0-Hecke algebras of finite Coxeter groups

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Abstract

We study the 0-Hecke algebra of an arbitrary finite Coxeter group, building on work of Norton [9]. We examine the correspondence between injective and projective modules, extensions between simple modules and (in type A) the structure of induced simple modules.

1 Introduction

Suppose that W is a Coxeter group, i.e. a group with a presentation of the form

$$W = \left\langle s_1, \dots, s_n \;\middle|\; (s_is_j)^{m_{ij}} = 1 \right\rangle$$

for some integer n and some symmetric n by n matrix (m_{ij}) with entries in $\mathbb{N} \cup \{\infty\}$ with $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$. Given a field $\mathbb{F}$ and an element q of $\mathbb{F}$, we define the *Iwahori–Hecke algebra* $\mathcal{H}_q(W)$ to be the associative algebra over $\mathbb{F}$ with generators $S_1, \dots, S_n$ and relations

$$S_i^2 = q + (q-1)S_i,$$
$$(S_iS_jS_i\dots)_{m_{ij}} = (S_jS_iS_j\dots)_{m_{ij}}$$

for all $i \neq j$, where $(aba\dots)_m$ denotes an alternating product of m terms. The Iwahori–Hecke algebra arises in the study of groups with (B, N)-pairs.

The algebra $\mathcal{H}_q(W)$ has been studied extensively in the case where q is non-zero, especially when W is of type A or B; in these cases, $\mathcal{H}_q(W)$ is cellular, and the representation theory is correspondingly well understood; however, this theory breaks down in the case $q = 0$. In [9], Norton studied the '0-Hecke algebra' $\mathcal{H} = \mathcal{H}_0(W)$; she classified the irreducible modules, decomposed the algebra into left ideals and described the Cartan invariants. In [2], Carter studied $\mathcal{H}$ in type A, i.e. where W is a symmetric group; he gave the decomposition numbers in this case. Krob and Thibon have also studied $\mathcal{H}$ in type A, giving a representation-theoretic interpretation of non-commutative symmetric functions [8]; this builds on earlier work of Duchamp, Krob, Leclerc and Thibon in [4]. Duchamp, Hivert and Thibon take this work further in [3], and that case prove some of the results

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in this paper. The author is grateful to the referee for pointing this reference out. In this paper we study the representation theory of $\mathcal{H}$ for W an arbitrary finite Coxeter group; we shall show that $\mathcal{H}$ is Frobenius, and classify those W for which $\mathcal{H}$ is symmetric. We calculate $\mathrm{Ext}^1_{\mathcal{H}}(M,N)$ for simple modules M and N, and finally we provide a 'branching rule' which describes (the submodule lattice of) a simple module induced from a 0-Hecke algebra of type A_{n-1} to a 0-Hecke algebra of type A_n.

2 Background and notation

From now on, we fix an arbitrary field $\mathbb{F}$ and an arbitrary finite Coxeter group W (with presentation as above), and write $\mathcal{H} = \mathcal{H}_0(W)$. We write l for the length function on W (in terms of the generators $s_1,\dots,s_n$). Basic facts about $\mathcal{H}$ can be found in Chapter 1 of Mathas's book [6]. Essential facts about finite Coxeter groups can be found in [7]; in particular, we shall use the Deletion and Exchange Conditions [7, §1.7] as well as the classification of finite Coxeter groups (with the notation of [7]).

We make a slight change of notation for $\mathcal{H}$, writing T_i for $-S_i$. This simply has the effect of removing the minus signs from the presentation of $\mathcal{H}$ given above (and from most of the rest of this paper). We have the following.

Theorem 2.1. [6, Lemma 1.12 & Theorem 1.13]
$\mathcal{H}$ has a basis $\{T_w \mid w \in W\}$ with $T_{s_i} = T_i$ and

$$T_iT_w = \begin{cases} T_{s_iw} & (l(s_iw) > l(w)) \\ T_w & (l(s_iw) < l(w)) \end{cases}$$

for all $i = 1,\dots,n$ and all $w \in W$.

Theorem 2.2. [9, §3]
Given a subset J of $\{1,\dots,n\}$, let M_J be the $\mathcal{H}$-module with basis $\{x\}$ and $\mathcal{H}$-action given by

$$T_ix = \begin{cases} x & (i \in J) \\ 0 & (i \notin J). \end{cases}$$

Then $\left\{M_J \;\middle|\; J \subseteq \{1,\dots,n\}\right\}$ is a complete set of irreducible modules for $\mathcal{H}$.

2.1 Finite Coxeter groups

Let W be a finite Coxeter group, and G the Coxeter graph of W. Since W is finite, it has a unique longest element, which we denote w_0. We shall use the following lemma repeatedly, often without comment.

Lemma 2.3. [7, §1.8]
For any $w \in W$, we have $l(ww_0) = l(w_0w) = l(w_0) - l(w)$. In particular, w_0 is an involution.

It wil be useful later to describe the automorphism of W induced by conjugation by w_0. It does not seem likely that the following result is new, though the author has been unable to find it in the literature.

Proposition 2.4. *The conjugation action of w_0 on W is given by $s_i \mapsto s_{\sigma(i)}$, where σ is the automorphism of G which fixes each connected component of G set-wise, and which restricts to:*

1. *the identity on each component of type A_1, $B_n(n \geqslant 2)$, $D_{2n}(n \geqslant 2)$, E_7, E_8, F_4, H_3, H_4 or $I_2(2m)(1 \leqslant m)$;*
2. *the unique non-trivial automorphism of each other connected component of G.*

In particular, w_0 is central in W if and only if every connected component of G is of one of the types listed in (1).

