u) = U$ implies

$$e_U\big([y:x]x + [y:x']x'\big)e_X = e_U\big([y:x]x + [y:x']x'\big)(ue_X) = 0,$$

where the last equality follows from Lemma 5.3. □

Lemma 6.5. *$\varphi : k\mathcal{Q} \to k\mathcal{F}$ is surjective.*

Proof. Since the elements e_X are orthogonal idempotents, to show that φ is surjective it suffices to show that $k\mathcal{F}e_X$ is in the image of φ for all $X \in \mathcal{L}$. It follows from Lemma 5.3 and Eq. (5.2) that the following is a basis of $k\mathcal{F}e_X$:

$$\big\{xe_X:\ x \in \mathcal{F},\ \operatorname{supp}(x) = X\big\}.$$

A proof of this can be found in [Sal07, Lemma 5.1] and [Sal08a, Lemma 6.1].

We proceed by induction on the rank of X in $\mathcal{L}$. If the rank is zero, then X is the intersection of all the hyperplanes in $\mathcal{A}$. There is only one face that has this support, the identity element of $\mathcal{F}$. Thus, $k\mathcal{F}e_X \subseteq \operatorname{im}(\varphi)$ since $e_X = \varphi(X)$.

Suppose $k\mathcal{F}e_Y \subseteq \operatorname{im}(\varphi)$ for all Y satisfying $\operatorname{rank}(Y) < r$. Let $X \in \mathcal{L}$ with $\operatorname{rank}(X) = r$. Let x and x' be two faces of support X that are separated by exactly one subspace Y of X having codimension one. We will prove that $(x - x')e_X \in \operatorname{im}(\varphi)$. Let y denote the face of support Y that is common to both x and x'. Then $x = yx$ and $x' = yx'$. Thus, up to a sign $(x - x')e_X$ is equal to

$$\begin{aligned}
\pm(x - x')e_X &= \big([y:x]x + [y:x']x'\big)e_X \\
&= y\big([y:x]x + [y:x']x'\big)e_X \\
&= y\ell_Y\big([y:x]x + [y:x']x'\big)e_X \\
&= y\varphi(X \to Y) \\
&= y\varphi(Y)\varphi(X \to Y) \\
&= (ye_Y)\varphi(X \to Y).
\end{aligned}$$

Here we used the identity $y\ell_Y = y$. Since Y is a proper subspace of X, $\operatorname{rank}(Y) < \operatorname{rank}(X) = r$. By the induction hypothesis, $ye_Y \in \operatorname{im}(\varphi)$. Hence, $xe_X - x'e_X \in \operatorname{im}(\varphi)$ for every pair x, x' of faces of support X sharing a common codimension one face.

For every pair of faces x and x' of support X, there exists a sequence of faces $x_0 = x, x_1, \dots, x_d = x'$ of support X such that x_{i-1} and x_i share a common codimension one face for each $1 \leqslant i \leqslant d$ [Bro89, Proposition 3 of §I.4E]. It follows that $xe_X - x'e_X \in \operatorname{im}(\varphi)$ for any pair of faces x, x' of support X. Since the sum of the coefficients of ℓ_X is nonzero, the elements $xe_X - x'e_X$, where $x, x' \in \mathcal{F}$ and $\operatorname{supp}(x) = \operatorname{supp}(x') = X$, together with $\ell_X e_X = e_X$ span the subspace $k\mathcal{F}e_X$. Since $e_X = \varphi(X)$, it follows that $k\mathcal{F}e_X \subseteq \operatorname{im}(\varphi)$. Thus, φ is surjective. □

Lemma 6.6. *The kernel of $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is generated as an ideal by the sum of all the paths of length two in $\mathcal{Q}$.*

Proof. Let ρ be the sum of all the paths of length two in $\mathcal{Q}$ and let $\mathcal{I}$ denote the ideal generated by ρ. If X and Y are two vertices of $\mathcal{Q}$ with $Y \leqslant X$ and $\dim(X) = \dim(Y) + 2$, then $Y\rho X$ is the sum of all the paths of length two that begin at X and end at Y. We begin by showing that these elements are in $\ker(\varphi)$.

Suppose $(X \to Y \to Z)$ is a path of length two in $\mathcal{Q}$. Let z be a face of support Z and y a face of support Y. Since $\operatorname{supp}(zy) = \operatorname{supp}(z) \vee \operatorname{supp}(Y)$, it follows that zy has support Y. By replacing y with zy, we can suppose that $z \lessdot y$. Thus,

$$\varphi(Y \to Z)\varphi(X \to Y) = \ell_Z\big([z:y]y + [z:y']y'\big)\ell_Y\big([y:x]x + [y:x']x'\big)e_X,$$

where y and y' are the two faces of support Y having z as a face, and x and x' are the two faces of support X having y as a face. Since y and y' have support Y, $y\ell_Y = y$ and $y'\ell_Y = y'$. Thus,

$$\varphi(Y \to Z)\varphi(X \to Y) = \ell_Z\big([z:y]y + [z:y']y'\big)\big([y:x]x + [y:x']x'\big)e_X. \tag{6.5}$$

So $\varphi(X \to Y \to Z)$ is a linear combination of elements of the form $\tilde{x}e_X$ with $\tilde{x}$ having support X, having a codimension one face of support Y, and having a codimension two face occurring in ℓ_Z with a nonzero coefficient.

Let z be a face occurring in ℓ_Z with a nonzero coefficient. There are exactly two codimension one faces of $\tilde{x}$ that contain z as a face [BD98, Lemma 2 of §5C]; call them y and u. So $\tilde{x}$ can only appear in $\varphi(X \to Y \to Z)$ and $\varphi(X \to U \to Z)$, where $Y = \operatorname{supp}(y)$ and $U = \operatorname{supp}(u)$. Moreover, in Eq. (6.5) exactly one of yx or yx' can be $\tilde{x}$; we can suppose that $yx = \tilde{x}$. So, $\tilde{x}e_X$ appears in $\varphi(X \to Y \to Z)$ with coefficient $[z:y][y:\tilde{x}]$. Similarly, $\tilde{x}e_X$ appears in $\varphi(X \to U \to Z)$ with coefficient $[z:u][u:\tilde{x}]$. It follows from Eq. (6.4) that the coefficient of $\tilde{x}e_X$ in the sum $\sum \varphi(X \to Y \to Z)$ is $[z:y][y:\tilde{x}] + [z:u][u:\tilde{x}] = 0$.

The above shows that $\mathcal{I} \subseteq \ker(\varphi)$. Let X and Y be two vertices in $\mathcal{Q}$, and let $M_{X,Y}$ be the subspace of the path algebra $k\mathcal{Q}$ spanned by elements of the form

$$\sum_{\{Z \in \mathcal{L}:\ U_{i+1} \lessdot Z \lessdot U_{i-1}\}} (U_0 \to U_1 \to \cdots \to U_{i-1} \to Z \to U_{i+1} \to \cdots \to U_{l-1} \to U_l),$$

where $0 < i < l$, $U_0 = X$ and $U_l = Y$. Note that $M_{X,Y}$ is a subspace of $Y\ker(\varphi)X$ since

$$\sum_{\{Z:\ U_{i+1} \lessdot Z \lessdot U_{i-1}\}} (U_{i-1} \to Z \to U_{i+1}) \in \ker(\varphi).$$

Thus, $\dim(M_{X,Y}) \leqslant \dim(Y\ker(\varphi)X)$. We show below that this is an equality, which implies that $Y\mathcal{I}X = Y(\ker\varphi)X$, from which it follows that $\ker\varphi = \mathcal{I}$.

We compute the dimension of the quotient space $Y(k\mathcal{Q})X/M_{X,Y}$ using results from *poset cohomology* [Wac07]. The poset obtained by reversing the order on $\mathcal{L}$ is a geometric lattice and so the dual of the poset $P = \{Z \in \mathcal{L}:\ Y \leqslant Z \leqslant X\}$ is also a geometric lattice [Sta07, Proposition 3.8]. The poset cohomology of P is isomorphic to the vector space $Y(k\mathcal{Q})X/M_{X,Y}$ and its dimension is known to be $|\mu(Y,X)|$, where μ is the Möbius function of $\mathcal{L}$ [Fol66,Bjö92]. This is also the dimension of $e_Yk\mathcal{F}e_X$ because $\sum_{Y \leqslant X} \dim(e_Yk\mathcal{F}e_X) = \dim(k\mathcal{F}e_X)$ counts the number of faces of support X (this follows from Lemma 5.3; for a proof see [Sal07, §12] or [Sal08a, Proposition 6.4]) and so does $\sum_{Y \leqslant X} |\mu(Y,X)|$ [Zas75]. Therefore, $\dim(Y(k\mathcal{Q})X/M_{X,Y}) \leqslant \dim(e_Yk\mathcal{F}e_X)$. In particular, $\dim(M_{X,Y}) \geqslant \dim(Y(k\mathcal{Q})X) - \dim(e_Yk\mathcal{F}e_X) = \dim(Y(\ker\varphi)X)$. $\square$

Lemma 6.7. *φ is W-equivariant.*

Proof. We need only show that $w(\varphi(P)) = \varphi(w(P))$ for every path P in $\mathcal{Q}$ and every $w \in W$.

If P is a path of length 0, then P is a vertex. Thus, $w(\varphi(P)) = w(e_P) = e_{w(P)} = \varphi(w(P))$ for all $w \in W$.

If $P = (X \to Y)$ is an arrow in $\mathcal{Q}$, then for all $w \in W$,

$$\begin{aligned} w\big(\varphi(X \to Y)\big) &= w\big(\ell_Y\big([y:x]x + [y:x']x'\big)e_X\big) \\ &= \ell_{w(Y)}\big([y:x]w(x) + [y:x']w(x')\big)e_{w(X)}. \end{aligned}$$

It follows directly from Eq. (6.1) and Eq. (6.2) that $[x:y]$ is equal to $\sigma_{\mathrm{supp}(x)}(w)\sigma_{\mathrm{supp}(y)}(w)[w(x):w(y)]$, so

$$\begin{aligned} &\big([y:x]w(x) + [y:x']w(x')\big) \\ &\quad = \sigma_Y(w)\sigma_X(w)\big(\big[w(y):w(x)\big]w(x) + \big[w(y):w(x')\big]w(x')\big). \end{aligned}$$

Hence, $w(\varphi(X \to Y)) = \varphi(w(X \to Y))$.

Since $w(X_0 \to \cdots \to X_p) = w(X_{p-1} \to X_p) \cdots w(X_0 \to X_1)$, the result follows. □

This establishes Theorem 6.2. As an immediate corollary, we get that $\mathcal{Q}$ is the quiver of $k\mathcal{F}$.

Corollary 6.8. *$\mathcal{Q}$ is the quiver of $k\mathcal{F}$.*

Proof. From Theorem 6.2, $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is a surjective k-algebra homomorphism that satisfies $0 = F^{n+1} \subseteq \ker(\varphi) \subseteq F^2$, where F is the ideal in $k\mathcal{Q}$ generated by the arrows and $n = \dim(V)$. Therefore, by Theorem 3.1, $\mathcal{Q}$ is the quiver of $k\mathcal{F}$. □

7. On the quiver of $(k\mathcal{F})^W$

Let Γ denote the quiver of $(k\mathcal{F})^W$. This section explores some implications of Theorems 5.2 and 6.2 for the structure of Γ. Since $\Sigma_k(W)$ is anti-isomorphic to $(k\mathcal{F})^W$, the quiver of $\Sigma_k(W)$ is Γ^*, the quiver obtained from Γ by reversing its arrows. So the results below also apply to $\Sigma_k(W)$ and Γ^*. In the next two sections we use Theorem 6.2 to compute the quiver of $(k\mathcal{F})^{\mathcal{S}_n}$ and the quiver of $(k\mathcal{F})^{\mathcal{B}_n}$.

