THE BIHECKE MONOID OF A FINITE COXETER GROUP AND ITS REPRESENTATIONS

FLORENT HIVERT, ANNE SCHILLING, AND NICOLAS M. THIÉRY

Abstract. For any finite Coxeter group W, we introduce two new objects: its cutting poset and its biHecke monoid. The cutting poset, constructed using a generalization of the notion of blocks in permutation matrices, almost forms a lattice on W. The construction of the biHecke monoid relies on the usual combinatorial model for the 0-Hecke algebra $H_0(W)$, that is, for the symmetric group, the algebra (or monoid) generated by the elementary bubble sort operators. The authors previously introduced the Hecke group algebra, constructed as the algebra generated simultaneously by the bubble sort and antisort operators, and described its representation theory. In this paper, we consider instead the *monoid* generated by these operators. We prove that it admits $|W|$ simple and projective modules. In order to construct the simple modules, we introduce for each $w \in W$ a combinatorial module T_w whose support is the interval $[1, w]_R$ in right weak order. This module yields an algebra, whose representation theory generalizes that of the Hecke group algebra, with the combinatorics of descents replaced by that of blocks and of the cutting poset.

Contents

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1. Introduction

In this paper we introduce two novel objects for any finite Coxeter group W: its *cutting poset* and its *biHecke monoid*. The cutting poset is constructed using a generalization of blocks in permutation matrices to any Coxeter group and is almost a lattice. The biHecke monoid is generated simultaneously by the sorting and antisorting operators associated to the combinatorial model of the 0-Hecke algebra $H_0(W)$. It turns out that the representation theory of the biHecke monoid, and in particular the construction of its simple modules, is closely tied to the cutting poset.

The study of these objects combines methods from and impacts several areas of mathematics: Coxeter group theory, monoid theory, representation theory, combinatorics (posets, permutations, descent sets), as well as computer algebra. The guiding principle is the use of representation theory, combined with computer exploration, to extract combinatorial structures from an algebra, and in particular a monoid algebra, often in the form of posets or lattices. This includes the structures associated to monoid theory (such as for example Green relations), but also goes beyond. For example, we find connections between the classical orders of Coxeter groups (left, right, and left-right weak order and Bruhat order) and the Green relations on our monoids ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{H}$-order and ordered monoids), and these orders play a crucial rôle in the combinatorics and representation theory of the biHecke monoid.

The usual combinatorial model for the 0-Hecke algebra $H_0(\mathfrak{S}_n)$ of the symmetric group is the algebra (or monoid) generated by the (anti) bubble sort operators $\pi_1, \dots, \pi_{n-1}$, where π_i acts on words of length n and sorts the letters in positions i and $i+1$ decreasingly. By symmetry, one can also construct the bubble sort operators $\overline{\pi}_1, \dots, \overline{\pi}_{n-1}$, where $\overline{\pi}_i$ acts by sorting increasingly, and this gives an isomorphic construction $\overline{H}_0$ of the 0-Hecke algebra. This construction generalizes naturally to any finite Coxeter group W. Furthermore, when W is a Weyl group, and hence can be affinized, there is an additional operator π_0 projecting along the highest root.

In [HT09] the first and last author constructed the *Hecke group algebra* $\mathcal{H}W$ by gluing together the 0-Hecke algebra and the group algebra of W along their right regular representation. Alternatively, $\mathcal{H}W$ can be constructed as the *biHecke algebra* of W, by gluing together the two realizations $H_0(W)$ and $\overline{H}_0(W)$ of the 0-Hecke algebra. $\mathcal{H}W$ admits a more conceptual description as the algebra of all operators on $\mathbb{K}W$ preserving left antisymmetries; the representation theory of $\mathcal{H}W$ follows, governed by the combinatorics of descents. In [HST09], the authors further proved that, when W is a Weyl group, $\mathcal{H}W$ is a natural quotient of the affine Hecke algebra.

In this paper, following a suggestion of Alain Lascoux, we study the *biHecke monoid* $M(W)$, obtained by gluing together the two 0-Hecke *monoids*. This involves the combinatorics of the usual poset structures on W (left, right, left-right, Bruhat order), as well as the new cutting poset. Building upon the extensive study of the representation theory of the 0-Hecke algebra [Nor79, Car86, Den10a, Den10b], we explore the representation theory of the biHecke monoid. In the process, we prove that the biHecke monoid is aperiodic and its Borel submonoid fixing the identity is $\mathcal{J}$-trivial. This sparked our interest in the representation theory of $\mathcal{J}$-trivial and aperiodic monoids, and the general results we found along the way are presented in [DHST10] and an upcoming sequel.

We further prove that the simple and projective modules of M are indexed by the elements of W. In order to construct the simple modules, we introduce for each $w \in W$ a combinatorial module T_w whose support is the interval $[1, w]_R$ in right weak order. This module yields an algebra, whose representation theory generalizes that of the Hecke group algebra, with the combinatorics of descents replaced by that of blocks and of the cutting poset.

Let us finish by giving some additional motivation for the study of the biHecke monoid. In type A, the tower of algebras $(\mathbb{K}M(\mathfrak{S}_n))_{n\in\mathbb{N}}$ possesses long sought-after properties. Indeed, it is well-known that several combinatorial Hopf algebras arise as Grothendieck rings of towers of algebras. The prototypical example is the tower of algebras of the symmetric groups which gives rise to the Hopf algebra Sym of symmetric functions, on the Schur basis [Mac95, Zel81]. Another example, due to Krob and Thibon [KT97], is the tower of the 0-Hecke algebras of the symmetric groups which gives rise to the Hopf algebra QSym of quasi-symmetric functions of [Ges84], on the F_I basis. The product rule on the F_I's is naturally lifted through the descent map to a product on permutations, leading to the Hopf algebra FQSym of free quasi-symmetric functions [DHT02]. This calls for the existence of a tower of algebras $(A_n)_{n\in\mathbb{N}}$, such that each A_n contains $H_0(\mathfrak{S}_n)$ and has its simple modules indexed by the elements of $\mathfrak{S}_n$. The biHecke monoids $M(\mathfrak{S}_n)$, and their Borel submonoids $M_1(\mathfrak{S}_n)$ and $M_{w_0}(\mathfrak{S}_n)$, satisfy these properties, and are therefore expected to yield new representation theoretical interpretations of the bases of FQSym.

In the remainder of this introduction, we briefly review Coxeter groups and their 0-Hecke monoids, introduce the biHecke monoid which is our main objects of study, and outline the rest of the paper.

1.1. **Coxeter groups.** Let (W,S) be a Coxeter group, that is, a group W with a presentation

$$W = \langle\, S \;\mid\; (ss')^{m(s,s')},\ \forall s,s'\in S\,\rangle\,, \tag{1.1}$$

with $m(s,s')\in\{1,2,\dots,\infty\}$ and $m(s,s)=1$. The elements $s\in S$ are called *simple reflections*, and the relations can be rewritten as:

$$\begin{aligned} s^2 &= 1 && \text{for all } s\in S\,,\\ \underbrace{ss'ss's\cdots}_{m(s,s')} &= \underbrace{s'ss'ss'\cdots}_{m(s,s')} && \text{for all } s,s'\in S\,, \end{aligned} \tag{1.2}$$

where 1 denotes the identity in W.

Most of the time, we just write W for (W,S). In general, we follow the notation of [BB05], and we refer to this monograph and to [Hum90] for details on Coxeter groups and their Hecke algebras. Unless stated otherwise, we always assume that W is finite, and denote its generators by $S=(s_i)_{i\in I}$, where $I=\{1,2,\dots,n\}$ is the *index set* of W.