Proof. We have

$$\begin{aligned} l(w_0 s_i w_0) &= l(w_0) - (l(s_i w_0)) \\ &= l(w_0) - (l(w_0) - 1) \\ &= 1, \end{aligned}$$

so that $w_0 s_i w_0 = s_{\sigma(i)}$ for some σ. σ must be an automorphism of G (as a labelled graph), since m_{ij} is the multiplicative order of $s_i s_j$. Furthermore, since W is the direct product of the Coxeter groups corresponding to the connected components of G, σ must fix each connected component set-wise. So we may assume that W is irreducible.

The cases listed are precisely those for which all the degrees of (the elementary invariant polynomials of) W are even [7, §3.7]. In these cases, we have by [7, Corollary 3.19] that w_0 maps to $-I$ in the standard reflection representation of W. Since this representation is faithful, w_0 must be central. In the remaining cases, it is easy to find some s_i with which w_0 does not commute. Hence conjugation by w_0 induces a non-trivial automorphism of G; by checking the Coxeter graphs in these cases, it may be verified that there is a unique non-trivial automorphism of G in each case. □

3 Automorphisms of $\mathcal{H}$ and duality

In this section, we describe some automorphisms and anti-automorphsisms of $\mathcal{H}$, and examine the induced self-equivalences of the module category of $\mathcal{H}$. We begin with a lemma which we shall use several times; it appears in the proof of [9, Lemma 4.3].

Lemma 3.1. *For any i, j and any $n \geqslant 1$ we have*

$$\Big((T_i-1)(T_j-1)(T_i-1)\dots\Big)_n = (T_iT_jT_i\dots)_n + \sum_{m=1}^{n-1}(-1)^{m-n}\Big((T_iT_jT_i\dots)_m + (T_jT_iT_j\dots)_m\Big) + (-1)^n.$$

In particular, we have

$$\Big((T_i-1)(T_j-1)(T_i-1)\dots\Big)_{m_{ij}} = \Big((T_j-1)(T_i-1)(T_j-1)\dots\Big)_{m_{ij}}.$$

Proof. This is a simple induction on n. □

Proposition 3.2.

- *There is an automorphism θ of $\mathcal{H}$ defined by*

$$\theta : T_i \longmapsto 1 - T_i$$

for all i.

- *There is an automorphism ϕ of $\mathcal{H}$ defined by*

$$\phi : T_i \longmapsto T_{w_0 s_i w_0}$$

for all i.

- *There is an anti-automorphism χ of $\mathcal{H}$ defined by*

$$\chi : T_i \longmapsto T_i$$

for all i.

Furthermore, θ, ϕ and χ commute and each has order 1 *or* 2.

Proof. It is trivial that θ^2, ϕ^2 and χ^2 are all the identity map, and in particular that θ, ϕ and χ are all invertible; it is also clear that they commute. It remains to verify the defining relations of $\mathcal{H}$, which is routine for ϕ and χ. For θ, we have

$$(1 - T_i)^2 = 1 - 2T_i + T_i^2 = 1 - T_i,$$

while the braid relations follow from Lemma 3.1. □

The involution θ is also discussed in [8].

Now suppose M is an $\mathcal{H}$-module. We define $\overline{M}$ to be the module with the same underlying vector space as M, and with action

$$h \cdot m = \theta(h)m$$

for $h \in \mathcal{H}$ and $m \in M$. We define $\widehat{M}$ to be the module with the same underlying vector space as M, and with action

$$h \cdot m = \phi(h)m$$

for $h \in \mathcal{H}$ and $m \in M$. Then $M \mapsto \overline{M}$ and $M \mapsto \widehat{M}$ define self-equivalences of $\mathrm{mod}(\mathcal{H})$ of order 1 or 2.

We also define $M^\diamond$ to be the module to be the vector space dual to M with $\mathcal{H}$-action given by

$$(h \cdot f)(m) = f(\chi(h)m)$$

for $h \in \mathcal{H}$, $f \in M^*$ and $m \in M$. Finally, we define $M^\circ = (\widehat{M})^\diamond \cong \widehat{M^\diamond}$. $M \mapsto M^\circ$ defines an equivalence of categories $\mathrm{mod}(\mathcal{H}) \to (\mathrm{mod}(\mathcal{H}))^{\mathrm{op}}$.

The effect of these functors on simple modules is easily found. For $J \subseteq \{1, \dots, n\}$, write $\overline{J}$ for its complement. Recall also the automorphism σ of the Coxeter graph of W from Proposition 2.4.

Proposition 3.3. *We have*

$$\overline{M_J} \cong M_{\overline{J}},$$
$$(M_J)^\diamond \cong M_J,$$
$$\widehat{M_J} \cong (M_J)^\circ \cong M_{\sigma(J)}.$$

It turns out that M° is a good definition of a 'dual module' to M; in particular, we shall see that any projective module is self-dual with this definition, and that induction from type A_{n-1} to type A_n preserves this notion of duality.