Our first result deals with the vertices of Γ. Since they correspond to idempotents in a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$, Theorem 5.2 implies that Γ has one vertex for each orbit $\mathcal{O} \in \mathcal{L}/W$.

Proposition 7.1. *Γ has exactly one vertex for each W-orbit of elements in $\mathcal{L}$, where $\mathcal{L}$ is the intersection lattice of the reflection arrangement of W.*

Combined with Remark 5.1, this implies that the quiver of $\Sigma_k(W)$ has exactly one vertex for each equivalence class of subsets of S.

The next observation will be the main tool in the remainder of this section. It gives a sufficient condition for there to be no arrow between 2 given vertices in Γ.

Lemma 7.2. *Let $\mathcal{O}, \mathcal{O}' \in \mathcal{L}/W$ be vertices of Γ. If for every path P in $\mathcal{Q}$ that begins at a vertex in $\mathcal{O}'$ and ends at a vertex in $\mathcal{O}$ there exists $w \in W$ such that $w(P) = -P$, then there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$ in Γ.*

Proof. It follows from the definition of the quiver of an algebra (Section 3) that if the vector space $\varepsilon_{\mathcal{O}}(k\mathcal{F})^W\varepsilon_{\mathcal{O}'}$ is the zero vector space, then there is no arrow $\mathcal{O}' \to \mathcal{O}$. We will show that this vector space is zero if the hypothesis holds.

It follows from Theorem 6.2 that φ restricts to a surjection $\nu_{\mathcal{O}}(k\mathcal{Q})^W\nu_{\mathcal{O}'} \twoheadrightarrow \varepsilon_{\mathcal{O}}(k\mathcal{F})^W\varepsilon_{\mathcal{O}'}$, where $\nu_{\mathcal{O}} = \sum_{X \in \mathcal{O}} X$ for each $\mathcal{O} \in \mathcal{L}/W$. We will show $\nu_{\mathcal{O}}(k\mathcal{Q})^W\nu_{\mathcal{O}'} = 0$. This subspace is spanned by elements of the form $\sum_{u \in W} u(P)$, where P is a path of $\mathcal{Q}$ that begins at a vertex in $\mathcal{O}'$ and ends at a vertex in $\mathcal{O}$. The hypothesis states that $w(P) = -P$ for some $w \in W$, so

$$\sum_{u \in W} u(P) = \sum_{u \in W} u\big(w(P)\big) = -\Big(\sum_{u \in W} u(P)\Big).$$

Therefore, $\sum_{u \in W} u(P) = 0$. So $\nu_{\mathcal{O}}(k\mathcal{Q})^W \nu_{\mathcal{O}'} = 0$. $\square$

Our first consequence of this lemma is that Γ contains no oriented cycles.

Proposition 7.3. *If $\mathcal{O}' \to \mathcal{O}$ is an arrow in Γ, then $\mathcal{O} < \mathcal{O}'$ in $\mathcal{L}/W$. In particular, Γ does not contain any oriented cycles.*

Proof. If $(X_0 \to \cdots \to X_l)$ is a path in $\mathcal{Q}$, then $X_l \leqslant X_0$. In particular, $\mathcal{O}_{X_l} \leqslant \mathcal{O}_{X_0}$. So if $\mathcal{O} \not< \mathcal{O}'$, then the condition of Lemma 7.2 is vacuously satisfied since there are no paths in $\mathcal{Q}$ from a vertex in $\mathcal{O}'$ to a vertex in $\mathcal{O}$. Therefore, there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$ in Γ. It follows that Γ cannot contain an oriented cycle. $\square$

Corollary 7.4. *The algebra $(k\mathcal{F})^W$ is a quasi-hereditary algebra.*

This result follows from the definition of a quasi-hereditary algebra since Γ contains no oriented cycles (for an introduction to quasi-hereditary algebras, see Vlastimil Dlab's appendix to [DK94]). Associated to every quasi-hereditary algebra A is a distinguished module T, called the *characteristic tilting module* of A, and the *Ringel dual* of A is the algebra $\mathrm{End}_A(T)$; it develops that the Ringel dual of A is Morita equivalent to A [Rin91]. It would be interesting to identify the characteristic tilting module and the Ringel dual of $(k\mathcal{F})^W$ and $\Sigma_k(W)$.

Our next result shows that Γ contains at least one isolated vertex.

Proposition 7.5. *There are no arrows in Γ beginning at the vertex $\mathcal{O}_V$, where $\mathcal{O}_V = \{V\}$ is the W-orbit of the ambient vector space V of the reflection arrangement of W.*

Proof. Let $(X_0 \to \cdots \to X_l)$ be a path in $\mathcal{Q}$ with $X_0 = V$. Let $w \in W$ denote the reflection in the hyperplane X_1. Then

$$\begin{aligned} w(X_0 \to \cdots \to X_l) &= \sigma_{X_0}(w)\sigma_{X_l}(w)\big(w(X_0) \to \cdots \to w(X_l)\big) \\ &= -(X_0 \to \cdots \to X_l) \end{aligned}$$

since w fixes pointwise all the subspaces $X_1, X_2, \ldots, X_l$ and changes the orientation of X_0. By Lemma 7.2, there is no arrow in Γ beginning at $\mathcal{O}_V = \{V\}$. $\square$

The poset $\mathcal{L}/W$ is a ranked poset, with the rank of an element $\mathcal{O} \in \mathcal{L}/W$ equal to the rank of any $X \in \mathcal{O}$ as an element of $\mathcal{L}$ (which is $\dim(X) - \dim(\bigcap_{H \in \mathcal{A}} H)$). As we will see in Theorem 8.1, if W is the symmetric group $\mathcal{S}_n$, $n \geqslant 2$, then the existence of an arrow $\mathcal{O}' \to \mathcal{O}$ in the quiver of $\Sigma_k(W)$ implies that $\mathcal{O} \lessdot \mathcal{O}'$ in $\mathcal{L}/W$. The next result shows that this is not necessarily true for other W.

Proposition 7.6. *If W is a finite Coxeter group of type A_1, B_n, D_{2n}, E_7, E_8, F_4, $I_2(2n)$, H_3 or H_4, then there is no arrow in Γ from $\mathcal{O}'$ to $\mathcal{O}$ if the difference between their ranks in $\mathcal{L}/W$ is odd. In particular, if $\mathcal{O} \lessdot \mathcal{O}'$, then there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$ in Γ.*

Proof. If the type of W is one of those listed above, then W contains the transformation $w(\vec{v}) = -\vec{v}$ for $\vec{v} \in V$ [Kan01, Lemma 27.2]. Since $\sigma_X(w) = (-1)^{\dim(X)}$, we have $\sigma_Y(w)\sigma_X(w) = -1$ if and only if $\dim(X) + \dim(Y)$ is even.

If $(X_0 \to \cdots \to X_l)$ is a path from $X_0 \in \mathcal{O}'$ to $X_l \in \mathcal{O}$, then the hypothesis on the difference between the ranks of $\mathcal{O}'$ and $\mathcal{O}$ in $\mathcal{L}/W$ implies that $\dim(X_0) + \dim(X_l)$ is odd. Therefore,

$$w(X_0 \to \cdots \to X_l) = \sigma_{X_0}(w)\sigma_{X_l}(w)(X_0 \to \cdots \to X_l) = -(X_0 \to \cdots \to X_l).$$

The result now follows from Lemma 7.2. □

Combined with Proposition 7.5 the above result implies the following.

Corollary 7.7. *If the type of W is one of those listed in Proposition* 7.6, *then Γ contains at least three connected components.*

Recall that the ***Loewy length*** of an algebra A is the smallest integer ℓ such that $\operatorname{rad}^{\ell}(A) = 0$. If W belongs to one of the types listed in Proposition 7.6, then that result can be used to give an upper bound on the Loewy length of $(k\mathcal{F})^W$.

Proposition 7.8. *Let (W, S) be a Coxeter system and let $n = |S|$. If W is of type A_1, B_m, D_{2m}, E_7, E_8, F_4, $I_2(2m)$, H_3 or H_4, then the Loewy length of $\Sigma_k(W)$ is at most $\frac{n+1}{2}$.*

Proof. Since Γ contains no oriented cycles, *one plus* the length of the longest path in Γ is an upper bound on the Loewy length of $(k\mathcal{F})^W$. So we bound the length of the longest path in Γ. Suppose $\mathcal{O}_0 \to \mathcal{O}_1 \to \cdots \to \mathcal{O}_l$ is a path in Γ with $l \geqslant 1$. Since $n = |S|$ is the rank of the poset $\mathcal{L}$, Proposition 7.5 implies that $\operatorname{rank}(\mathcal{O}_0) \leqslant n-1$. Combined with Proposition 7.6, we obtain that

$$n - 1 \geqslant \bigl(\operatorname{rank}(\mathcal{O}_0) - \operatorname{rank}(\mathcal{O}_l)\bigr) \geqslant \sum_{i=1}^{l} \bigl(\operatorname{rank}(\mathcal{O}_{i-1}) - \operatorname{rank}(\mathcal{O}_i)\bigr) \geqslant 2l.$$

Thus, $l \leqslant \frac{n-1}{2}$, and so the Loewy length of $\Sigma_k(W)$ is at most $\frac{n+1}{2}$. □

These upper bounds are in fact equalities [BP08]. This approach of bounding the length of the longest path in the quiver was also used in [Sal08b] to determine the Loewy length of the descent algebra of type D_{2m+1}, the only case not covered by earlier results [BP08].

We have seen that the surjection $\varphi : (k\mathcal{Q})^W \twoheadrightarrow (k\mathcal{F})^W$ plays an important role in deducing information about Γ. The next result, in conjunction with Theorem 3.1, explains why this is the case.

Theorem 7.9. *Suppose $\psi : k\Gamma \twoheadrightarrow (k\mathcal{F})^W$ is a surjection of k-algebras with an admissible kernel and let $\varphi : k\mathcal{Q} \twoheadrightarrow k\mathcal{F}$ denote the k-algebra surjection of Theorem* 6.2. *Then ψ factors through φ.*

Proof. For each $\mathcal{O} \in \mathcal{L}/W$, let $\nu_{\mathcal{O}} = \sum_{X \in \mathcal{O}} X \in k\mathcal{Q}$. The elements $\varepsilon_{\mathcal{O}} = \varphi(\nu_{\mathcal{O}})$ form a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$ (Theorem 5.2), and so do the elements $f_{\mathcal{O}} = \psi(\mathcal{O})$. Since $\varepsilon_{\mathcal{O}}$ and $f_{\mathcal{O}}$ lift the same idempotent in $(k\mathcal{F})^W/\operatorname{rad}(k\mathcal{F})^W \cong (k\mathcal{L})^W$, there exists $u_{\mathcal{O}} \in (k\mathcal{F})^W$ such that $\varepsilon_{\mathcal{O}} = u_{\mathcal{O}} f_{\mathcal{O}} u_{\mathcal{O}}^{-1}$ [Ben98, Theorem 1.7.3]. Let $u = \sum_{\mathcal{O}} \varepsilon_{\mathcal{O}} u_{\mathcal{O}} f_{\mathcal{O}}$. Then $u^{-1} = \sum_{\mathcal{O}} f_{\mathcal{O}} u_{\mathcal{O}}^{-1} \varepsilon_{\mathcal{O}}$ and $\psi(\mathcal{O}) = f_{\mathcal{O}} = u^{-1}\varepsilon_{\mathcal{O}} u = u^{-1}\varphi(\nu_{\mathcal{O}})u$ for all $\mathcal{O} \in \mathcal{L}/W$.