The prototypical example is the Coxeter group of type A_{n-1} which is the n-th symmetric group $(W,S):=(\mathfrak{S}_n,\{s_1,\dots,s_{n-1}\})$, where s_i denotes the *elementary transposition* which exchanges i and $i+1$. The relations are given by:

$$\begin{aligned} s_i^2 &= 1 && \text{for } 1\le i\le n-1\,,\\ s_is_j &= s_js_i && \text{for } |i-j|\ge 2\,,\\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} && \text{for } 1\le i\le n-2\,; \end{aligned} \tag{1.3}$$

the last two relations are called the *braid relations*. When writing a permutation $\mu\in\mathfrak{S}_n$ explicitly, we use *one-line notation*, that is the sequence $\mu_1\mu_2\dots\mu_n$, where $\mu_i:=\mu(i)$.

A *reduced word* $i_1\dots i_k$ for an element $w\in W$ corresponds to a decomposition $w=s_{i_1}\cdots s_{i_k}$ of w into a product of generators in S of minimal length $k=\ell(w)$. A *(right) descent* of w is an element $i\in I$ such that $\ell(ws_i)<\ell(w)$. If w is a permutation, this translates into $w_i>w_{i+1}$. *Left descents* are defined analogously. The sets of left and right descents of w are denoted by $\mathrm{D}_L(w)$ and $\mathrm{D}_R(w)$, respectively.

For $J\subseteq I$, we denote by $W_J=\langle s_j\mid j\in J\rangle$ the subgroup of W generated by s_j with $j\in J$. Furthermore, the longest element in W_J (resp. W) is denoted by s_J (resp. w_0). Any finite Coxeter group $W:=\langle s_i\mid i\in I\rangle$ can be realized as a finite reflection group (see for example [Hum90, Chapter 5.6] and [BB05, Chapter 4]). The generators s_i of W can be interpreted as reflections on hyperplanes in some $|I|$-dimensional vector space V. The simple roots α_i for $i\in I$ form a basis for V; the set of all roots is given by $\Phi:=\{w(\alpha_i)\mid i\in I, w\in W\}$. One can associate reflections s_α to all roots $\alpha \in \Phi$. If $\alpha, \beta \in \Phi$ and $w \in W$, then $w(\alpha) = \beta$ if and only if $w s_\alpha w^{-1} = s_\beta$ (see [Hum90, Chapter 5.7]).

1.2. **The 0-Hecke monoid.** The *0-Hecke monoid* $H_0(W) = \langle \pi_i \mid i \in I \rangle$ of a Coxeter group W is generated by the *simple projections* π_i with relations

$$\begin{aligned} \pi_i^2 &= \pi_i && \text{for all } i \in I, \\ \underbrace{\pi_i \pi_j \pi_i \pi_j \cdots}_{m(s_i,s_j)} &= \underbrace{\pi_j \pi_i \pi_j \pi_i \cdots}_{m(s_i,s_j)} && \text{for all } i,j \in I\ . \end{aligned} \tag{1.4}$$

Thanks to these relations, the elements of $H_0(W)$ are canonically indexed by the elements of W by setting $\pi_w := \pi_{i_1} \cdots \pi_{i_k}$ for any reduced word $i_1 \dots i_k$ of w. We further denote by π_J the longest element of the *parabolic submonoid* $H_0(W_J) := \langle \pi_i \mid i \in J \rangle$.

As mentioned before, all finite Coxeter groups W can be realized as finite reflection groups. The generators s_i of W can be interpreted as reflections on hyperplanes. The generators π_i of the 0-Hecke monoid act as a reflection away from some fundamental domain and as the identity otherwise.

The right regular representation of $H_0(W)$ induces a concrete realization of $H_0(W)$ as a monoid of operators acting on W, with generators $\pi_1, \dots, \pi_n$ defined by:

$$w.\pi_i := \begin{cases} w & \text{if } i \in \mathrm{D}_R(w), \\ w s_i & \text{otherwise.} \end{cases} \tag{1.5}$$

In type A, π_i sorts the letters at positions i and $i+1$ decreasingly, and for any permutation w, $w.\pi_{w_0} = n \cdots 21$. This justifies naming π_i an *elementary bubble antisorting operator*.

Another concrete realization of $H_0(W)$ can be obtained by considering instead the *elementary bubble sorting operators* $\overline{\pi}_1, \dots, \overline{\pi}_n$, whose action on W are defined by:

$$w.\overline{\pi}_i := \begin{cases} w s_i & \text{if } i \in \mathrm{D}_R(w), \\ w & \text{otherwise.} \end{cases} \tag{1.6}$$

In type A, and for any permutation w, one has $w.\overline{\pi}_{w_0} = 12 \cdots n$.

Remark 1.1. *For a given $w \in W$, define v by $wv = w_0$, where w_0 is the longest element of W. Then*

$$i \in \mathrm{D}_R(w) \iff i \notin \mathrm{D}_L(v) \iff i \notin \mathrm{D}_R(v^{-1}) = \mathrm{D}_R(w_0 w).$$

Hence, the action of $\overline{\pi}_i$ on W can be expressed by the action of π_i on W using w_0:

$$w.\overline{\pi}_i = w_0[(w_0 w).\pi_i].$$

1.3. **The biHecke monoid $M(W)$.** We now introduce our main object of study, the biHecke monoid.

Definition 1.2. *Let W be a finite Coxeter group. The* biHecke monoid *is the submonoid of functions from W to W generated simultaneously by the elementary bubble sorting and antisorting operators of* (1.5) *and* (1.6)*:*

$$M := M(W) := \langle \pi_1, \pi_2, \dots, \pi_n, \overline{\pi}_1, \overline{\pi}_2, \dots, \overline{\pi}_n \rangle\ .$$

As mentioned in [HT09, HST09] this monoid admits several natural variants, depending on the choice of the generators:

$$\langle \pi_1, \pi_2, \dots, \pi_n, \overline{\pi}_1, \overline{\pi}_2, \dots, \overline{\pi}_n \rangle,$$
$$\langle \pi_1, \pi_2, \dots, \pi_n, s_1, s_2, \dots, s_n \rangle,$$
$$\langle \pi_0, \pi_1, \pi_2, \dots, \pi_n \rangle,$$

where π_0 is defined when W is a Weyl group and hence can be affinized. Another close variant is the monoid of all strictly order preserving functions on the Boolean lattice [Gau10]. All those monoids, and their representation theory, remain to be studied.

1.4. **Outline.** The remainder of this paper consists of two parts: we first introduce and study the new cutting poset structure on finite Coxeter groups, and then proceed to the biHecke monoid and its representation theory.

In Section 2, we recall some basic facts, definitions, and properties about posets, Coxeter groups, monoids, and representation theory that are used throughout the paper.

In Section 3, we generalize the notion of blocks of permutation matrices to any Coxeter group, and use it to define a new poset structure on W, which we call the *cutting poset*; we prove that it is (almost) a lattice, and derive that its Möbius function is essentially that of the hypercube.

In Section 4, we study the combinatorial properties of $M(W)$. In particular, we prove that it preserves left and Bruhat order, derive consequences on the fibers and image sets of its elements, prove that it is aperiodic, and study Green relations and idempotents.

In Section 5, our strategy is to consider a "Borel" triangular submonoid of $M(W)$ whose representation theory is simpler, but with the same number of simple modules, to later induce back information about the representation theory of $M(W)$. Namely, we study the submonoid $M_1(W)$ of the elements fixing 1 in $M(W)$. This monoid not only preserves Bruhat order, but furthermore is regressive. It follows that it is $\mathcal{J}$-trivial which is the desired triangularity property. It is for example easily derived that $M_1(W)$ has $|W|$ simple modules, all of dimension 1. In fact most of our results about M_1 generalize to any $\mathcal{J}$-trivial monoid, which is the topic of a separate paper on the representation theory of $\mathcal{J}$-trivial monoids [DHST10].