Proposition 3.4. *Consider $\mathcal{H}$ as an $\mathcal{H}$-module. Then*

$$\overline{\mathcal{H}} \cong \widehat{\mathcal{H}} \cong \mathcal{H}^\diamond \cong \mathcal{H}^\circ \cong \mathcal{H}.$$

Proof. The fact that θ and ϕ are automorphisms implies that $\overline{\mathcal{H}} \cong \widehat{\mathcal{H}} \cong \mathcal{H}$ and $\mathcal{H}^\diamond \cong \mathcal{H}^\circ$, so we need only show that $\mathcal{H}^\diamond \cong \mathcal{H}$. Let $\{f_w \mid w \in W\}$ be the basis for $\mathcal{H}^*$ dual to the the basis $\{T_w \mid w \in W\}$ for $\mathcal{H}$. Then Theorem 2.1 implies that

$$T_i f_w = \begin{cases} f_w + f_{s_i w} & (l(s_i w) > l(w)) \\ 0 & (l(s_i w) < l(w)). \end{cases}$$

We shall find a basis for $\mathcal{H}$ which gives the same $\mathcal{H}$-action. Given $w \in W$, let $s_{i_1} \dots s_{i_r}$ be any reduced expression for w, and define

$$X_w = (T_{s_{i_1}} - 1) \dots (T_{s_{i_r}} - 1).$$

As pointed out in the proof of [9, Lemma 4.3], X_w does not depend on the reduced expression chosen: since any reduced expression for w can be transformed into any other by means of the braid relations, we can apply Lemma 3.1. To show that $\{X_w \mid w \in W\}$ is a basis for $\mathcal{H}$, we prove linear independence: if $\sum_{w \in W} \lambda_w X_w = 0$, take w_1 of maximal length such that $\lambda_{w_1} \neq 0$. Then when we express $\sum_{w \in W} \lambda_w X_w$ in terms of the basis $\{T_w\}$, we find that the coefficient of T_{w_1} is λ_{w_1}; contradiction.

It remains to prove that

$$T_i X_w = \begin{cases} X_w + X_{s_i w} & (l(s_i w) > l(w)), \\ 0 & (l(s_i w) < l(w)); \end{cases}$$

if $l(s_i w) > l(w)$, then $s_i s_{i_1} \dots s_{i_r}$ is a reduced expression for $s_i w$, and so we have

$$X_{s_i w} = (T_i - 1) X_w$$

as required. If $l(s_i w) < l(w)$, then by the Exchange Condition there is a reduced expression $s_{i_1} \dots s_{i_r}$ for w with $i_1 = i$. So

$$T_i X_w = T_i (T_i - 1) W_{s_i w} = 0. \qquad \square$$

4 Injective and projective modules for $\mathcal{H}$

Recall that an algebra A over $\mathbb{F}$ if *Frobenius* if there is a linear map $\lambda : A \to \mathbb{F}$ whose kernel contains no right or left ideal of A. If in addition we have

$$\lambda(ab) = \lambda(ba)$$

for all $a, b \in A$, we say that A is *symmetric*.

Proposition 4.1. *$\mathcal{H}$ is Frobenius.*

Proof. Define $\lambda : \mathcal{H} \to \mathbb{F}$ by mapping

$$T_w \longmapsto \begin{cases} 1 & (w = w_0) \\ 0 & (w \neq w_0). \end{cases}$$

We must show that for any $0 \neq h \in \mathcal{H}$, there are $j, k \in \mathcal{H}$ such that $\lambda(jh)$ and $\lambda(hk)$ are non-zero. Express h in terms of the basis $\{T_w\}$, and let w be an element of maximal length such that T_w occurs with non-zero coefficient. Now define $j = T_{w_0 w^{-1}}$ and $k = T_{w^{-1} w_0}$. We claim that $jT_w = T_{w_0} = T_w k$, while $\lambda(jT_x) = 0 = \lambda(T_x k)$ for any $x \neq w$ with $l(x) \leqslant l(w)$, which is sufficient. To prove the claim, we notice that for any $x, y \in W$, $T_x T_y$ is of the form T_z, where $l(x) \leqslant l(x) + l(y)$, with equality if and only if $l(xy) = l(x) + l(y)$ (in which case $z = xy$). □

Remark. Proposition 4.1 is proved in type A in [3].

$\mathcal{H}$ is not necessarily symmetric, but it is 'quasi-symmetric' in the following sense.

Proposition 4.2. *Let $\lambda : \mathcal{H} \to \mathbb{F}$ be as in the proof of Proposition 4.1. Then for any a and b in $\mathcal{H}$ we have*

$$\lambda(ab) = \lambda(\phi(b)a).$$

Proof. By linearity, it suffices to consider the case where $a = T_w$ and $b = T_x$ for $w, x \in W$. Fix w, and choose a reduced expression $u_1 \dots u_r$, where each u_i equals some s_k. Say that a sub-expression $u_{j_1} \dots u_{j_t}$ (where $1 \leqslant j_1 < \dots < j_t \leqslant r$) is *good* if

- it is a reduced expression, and
- for any i, k such that $j_{k-1} < i < j_k$, we have

$$l(u_{j_1} \dots u_{j_{k-1}} u_i) > l(u_{j_1} \dots u_{j_{k-1}}).$$

Lemma 4.3. *$T_w T_x$ equals T_{w_0} if and only if $x = u_{j_t} \dots u_{j_1} w_0$ for some good sub-expression $u_{j_1} \dots u_{j_t}$ of $u_1 \dots u_r$.*