If $\mathcal{O}' \to \mathcal{O}$ is an arrow in Γ, then $\psi(\mathcal{O}' \to \mathcal{O})$ is a nonzero element of the subspace $f_{\mathcal{O}}(k\mathcal{F})^W f_{\mathcal{O}'} = u^{-1}(\varepsilon_{\mathcal{O}}(k\mathcal{F})^W \varepsilon_{\mathcal{O}'})u$. Since φ is surjective, there exists an element $\rho_{(\mathcal{O}' \to \mathcal{O})}$ in $\nu_{\mathcal{O}} k\mathcal{Q} \nu_{\mathcal{O}'}$ such that $\psi(\mathcal{O}' \to \mathcal{O}) = u^{-1}\varphi(\rho_{(\mathcal{O}' \to \mathcal{O})})u$, and there exists $U \in k\mathcal{Q}$ such that $\varphi(U) = u$. Since $\mathcal{Q}$ contains no oriented cycles, $\varphi(U)$ is invertible if and only if U is invertible. Thus, U is invertible. Let $\xi : k\Gamma \to k\mathcal{Q}$ be the homomorphism defined on the vertices and arrows of Γ by $\xi(\mathcal{O}) = U^{-1}\nu_{\mathcal{O}} U$ and $\xi(\mathcal{O}' \to \mathcal{O}) = U^{-1}\rho_{(\mathcal{O}' \to \mathcal{O})}U$. It follows that $\psi(P) = (\varphi \circ \xi)(P)$ for all $P \in k\Gamma$. □

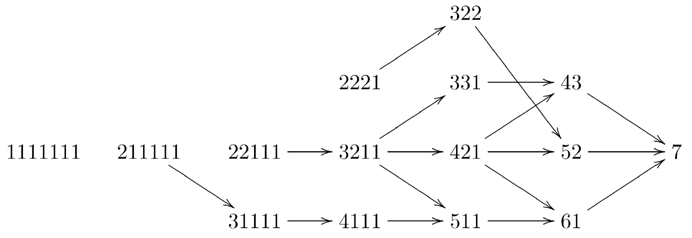

Fig. 1. The quiver of $(k\mathcal{F})^{\mathcal{S}_7}$.

8. The quiver of $(k\mathcal{F})^{\mathcal{S}_n}$

In this section we determine the quiver of $(k\mathcal{F})^{\mathcal{S}_n}$. We begin by fixing notation. Throughout, let $\mathcal{A}$ be the reflection arrangement of the symmetric group $\mathcal{S}_n$, let $k\mathcal{F}$ and $\mathcal{L}$ be the face semigroup algebra and the intersection lattice of $\mathcal{A}$, respectively, and let $\varphi : kQ \to k\mathcal{F}$ be the map defined in Theorem 6.2. Recall from Section 2.4 that an ***integer partition*** of $n \in \mathbb{N}$ is a collection of positive integers that sum to n.

Theorem 8.1. *The quiver of $(k\mathcal{F})^{\mathcal{S}_n}$ is the directed graph with one vertex ν_p for each integer partition p of n and exactly one arrow $\nu_p \to \nu_q$ if and only if q is obtained from p by adding two distinct elements of p.*

The quiver of $(k\mathcal{F})^{\mathcal{S}_7}$ is illustrated in Fig. 1.

Proof. Let Γ be the quiver defined in the statement of the theorem. We define a homomorphism of k-algebras $\psi : k\Gamma \to (k\mathcal{F})^{\mathcal{S}_n}$ with an admissible kernel. It then follows from Theorem 3.1 that Γ is the quiver of $(k\mathcal{F})^{\mathcal{S}_n}$. See Section 2.4 for definitions.

Definition of ψ. For $X \in \mathcal{L}$, write $\pi(X) = \{B_1, \ldots, B_r\}$, where $|B_1| \geqslant \cdots \geqslant |B_r|$, for the set partition associated to X, and let $\rho(X) = (|B_1|, |B_2|, \ldots, |B_r|)$. Note that two elements X and X' in $\mathcal{L}$ are in the same S_n-orbit if and only if $\rho(X) = \rho(X')$.

Define ψ on the vertices ν_p of Γ by

$$\psi(\nu_p) = \sum_{\substack{X \in \mathcal{L} \\ \rho(X) = p}} \varphi(X).$$

If $\nu_p \to \nu_q$ is an arrow in Γ, then fix an arrow $X \to Y$ in Q with $\rho(X) = p$ and $\rho(Y) = q$, and define

$$\psi(\nu_p \to \nu_q) = \sum_{w \in \mathcal{S}_n} \varphi\big(w(X \to Y)\big).$$

We argue that ψ extends to a unique k-algebra homomorphism $\psi : \Gamma \to (k\mathcal{F})^W$. By Theorem 3.2, we need to show that $\psi(\nu_p \to \nu_q) = \psi(\nu_q)\psi(\nu_p \to \nu_q)\psi(\nu_p)$ for all arrows $\nu_p \to \nu_q$ in Γ. Well,

$$\begin{aligned}
\psi(\nu_p \to \nu_q)\psi(\nu_p) &= \sum_{w \in \mathcal{S}_n} \varphi\left(w(X \to Y) \sum_{\rho(Z)=p} Z \right) \\
&= \sum_{w \in \mathcal{S}_n} \varphi\left(w(X \to Y) \sum_{\rho(Z)=p} w(Z) \right) \\
&= \sum_{w \in \mathcal{S}_n} (\varphi \circ w)\left((X \to Y) \sum_{\rho(Z)=p} Z \right)
\end{aligned}$$

$$= \sum_{w \in \mathcal{S}_n} (\varphi \circ w)\big((X \to Y)X\big)$$
$$= \sum_{w \in \mathcal{S}_n} \varphi\big(w(X \to Y)\big) = \psi(\nu_p \to \nu_q).$$

Similarly, $\psi(\nu_q)\psi(\nu_p \to \nu_q) = \psi(\nu_p \to \nu_q)$.

The kernel of ψ is admissible. We next argue that the kernel of ψ is an admissible ideal of $k\Gamma$. Recall that an ideal of the path algebra is admissible if every element in the ideal is a linear combination of paths of length at least two. Suppose $a \in \ker(\psi)$. By multiplying a on the left and right by vertices of Γ, we can suppose that a is a linear combination of paths that begin at ν_p and end at ν_q. If $\nu_p = \nu_q$, then a is a scalar multiple of a vertex. This cannot happen as $\psi(\nu_q)$ is nonzero because it is part of a complete system of primitive orthogonal idempotents (Theorem 5.2). If $\nu_p \to \nu_q$ is an arrow, then $\psi(\nu_p \to \nu_q) = \sum_w w(X \to Y)$. This is zero if and only if there exists $w \in \mathcal{S}_n$ such that $w(X \to Y) = -(X \to Y)$. We show this happens if and only if $q = \rho(Y)$ is obtained from $p = \rho(X)$ by adding two equal parts of p. Then we are done, since if such a w exists, then $\nu_p \to \nu_q$ is not an arrow of Γ.

Let $\pi(X) = \{B_1, \ldots, B_r\}$ and suppose $|B_i| = p_i$ for all $1 \leqslant i \leqslant r$. Since $Y \lessdot X$, the set partition $\pi(Y)$ is obtained from $\pi(X)$ by merging two blocks B_i and B_j. By re-indexing we can suppose $i = 1$ and $j = 2$. If $p_1 = p_2$, then any permutation $\omega \in \mathcal{S}_n$ that maps B_1 to B_2 and B_2 to B_1 while fixing the other blocks of $\pi(X)$ will satisfy $\omega(X \to Y) = -(X \to Y)$. Suppose instead that $p_1 \neq p_2$. If $\omega \in \mathcal{S}_n$ with $\omega(X) = X$ and $\omega(Y) = Y$, then ω permutes the blocks of $\pi(X)$ and the blocks of $\pi(Y)$. It follows that $\omega(B_1) = B_1$ and $\omega(B_2) = B_2$ since $p_1 \neq p_2$. Let x and y be the set compositions $(B_1, B_2, B_3, \ldots, B_m)$ and $(B_1 \cup B_2, B_3, \ldots, B_m)$, respectively. Then, $y\omega(x) = x$. So $\omega(x)$ and x correspond to faces of support X that lie on the same side of Y. Since ω does not swap the two half spaces of X determined by Y, the discussion following Theorem 6.2 implies $\omega(X \to Y) = (X \to Y)$.

Thus, a is a linear combination of paths of length at least two, so $\ker(\psi)$ is an admissible ideal of $k\Gamma$.

ψ is surjective. We show that $\psi(k\Gamma) + \operatorname{rad}^2(k\mathcal{F})^{\mathcal{S}_n} = (k\mathcal{F})^{\mathcal{S}_n}$; the result then follows from standard ring theory: *if A is a k-algebra and A' is a k-subalgebra of A such that $A' + \operatorname{rad}^2(A) = A$, then $A' = A$* [Ben98, Proposition 1.2.8]. To do this we will use the following result of Manfred Schocker [Sch06, Theorem 9.10]: $\operatorname{rad}^2(k\mathcal{F})^{\mathcal{S}_n} = \operatorname{rad}^2(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{S}_n}$. (This can be proved using results of this paper; such a proof is outlined in Theorem 8.2.)

Since $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is surjective (Theorem 6.2), it follows that the elements $\sum_{w \in \mathcal{S}_n} w(\varphi(P))$, where P is a path in $\mathcal{Q}$, span $(k\mathcal{F})^{\mathcal{S}_n}$. Furthermore, $\operatorname{rad}^2(k\mathcal{F})$ is spanned by the elements $\varphi(P)$, where P is of length at least two. Thus, if P has length at least two, then $\sum_w w(\varphi(P))$ is in $\operatorname{rad}^2(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{S}_n} = \operatorname{rad}^2(k\mathcal{F})^{\mathcal{S}_n}$. If P has length zero, then $P = X$ is a vertex and

$$\sum_{w \in \mathcal{S}_n} w\big(\varphi(X)\big) = \varphi\Bigg(\sum_{w \in \mathcal{S}_n} w(X)\Bigg) = \lambda\varphi\Bigg(\sum_{\substack{Y \in \mathcal{L} \\ \rho(Y) = \rho(X)}} Y\Bigg) = \lambda\psi(\nu_{\rho(X)}),$$

where $\lambda = |\{w \in W\colon\ w(X) = X\}|$.