In Section 6, we construct, for each $w \in W$, the *translation module* T_w by induction of the corresponding simple $M_1(W)$-module. It is a quotient of the indecomposable projective module P_w of $M(W)$, and therefore admits the simple module S_w of $M(W)$ as top. It further admits a simple combinatorial model using the right classes with the interval $[1, w]_R$ as support, and which passes down to S_w. We derive a formula for the dimension of S_w, using an inclusion-exclusion on the sizes of intervals in $(W, \leq_R)$ along the cutting poset. On the way, we study the algebra $\mathcal{H}W^{(w)}$ induced by the action of $M(W)$ on T_w. It turns out to be a natural w-analogue of the Hecke group algebra, acting not anymore on the full Coxeter group, but on the interval $[1, w]_R$ in right order. All the properties of the Hecke group algebra pass through this generalization, with the combinatorics of descents being replaced by that of blocks and of the cutting poset. In particular, $\mathcal{H}W^{(w)}$ is Morita equivalent to the incidence algebra of the sublattice induced by the cutting poset on the interval $[1, w]_{\sqsubseteq}$.

In Section 7, we apply the findings of Sections 4, 5, and 6 to derive results on the representation theory of $M(W)$. We conclude in Section 8 with discussions on further research in progress.

There are two appendices. Appendix A summarizes some results on colored graphs which are used in Section 4 to prove properties of the fibers and image sets of elements in the biHecke monoid. In Appendix B we present tables of q-Cartan invariant and decomposition matrices for $M(\mathfrak{S}_n)$ for $n = 2, 3, 4$.

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2. Background

We review some basic facts about partial orders and finite posets in Section 2.1, finite lattices and Birkhoff's theorem in Section 2.2, order-preserving functions in Section 2.3, the usual partial orders on Coxeter groups (left and right weak order, Bruhat order) in Section 2.4, and the notion of $\mathcal{J}$-order (and related orders) and aperiodic monoids in Section 2.5. We also prove a result in Proposition 2.4 about the image sets of order-preserving and regressive idempotents on a poset that will be used later in the study of idempotents of the biHecke monoid. Sections 2.6 and 2.7 contain reviews of some representation theory of algebras and monoids that will be relevant in our study of translation modules.

2.1. **Finite posets.** For a general introduction to posets and lattices, we refer the reader to e.g. [Pou10, Sta97] or [Wik10, Poset, Lattice]. Throughout this paper, all posets are finite.

A partially ordered set (or *poset* for short) $(P, \preceq)$ is a set P with a binary relation $\preceq$ so that for all $x, y, z \in P$:

(i) $x \preceq x$ (reflexivity);
(ii) if $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry);
(iii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity).

When we exclude the possibility that $x = y$, we write $x \prec y$.

If $x \preceq y$ in P, we define the *interval*

$$[x, y]_P := \{z \in P \mid x \preceq z \preceq y\}.$$

A pair (x, y) such that $x \prec y$ and there is no $z \in P$ such that $x \prec z \prec y$ is called a *covering*. We denote coverings by $x \to y$. The *Hasse diagram* of $(P, \preceq)$ is the diagram where the vertices are the elements $x \in P$, and there is an upward-directed edge between x and y if $x \to y$.

Definition 2.1. *Let $(P, \preceq)$ be a poset and $X \subseteq P$.*

(i) X is convex *if for any $x, y \in X$ with $x \preceq y$ we have $[x, y] \subseteq X$.*
(ii) X is connected *if for any $x, y \in X$ with $x \prec y$ there is a path in the Hasse diagram $x = x_0 \to x_1 \to \cdots \to x_k = y$ such that $x_i \in X$ for $0 \le i \le k$.*

The Möbius inversion formula [Sta97, Proposition 3.7.1] generalizes the inclusion-exclusion principle to any poset. Namely, there exists a unique function μ, called the *Möbius function* of P, which assigns an integer to each ordered pair $x \preceq y$ and enjoys the following property: for any two functions $f, g : P \to G$ taking values in an additive group G,

$$g(x) = \sum_{y \preceq x} f(y) \quad \text{if and only if} \quad f(y) = \sum_{x \preceq y} \mu(x, y)\ g(x)\,. \tag{2.1}$$

The Möbius function can be computed thanks to the following recursion:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \preceq z \prec y} \mu(x, z), & \text{for } x \prec y. \end{cases}$$

2.2. **Finite lattices and Birkhoff's theorem.** Let $(P, \preceq)$ be a poset. The *meet* $z = \bigwedge A$ of a subset $A \subseteq P$ is an element such that (1) $z \preceq x$ for all $x \in A$ and (2) $u \preceq x$ for all $x \in A$ implies that $u \preceq z$. When the meet exists, it is unique and is denoted by $\bigwedge A$. The meet of the empty set $A = \{\}$ is the largest element of the poset, if it exists. The meet of two elements $x, y \in P$ is denoted by $x \wedge y$. A poset $(P, \preceq)$ for which every pair of elements has a meet is called a *meet-semilattice*. In that case, P endowed with the meet operation is a commutative $\mathcal{J}$-trivial semigroup, and in fact a monoid with unit the maximal element of P, if the latter exists.

Reversing all comparisons, one can similarly define the *join* $\bigvee A$ of a subset $A \subseteq P$ or $x \vee y$ of two elements $x, y \in P$, and *join-semilattices*. A *lattice* is a poset for which both meets and joins exist for pair of elements. Recall that we only consider finite posets, so we do not have to worry about the distinction between lattices and complete lattices.

A lattice $(L, \vee, \wedge)$ is *distributive* if the following additional identity holds for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,.$$

This condition is equivalent to its dual:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\,.$$

Birkhoff's representation theorem (see e.g. [Wik10, Birkhoff's representation theorem], or [Sta97, Theorem 3.4.1]) states that any finite distributive lattice can be represented as a sublattice of a Boolean lattice, that is a collection of sets stable under union and intersection. Furthermore, there is a canonical such representation which we construct now.

An element z in a lattice L is called *join-irreducible* if z is not the smallest element in L and $z = x \vee y$ implies $z = x$ or $z = y$ for any $x, y \in L$ (and similarly for meet-irreducible). Equivalently, since L is finite, z is join-irreducible if and only if it covers exactly one element in L. We denote by $I(L)$ the *poset of join-irreducible elements* of L, that is the restriction of L to its join-irreducible elements. Note that this definition still makes sense for nonlattices. From a monoid point of view, $I(L)$ is the minimal generating set of L.