Proof. First suppose that $u_{j_1} \dots u_{j_t}$ is good. Since $u_{j_t} \dots u_{j_1}$ is reduced, we have

$$l(u_{j_{k-1}} \dots u_{j_1}) < l(u_{j_k} \dots u_{j_1}),$$

so that

$$T_{u_{j_k}} T_{u_{j_{k_1}} \dots u_{j_1} w_0} = T_{u_{j_k} u_{j_{k-1}} \dots u_{j_1} w_0}.$$

For $j_{k-1} < i < j_k$, we have $l(u_i u_{j_{k-1}} \dots u_{j_1}) > l(u_{j_{k-1}} \dots u_{j_1})$, so that

$$T_{u_i} T_{u_{j_{k-1}} \dots u_{j_1} w_0} = T_{u_{j_{k-1}} \dots u_{j_1} w_0}.$$

Hence if $x = u_{j_t} \dots u_{j_1} w_0$ we have

$$T_w T_x = T_{u_1} \dots T_{u_r} T_{u_{j_t} \dots u_{j_1} w_0} = T_{w_0}.$$

Conversely, suppose that

$$T_{w_0} = T_w T_x = T_{u_1} \dots T_{u_r} T_x.$$

Let $j_1 < \dots < j_t$ be those values of j for which

$$T_{u_j} T_{u_{j+1}} \dots T_{u_r} T_x \neq T_{u_{j+1}} \dots T_{u_r} T_x.$$

Then we have $T_w T_x = T_{u_{j_1}} \dots T_{u_{j_t}} T_x = T_{u_{j_1} \dots u_{j_t} w_0}$, so that $x = u_{j_t} \dots u_{j_1} w_0$; the fact that $u_{j_1} \dots u_{j_t}$ is good follows from the definition of $j_1, \dots, j_t$. □

Now we show that the 'good' condition is a red herring.

Lemma 4.4. *The set of elements of W equal to $u_{j_1} \dots u_{j_t}$ for a good sub-expression $u_{j_1} \dots u_{j_t}$ of $u_1 \dots u_r$ equals the set of elements of W equal to $u_{j_1} \dots u_{j_t}$ for* any *sub-expression $u_{j_1} \dots u_{j_t}$ of $u_1 \dots u_r$.*

Proof. Given a sub-expression $u_{j_1} \dots u_{j_t}$ which is not good, we shall transform it into a good sub-expression without changing the element of W it represents. We proceed by induction on t, and for fixed t, we proceed by reverse induction on $j_1 + \dots + j_t$.

First suppose $u_{j_1} \dots u_{j_t}$ is not reduced. Then by the Deletion Condition, we may delete two entries in this subexpression without changing the element of W it represents. We are then done by induction on t.

Now suppose $u_{j_1} \dots u_{j_t}$ is reduced but not good. Then there exist k, i such that $j_{k-1} < i < j_k$ and

$$l(u_{j_1} \dots u_{j_{k-1}} u_i) < l(u_{j_1} \dots u_{j_{k-1}}).$$

By the Exchange Condition, there is some $c \leqslant k-1$ such that

$$u_{j_1} \dots \widehat{u_{j_c}} \dots u_{j_{k-1}} u_i = u_{j_1} \dots u_{j_{k-1}}.$$

Hence

$$u_{j_1} \dots u_{j_t} = u_{j_1} \dots \widehat{u_{j_c}} \dots u_{j_{k-1}} u_i u_{j_k} \dots u_{j_t},$$

so we may replace u_{j_c} with u_i in our sub-expression, and we are done by induction on $j_1 + \dots + j_t$. □

We conclude that T_wT_x equals T_{w_0} if and only if $x = u_{j_t} \dots u_{j_1} w_0$ for some sub-expression $u_{j_1} \dots u_{j_t}$ of $u_1 \dots u_r$. Similarly, we find that $\phi(T_x)T_w$ equals T_{w_0} if and only if

$$\phi(T_x) = T_{w_0 u_{j_t} \dots u_{j_1}}$$

for some sub-expression $u_{j_1} \dots u_{j_t}$. But $\phi(T_x) = T_{w_0 x w_0}$, and so $\phi(T_x)T_w$ equals T_{w_0} if and only if T_wT_x does. □

Now we discuss the consequences for injective and projective modules. Given an $\mathcal{H}$-module M, let $P(M)$ and $I(M)$ denote its projective cover and injective hull, respectively.

Proposition 4.5. *$\mathcal{H}$ is self-injective, with*

$$P(M_J) \cong I(\widehat{M_J})$$

for all $J \subseteq \{1, \dots, n\}$. Hence $P^\circ \cong P$ for any projective $\mathcal{H}$-module P. $\mathcal{H}$ is symmetric if and only if each connected component of G is of one of the types listed in Proposition 2.4.

Proof. Since $\mathcal{H}$ is Frobenius, it is self-injective [1, Proposition 1.6.2]. Hence $P = P(M_J)$ is isomorphic to the injective hull of some simple module. Let e be an idempotent such that $P(M_J) \cong \mathcal{H}e$ (Norton [9] describes such an idempotent explicitly). Then $\mathcal{H}\phi(e) \cong \widehat{P} \cong P(\widehat{M_J})$. Also, $\mathrm{soc}(P)e$ is a left ideal in $\mathcal{H}$ and so there is some $x \in \mathrm{soc}(P)$ such that

$$0 \neq \lambda(xe) = \lambda(\phi(e)x),$$

so

$$0 \neq \phi(e)\,\mathrm{soc}(P) \cong \mathrm{Hom}_{\mathcal{H}}(\widehat{P}, \mathrm{soc}(P)),$$

and we must have $\mathrm{soc}(P) \cong \widehat{M_J}$.