It remains to show that $\sum_w \varphi(w(P)) \in \operatorname{im}(\psi)$ if P is an arrow. We first show that if $X \to Y$ and $X' \to Y'$ are two arrows with X and X' in the same $\mathcal{S}_n$-orbit and Y and Y' in the same $\mathcal{S}_n$-orbit, then there exists a permutation u such that $u(X' \to Y') = \pm(X \to Y)$. Let $\pi(X) = \{B_1, B_2, \ldots, B_r\}$ and $\pi(X') = \{B'_1, B'_2, \ldots, B'_r\}$. Since X and X' are in the same orbit, there exists a permutation w mapping X' to X. So we can assume that $X' = X$. Up to a re-indexing of the blocks, $B_1 \cup B_2$ is a block of $\pi(Y)$ and $B_3 \cup B_4$ is a block of $\pi(Y')$. Since Y and Y' are in the same orbit, it follows that $|B_1| = |B_3|$ and $|B_2| = |B_4|$. Therefore, any permutation that swaps B_1 with B_3 and B_2 with B_4 will map X to X and Y' to Y.

Let $X \to Y$ be an arrow in $\mathcal{Q}$. If there is a $w \in \mathcal{S}_n$ such that $w(X \to Y) = -(X \to Y)$, then $\sum_w w(X \to Y) = 0$. Otherwise, $\sum_w \varphi(w(X \to Y)) = \pm\psi(\mathcal{O}_X \to \mathcal{O}_Y)$ by the above. □

Since the descent algebra $\Sigma_k(\mathcal{S}_n)$ is isomorphic to the opposite algebra of $(k\mathcal{F})^{\mathcal{S}_n}$ (Theorem 2.1), its quiver is obtained by reversing the arrows in Theorem 8.1. This quiver, as a directed graph, appears in the work of Adriano Garsia and Christophe Reutenauer [GR89]; see especially Section 5 of [GR89] and the figures contained therein. Manfred Schocker [Sch04, Theorem 5.1] was the first to show that this is the quiver of $\Sigma_k(\mathcal{S}_n)$ by using results of Dieter Blessenohl and Hartmut Laue [BL96,BL02].

We also remark that the argument presented above can be used to find the quiver of $(k\mathcal{F})^W$ for arbitrary finite Coxeter groups W once the relationship between $\operatorname{rad}^2((k\mathcal{F})^W)$ and $\operatorname{rad}^p(k\mathcal{F}) \cap (k\mathcal{F})^W$ is understood. We do this in Section 9 for the finite Coxeter group of type B.

8.1. Descending Loewy series of $(k\mathcal{F})^{\mathcal{S}_n}$

The proof of Theorem 8.1 relied on the case $m = 2$ of the following result of Manfred Schocker.

Theorem 8.2. *(See Theorem 9.10 of [Sch06].) Let $k\mathcal{F}$ be the face semigroup algebra of the reflection arrangement of the symmetric group $\mathcal{S}_n$. For all $m \in \mathbb{N}$,*

$$\operatorname{rad}^m(k\mathcal{F})^{\mathcal{S}_n} = \operatorname{rad}^m(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{S}_n}.$$

Manfred Schocker proved this by constructing a basis of $\operatorname{rad}^m(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{S}_n}$ and noting that the basis coincides, under an anti-isomorphism $(k\mathcal{F})^{\mathcal{S}_n} \cong \Sigma_k(\mathcal{S}_n)$, to a basis of $\operatorname{rad}^m \Sigma_k(\mathcal{S}_n)$ constructed by Dieter Blessenohl and Hartmut Laue [BL96].

This result can also be proved using just the theory developed in this paper. In fact, in Theorem 9.2, we prove the corresponding result for the hyperoctahedral group, which is new. That proof can be adapted to prove the above. We provide a very rough outline of the argument and leave the details to the interested reader.

Outline of a proof of Theorem 8.2. To prove this, a lemma corresponding to Lemma 9.3—and proved by arguing in the same way—is needed:

$$\begin{aligned}\alpha\mathcal{N}(X_0 \to \cdots \to X_m) &= \mathcal{N}(X_1 \to \cdots \to X_m)\mathcal{N}(X_0 \to X_1) \\ &\quad - \sum_{\substack{t \in \mathcal{S}_n,\, t(X_1)=X_1, \\ t(A\cup B)\neq A\cup B}} \sigma_{X_1}(t)\sigma_{X_m}(t)\mathcal{N}\big(X_0 \to t(X_1) \to \cdots \to t(X_m)\big),\end{aligned}$$

where $(X_0 \to \cdots \to X_m)$ is a path in $\mathcal{Q}$ of length $m \geqslant 2$, the set partition $\pi(X_1)$ is obtained by merging two blocks A and B of $\pi(X_0)$, and α is the number of permutations that fix X_1 and $A \cup B$.

Begin by reducing to the case $m = 2$ by mimicking the proof of Theorem 9.2. To prove the case $m = 2$, first establish the containment $\operatorname{rad}^2(k\mathcal{F})^{\mathcal{S}_n} \subseteq \operatorname{rad}^2(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{S}_n}$. For the reverse containment, argue by contradiction: suppose that there exists a path $P = (X_0 \to \cdots \to X_m)$ in $\mathcal{Q}$ of length at least two such that $\varphi(\mathcal{N}(P)) \notin \operatorname{rad}^2(k\mathcal{F})^{\mathcal{S}_n}$; and of all such paths (that begin at X_0), pick P such that $|A| + |B|$ is maximal, where A and B are the blocks of $\pi(X_0)$ that are merged to get $\pi(X_1)$. Then argue as in Step 3 of the proof of Theorem 9.2 that $A \cup B$ is not a block of $\pi(X_m)$. This means that $A \cup B$ is merged with some other block at some point. Argue as in Step 2 of the proof to show (using the relations in the partition lattice) that we can suppose that $\pi(X_2)$ is obtained by merging two blocks C and D, where $|C| = |A| + |B|$. Then derive a contradiction as in Step 4 of the proof, by examining the three cases: $C, D \neq A \cup B$; $D = A \cup B \neq C$; $C = A \cup B \neq D$. □

9. The quiver of $(k\mathcal{F})^{\mathcal{B}_n}$

In this section we determine the quiver of $(k\mathcal{F})^{\mathcal{B}_n}$. Throughout, let $\mathcal{A}$ be the reflection arrangement of the hyperoctahedral group $\mathcal{B}_n$ (defined below), let $k\mathcal{F}$ and $\mathcal{L}$ be the face semigroup algebra and the intersection lattice of $\mathcal{A}$, respectively, and let $\varphi : k\mathcal{Q} \to k\mathcal{F}$ be the map defined in Theorem 6.2.

9.1. The Coxeter group of type B

Let $n \in \mathbb{N}$. The ***Coxeter group of type B and rank n***, denoted by $\mathcal{B}_n$, is the finite group of orthogonal transformations of $\mathbb{R}^n$ generated by reflections in the hyperplanes

$$\{\vec{v} \in \mathbb{R}^n\colon\ v_i = 0\}, \quad \{\vec{v} \in \mathbb{R}^n\colon\ v_i = v_j\}, \quad \{\vec{v} \in \mathbb{R}^n\colon\ v_i = -v_j\},$$

where $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$. This set of hyperplanes is the ***reflection arrangement*** of $\mathcal{B}_n$. We identify $\mathcal{B}_n$ with the group of *signed permutations* as follows. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \dots, n\}$ and let $[\pm n] = [n] \cup (-[n])$. A ***signed permutation*** of $[\pm n]$ is a permutation w of the set $[\pm n]$ satisfying $w(-i) = -w(i)$ for all $i \in [n]$. Every signed permutation w induces an orthogonal transformation of $\mathbb{R}^n$ by permuting and negating coordinates. Moreover, any transformation in $\mathcal{B}_n$ arises in this fashion.

For any $A \subseteq [\pm n]$ let $\overline{A} = \{-i\colon\ i \in A\}$. Under the above identification the intersection lattice of the type B arrangement is identified with the sublattice Π_n^B of set partitions of $[\pm n]$ of the form $\{B_1, \dots, B_r, Z, \overline{B}_r, \dots, \overline{B}_1\}$, and where Z can be empty and satisfies $\overline{Z} = Z$ [BI99, Theorem 4.1].

To simplify notation, we let $\pi(X)$ denote the set partition of $[\pm n]$ induced by $X \in \mathcal{L}$, and we let $\{B_1, \dots, B_r; Z\}$ denote the set partition $\{B_1, \dots, B_r, Z, \overline{B}_r, \dots, \overline{B}_1\}$. The set Z is called the ***zero block*** and the other sets are called ***nonzero blocks***. Under this isomorphism the action of $\mathcal{B}_n$ on $X \in \mathcal{L}$ is given by permuting the elements of $\pi(X)$. That is, $\pi(w(X)) = w(\pi(X))$ for all $w \in \mathcal{B}_n$ and $X \in \mathcal{L}$.

The intervals of length two in Π_n^B play an important role in what follows. So we quickly describe them. If $P' \lessdot P$ is a cover relation in Π_n^B, then either P' is obtained from P by merging two distinct nonzero blocks of P, or P' is obtained from P by merging a nonzero block B with $\overline{B}$ and the zero block of P. It follows that there are four types of intervals of length two in Π_n^B, which are illustrated in Fig. 2.

9.2. The quiver of $(k\mathcal{F})^{\mathcal{B}_n}$

We now describe the quiver of $(k\mathcal{F})^{\mathcal{B}_n}$.

Theorem 9.1. *The quiver of $(k\mathcal{F})^{\mathcal{B}_n}$ contains one vertex ν_p for each integer partition p of $0, 1, \dots, n$, and $m_{p,q}$ arrows from ν_p to ν_q, where*

$$m_{p,q} = \begin{cases} 2, & \textit{if } q \textit{ is obtained by adding } 3 \textit{ distinct parts of } p, \\ 1, & \textit{if } q \textit{ is obtained by adding } 3 \textit{ parts of } p,\ 2 \textit{ of which are distinct}, \\ 1, & \textit{if } q \textit{ is obtained by deleting } 2 \textit{ distinct parts of } p, \\ 0, & \textit{otherwise.} \end{cases}$$

The quiver of $(k\mathcal{F})^{\mathcal{B}_6}$ is illustrated in Fig. 3.

Proof. Let Γ be the quiver with one vertex ν_p for each integer partition p of $0, 1, \dots, n$, and $m_{p,q}$ arrows from the vertex ν_p to ν_q. We will use Theorem 3.1 to show that Γ is the quiver of $(k\mathcal{F})^{\mathcal{B}_n}$ by constructing a surjective k-algebra morphism $\psi : k\Gamma \twoheadrightarrow (k\mathcal{F})^{\mathcal{B}_n}$ that has an admissible kernel.

For each $X \in \mathcal{L}$, let $\pi(X) = \{A_1, \dots, A_r; Z\}$ with $|A_1| \geqslant \cdots \geqslant |A_r|$, and let $\rho(X)$ be the integer partition $(|A_1|, |A_2|, \dots, |A_r|)$. It follows that X and Y are in the same $\mathcal{B}_n$-orbit if and only if $\rho(X) = \rho(Y)$.