A *lower set* of a poset P is a subset Y of P such that, for any pair $x \leq y$ of comparable elements of P, x is in Y whenever y is. *Upper sets* are defined dually. The family of lower sets of P ordered by inclusion is a distributive lattice, the *lower sets lattice* $O(P)$. Birkhoff's representation theorem states that any finite distributive lattice is isomorphic to the lattice $O(I(L))$ of lower sets of the poset $I(L)$ of its join-irreducible elements, via the reciprocal isomorphisms:

$$\begin{cases} L & \to O(I(L)) \\ x & \mapsto \{y \in I(L) \mid y \leq x\} \end{cases} \qquad \text{and} \qquad \bigvee : \begin{cases} O(I(L)) & \to L \\ I & \mapsto \bigvee I\,. \end{cases}$$

2.3. **Order-preserving functions.**

Definition 2.2. *Let $(P, \preceq)$ be a poset and $f : P \to P$ a function.*

(i) f is called order-preserving *if $x \preceq y$ implies $f(x) \preceq f(y)$. We also say f preserves the order $\preceq$.*
(ii) f is called regressive *if $f(x) \preceq x$ for all $x \in P$.*
(iii) f is called extensive *if $x \preceq f(x)$ for all $x \in P$.*

Lemma 2.3. *Let $(P, \preceq)$ be a poset and $f : P \to P$ an order-preserving map. Then, the preimage $f^{-1}(C)$ of a convex subset $C \subseteq P$ is convex. In particular, the preimage of a point is convex.*

Proof. Let $x, y \in f^{-1}(C)$ with $x \preceq y$. Since f is order-preserving, for any $z \in [x, y]$, we have $f(x) \preceq f(z) \preceq f(y)$, and therefore $f(z) \in C$. □

Proposition 2.4. *Let $(P, \preceq)$ be a poset and $f : P \to P$ be an order-preserving and regressive idempotent. Then, f is determined by its image set. Namely, for $u \in P$ we have:*

$$f(u) = \sup_{\preceq}\big({\downarrow} u \cap \operatorname{im}(f)\big),$$

the supremum being always well-defined. Here ${\downarrow} u = \{x \in P \mid x \preceq u\}$.

An equivalent statement is that, for $v \in \operatorname{im}(f)$,

$$f^{-1}(v) = {\uparrow} v \setminus \bigcup_{\substack{v' \in \operatorname{im}(f) \\ v' \succ v}} {\uparrow} v'\,,$$

where ${\uparrow} v = \{x \in P \mid x \succeq v\}$.

Proof. We first prove that ${\downarrow} u \cap \operatorname{im}(f) = f({\downarrow} u)$. The inclusion $\supseteq$ follows from the fact that f is regressive: taking $v \in {\downarrow} u$, we have $f(v) \preceq v \preceq u$ and therefore $f(v) \in {\downarrow} u \cap \operatorname{im}(f)$. The inclusion $\subseteq$ follows from the assumption that f is an idempotent: for $v \in \operatorname{im}(f)$ with $v \preceq u$, one has $v = f(v)$, so $v \in f({\downarrow} u)$.

Since f is order-preserving, $f(\downarrow u)$ has a unique maximal element, namely $f(u)$. The first statement of the proposition follows. The second statement is a straightforward reformulation of the first one. □

An *interior operator* is a function $L \to L$ on a lattice L which is order-preserving, regressive and idempotent (see e.g. [Wik10, Moore Family]). A subset $A \subseteq L$ is a *dual Moore family* if it contains the smallest element $\bot_L$ of L and is stable under joins. The image set of an interior operator is a *dual Moore family.* Reciprocally, any dual Moore family A defines an interior operator by:

$$\begin{aligned} L &\longrightarrow L \\ x &\longmapsto \operatorname{red}(x) := \bigvee_{a\in A, a\preceq x} a\ , \end{aligned} \tag{2.2}$$

where $\bigvee_{\{\}} = \bot_L$ by convention.

A (dual) Moore family is itself a lattice with the order and join inherited from L. The meet operation usually differs from that of L and is given by $x \wedge_A y = \operatorname{red}(x \wedge_L y)$.

2.4. **Classical partial orders on Coxeter groups.** A Coxeter group $W = \langle s_i \mid i \in I\rangle$ comes endowed with several natural partial orders: left (weak) order, right (weak) order, left-right (weak) order, and Bruhat order. All of these play an important role for the representation theory of the biHecke monoid $M(W)$.

Fix $u, w \in W$. Then, in *right (weak) order*,

$$u \leq_R w \quad \text{if } w = us_{i_1}\cdots s_{i_k} \text{ for some } i_j \in I \text{ and } \ell(w) = \ell(u) + k.$$

Similarly, in *left (weak) order*,

$$u \leq_L w \quad \text{if } w = s_{i_1}\cdots s_{i_k}u \text{ for some } i_j \in I \text{ and } \ell(w) = \ell(u) + k,$$

and in *left-right (weak) order*,

$$u \leq_{LR} w \quad \text{if } w = s_{i_1}\cdots s_{i_k}us_{i'_1}\cdots s_{i'_\ell} \text{ for some } i_j, i'_j \in I \text{ and } \ell(w) = \ell(u) + k + \ell.$$

Note that left-right order is the transitive closure of the union of left and right order. Thanks to associativity, this is equivalent to the existence of a $v \in W$ such that $u \leq_L v$ and $v \leq_R w$.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ be a reduced expression for w. Then, in *Bruhat order*,

$$\begin{aligned} u \leq_B w \quad & \text{if there exists a reduced expression } u = s_{j_1}\cdots s_{j_k} \\ & \text{where } j_1 \dots j_k \text{ is a subword of } i_1 \dots i_\ell. \end{aligned}$$

For any finite Coxeter group W, the posets $(W, \leq_R)$ and $(W, \leq_L)$ are graded lattices [BB05, Section 3.2]. The following proposition states that any interval is isomorphic to some interval starting at 1:

Proposition 2.5. [BB05, Proposition 3.1.6] *Let $\mathcal{O} \in \{L, R\}$, $u \leq_{\mathcal{O}} w \in W$. Then $[u, w]_{\mathcal{O}} \cong [1, t]_{\mathcal{O}}$ where $t = wu^{-1}$.*

This motivates the following definition:

Definition 2.6. *The* type *of an interval in left (resp. right) order is defined to be* $\operatorname{type}([u, w]_L) := wu^{-1}$ *(resp.* $\operatorname{type}([u, w]_R) := u^{-1}w$*).*

It is easily shown that, if $\mathcal{O}$ is considered as a colored poset, then the converse of Proposition 2.5 holds as well:

Remark 2.7. *Fix a type t. Then, the collection of all intervals in left weak order of type t is in bijection with $[1, t^{-1}w_0]_R$, and the operators π_i and $\overline{\pi}_i$ act transitively on the right on this collection. More precisely: π_a induces an isomorphism from $[1, ba^{-1}]_L$ to $[a, b]_L$, and $\overline{\pi}_{a^{-1}}$ induces an isomorphism from $[a, b]_L$ to $[1, ba^{-1}]_L$.*

Proof. Take $u \in [a, b]_L$, and let $s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of a. Let $s_{j_1} \cdots s_{j_\ell}$ be a reduced decomposition of $ua^{-1} = us_{i_k} \cdots s_{i_1}$. Then

$$u = (s_{j_1} \cdots s_{j_\ell})(s_{i_1} \cdots s_{i_k})$$

is a reduced decomposition of u and $u.\overline{\pi}_{a^{-1}} = s_{j_1} \cdots s_{j_\ell} = ua^{-1}$. Reciprocially, applying π_a to an element $u \in [1, ba^{-1}]_L$ progressively builds up a reduced word for a. The result follows. □

2.5. **Preorders on monoids.** In 1951 Green [Gre51] introduced several preorders on monoids which are essential for the study of their structures (see for example [Pin10, Chapter V]). Throughout this paper, we only consider finite monoids M. Define $\le_{\mathcal{R}}, \le_{\mathcal{L}}, \le_{\mathcal{J}}, \le_{\mathcal{H}}$ for $x, y \in M$ as follows:

$$\begin{aligned}
x \le_{\mathcal{R}} y \quad &\text{if and only if } x = yu \text{ for some } u \in M\\
x \le_{\mathcal{L}} y \quad &\text{if and only if } x = uy \text{ for some } u \in M\\
x \le_{\mathcal{J}} y \quad &\text{if and only if } x = uyv \text{ for some } u, v \in M\\
x \le_{\mathcal{H}} y \quad &\text{if and only if } x \le_{\mathcal{R}} y \text{ and } x \le_{\mathcal{L}} y.
\end{aligned}$$