Since $\mathcal{H}^\circ \cong \mathcal{H}$ and $P(M_J) \cong I(M_J^\circ)$, we find that any projective module is self-dual. Proposition 4.2 says that $\mathcal{H}$ is symmetric when ϕ is the identity; on the other hand, for a symmetric algebra, $P(S) \cong I(S)$ for a simple module S, so $\mathcal{H}$ is *not* symmetric when ϕ is not the identity. □

Remark. The correspondence between injective and projective modules also follows (once we have self-injectivity) from [9, Lemma 4.23], in which the socle of each indecomposable left ideal of $\mathcal{H}$ is found explicitly.

5 Extensions of simple modules

In this section, we calculate the space $\mathrm{Ext}^1_{\mathcal{H}}(M, N)$ for simple $\mathcal{H}$-modules M and N. Since all simple $\mathcal{H}$-modules are one-dimensional, the easiest way to do this is simply to classify two-dimensional modules. This gives the following result (which is also proved, in type A, in [3]).

Theorem 5.1. *Suppose $J, K \subseteq \{1, \dots, n\}$. Then $\dim_{\mathbb{F}} \mathrm{Ext}^1_{\mathcal{H}}(M_J, M_K)$ is 1 if*

- *neither of J and K is contained in the other, and*

- *for any $j \in J \setminus K$ and $k \in K \setminus J$, we have $m_{jk} \geqslant 3$,*

and 0 *otherwise.*

Proof. Suppose we have a two-dimensional module M which is an extension of M_J by M_K. Let $\{e_2\}$ be a basis for a submodule isomorphic to M_K, and extend to a basis $\{e_1, e_2\}$ for M. If we let $J_i = \mathbb{1}(i \in J)$ and $K_i = \mathbb{1}(i \in K)$, then T_i acts on M by the matrix

$$A_i = \begin{pmatrix} J_i & 0 \\ a_i & K_i \end{pmatrix}$$

for some a_i. We must check the defining relations of $\mathcal{H}$.

The fact that T_i is idempotent simply means that $a_i = 0$ whenever $J_i = K_i$. Now we check the braid relations

$$(A_j A_k A_j \dots)_{m_{jk}} = (A_k A_j A_k \dots)_{m_{jk}}.$$

if either $J_j = K_j$ or $J_k = K_k$ then one of A_j, A_k is either 0 or the identity matrix, and the braid relation is immediate. In the case where $J_j = J_k = 1$, $K_j = K_k = 0$, we have

$$(A_j A_k A_j \dots)_m = A_j$$

for any $m > 0$, so we must have $a_j = a_k$. Similarly if $J_j = J_k = 0$, $K_j = K_k = 1$, we have $a_j = a_k$. If $J_j = K_k = 1$, $K_j = J_k = 0$, then we have

$$(A_j A_k A_j \dots)_m = 0$$

for all $m \geqslant 2$, while

$$(A_k A_j A_k \dots)_m = \begin{cases} \begin{pmatrix} 0 & 0 \\ a_j + a_k & 0 \end{pmatrix} & (m = 2) \\ 0 & (m \geqslant 3). \end{cases}$$

We conclude that M affords a representation of $\mathcal{H}$ if and only if there exist $a, b \in \mathbb{F}$ such that each A_i is one of the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix},$$

and such that $a + b = 0$ if there exist $j \in J \setminus K$, $k \in K \setminus J$ such that $m_{jk} = 2$.

If $a + b = 0$, then these four matrices can be simultaneously conjugated to diagonal matrices, and so M is a split extension. If $a + b \neq 0$, then the extension is non-split. But simultaneous conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$ takes the pair (a, b) to the pair $(da + c, db - c)$, and so all non-split extensions are isomorphic. The result follows. □

Remark. Theorem 5.1 affords a slightly quicker classification of the blocks of $\mathcal{H}$ in the case where W is irreducible than in [9, Theorem 5.2]. Given a proper non-empty subset J of $\{1,\dots,n\}$, we wish to show that M_J lies in the same block of $\mathcal{H}$ as $M_{\{1\}}$; we do this by exhibiting a sequence $J = J_0, J_1, \dots, J_r = \{1\}$ of subsets with $\mathrm{Ext}^1_{\mathcal{H}}(M_{J_{i-1}}, M_{J_i}) \neq 0$ for all i. By Theorem 5.1, we can construct J_i from J_{i-1} by replacing $j \in J_{i-1}$ with some $k \notin J_{i-1}$ which is adjacent to j in the Coxeter graph, or by replacing $j, j' \in J_{i-1}$ with some $k \notin J_{i-1}$ which is adjacent to both j and j' in the Coxeter graph. Since the Coxeter graph is connected, it is easily seen that we can get to $J_r = \{1\}$ in this way.