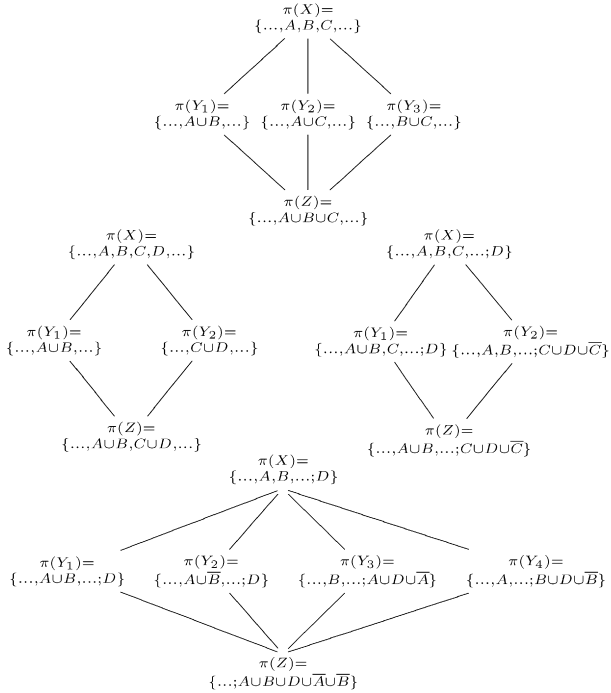

Fig. 2. The four types of intervals of length two in the lattice Π_n^B of set partitions of type B.

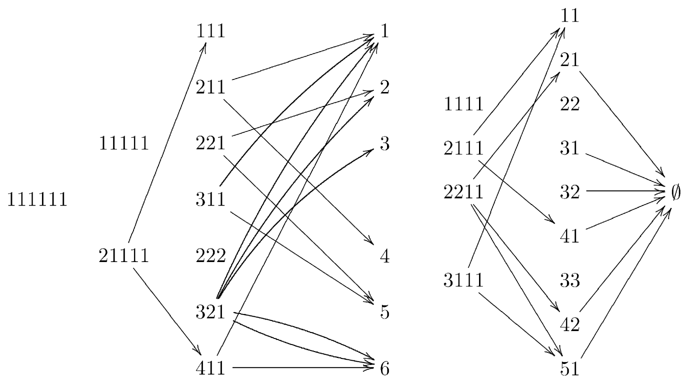

Fig. 3. The quiver of $(k\mathcal{F})^{\mathcal{B}_6}$.

Definition of ψ on vertices. Let $\varphi : k\mathcal{Q} \to k\mathcal{F}$ denote the k-algebra homomorphism of Theorem 6.2. Define a function ψ on the vertices of Γ by

$$\psi(\nu_p) = \sum_{\rho(X)=p} \varphi(X),$$

where p is an integer partition of some $m \in \{0, 1, \ldots, n\}$.

Definition of ψ on arrows. We define ψ on the three types of arrows of Γ individually. See Fig. 2 for the different types of intervals of length two in Π_n^B.

Suppose q is obtained by adding three distinct parts p_1, p_2 and p_3 of p, where $p_1 > p_2 > p_3$, and let $\alpha_1^{p,q}$ and $\alpha_2^{p,q}$ be the two arrows in Γ from ν_p to ν_q. Let $X \in \mathcal{L}$ with $\rho(X) = p$, and let A, B and C be three distinct blocks of $\pi(X)$ with $|A| = p_1$, $|B| = p_2$ and $|C| = p_3$. Let $\pi(Y_1)$ be the set partition obtained from $\pi(X)$ by merging A and B, let $\pi(Y_2)$ be the set partition obtained from $\pi(X)$ by merging A with C and let $\pi(Z)$ be the set partition obtained from $\pi(X)$ by merging A, B and C. For $i \in \{1, 2\}$, define

$$\psi\big(\alpha_i^{p,q}\big) = \sum_{w \in \mathcal{B}_n} \varphi\big(w(X \to Y_i \to Z)\big).$$

Suppose q is obtained by adding three parts p_1, p_2 and p_3 of p with $p_1 \neq p_2 = p_3$, and let $\beta_{p,q}$ be the arrow in Γ from ν_p to ν_q. Let X, Y_1, Y_2 and Z be as above. Define

$$\psi(\beta_{p,q}) = \sum_{w \in \mathcal{B}_n} \varphi\big(w(X \to Y_1 \to Z)\big).$$

Finally, suppose that q is obtained by deleting two distinct parts p_1 and p_2 of p, and let $\gamma_{p,q}$ be the arrow in Γ from ν_p to ν_q. Let $X \in \mathcal{L}$ with $\rho(X) = p$, and let A and B be two distinct blocks of $\pi(X)$ with $|A| = p_1$ and $|B| = p_2$. Let $\pi(Y_1)$ be the set partition obtained from $\pi(X)$ by merging A and B, and let $\pi(Z)$ be the set partition obtained from $\pi(X)$ by merging A, B, $\overline{A}$, $\overline{B}$ and the zero block of $\pi(X)$. Define

$$\psi(\gamma_{p,q}) = \sum_{w \in \mathcal{B}_n} \varphi\big(w(X \to Y_1 \to Z)\big).$$

Extension of ψ to an algebra homomorphism. By Theorem 3.2, ψ extends to a unique k-algebra homomorphism $\psi : k\Gamma \to (k\mathcal{F})^{\mathcal{B}_n}$ if the elements $\psi(\nu_p)$ form a complete system of primitive orthogonal idempotents and if $\psi(\nu_q)\psi(\nu_p \to \nu_q)\psi(\nu_p) = \psi(\nu_p \to \nu_q)$ for every arrow $\nu_p \to \nu_q$ in Γ. The first condition follows from Theorem 6.2. Write $\psi(\nu_p \to \nu_q) = \sum_w w(X \to Y \to Z)$ and note that

$$\begin{aligned}
\psi(\nu_p \to \nu_q)\psi(\nu_p) &= \sum_{w \in \mathcal{B}_n} \varphi\bigg(w(X \to Y \to Z) \sum_{\rho(X')=p} X'\bigg) \\
&= \sum_{w \in \mathcal{B}_n} \varphi\bigg(w(X \to Y \to Z) \sum_{\rho(X')=p} w(X')\bigg) \\
&= \sum_{w \in \mathcal{B}_n} \varphi\bigg(w\bigg(\sum_{\rho(X')=p} (X \to Y \to Z)X'\bigg)\bigg) \\
&= \sum_{w \in \mathcal{B}_n} \varphi\big(w(X \to Y \to Z)\big) = \psi(\nu_p \to \nu_q).
\end{aligned}$$

Similarly, $\psi(\nu_q)\psi(\nu_p \to \nu_q) = \psi(\nu_p \to \nu_q)$.

ψ is surjective. Next we prove that $\psi : k\Gamma \to (k\mathcal{F})^{\mathcal{B}_n}$ is surjective. We show that $\psi(k\Gamma) + \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n}) = (k\mathcal{F})^{\mathcal{B}_n}$; the result then follows from standard ring theory: *if A is a k-algebra and A' is a k-subalgebra of A such that $A' + \operatorname{rad}^2(A) = A$, then $A' = A$* [Ben98, Proposition 1.2.8]. In order to do this we will use a fact whose proof we defer to later (Theorem 9.2): that $\operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n}) = \operatorname{rad}^4(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$.

Since $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is surjective (Theorem 6.2), the images of the paths P of $\mathcal{Q}$ span $k\mathcal{F}$. It follows that the elements $\varphi(\mathcal{N}(P))$, form a spanning set for $(k\mathcal{F})^{\mathcal{B}_n}$ (recall that $\mathcal{N}(P) = \sum_{w \in \mathcal{B}_n} w(P)$).

Furthermore, $\operatorname{rad}^4(k\mathcal{F})$ is spanned by elements $\varphi(\mathcal{N}(P))$, where P is of length at least four. So if P has length at least 4, then $\varphi(\mathcal{N}(P))$ is in $\operatorname{rad}^4(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$, so it is in $\operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$ (Theorem 9.2). It remains to prove that $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$ if the length of P is less than 4. If the length of P is odd, then the signed permutation $i \mapsto -i$ for all $i \in [\pm n]$ maps P to $-P$, so $\mathcal{N}(P) = 0$. It remains to prove this for vertices and for paths of length 2.

Suppose $P = X$ is a vertex of $\mathcal{Q}$. If $\lambda = |\{w \in \mathcal{B}_n:\ w(X) = X\}|$, then

$$\varphi(\mathcal{N}(X)) = \sum_{w \in \mathcal{B}_n} w(\varphi(X)) = \lambda \sum_{\rho(Y)=\rho(X)} \varphi(Y) = \lambda \psi(\nu_{\rho(X)}).$$

So $\varphi(\mathcal{N}(X)) \in \operatorname{im}(\psi)$.

Suppose that $P = (X \to Y \to Z)$ is path of length two in $\mathcal{Q}$. Let $p = \rho(X)$ and $q = \rho(Z)$. There are four cases to consider, corresponding to the four types of intervals illustrated in Fig. 2.

Case 1. *Suppose q is obtained from p by adding p_1 to p_2 and deleting p_3, where p_1, p_2 and p_3 are three parts of p.* Since $\rho(Z) = q$, a nonzero block C of $\pi(X)$ is contained in the zero block of $\pi(Z)$. The signed permutation that negates the elements of C maps P to $-P$, and so $\varphi(\mathcal{N}(P)) = 0 \in \operatorname{im}(\psi)$.

Case 2. *Suppose q is obtained from p by deleting p_1 and p_2, where p_1 and p_2 are two parts of p.* Then there are two nonzero blocks A and B of $\pi(X)$, of sizes p_1 and p_2, respectively, that are contained in the zero block of $\pi(Z)$. We will show that if $P' = (X' \to Y' \to Z')$ is a path with $\rho(X') = p$ and $\rho(Z') = q$, then P' is in the $\mathcal{B}_n$-orbit of a path from X to Z. If $\rho(X') = p$, then there exists $w \in \mathcal{B}_n$ such that $w(P')$ begins at X. Since $w(\pi(Z'))$ is obtained from $\pi(X)$ by merging two blocks A' and B' of sizes p_1 and p_2, respectively, with the zero block of $\pi(X)$, it follows that the signed permutation u that swaps A' with A and B' with B maps $w(P')$ to a path that begins at X and ends at Z.

There are exactly four paths $P_i = (X \to Y_i \to Z)$, where $i \in \{1,2,3,4\}$, in $\mathcal{Q}$ that begin at X and end at Z: $\pi(Y_1)$ contains the block $A \cup B$; $\pi(Y_2)$ contains the block $A \cup \overline{B}$; the zero block of $\pi(Y_3)$ contains A; the zero block of $\pi(Y_4)$ contains B. The signed permutation that negates A maps P_3 to $-P_3$, so $\mathcal{N}(P_3) = 0$. Similarly, $\mathcal{N}(P_4) = 0$. Since $P_1 + P_2 + P_3 + P_4 \in \ker(\varphi)$ (Lemma 6.6), we have $\varphi(\mathcal{N}(P_1)) = -\varphi(\mathcal{N}(P_2))$.

If $\varphi(\mathcal{N}(P_1)) = 0$, then $\mathcal{N}(P_1) \in \ker(\varphi)$, so $\mathcal{N}(P_1)$ is a scalar multiple of $P_1 + P_2 + P_3 + P_4$. Since P_3 is not in the orbit of P_1, it follows that $\mathcal{N}(P_1) = 0$. So there exists a signed permutation that maps P_1 to $-P_1$. This happens if and only if $|A| = |B|$. So if $p_1 = p_2$, then $\varphi(\mathcal{N}(P_1)) = 0$, and if $p_1 \neq p_2$, then $\varphi(\mathcal{N}(P_1)) = \pm\psi(\gamma_{p,q})$. Since $P \in \{P_1, P_2, P_3, P_4\}$, it follows that $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$.