These preorders give rise to equivalence relations:

$$\begin{aligned}
x \,\mathcal{R}\, y \quad &\text{if and only if } xM = yM\\
x \,\mathcal{L}\, y \quad &\text{if and only if } Mx = My\\
x \,\mathcal{J}\, y \quad &\text{if and only if } MxM = MyM\\
x \,\mathcal{H}\, y \quad &\text{if and only if } x \,\mathcal{R}\, y \text{ and } x \,\mathcal{L}\, y.
\end{aligned}$$

We further add the relation $\le_{\mathcal{B}}$ (and its associated equivalence relation $\mathcal{B}$) defined as the finest preorder such that $x \le_{\mathcal{B}} 1$, and

$$x \le_{\mathcal{B}} y \text{ implies that } uxv \le_{\mathcal{B}} uyv \text{ for all } x, y, u, v \in M.$$

(One can view $\le_{\mathcal{B}}$ as the intersection of all preorders with the above property. There exists at least one such preorder, namely $x \le y$ for all $x, y \in M$).

Beware that 1 is the largest element of those (pre)-orders. This is the usual convention in the semigroup community, but is the converse convention from the closely related notions of left/right/left-right/Bruhat order in Coxeter groups as introduced in Section 2.4.

Example 2.8. For the 0-Hecke monoid introduced in Section 1.2, $\mathcal{K}$-order for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{B}\}$ corresponds to the reverse of right, left, left-right and Bruhat order of Section 2.4. More precisely for $x, y \in H_0(W)$, $x \le_{\mathcal{K}} y$ if and only if $x \ge_K y$ for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{B}\}$ and $K \in \{R, L, LR, B\}$ the corresponding letter.

Definition 2.9. *Elements of a monoid M in the same $\mathcal{K}$-equivalence class are called $\mathcal{K}$-classes, where $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{B}\}$. The $\mathcal{K}$-class of $x \in M$ is denoted by $\mathcal{K}(x)$.*

A monoid M is called $\mathcal{K}$-trivial *if all $\mathcal{K}$-classes are of cardinality one.*

An element $x \in M$ is called regular *if it is $\mathcal{J}$-equivalent to an idempotent.*

An equivalent formulation of $\mathcal{K}$-triviality is given in terms of *ordered* monoids. A monoid M is called:

right-ordered	if $xy \leq x$ for all $x, y \in M$
left-ordered	if $xy \leq y$ for all $x, y \in M$
left-right-ordered	if $xy \leq x$ and $xy \leq y$ for all $x, y \in M$
two-sided-ordered	if $xy = yz \leq y$ for all $x, y, z \in M$ with $xy = yz$
ordered with 1 *on top*	if $x \leq 1$, and $x \leq y$ implies $uxv \leq uyv$ for all $x, y, u, v \in M$

for some partial order $\leq$ on M.

Proposition 2.10. *M is right-ordered (resp. left-ordered, left-right-ordered, two-sided-ordered, ordered with* 1 *on top) if and only if M is $\mathcal{R}$-trivial (resp. $\mathcal{L}$-trivial, $\mathcal{J}$-trivial, $\mathcal{H}$-trivial, $\mathcal{B}$-trivial).*

When M is $\mathcal{K}$-trivial for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{B}\}$, the partial order $\leq$ is finer than $\leq_{\mathcal{K}}$; that is for any $x, y \in M$, $x \leq_{\mathcal{K}} y$ implies $x \leq y$.

Proof. We give the proof for right-order as the other cases can be proved in a similar fashion.

Suppose M is right-ordered and that $x, y \in M$ are in the same $\mathcal{R}$-class. Then $x = ya$ and $y = xb$ for some $a, b \in M$. This implies that $x \leq y$ and $y \leq x$ so that $x = y$. Conversely, suppose that all $\mathcal{R}$-classes are singletons. Then $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$ imply that $x = y$, so that the $\mathcal{R}$-preorder turns into a partial order. Hence M is right-ordered using $xy \leq_{\mathcal{R}} x$. □

Definition 2.11. *A monoid M is* aperiodic *if there is an integer $N > 0$ such that for each $x \in M$, $x^N = x^{N+1}$.*

From this definition it is clear that, for an aperiodic monoid M, the sequence $(x^n)_{n \in \mathbb{N}}$ eventually stabilizes for every $x \in M$. We write x^ω for the stable element and $E(M) := \{x^\omega \mid x \in M\}$ for the set of idempotents.

An equivalent characterization of aperiodic monoids is that they are $\mathcal{H}$-trivial, or that the groups in M are trivial (see for example [Pin10, VII, 4.2, Aperiodic monoids]). In this sense, the notion of aperiodic monoids is orthogonal to that of groups as they contain no group-like structure (there are no elements with partial inverses). On the same token, their representation theory is orthogonal to that of groups.

As we will see in Section 4.4, the biHecke monoid $M(W)$ of Definition 1.2 is aperiodic. Its Borel submonoid $M_1(W)$ of functions fixing the identity is $\mathcal{J}$-trivial (see Section 5).

2.6. **Representation theory of algebras.** We refer to [CR06] for an introduction to representation theory, and to [Ben91] for more advanced notions such as Cartan matrices and quivers. Here we mostly review composition series and characters.

Let A be a finite-dimensional algebra. Given an A-module X, any strictly increasing sequence $(X_i)_{i \leq k}$ of submodules

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

is called a *composition series for X*. A composition series $(Y_j)_{i \leq \ell}$ such that, for any i, $Y_i = X_j$ for some j is called a *refinement of* $(X_i)_{i \leq k}$. A composition series $(X_i)_{i \leq k}$ without a non-trivial refinement is called a *maximal composition series.* For any maximal composition series the quotient module X_j / X_{j-1} is simple and the simple module X_j/X_{j-1} is called a *composition factor*. The multiplicity of a simple module S in the composition series is the number of indices j such that X_j/X_{j-1} is isomorphic to S. The Jordan-Hölder theorem states that this multiplicity does not depend on the choice of the maximal composition series. Hence, we may define the *generalized character* (or *character* for short) of a module X as the formal sum

$$[X] := \sum_{i \in I} c_i [S_i]\,,$$

where I indexes the simple modules of A and c_i is the multiplicity of the simple module S_i in any composition series for X.

The additive group of formal sums $\sum_{i\in I} m_i[S_i]$, with $m_i \in \mathbb{Z}$, is called the *Grothendieck group of the category of A-modules* and is denoted by $G_0(A)$. By definition, the character verifies that, for any exact sequence

$$0 \to X \to Y \to Z \to 0$$

the following equality holds in the Grothendieck group

$$[X] = [Y] + [Z]\,.$$

See [Ser77] for more information about Grothendieck groups.

Suppose that B is a subalgebra of A. Any A-module X naturally inherits an action from B. The thereby constructed B-module is called the *restriction of X to B* and its B-character $[X]_B$ depends only on its A-character $[X]_A$. Indeed, any A-composition series can be refined to a maximal B-composition series and the resulting multiplicities depend only on those in the A-composition series and in the composition series of the simple modules of A restricted to B. This defines a $\mathbb{Z}$-linear map $[X]_A \mapsto [X]_B$, called the *decomposition map.* Let $(S_i^A)_{i\in I}$ and $(S_j^B)_{j\in J}$ be complete families of simple module representatives for A and B, respectively. The matrix of the decomposition map is called the *decomposition matrix* of A over B; its coefficient (i,j) is the multiplicity of S_j^B as a composition factor of S_i^A, viewed as a B-module.