6 Branching of induced representations in type A

In this section, we specialise to 0-Hecke algebras of type A. Let $\mathcal{H}_n$ denote the 0-Hecke algebra for the Coxeter group of type A_n, with generators $s_1, \dots, s_n$ and

$$m_{ij} = \begin{cases} 3 & (|i-j| = 1) \\ 2 & (|i-j| > 1). \end{cases}$$

By Proposition 2.4, the automorphism ϕ is given by $T_i \mapsto T_{n+1-i}$.

$\mathcal{H}_{n-1}$ is naturally a subalgebra of $\mathcal{H}_n$, and $\mathcal{H}_n$ is free as an $\mathcal{H}_{n-1}$-module. Given a simple module M_J for $\mathcal{H}_{n-1}$, we wish to study the structure of the induced module

$$\mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} M_J = \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} M_J.$$

We shall show that this module is multiplicity-free and describe its composition factors and submodule lattice.

In [8, §5], the induction of simple and projective modules from $\mathcal{H}_{n-1}$ to $\mathcal{H}_n$ is discussed; the authors of that paper look at the more general situation $\mathcal{H}_0(\mathfrak{S}_{n-m} \times \mathfrak{S}_m) \leqslant \mathcal{H}_0(\mathfrak{S}_n)$, and describe the composition factors of an induced simple module, via quasi-symmetric functions. In fact, they consider the filtration on an induced simple module which arises from the length filtration on $\mathcal{H}_0(\mathfrak{S}_n)$, and give a 'graded characteristic' which decribes the composition factors of the layers of this filtration. But they do not describe in full the submodule lattice of an induced simple module, which is our task.

Given a multiplicity-free module M (or indeed any module whose submodule lattice is distributive), we may encode its submodule lattice simply by imposing a partial order on the set of composition factors: for composition factors S, T, we write $S \geqslant_M T$ if every submodule of M with S as a composition factor also has T as a composition factor. Equivalently, we may simply write down the poset of those submodules of M with simple cosocles, ordered by inclusion, and label each such submodule by the isomorphism class of its cosocle.

We make a slight change of notation for simple modules: given $J \subseteq \{1,\dots,n\}$, we write $J_i = 1$ if $i \in J$ and 0 otherwise, as before. Then we write

$$M_J = M(J_1, \dots, J_n).$$

Now for $J \subseteq \{1,\dots,n-1\}$ we examine the structure of $M = \mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} M_J$. It is easy to find a filtration of M by simple modules. If $\{x\}$ is a basis for M_J, then let

$$\begin{aligned} x_n &= 1 \otimes x, \\ x_{n-1} &= T_n \otimes x, \\ x_{n-2} &= T_{n-1}T_n \otimes x, \\ &\vdots \\ x_0 &= T_1 T_2 \dots T_n \otimes x. \end{aligned}$$

Proposition 6.1. $\{x_0, \dots, x_n\}$ *is a basis for M. Moreover, for $i = 0, \dots, n$, the subspace*

$$M_i = \langle x_0, \dots, x_i \rangle$$

is a submodule of M, and we have

$$\begin{aligned} M_n/M_{n-1} &\cong M(J_1, \dots, J_{n-1}, 0), \\ M_{n-1}/M_{n-2} &\cong M(J_1, \dots, J_{n-2}, 0, 1), \\ M_{n-2}/M_{n-3} &\cong M(J_1, \dots, J_{n-3}, 0, 1, J_{n-1}), \\ &\vdots \\ M_2/M_1 &\cong M(J_1, 0, 1, J_3, \dots, J_{n-1}), \\ M_1/M_0 &\cong M(0, 1, J_2, \dots, J_{n-1}), \\ M_0 &\cong M(1, J_1, \dots J_{n-1}). \end{aligned}$$

In particular, M is multiplicity-free.

Proof. Given $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, we have

$$T_i x_j = \begin{cases} J_i x_j & (i < j) \\ x_{j-1} & (i = j) \\ x_j & (i = j+1) \\ J_{i-1} x_j & (i > j+1). \end{cases}$$

So $x_0, \dots, x_n$ certainly span M. The fact that M_i is a submodule can also be seen from this action, as can the eigenvalues of $T_1, \dots, T_n$ on the quotients M_i/M_{i-1}. These quotients are then seen to be non-isomorphic: if

$$M(J_1, \dots, J_{i-1}, 0, 1, J_{i+1}, \dots, J_n) = M(J_1, \dots, J_{j-1}, 0, 1, J_{j+1}, \dots, J_n)$$

with $i < j$, then we have

$$1 = J_{i+1} = J_{i+2} = \dots = J_{j-2} = J_{j-1} = 0.$$

So M is multiplicity-free, and has $n+1$ composition factors. So $\dim_{\mathbb{F}} M \geqslant n+1$, and $\{x_0, \dots, x_n\}$ is a basis. □

Remark. The action of T_i on M given in the above proof shows that M is a *combinatorial module*, as defined in [3, §2.2].

We impose a total order on the composition factors of M according to this filtration:

$$M(J_1,\dots,J_n,0) > M(J_1,\dots,J_{n-1},0,1) > \dots > M(0,1,J_2,\dots,J_{n-1}) > M(1,J_2,\dots,J_{n-1}).$$

Then the partial order $\geqslant_M$ which encodes the submodule lattice of M is a sub-partial order of $\geqslant$. Our main result is as follows.