Case 3. *Suppose q is obtained by adding three parts p_1, p_2 and p_3 of p.* Since p and q are partitions of the same integer, the zero blocks of $\pi(X)$ and $\pi(Z)$ are the same. So there are two possibilities for $\pi(Z)$: either $\pi(Z)$ contains the nonzero blocks $A \cup B$ and $C \cup D$, or $\pi(Z)$ contains the nonzero block $A \cup B \cup C$, where A, B, C and D are (nonzero) blocks of $\pi(X)$. In the first case the signed permutation that negates the elements of $A \cup B$ maps P to $-P$, thus $\mathcal{N}(P) = 0$.

So suppose $\pi(Z)$ contains the nonzero block $A \cup B \cup C$, and that $|A| = p_1$, $|B| = p_2$ and $|C| = p_3$. Let $P' = (X' \to Y' \to Z')$ be another path in $\mathcal{Q}$ with $\rho(X') = p$ and $\rho(Z') = q$. Then either $\pi(Z')$ contains the nonzero block $A' \cup B' \cup C'$, where A', B', C' are blocks of $\pi(X')$ with $|A'| = |A|$, $|B'| = |B|$ and $|C'| = |C|$, or $\mathcal{N}(P') = 0$ (as above). In the former situation we have, by arguing as in Case 2, that P' is in the $\mathcal{B}_n$-orbit of a path that begins at X and ends at Z. This implies that $\psi(\nu_p \to \nu_q) = \pm\varphi(\mathcal{N}(P'))$ for some path P' beginning at X and ending at Z.

Let $P_i = (X \to Y_i \to Z)$ for $i \in \{1,2,3\}$ be the three paths in $\mathcal{Q}$ from X to Z, where $A \cup B$ is a block of $\pi(Y_1)$, $A \cup C$ is a block of $\pi(Y_2)$, and $B \cup C$ is a block of $\pi(Y_3)$. If $p_1 > p_2 > p_3$, then the previous paragraph implies that $\psi(\alpha_i^{p,q}) = \pm\varphi(\mathcal{N}(P_i))$ for $i \in \{1,2\}$. Hence, $\varphi(\mathcal{N}(P_1))$ and $\varphi(\mathcal{N}(P_2))$ are in $\operatorname{im}(\psi)$, and so $\varphi(\mathcal{N}(P_3)) \in \operatorname{im}(\psi)$ since $P_1 + P_2 + P_3 \in \ker(\varphi)$ (Theorem 6.2). Therefore, $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$ since $P \in \{P_1, P_2, P_3\}$.

Suppose $p_1 = p_2 \neq p_3$ and suppose $\pi(Y_1)$ contains the block $A \cup B$. It follows that $\psi(\beta_{p,q}) = \pm\varphi(\mathcal{N}(P_i))$ for some $i \in \{2,3\}$. The signed permutation that swaps A and B maps P_1 to $-P_1$ and so $\mathcal{N}(P_1) = 0$. Since $P_1 + P_2 + P_3 \in \ker(\varphi)$ (Lemma 6.6), it follows that $\varphi(\mathcal{N}(P_2)) = -\varphi(\mathcal{N}(P_3)) = \pm\psi(\beta_{p,q})$. Thus, $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$.

If $p_1 = p_2 = p_3$, then the argument in the previous paragraph implies that $\mathcal{N}(P_i) = 0$ for all $i \in \{1, 2, 3\}$. Hence, $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$.

Case 4. *Suppose that q is obtained from p by adding p_1 to p_2 and by adding p_3 to p_4, where p_1, p_2, p_3 and p_4 are parts of p.* If q can also be obtained from p by merging three parts of p, then we can apply the argument of the previous case. On the other hand, suppose q is not obtained from p by merging three parts of p. Then $\pi(Z)$ contains the nonzero blocks $A \cup B$ and $C \cup D$, where A, B, C, and D are (nonzero) blocks of $\pi(X)$ and $|A| = p_1$, $|B| = p_2$, $|C| = p_3$ and $|D| = p_4$. The signed permutation that negates the elements of $A \cup B$ maps P to $-P$, so $\mathcal{N}(P) = 0$. Hence, $\varphi(\mathcal{N}(P)) \in \operatorname{im}(\psi)$.

The kernel of ψ is admissible. To complete the proof we need to show that $\ker(\psi)$ is an admissible ideal of $k\Gamma$. Recall that an ideal of the path algebra is admissible if every element in the ideal is a linear combination of paths of length at least two. Suppose $a \in \ker(\psi)$. By multiplying a on the left and right by vertices of Γ, we can suppose that a is a linear combination of paths that begin at ν_p and end at ν_q. If $\nu_p = \nu_q$, then a is a scalar multiple of a vertex. This implies $a = 0$ because $\psi(\nu_q)$ is nonzero: it belongs to a complete system of primitive orthogonal idempotents (Theorem 6.2). If a is a linear combination of arrows that begin at ν_p and end at ν_q, then there are three cases to consider depending on the type of the arrows. We will show that $\psi(\alpha_1^{p,q})$ and $\psi(\alpha_2^{p,q})$ are linearly independent—that $\psi(\beta_{p,q})$ and $\psi(\gamma_{p,q})$ are nonzero can be proved using a similar argument.

Suppose q is obtained from p by adding three distinct parts of p. For $i \in \{1, 2\}$, let $P_i = (X \to Y_i \to Z)$ be the paths used to define $\psi(\alpha_i^{p,q}) = \varphi(\mathcal{N}(P_i))$ above. If $\lambda_1\psi(\alpha_1^{p,q}) = \lambda_2\psi(\alpha_2^{p,q})$, then $\lambda_1\mathcal{N}(P_1) - \lambda_2\mathcal{N}(P_2)$ is an element of $\ker(\varphi)$. Thus, $Z(\lambda_1\mathcal{N}(P_1) - \lambda_2\mathcal{N}(P_2))X \in Z(\ker\varphi)X$. By Theorem 6.2, $Z(\ker\varphi)X$ is spanned by $P_1 + P_2 + P_3$, where P_3 is the third path from X to Z. Hence, either P_3 is in the orbit of P_1 or P_2, or $\mathcal{N}(P_3) = 0$. The latter happens if and only if $|B| = |C|$, contradicting that $|B| = p_2 \neq p_3 = |C|$. The former happens if and only if $|A| = |B|$ or $|A| = |C|$. This is again a contradiction. So $\lambda_1\psi(\alpha_1^{p,q}) \neq \lambda_2\psi(\alpha_2^{p,q})$.

Therefore, if $a \in \ker(\psi)$, then a is a linear combination of paths of length at least two. So $\ker(\psi)$ is an admissible ideal of $k\Gamma$. □

9.3. *Descending Loewy Series of* $(k\mathcal{F})^{\mathcal{B}_n}$

Here we prove the following result on the square of the radical of $(k\mathcal{F})^{\mathcal{B}_n}$ that was used in the proof of Theorem 9.1. The proof of this result can be adapted to prove the corresponding result in type A. See Section 8.1 for more details.

Theorem 9.2. *Let $k\mathcal{F}$ be the face semigroup algebra of the reflection arrangement of $\mathcal{B}_n$. Then for all $m \in \mathbb{N}$,*

$$\operatorname{rad}^m\big((k\mathcal{F})^{\mathcal{B}_n}\big) = \operatorname{rad}^{2m}(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}.$$

During the course of the proof we will need bases of the subspaces $X \in \mathcal{L}$. We use a basis described by the set partition $\pi(X) = \{B_1, \dots, B_r; Z\}$: for $i \in [r]$, let

$$\boldsymbol{\beta}_{B_i} = \sum_{j \in B_i} \vec{e}_j,$$

where $\vec{e}_1, \dots, \vec{e}_n$ is the standard basis of $\mathbb{R}^n$ and $\vec{e}_{-j} = -\vec{e}_j$ for $j \in [n]$. We call $\boldsymbol{\beta}_{B_1}, \boldsymbol{\beta}_{B_2}, \dots, \boldsymbol{\beta}_{B_r}$ the ***standard basis*** of X.

We begin with the following lemma, which we will use several times in the proof. Throughout this section, let $\mathcal{N}(P) = \sum_{w \in \mathcal{B}_n} w(P)$ for any path P of $\mathcal{Q}$.

Lemma 9.3. *If $\pi(X_1)$ is obtained from $\pi(X_0)$ by merging two nonzero blocks A and B and if $\pi(X_2)$ is obtained from $\pi(X_1)$ by merging $A \cup B$ with a nonzero block C, then*

$$\lambda\mathcal{N}(X_0 \to \cdots \to X_m) = \mathcal{N}(X_2 \to \cdots \to X_m)\mathcal{N}(X_0 \to X_1 \to X_2) - \sum_{\substack{t \in \mathcal{B}_n \\ t(X_2)=X_2 \\ t(A\cup B\cup C) \neq \pm(A\cup B\cup C)}} \sigma_{X_2}(t)\sigma_{X_m}(t)\mathcal{N}\big(X_0 \to X_1 \to t(X_2) \to \cdots \to t(X_m)\big),$$

where λ is the cardinality of $\{t \in \mathcal{B}_n\colon t(X_2) = X_2$ and $t(A \cup B \cup C) = \pm(A \cup B \cup C)\}$.

Proof. Note that

$$\begin{aligned}\mathcal{N}(X_2 \to \cdots \to X_m)\mathcal{N}(X_0 \to X_1 \to X_2) &= \mathcal{N}\big(\mathcal{N}(X_2 \to \cdots \to X_m)(X_0 \to X_1 \to X_2)\big) \\ &= \sum_{\substack{t \in \mathcal{B}_n \\ t(X_2)=X_2}} \sigma_{X_2}(t)\sigma_{X_m}(t)\mathcal{N}\big(X_0 \to X_1 \to t(X_2) \to \cdots \to t(X_m)\big).\end{aligned}$$

Therefore, we need only show that if $t(X_2) = X_2$ and $t(A \cup B \cup C) = \pm(A \cup B \cup C)$, then the summand in the above sum is $\mathcal{N}(X_0 \to \cdots \to X_m)$.

Suppose $t(X_2) = X_2$ and suppose that $t(A \cup B \cup C) = \varepsilon(A \cup B \cup C)$, where $\varepsilon = \pm 1$. Let s be the signed permutation defined by

$$s(i) = \begin{cases} \varepsilon i, & \text{if } i \in A \cup B \cup C \cup \overline{A} \cup \overline{B} \cup \overline{C}, \\ t(i), & \text{otherwise.} \end{cases}$$

Then $s(A) = \varepsilon A$, $s(B) = \varepsilon B$ and $s(C) = \varepsilon C$. Hence, $s(X_i) = X_i$ for $i \in \{0, 1, 2\}$ and $s(X_j) = t(X_j)$ for all $j \in \{2, \ldots, m\}$.