The adjoint construction of restriction is called *induction*: for any right B-module X the space

$$X{\uparrow}_B^A := X \otimes_B A$$

is naturally endowed with a right A-module structure by right multiplication by elements of A, and is called the *module induced by X from B to A*.

2.7. **Representation theory of monoids.** Although representation theory started at the beginning of the 20th century with groups before being extended to more general algebraic structures such as algebras, one has to wait until 1942 [Cli42] for first results on the representation theory of semigroups and monoids. Renewed interest in this subject was sparked more recently by the emergence of connections with probability theory and combinatorics (see e.g. [Bro00, Sal07]). Compared to groups, only a few results are known, the most important one being the construction of the simple modules. It is originally due to Clifford, Munn, and Ponizovskiĭ, and we recall here the construction of [GMS09] (see also the historical references therein): Let M be a finite monoid and $\mathcal{J}(M)$ its set of $\mathcal{J}$-classes. For any $\mathcal{J}$-class J, consider the sets

$$M_{\geq J} := \bigcup_{K\in\mathcal{J}(M),\ K\geq_{\mathcal{J}} J} K \qquad \text{and} \qquad I_J := M - M_{\geq J}\,.$$

Then, I_J is an ideal of M so that $M_{\geq J} = M/I_J$ is a quotient monoid of M. As a consequence any $M_{\geq J}$-module is an M-module.

Recall that a $\mathcal{J}$-class is called *regular* if it contains an idempotent and call $\mathcal{U}(M)$ the set of regular $\mathcal{J}$-classes. Fix an idempotent traversal $E = \{e_J \mid J \in \mathcal{U}(M)\}$ and let

$$G_J := G_{e_J} = e_J M e_J \cap J = e_J M_{\geq J} e_J \,,$$

for any $J \in \mathcal{U}(M)$. Then, G_J is a group which does not depend on the choice of e_J. More precisely, if $e, f \in J$, the ideal MeM and MfM are equal and the group G_e and G_f are conjugate and isomorphic. Then the simple M-module can be constructed as follows:

Theorem 2.12 (Clifford, Munn, Ponizovskiĭ, see [GMS09] Theorem 7)**.** *Let M be a monoid and G_J be the maximal group constructed as above. For all $J \in \mathcal{U}(M)$, let $S_1^J, \dots, S_{n_J}^J$ be a complete family of simple G_J-modules, and set*

$$X_i^J := \operatorname{top}(S_i^J{\uparrow}_{G_J}^{M_{\geq J}}) = \operatorname{top}(S_J \otimes_{\mathbb{K}G_J} \mathbb{K}M_{\geq J}) \,, \tag{2.3}$$

where $\operatorname{top}(X) := X/\operatorname{rad} X$ *is the semi-simple quotient of the module X. Then, (X_i^J for $J \in \mathcal{U}(M)$ and $i = 1, \dots, n_J$) is a complete family of simple M-modules.*

In the present paper we only need the very particular case of aperiodic monoids. The key point is that a monoid is aperiodic if and only if all the groups G_J are trivial [Pin10, Proposition 4.9]: $G_J = \{e_J\}$. As a consequence, the only G_J-module is the trivial one, 1, so that the previous construction boils down to the following theorem:

Theorem 2.13. *Let M be an aperiodic monoid. Choose an idempotent traversal $E = \{e_J \mid J \in \mathcal{U}(M)\}$ of the regular $\mathcal{J}-$classes. Further set*

$$X^J := \operatorname{top}(1{\uparrow}_{e_J}^{M_{\geq J}}) = \operatorname{top}(e_J M_{\geq J}) \,. \tag{2.4}$$

Then, the family $(X^J)_{J \in \mathcal{U}(M)}$ is a complete family of representatives of simple M-modules. In particular, there are as many isomorphic types of simple modules as regular $\mathcal{J}$-classes.

Moreover, it is easy to see that $e_J M_{\geq J}$ is nothing but the $\mathcal{R}$-class of e_J.

3. Blocks of Coxeter group elements and the cutting poset

In this section, we develop the combinatorics underlying the representation theory of the translation modules studied in Section 6. The key question is: given $w \in W$, for which subsets $J \subseteq I$ does the canonical bijection between a Coxeter group W and the Cartesian product $W_J \times {}^J W$ of a parabolic subgroup W_J by its set of coset representatives ${}^J W$ in W restrict properly to an interval $[1, w]_R$ in right order (see Figure 1)? In type A, the answer is given by the so-called blocks in the permutation matrix of w, and we generalize this notion to any Coxeter group.

We start with some results on parabolic subgroups and quotients in Section 3.1, which are used to define *blocks* and *cutting points* of Coxeter group elements in Section 3.2. Then, we illustrate the notion of blocks in type A in Section 3.3, recovering the usual blocks in permutation matrices. In Section 3.4 it is shown that $(W, \sqsubseteq)$ with the cutting order $\sqsubseteq$ is a poset (see Theorem 3.19). In Section 3.5 we show that blocks are closed under unions and intersections, and relate these to meets and joins in left and right order, thereby endowing the set of cutting points of a Coxeter group element with the structure of a distributive lattice (see Theorem 3.26). In Section 3.6, we discuss various indexing sets for cutting points, which leads to the notion of w-analogues of descent sets in Section 3.7. Properties of the *cutting poset* are studied in Section 3.8 (see Theorem 3.41, which also recapitulates the previous theorems).

Throughout this section $W := \langle s_i \mid i \in I\rangle$ denotes a finite Coxeter group.

3.1. **Parabolic subgroups and cosets representatives.** For a subset $J \subseteq I$, the *parabolic subgroup* W_J of W is the Coxeter subgroup of W generated by s_j for $j \in J$. A complete system of minimal length representatives of the right cosets $W_J w$ (resp. of the left cosets wW_J) are given respectively by:

$$\begin{aligned} {}^{J}W &:= \{x \in W \mid \mathrm{D}_L(x) \cap J = \emptyset\}\,, \\ W^J &:= \{x \in W \mid \mathrm{D}_R(x) \cap J = \emptyset\}\,. \end{aligned}$$

Every $w \in W$ has a unique decomposition $w = w_J\,{}^{J}w$ with $w_J \in W_J$ and ${}^{J}w \in {}^{J}W$. Similarly, there is a unique decomposition $w = w^K\,{}_Kw$ with ${}_Kw \in {}_KW = W_K$ and $w^K \in W^K$.

Lemma 3.1. *Take $w \in W$.*

(i) For $J \subseteq I$ consider the unique decomposition $w = uv$ where $u = w_J$ and $v = {}^{J}w$. Then, the unique decomposition of ws_k is $ws_k = (us_j)v$ if vs_kv^{-1} is a simple reflection s_j with $j \in J$ and $ws_k = u(vs_k)$ otherwise.

(ii) For $K \subseteq I$ consider the unique decomposition $w = vu$ where $u = {}_Kw$ and $v = w^K$. Then, the unique decomposition of s_jw is $s_jw = v(s_ku)$ if $v^{-1}s_jv$ is a simple reflection s_k with $k \in K$ and $s_jw = (s_jv)u$ otherwise.

Proof. This follows directly from [BB05, Lemma 2.4.3 and Proposition 2.4.4]. □

Note in particular that, if we are in case (i) of Lemma 3.1, we have:

- If k is a right descent of w, then $(ws_k)_J \in [1, w_J]_R$ and ${}^{J}(ws_k) \in [1, {}^{J}ws_k]_R$.
- If k is not a right descent of w, then either s_k skew commutes with ${}^{J}w$, or ${}^{J}(ws_k) = {}^{J}ws_k$. In particular, ${}^{J}(ws_k) \leq_R {}^{J}ws_k$.