Theorem 6.2. *Suppose M_K and M_L are composition factors of M. Then $M_K >_M M_L$ if and only if $M_K > M_L$ and neither of K, L is contained in the other.*

The proof is slightly complicated. First we show that induction is well-behaved with respect to the functors $N \mapsto \overline{N}$ and $N \mapsto N^\circ$.

Lemma 6.3. *Let N be any $\mathcal{H}_{n-1}$-module. Then $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} \overline{N} \cong \overline{\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N}$.*

Proof. $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N$ is spanned by elements

$$T_{j+1}T_{j+2}\dots T_n \otimes m$$

for $m \in N$. Likewise, $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} \overline{N}$ is spanned by elements

$$(1-T_{j+1})(1-T_{j+2})\dots(1-T_n)\otimes m.$$

We define a map $\overline{\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N} \to \operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} \overline{N}$ via

$$T_{j+1}T_{j+2}\dots T_n \otimes m \longmapsto (1-T_{j+1})(1-T_{j+2})\dots(1-T_n)\otimes m$$

for all j and all $m \in N$. The fact that θ is an automorphism of $\mathcal{H}$ shows that this is a module isomorphism. □

Lemma 6.4. *Let N be any $\mathcal{H}_{n-1}$-module. Then $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N^\circ \cong (\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N)^\circ$.*

Proof. Let $\{e_1,\dots,e_r\}$ and $\{\epsilon_1,\dots,\epsilon_r\}$ be dual bases for N and N°, so that if $\langle\,,\,\rangle$ is the bilinear form given by $\langle e_i, \epsilon_j\rangle = \delta_{ij}$, then

$$\langle T_i m, \mu\rangle = \langle m, T_{n-i}\mu\rangle$$

for all $m \in N, \mu \in N^\circ$. Then we claim that

$$\{T_{j+1}\dots T_n \otimes e_k \mid 0 \leqslant j \leqslant n, 1 \leqslant k \leqslant r\}$$

is a basis for $\operatorname{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N$; this follows as in the proof of Proposition 6.1. Similarly,

$$\{(1-T_{j+1})\dots(1-T_n)\otimes \epsilon_k \mid 0 \leqslant j \leqslant n, 1 \leqslant k \leqslant r\}$$

is a basis for $\mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N^\circ$. Hence so is

$$\{(T_{j+1}-1)\dots(T_n-1)\otimes\epsilon_k \mid 0\leqslant j\leqslant n, 1\leqslant k\leqslant r\}.$$

Now we make these bases dual in such a way as to respect the $\mathcal{H}_n$-action: let $(\,,\,)$ be the bilinear form given by

$$(T_{j+1}\dots T_n\otimes e_k, (T_{s+1}-1)\dots(T_n-1)\otimes\epsilon_t) = \delta_{kt}\delta_{j(n-s)}.$$

Then we claim

$$(T_i m,\mu) = (m, T_{n+1-i}\mu)$$

for all $m\in\mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N$, $\mu\in\mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} N^\circ$, which is what we want. The claim follows by explicitly considering the action of T_i on these basis elements. Specifically, we have

$$T_i(T_{j+1}\dots T_n\otimes e_k) = \begin{cases} T_{j+1}\dots T_n\otimes T_i e_k & (i<j)\\ T_jT_{j+1}\dots T_n\otimes e_k & (i=j)\\ T_{j+1}\dots T_n\otimes e_k & (i=j+1)\\ T_{j+1}\dots T_n\otimes T_{i-1}e_k & (i>j+1).\end{cases}$$

and

$$T_i(T_{j+1}-1)\dots(T_n-1)\otimes\epsilon_k = \begin{cases} (T_{j+1}-1)\dots(T_n-1)\otimes T_i\epsilon_k & (i<j)\\ (T_j-1)\dots(T_n-1)\otimes\epsilon_k + (T_{j+1}-1)\dots(T_n-1)\otimes\epsilon_k & (i=j)\\ 0 & (i=j+1)\\ (T_{j+1}-1)\dots(T_n-1)\otimes T_{i-1}\epsilon_k & (i>j+1);\end{cases}$$

the claim may now be checked. □

Proof of Theorem 6.2. We proceed by induction on n; small cases may be easily checked, so assume now that $n\geqslant 4$. The inductive step is based on the following.

Claim. Given the inductive hypothesis, M has a submodule M^- such that

- $M/M^-\cong M(J_1,\dots,J_{n-1},1-J_{n-1})$;
- For any composition factors M_K, M_L of M^-, we have $M_K >_{M^-} M_L$ if and only if $M_K > M_L$ and neither of K, L is contained in the other.

Proof. By Lemma 6.3, we may assume that $J_{n-1}=1$. Then we may put $M^- = M_{n-1}$ as defined in Proposition 6.1. By the module action given in the proof of Proposition 6.1, M^- is isomorphic as an $\mathcal{H}_{n-1}$-module to $\mathrm{Ind}_{\mathcal{H}_{n-2}}^{\mathcal{H}_{n-1}} M(J_1,\dots,J_{n-2})$, while T_n acts on M^- as the identity. Hence by induction we know the submodule lattice of M^-; since $n\in K$ for all composition factors M_K of M^-, we have $K\subset L$ if and only if $K\setminus\{n\}\subseteq L\setminus\{n\}$, and the result follows. □

By taking dual modules and using Lemma 6.4 (or simply by a similar argument to that used to justify the above claim), we deduce the following.