Next we argue that $\sigma_{X_2}(t)\sigma_{X_m}(t) = \sigma_{X_0}(s)\sigma_{X_m}(s)$. Let $\pi(X_0) = \{A, B, C, D_1, \ldots, D_r\}$ and let $\boldsymbol{\beta}_A, \boldsymbol{\beta}_B, \boldsymbol{\beta}_C, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r$ denote the standard basis for X_0. Then the standard basis for X_2 is $\boldsymbol{\beta}_A + \boldsymbol{\beta}_B + \boldsymbol{\beta}_C, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r$. Since both s and t induce the same permutation on the standard basis vectors of X_2 and X_m, it follows that $\sigma_{X_2}(s) = \sigma_{X_2}(t)$ and $\sigma_{X_m}(s) = \sigma_{X_m}(t)$. And since s either fixes or negates all three vectors $\boldsymbol{\beta}_A, \boldsymbol{\beta}_B, \boldsymbol{\beta}_C$ it follows that $\sigma_{X_0}(s) = \sigma_{X_2}(s)$. Thus,

$$\begin{aligned}&\sigma_{X_2}(t)\sigma_{X_m}(t)\mathcal{N}\big(X_0 \to X_1 \to t(X_2) \to \cdots \to t(X_m)\big) \\ &\quad = \sigma_{X_0}(s)\sigma_{X_m}(s)\mathcal{N}\big(s(X_0) \to s(X_1) \to s(X_2) \to \cdots \to s(X_m)\big) \\ &\quad = \mathcal{N}\big(s(X_0 \to \cdots \to X_m)\big) = \mathcal{N}(X_0 \to \cdots \to X_m). \qquad \square\end{aligned}$$

Proof of Theorem 9.2. We first argue that we need only prove the cases $m = 1, 2$.

Reduction to the cases $m = 1, 2$. Let $\psi : k\Gamma \to (k\mathcal{F})^{\mathcal{B}_n}$ be the k-algebra homomorphism defined in the proof of Theorem 9.1. Note that $\psi = \varphi \circ \xi$, where ξ maps paths of length l in Γ to paths of length $2l$ in $\mathcal{Q}$. The case $m = 2$ was what was needed to prove that ψ is surjective. Hence, if $a \in \operatorname{rad}^{2m}(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$ for some $m \in \mathbb{N}$, then there exists $c \in k\Gamma$ such that $\psi(c) = a$. We will argue that $c \in \operatorname{rad}^m(k\Gamma)$, thus showing that $a = \psi(c) \in \psi(\operatorname{rad}^m(k\Gamma)) \subseteq \operatorname{rad}^m(k\mathcal{F})^{\mathcal{B}_n}$. (The reverse containment is immediate.)

Since $\varphi(\operatorname{rad}^p(k\mathcal{Q})) = \operatorname{rad}^p(k\mathcal{F})$ for all $p \in \mathbb{N}$ (this follows from the fact that $\mathcal{Q}$ is the quiver of $k\mathcal{F}$ and contains no oriented cycles [ASS06, Corollary II.2.11]), it follows that $\xi(c)$ is a linear combination of paths of $\mathcal{Q}$ having length at least $2m$. Hence, c is a linear combination of paths of Γ having length at least m, so $c \in \operatorname{rad}^m(k\Gamma)$.

The case $m = 1$. In the proof of Proposition 4.1 we argued that $\mathrm{rad}((k\mathcal{F})^W) = \mathrm{rad}(k\mathcal{F}) \cap (k\mathcal{F})^W$ for any finite Coxeter group W. So we need only show that $\mathrm{rad}(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n} = \mathrm{rad}^2(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$. Let $\varphi : k\mathcal{Q} \to k\mathcal{F}$ denote the surjection of Theorem 6.2. Then $\mathrm{rad}^i(k\mathcal{F})$ is spanned by the elements $\varphi(P)$, where P is a path of length at least i. Since the transformation $\vec{v} \mapsto -\vec{v}$ is an element of $\mathcal{B}_n$, it follows that $\sum_{w \in \mathcal{B}_n} w(P) = 0$ if P is a path of odd length (see Proposition 7.6 and its proof). So $\mathrm{rad}^{2i}(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n} = \mathrm{rad}^{2i-1}(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$ for $i \geqslant 1$ since both are spanned by the elements $\varphi(\sum_{w \in \mathcal{B}_n} w(P))$, where P is a path of length at least $2i$.

The case $m = 2$. We first argue that $\mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n}) \subseteq \mathrm{rad}^4(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$. Let $a \in \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$. Then $a = bc$ for two elements $b, c \in \mathrm{rad}(k\mathcal{F})^{\mathcal{B}_n} = \mathrm{rad}^2(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$. Thus, bc is an element of $\mathrm{rad}^4(k\mathcal{F})$ and $(k\mathcal{F})^{\mathcal{B}_n}$.

We prove the reverse containment by contradiction. Suppose $\mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n}) \subsetneq \mathrm{rad}^4(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$. Since $\mathrm{rad}^4(k\mathcal{F}) \cap (k\mathcal{F})^{\mathcal{B}_n}$ is spanned by elements of the form $\varphi(\mathcal{N}(P))$, where P is a path in $\mathcal{Q}$ of length at least $m \geqslant 4$, it follows that there exists a path $P = (X_0 \to \cdots \to X_m)$ such that $m \geqslant 4$ and $\varphi(\mathcal{N}(P)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$. We first argue that we can assume that P satisfies the following: $\pi(X_1)$ contains a nonzero block $A \cup B$, where $A \neq B$ are blocks of $\pi(X_0)$ (Step 1); $\pi(X_2)$ contains the nonzero block $A \cup B \cup C$, where C is a block of $\pi(X_1)$ (Step 2); $\pi(X_3)$ contains a nonzero block $D \cup E$, where D and E are blocks of $\pi(X_2)$ and $|D| = |A \cup B \cup C|$ (Step 3). Then we derive a contradiction (Step 4).

Step 1. We argue that $\pi(X_1)$ contains a nonzero block $A \cup B$, where $A \neq B$ are blocks of $\pi(X_0)$.

If not, then the zero block Z_1 of $\pi(X_1)$ is $Z_1 = B \cup Z_0 \cup \overline{B}$ for some nonzero block $B \in \pi(X_0)$, where Z_0 is the zero block of $\pi(X_0)$. Let t be the signed permutation that negates the elements of B and fixes the other elements. Then $\mathcal{N}(P) = \mathcal{N}(t(P)) = -\mathcal{N}(P)$. Hence, $\mathcal{N}(P) = 0$, contradicting that $\varphi(\mathcal{N}(P)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$.

Step 2. We argue that $\pi(X_2)$ contains the nonzero block $A \cup B \cup C$, where C is a block of $\pi(X_1)$.

First we show that $A \cup B$ is not a block of $\pi(X_m)$. If $A \cup B$ is a block of $\pi(X_m)$, then it is a block of $\pi(X_j)$ for all $j \in \{1, \ldots, m\}$. Let t be the signed permutation that negates the elements of $A \cup B$. Then $t(P) = -P$, so $\mathcal{N}(P) = 0$, contradicting that $\varphi(\mathcal{N}(P)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$.

This implies that there exists $j \in [m]$ such that $\pi(X_j)$ is obtained from $\pi(X_{j-1})$ by merging $A \cup B$. We argue that we can assume $j = 2$. From Theorem 6.2, it follows that

$$P \quad + \sum_{\substack{Y \neq X_{j-1} \\ X_{j-2} \to Y \to X_j}} (X_0 \to \cdots \to X_{j-2} \to Y \to X_j \to \cdots \to X_m) \in \ker(\varphi).$$

Since $\varphi(\mathcal{N}(P)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$, there must exist at least one $Y \neq X_{j-1}$ such that $\varphi(\mathcal{N}(X_0 \to \cdots \to X_{j-2} \to Y \to X_j \to \cdots \to X_m)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$. By examining the intervals of length two in Π_n^B (see Fig. 2), we note that $\pi(Y)$ is obtained from $\pi(X_{j-2})$ by merging $A \cup B$ with some other block of $\pi(X_{j-2})$. By replacing P with this path, noticing that this new path still begins with $X_0 \to X_1$, and repeating this argument until $j = 2$, we have that $\pi(X_2)$ is obtained from $\pi(X_1)$ by merging $A \cup B$.

If the zero block of $\pi(X_2)$ is $Z_2 = A \cup B \cup Z_0 \cup \overline{A} \cup \overline{B}$, where Z_0 is the zero block of $\pi(X_0)$, then any $t \in \mathcal{B}_n$ that fixes X_2 also fixes Z_2. This implies, by appealing to the argument in the proof of Lemma 9.3, that $\mathcal{N}(P)$ is a scalar multiple of $\mathcal{N}(X_2 \to \cdots \to X_m)\mathcal{N}(X_0 \to X_1 \to X_2)$, contradicting that $\varphi(\mathcal{N}(P)) \notin \mathrm{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$.

Step 3. We argue that we can suppose that $\pi(X_3)$ contains a nonzero block $D \cup E$, where D and E are blocks of $\pi(X_2)$ and $|D| = |A \cup B \cup C|$.

We first argue that not all of the blocks of $\pi(X_2)$ of size $\lambda = |A \cup B \cup C|$ are also blocks of $\pi(X_m)$. We do this by showing that we can factor $\mathcal{N}(P)$. By Lemma 9.3, we need only show that if $t(X_2) = X_2$, then

$$\begin{tikzcd}
\begin{array}{c}\pi(X_0)=\\ \{\ldots,A,B,C,D,E,\ldots\}\end{array} \arrow[d] \arrow[dr] & \\
\begin{array}{c}\pi(X_1)=\\ \{\ldots,A\cup B,C,D,E,\ldots\}\end{array} \arrow[d] \arrow[dr] & \begin{array}{c}\pi(Y_1)=\\ \{\ldots,A,B,C,D\cup E,\ldots\}\end{array} \arrow[d] \\
\begin{array}{c}\pi(X_2)=\\ \{\ldots,A\cup B\cup C,D,E,\ldots\}\end{array} \arrow[d] & \begin{array}{c}\pi(Y_2)=\\ \{\ldots,A\cup B,C,D\cup E,\ldots\}\end{array} \arrow[dl] \\
\begin{array}{c}\pi(X_3)=\\ \{\ldots,A\cup B\cup C,D\cup E,\ldots\}\end{array} &
\end{tikzcd}$$

Fig. 4. In Case 1, D and E are not $\pm(A \cup B \cup C)$. Note that $|D \cup E| > |A \cup B|$.

$$\sigma_{X_2}(t)\sigma_{X_m}(t)\mathcal{N}\big(X_0 \to X_1 \to t(X_2) \to \cdots \to t(X_m)\big) = \mathcal{N}(P).$$

Write $\pi(X_2) = \{B_1, \ldots, B_k, C_1, \ldots, C_l; Z_0\}$, where $|C_i| = \lambda$, $|B_j| \neq \lambda$, and write $\pi(X_m) = \{D_1, \ldots, D_h, C_1, \ldots, C_l; Z_m\}$. Suppose $t \in \mathcal{B}_n$ such that $t(X_2) = X_2$. Then t permutes the blocks $\pm B_1, \ldots, \pm B_k$, as well as the blocks $\pm C_1, \ldots, \pm C_l$. Define $s \in \mathcal{B}_n$ by $s|_{B_i} = t, s|_{Z_0} = t$ and $s|_{C_j} = 1$. Then $s(X_0) = X_0$, $s(X_1) = X_1$ and $s(X_j) = t(X_j)$ for all $j \in \{2, \ldots, m\}$. It remains to show that $\sigma_{X_0}(s)\sigma_{X_m}(s) = \sigma_{X_2}(t)\sigma_{X_m}(t)$. This follows by comparing the actions of s and t on the standard basis $\boldsymbol{\beta}_{B_1}, \ldots, \boldsymbol{\beta}_{B_k}$, $\boldsymbol{\beta}_{C_1}, \ldots, \boldsymbol{\beta}_{C_l}$ of X_2.