Definition 3.2. *A subset $J \subseteq I$ is* left reduced *with respect to w if $J' \subsetneq J$ implies ${}^{J}w <_L {}^{J'}w$ (or equivalently, if for any $j \in J$, s_j appears in some and hence all reduced words for w_J).*

Similarly, $K \subseteq I$ is right reduced *with respect to w if $K' \subsetneq K$ implies $w^K <_R w^{K'}$.*

Lemma 3.3. *Let $w \in W$ and $J \subseteq I$ be left reduced with respect to w. Then*

(i) $v = {}^{J}w \leq_R w$ if and only if there exists $K \subseteq I$ and a bijection $\phi_R : J \to K$ such that $s_jv = vs_{\phi_R(j)}$ for all $j \in J$.

For $K \subseteq I$ right reduced with respect to w, we have

(ii) $v = w^K \leq_L w$ if and only if there exists $J \subseteq I$ and a bijection $\phi_L : K \to J$ such that $vs_k = s_{\phi_L(k)}v$ for all $k \in K$.

Proof. Assume first that the bijection ϕ_R exists, and write $w = s_{j_1}\cdots s_{j_\ell}v$, where the product is reduced and $j_i \in J$. Then,

$$w = s_{j_1}\cdots s_{j_\ell}v = s_{j_1}\cdots s_{j_{\ell-1}}vs_{\phi_R(j_\ell)} = vs_{\phi_R(j_1)}\cdots s_{\phi_R(j_\ell)}\,,$$

where the last product is reduced. Therefore $v \leq_R w$.

Assume conversely that $v = {}^J w \leq_R w$, write the reduced expression $w = vs_{k_1} \cdots s_{k_\ell} \geq_R v$, and set $K = \{k_1, \dots, k_\ell\}$. By Lemma 3.1, the sequence

$$v = {}^J v, {}^J(vs_{k_1}), \dots, {}^J(vs_{k_1} \cdots s_{k_\ell}) = {}^J w = v$$

preserves right order, and therefore is constant. Hence, at each step i

$${}^J(vs_{k_1} \cdots s_{k_i}) = {}^J({}^J(vs_{k_1} \cdots s_{k_{i-1}})s_{k_i}) = {}^J(vs_{k_i}) = v\,.$$

Applying Lemma 3.1 again, it follows that there is a subset $J' \subseteq J$, and a bijective map $\phi_R : J' \to K$ such that $s_j v = vs_{\phi_R(j)}$ for all $j \in J'$. Then, $w = s_{\phi_R^{-1}(k_1)} \cdots s_{\phi_R^{-1}(k_\ell)} v$, and by minimality of J, $J = J'$.

The second part is the symmetric statement. □

By Lemma 3.1, for any $w \in W$ and $J \subseteq I$ we have $[1, w]_R \subseteq [1, w_J]_R [1, {}^J w]_R$ and similarly for any $K \subseteq I$ we have $[1, w]_L \subseteq [1, w^K]_L [1, {}_K w]_L$.

Lemma 3.4. *Take $w \in W$, $K \subseteq I$, and assume that $s_i w = ws_k$ for $i \in I$ and $k \in K$, where the products are reduced. Then, there exists $k' \in K$ such that $s_i w^K = w^K s_{k'}$, where the products are again reduced.*

Proof. We have $w^K = (ws_k)^K = (s_i w)^K = (s_i w^K)^K$. Hence, by Lemma 3.1 (ii) there exists $k' \in K$ such that $w^K s_{k'} = s_i w^K$, as desired. □

3.2. **Definition and characterizations of blocks and cutting points.** We now come to the definition of *blocks* of Coxeter group elements and associated *cutting points*. They will lead to a new poset on the Coxeter group W, which we coin the *cutting poset* in Section 3.4.

Definition 3.5 (Blocks and cutting points)**.** *Let $w \in W$. We call $K \subseteq I$ a* right block *(resp. $J \subseteq I$ a* left block*) of w, if there exists $J \subseteq I$ (resp. $K \subseteq I$) such that*

$$W_J w = w W_K\,.$$

In that case, $v := w^K$ is called a cutting point *of w, which we denote by $v \sqsubseteq w$. Furthermore, K is* proper *if $K \neq \emptyset$ and $K \neq I$; it is* nontrivial *if $w^K \neq w$ (or equivalently ${}_K w \neq 1$); analogous definitions are made for left blocks.*

We denote by $\mathcal{B}_{\mathcal{R}}(w)$ the set of all right blocks for w, and by $\mathcal{RB}_{\mathcal{R}}(w)$ the set of all (right) reduced (see definition 3.2) right blocks for w. The sets $\mathcal{B}_{\mathcal{L}}(w)$, $\mathcal{RB}_{\mathcal{L}}(w)$ are similarly defined on the left.

Here is an equivalent characterization of blocks which also shows that cutting points can be equivalently defined using ${}^J w$ instead of w^K.

Proposition 3.6. *Let $w \in W$ and $J, K \subseteq I$. Then, the following are equivalent:*

(i) $W_J w = w W_K$;

(ii) There exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} w^K$ (or equivalently $w^K(\alpha_k) = \alpha_{\phi(k)}$) for all $k \in K$.

Furthermore, when any, and therefore all, of the above hold then,

(iii) $w^K = {}^J w$.

Proof. Suppose (i) holds. Then $W_J {}^J w = w^K W_K$. Since ${}^J w$ has no left descents in J and w^K has no right descents in K, we know that on both sides ${}^J w$ and w^K are the shortest elements and hence have to be equal: ${}^J w = w^K$; this proves (iii). Furthermore, every reduced expression $w^K s_k$ with $k \in K$ must correspond to some reduced expression $s_j {}^J w$ for some $j \in J$, and vice versa. Hence there exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} {}^J w = s_{\phi(k)} w^K$. Therefore point (ii) holds.

Suppose now that point (ii) holds. Then, for any expression $s_{k_1} \cdots s_{k_\ell} \in W_K$, we have

$$w^K s_{k_1} \cdots s_{k_\ell} = s_{\phi(k_1)} w^K s_{k_2} \cdots s_{k_\ell} = \cdots = s_{\phi(k_1)} \cdots s_{\phi(k_\ell)} w^K .$$

It follows that

$$w^K W_K = W_J w^K .$$

In particular $w \in W_J w^K$ and therefore

$$W_J w = W_J w^K = w^K W_K = w W_K . \qquad \square$$

In general, condition (iii) of Proposition 3.6 is only a necessary, but not sufficient condition for K to be a block. See Example 3.12.

Proposition 3.7. *If K is a right block (or more generally if $w^K = w^{K'}$ with K' a right block), then the bijection*

$$\begin{cases} W^K \times {}_K W & \to W \\ (v,u) & \mapsto vu \end{cases}$$

restricts to a bijection $[1, w^K]_L \times [1, {}_K w]_L \to [1, w]_L$.

Similarly, if J is a left block (or more generally if ${}^J w = {}^{J'} w$ with J' a left block), then the bijection

$$\begin{cases} W_J \times {}^J W & \to W \\ (u,v) & \mapsto uv \end{cases}$$

restricts to a bijection $[1, w_J]_R \times [1, {}^J w]_R \to [1, w]_R$ (see Figure 1).

Proof. By Proposition 3.6 we know that, if K is a right block, then there exists a bijection $\phi : K \to J$ such that $w^K s_k = s_{\phi(k)} w^K$. Hence the map $y \mapsto w^K y$ induces a skew-isomorphism between $[1, {}_K w]_L$ and $[w^K, w]_L$, where an edge k is mapped to edge $\phi(k)$. It follows in particular that $uv \leq_L w^K \ v \leq_L w^K \ {}_K w = w$ for any $u \in [1, w^K]_L$ and $v \in [1, {}_K w]_L$, as desired.