Claim. Given the inductive hypothesis, M has a submodule S isomorphic to $M(1-J_1, J_1, \ldots, J_{n-1})$, and for any two composition factors M_K, M_L of M/S we have $M_K >_{M/S} M_L$ if and only if $M_K > M_L$ and neither of K, L is contained in the other.

This is almost enough to determine the submodule lattice of M: given composition factors $M_K > M_L$, we now know whether $M_K >_M M_L$ except in the case

$$M_K = M/M^- \cong M(J_1, \ldots, J_{n-1}, 1 - J_{n-1}), \qquad M_L = S \cong M(1 - J_1, J_1, \ldots, J_{n-1}).$$

But we claim that there is a composition factor M_N of M^-/S such that

$$M(J_1, \ldots, J_{n-1}, 1 - J_{n-1}) >_M M_N >_M M(1 - J_1, J_1, \ldots, J_{n-1}); \tag{$*$}$$

this will then imply that $M(J_1, \ldots, J_{n-1}, 1 - J_{n-1}) >_M M(1 - J_1, J_1, \ldots, J_{n-1})$, and the theorem will be proved.

By Proposition 6.1, the composition factors of M^-/S are

$$\begin{aligned}
&M(J_1, \ldots, J_{n-2}, 0, J_{n-1}),\\
&M(J_1, \ldots, J_{n-3}, 0, 1, J_{n-1}),\\
&M(J_1, \ldots, J_{n-4} 0, 1, J_{n-2}, J_{n-1}),\\
&\vdots\\
&M(J_1, J_2, 0, 1, J_4, \ldots, J_{n-1}),\\
&M(J_1, 0, 1, J_3, \ldots, J_{n-1}),\\
&M(J_1, 1, J_2, \ldots, J_{n-1}).
\end{aligned}$$

So suppose $M_N = M(J_1, \ldots, J_{i-1}, 0, 1, J_{i+1}, \ldots, J_{n-1})$ for some $2 \leqslant i \leqslant n-2$, and that $(*)$ does not hold, i.e. one of $N \subseteq K$, $N \supseteq K$, $N \subseteq L$ or $N \supseteq L$ holds. These four possibilities are equivalent to

1. $1 \leqslant J_{i+1} \leqslant J_{i+2} \leqslant \ldots \leqslant J_{n-1} \leqslant 1 - J_{n-1}$,
2. $J_i = 0$ and $J_{i+1} \geqslant J_{i+2} \geqslant \ldots \geqslant J_{n-1} \geqslant 1 - J_{n-1}$,
3. $J_i = 1$ and $J_{i-1} \leqslant J_{i-2} \leqslant \ldots \leqslant J_1 \leqslant 1 - J_1$,
4. $0 \geqslant J_{i-1} \geqslant J_{i-2} \geqslant \ldots \geqslant J_1 \geqslant 1 - J_1$,

respectively. Neither (1) nor (4) can happen, so we have either

$$J_i = 0, J_{i+1} = \cdots = J_{n-1} = 1$$

or

$$J_i = 1, J_1 = \cdots = J_{i-1} = 0.$$

If there is no N such that $(*)$ holds, then this is true for all $2 \leqslant i \leqslant n-2$. This then implies that for some $1 \leqslant i \leqslant n-2$ we have

$$J_1 = \cdots = J_i = 0, \quad J_{i+1} = \cdots = J_{n-1} = 1.$$

But then we take $M_N = M(J_1, \ldots, J_{n-2}, 0, J_{n-1})$, and we are done. □

7 Further questions

Further questions about 0-Hecke algebras present themselves. Firstly, it would be nice to extend the results of Section 6, and find the structure of an induced simple module in types B and D, or more generally for any embedding of a Coxeter group of rank $n-1$ in a Coxeter group of rank n. Calculation of small cases in type B shows that we cannot hope that induced simple modules will be multiplicity-free in general, but it does seem plausible that the submodule lattice of an induced simple module is always distributive.

Another natural question is to ask what the centre of $\mathcal{H}$ is. It is easy enough to write down a condition in terms of length for a given element of $\mathcal{H}$ to be central, but this does not seem easy to apply.

Finally, one would like to know more about the structure of projective modules. It is tempting to wonder whether a result analogous to Martin's conjecture [5] for representations of symmetric groups holds for 0-Hecke algebras: recall that a module is *stable* if its radical filtration coincides with its socle filtration. In an earlier version of this paper, we conjectured that every indecomposable projective module for a 0-Hecke algebra is stable, and we are grateful to Maud de Visscher for pointing out that this conjecture fails for the Coxeter group of type A_4. So we make a different conjecture.

Conjecture 7.1. *Suppose W is a finite Coxeter group. Then every indecomposable projective module for $\mathcal{H}_0(W)$ is stable if and only if every irreducible component of W is of rank less than or equal to 3 or of type D_4. Furthermore, if W is irreducible of rank at most 3 or type D_4, then every indecomposable projective module in the non-trivial block of $\mathcal{H}_0(W)$ has Loewy length $h-1$, where h is the Coxeter number of W.*

It is easy to calculate from Theorem 5.1 that every irreducible component of W is of rank at most 3 or of type D_4 if and only if the ordinary quiver of $\mathcal{H}_0(W)$ is bipartite, and this provides a further link with Martin's conjecture. It would be routine but tedious to check the the second part of the conjecture, and we have not done this in detail.

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