This implies that some nonzero block D of $\pi(X_1)$ of size λ is merged to get $\pi(X_j)$ for some $j \in \{3, \ldots, m\}$. If the zero block of $\pi(X_j)$ is $Z_j = D \cup Z_{j-1} \cup \overline{D}$, where Z_{j-1} is the zero block of $\pi(X_{j-1})$, then let t be the signed permutation that negates the elements of D and fixes the other elements of $[\pm n]$. Since D is a block of $\pi(X_1)$ and $|D| = |A \cup B \cup C|$, we have either that D is a block of $\pi(X_0)$ or $D = A \cup B \cup C$. In both cases t negates an odd number of elements of the standard basis of X_0 and no elements of the standard basis of X_m. Thus, $\sigma_{X_0}(t) = -1$ and $\sigma_{X_m}(t) = 1$. It follows that $t(P) = -P$, and so $\varphi(\mathcal{N}(P)) = 0$, a contradiction.

By arguing as in Step 2, using the relations in $\ker(\varphi)$, we can assume that $j = 3$.

Step 4. We are now ready to conclude the proof. Let $P = (X_0 \to \cdots \to X_m)$ be a path of length $m \geqslant 4$ such that $\varphi(\mathcal{N}(P)) \notin \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$. From Step 1 we have that $\pi(X_1)$ contains a nonzero block $A \cup B$, where $A \neq B$ are blocks of $\pi(X_0)$. Of all such paths, pick P such that $|A| + |B|$ is maximal. That is, we suppose the following.

($\star$) If $P' = (X_0 \to Y_1 \to Y_2 \to \cdots \to Y_m)$ with $\varphi(\mathcal{N}(P')) \notin \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$ and if $\pi(Y_1)$ contains the nonzero block $A' \cup B'$, where $A' \neq B'$ are blocks of $\pi(X_0)$, then $|A' \cup B'| \leqslant |A \cup B|$.

In both Steps 2 and 3, we replaced P with other paths P' that begin with $X_0 \to X_1$ and such that $\varphi(\mathcal{N}(P')) \notin \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$. Therefore, we can assume that $\pi(X_2)$ contains the nonzero block $A \cup B \cup C$, where C is a block of $\pi(X_1)$ and that $\pi(X_3)$ contains a nonzero block $D \cup E$, where D and E are blocks of $\pi(X_2)$ and D has cardinality $\lambda = |A \cup B \cup C|$. Therefore, there are three cases to consider.

Case 1. Suppose $D, E \neq \pm(A \cup B \cup C)$. This case is illustrated in Fig. 4. The open interval $(X_3, X_1) = \{Y \in \mathcal{L}\colon\ X_3 < Y < X_1\}$ contains exactly two elements: X_2 and Y_2, where $\pi(Y_2)$ is obtained from $\pi(X_1)$ by merging D with E. The open interval (Y_2, X_0) also contains exactly two elements: X_1 and Y_1, where $\pi(Y_1)$ is obtained from $\pi(X_0)$ by merging D and E. If $P' = (X_0 \to Y_1 \to Y_2 \to X_3 \to \cdots \to X_m)$, then $\varphi(\mathcal{N}(P')) \notin \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$ since $P' - P \in \ker\varphi$ (Lemma 6.6), but $|D \cup E| > |A \cup B|$, contradicting ($\star$).

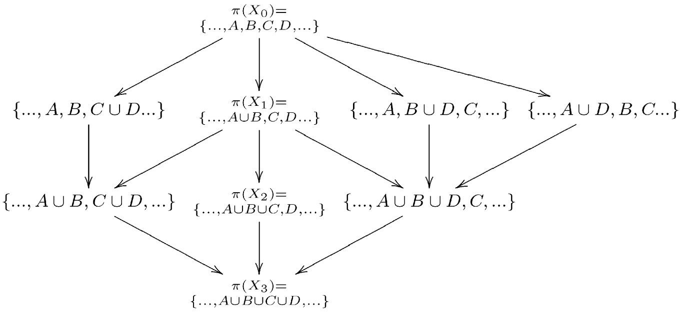

Fig. 5. In Case 2, $|D| = \lambda$, thus $|C \cup D|, |B \cup D|, |A \cup D| > |A \cup B|$.

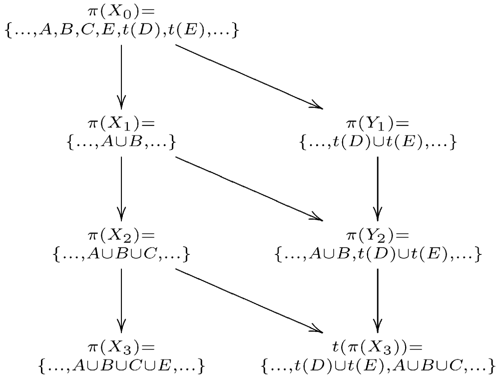

Fig. 6. In Case 3, $|E| \neq \lambda$. Note that $|t(D) \cup t(E)| > |A \cup B|$.

Case 2. Suppose $D \neq E = A \cup B \cup C$. This situation is illustrated in Fig. 5. Let $(X_0 \to Y_i \to Z_i \to X_3)$ for $i \in \{1, 2, 3\}$ be the three paths in Fig. 5 from X_0 to X_3 such that $Y_i \neq X_1$, and let $P_i = (X_0 \to Y_i \to Z_i \to X_3 \to \cdots \to X_m)$ for $i \in \{1, 2, 3\}$. Since $|D| > |A \cup B|$, the assumption ($\star$) implies that $\varphi(\mathcal{N}(P_i)) \in \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$ for $i \in \{1, 2, 3\}$. But $P - (P_1 + P_2 + P_3) \in \ker(\varphi)$ (Lemma 6.6), so this implies that $\mathcal{N}(P) \in \operatorname{rad}^2((k\mathcal{F})^{\mathcal{B}_n})$, a contradiction.

Case 3. Suppose $D = A \cup B \cup C \neq E$. If $|E| = \lambda$, then we can swap the roles of D and E and apply the argument from Case 2. So suppose that $|E| \neq \lambda$. By Lemma 9.3, there exists $t \in \mathcal{B}_n$ such that $t(X_2) = X_2$, $t(D) \neq \pm D$ and

$$\varphi\big(\mathcal{N}\big(X_0 \to X_1 \to X_2 \to t(X_3) \to \cdots \to t(X_m)\big)\big) \notin \operatorname{rad}^2\big((k\mathcal{F})^{\mathcal{B}_n}\big). \tag{9.1}$$

We argue that we are in the situation illustrated in Fig. 6. We first establish that $t(D), t(E) \in \pi(X_0)$. Since $t(X_2) = X_2$, it follows that t permutes the blocks of $\pi(X_2)$. Since $t(D) \neq \pm D$, it follows that $t(D)$ is a block of $\pi(X_2)$ different than $\pm D = \pm(A \cup B \cup C)$. And because all other blocks of $\pi(X_2)$ are blocks of $\pi(X_0)$, we have that $t(D)$ is a block of $\pi(X_0)$. Considering that $|E| \neq \lambda = |D|$, we have $t(E) \neq \pm D$, so the same reasoning implies that $t(E)$ is also a block of $\pi(X_0)$.

Since $t(X_2) = X_2$ and $\pi(X_3) \lessdot \pi(X_2)$, it follows that $t(\pi(X_3)) \lessdot \pi(X_2)$. Therefore, $t(\pi(X_3))$ is obtained from $\pi(X_2)$ by merging $t(D)$ and $t(E)$ since $t(D), t(E) \in \pi(X_2)$ and $t(D) \cup t(E) = t(A \cup B \cup C \cup E) \in t(\pi(X_3))$. There is exactly one other partition $\pi(Y_2)$ such that $t(\pi(X_3)) \lessdot \pi(Y_2) \lessdot \pi(X_1)$, the partition obtained from $\pi(X_1)$ by merging $t(D)$ with $t(E)$. There is exactly one other partition $\pi(Y_1)$ such that $\pi(Y_2) \lessdot \pi(Y_1) \lessdot \pi(X_0)$, the partition obtained from $\pi(X_0)$ by merging $t(D)$ with $t(E)$. So we are in the situation illustrated in the figure.

By Lemma 6.6, the following element is in $\ker(\varphi)$:

$$\big(X_0 \to X_1 \to X_2 \to t(X_3) \to \cdots \to t(X_m)\big) - \big(X_0 \to Y_1 \to Y_2 \to t(X_3) \to \cdots \to t(X_m)\big).$$

Together with (9.1), this implies that

$$\varphi\big(\mathcal{N}\big(X_0 \to Y_1 \to Y_2 \to t(X_3) \to \cdots \to t(X_m)\big)\big) \notin \operatorname{rad}^2\big((k\mathcal{F})^{\mathcal{B}_n}\big).$$

This contradicts our assumption ($\star$) because $\pi(Y_1)$ is obtained from $\pi(X_0)$ by merging the blocks $t(D)$ and $t(E)$, and $|t(D) \cup t(E)| = |t(A \cup B \cup C \cup E)| > |A \cup B|$. $\square$

10. Future directions

This article is part of an ongoing project to determine the quiver with relations of the descent algebras. There is still much to do.

The quivers of all the descent algebras have not yet been determined. The main outstanding case is the quiver of the descent algebra of type D as the exceptional types can be dealt with using computer algebra software [Pfe07]. It should be possible to adapt the proofs of Theorems 8.1 and 9.1 to this case as well, but given the similarity between these arguments, a general argument is more desirable. The main obstacle is to understand the relationship between $\operatorname{rad}^2((k\mathcal{F})^W)$ and $(k\mathcal{F})^W \cap \operatorname{rad}^p(k\mathcal{F})$. Indeed, if these two spaces are equal for some p, then there is no arrow from $\mathcal{O}'$ to $\mathcal{O}$, where $\mathcal{O}', \mathcal{O} \in \mathcal{L}/W$, if $\mathcal{O} < \mathcal{O}'$ and $\operatorname{rank}(\mathcal{O}') - \operatorname{rank}(\mathcal{O}) \geqslant p$; so only the intervals in $\mathcal{L}/W$ of length $p-1$ need to be studied. This is precisely what we did for types A and B, where p was 2 and 4, respectively.

Another task is to determine relations for some quiver presentation of the descent algebras. Very little is known here, even for type A.

Other representation theoretic questions also arise. As mentioned following Corollary 7.4, it would be interesting to determine the characteristic tilting module of each descent algebra as well as its Ringel dual. Also, the Cartan invariants of the descent algebras are not known in general. Formulas exist for type A (see [GR89], [BL96, Corollary 2.1], [KLT97, Section 3.6] and [Sch06, Section 9.4]) and a combinatorial interpretation for type B was given by Nantel Bergeron [Ber92, Theorem 3.3].

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