Assume now that K is not a block, but $w^K = w^{K'}$ with K' a block. Then, $[1, w^K]_L = [1, w^{K'}]_L$ and $[1, {}_K w]_L = [1, {}_{K'} w]_L$ and we are reduced to the previous case.

The second statement can be proved in the same fashion. $\square$

Due to Proposition 3.7, we also say that $[1, v]_R$ *tiles* $[1, w]_R$ if $v = {}^J w$ for some left block J (or equivalently $v = w^K$ for some right block K).

Proposition 3.8. *Let $w \in W$ and K be right reduced with respect to w. Then, the following are equivalent:*

(i) K is a reduced right block of w;
(ii) $w^K \leq_L w$.

The analogous statement can be made for left blocks.

See also Proposition 6.9 for yet another equivalent condition of reduced blocks.

Proof of Proposition 3.8. If K is a right block, then by Proposition 3.6 we have $w^K = {}^J w$, where J is the associated left block. In particular, $w^K = {}^J w \leq_L w$.

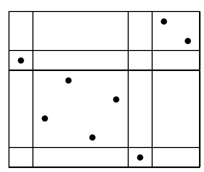
The converse statement follows from Lemma 3.3 and Proposition 3.6. $\square$

Example 3.9. For $w = w_0$, any $K \subseteq I$ is a reduced right block; of course $w_0^K \leq_L w_0$ and ${}_K w_0$ is the maximal element of the parabolic subgroup $W_K = {}_K W$. The cutting point $w^K \sqsubseteq w$ is the maximal element of the right descent class for the complement of K.

The associated left block is given by $J = \phi(K)$, where ϕ is the automorphism of the Dynkin diagram induced by conjugation by w_0 on the simple reflections. The tiling corresponds to the usual decomposition of W into right W_K cosets, or of W into left W_J cosets.

3.3. **Blocks of permutations.** In this section we illustrate the notion of blocks and cutting points introduced in the previous section for type A. We show that, for a permutation $w \in \mathfrak{S}_n$, the blocks of Definition 3.5 correspond to the usual notion of blocks of the permutation matrix of w (or unions thereof), and the cutting points w^K for right blocks K correspond to putting the identity in those blocks.

A *matrix-block* of a permutation w is an interval $[k', k'+1, \ldots, k]$ which is mapped to another interval. Pictorially, this corresponds to a square submatrix of the matrix of w which is again a permutation matrix (that of the *associated permutation*). For example, the interval $[2, 3, 4, 5]$ is mapped to the interval $[4, 5, 6, 7]$ by the permutation $w = 36475812 \in \mathfrak{S}_8$, and is therefore a matrix-block of w with associated permutation 3142. Similarly, $[7, 8]$ is a matrix-block with associated permutation 12:



For any permutation w, the singletons $[i]$ and the full set $[1, 2, \ldots, n]$ are always matrix-blocks; the other matrix-blocks of w are called *proper*. A permutation with no proper matrix-block, such as 58317462, is called *simple*. See [NMPR95, AAK03, AA05] for a review of simple permutations. Simple permutations are also strongly related to dimension 2 posets.

A permutation $w \in \mathfrak{S}_n$ is *connected* if it does not stabilize any subinterval $[1, \ldots, k]$ with $1 \leq k < n$, that is if w is not in any proper parabolic subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. Pictorially, this means that there are no diagonal matrix-blocks. A matrix-block is *connected* if the corresponding induced permutation is connected. In the above example, the matrix-block $[2, 3, 4]$ is connected, but the matrix-block $[7, 8]$ is not.

Proposition 3.10. *Let $w \in \mathfrak{S}_n$. The right blocks of w are in bijection with disjoint unions of (nonsingleton) matrix-blocks for w; each matrix-block with column set $[i, i+1, \ldots, k]$ contributes $\{i, i+1, \ldots, k-1\}$ to the right block; each matrix-block with row set $[i, i+1, \ldots, k]$ contributes $\{i, i+1, \ldots, k-1\}$ to the left block.*

In addition, trivial right blocks correspond to unions of identity matrix-blocks. Also, reduced right blocks correspond to unions of connected matrix-blocks.

Proof. Suppose $w \in \mathfrak{S}_n$ with a disjoint union of matrix-blocks with consecutive column sets $[i_1, \ldots, k_1]$ up to $[i_\ell, \ldots, k_\ell]$. Set $K_j = \{i_j, \ldots, k_j - 1\}$ for $1 \leq j \leq \ell$ and $K = K_1 \cup \cdots \cup K_\ell$. Define similarly J according to the rows of the blocks.

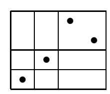
Then, multiplying w on the right by some element of W_K permutes some columns of w, while stabilizing each blocks. Therefore, the same transformation can be achieved by some permutation of the rows stabilising each block, that is by multiplication of w on the left by some element of W_J. Hence, using symmetry, $W_J w = wW_K$, that is J and K are corresponding left and right blocks for w.

Conversely, if K is a right block of w, then w^K maps each α_k with $k \in K$ to another simple root by Proposition 3.6. But then, splitting $K = K_1 \cup \cdots \cup K_\ell$ into consecutive subsets with $K_j = \{i_j, \ldots, k_j - 1\}$, the permutation w^K must contain the identity permutation in each matrix-block with column indices $[i_j, \ldots, k_j]$. This implies that w itself has matrix-blocks with column indices $[i_j, \ldots, k_j]$ for $1 \leq j \leq \ell$.

Note that, in the described correspondence, $w^K = w$ if and only if all matrix-blocks contain the identity. This proves the statement about trivial right blocks.

A reduced right block K has the property that for every $K' \subsetneq K$ we have $w^{K'} \neq w^K$. This implies that no matrix-block is in a proper parabolic subgroup, and hence they are all connected. $\square$

Example 3.11. As in Figure 1, consider the permutation 4312, whose permutation matrix is:



The reduced (right)-blocks are $K = \{\}$, $\{1\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. The cutting points are 4312, 3412, 4123, and 1234, respectively. The corresponding left blocks are $J = \{\}$, $\{3\}$, $\{1, 2\}$ and $\{1, 2, 3\}$, respectively. The nonreduced (right) blocks are $\{3\}$ and $\{1, 3\}$, as they are respectively equivalent to the blocks $\{\}$ and $\{1\}$. The trivial blocks are $\{\}$ and $\{3\}$.

Example 3.12. In general, condition (iii) of Proposition 3.6 is only a necessary, but not sufficient condition for K to be a block. For example, for $w = 43125$ (similar to 4312 of Example 3.11, but embedded in $\mathfrak{S}_5$), $J = \{3, 4\}$, and $K = \{1, 4\}$, one has ${}^J w = w^K$ yet neither J nor K are blocks. On the other hand (iii) of Proposition 3.6 becomes both necessary and sufficient for reduced blocks.

Remark 3.13. *It is obvious that the union and intersection of overlapping (possibly with a trivial overlap) matrix-blocks in $\mathfrak{S}_n$ are again matrix-blocks; we will see in Proposition 3.22 that this property generalizes to all types.*

Problem 3.14. *Fix $J \subseteq \{1, 2, \ldots, n-1\}$ and enumerate the permutations $w \in \mathfrak{S}_n$ for which J is a left block.*

3.4. **The cutting poset.** In this section, we show that $(W, \sqsubseteq)$ indeed forms a poset. We start by showing that for a fixed $u \in W$, the set of elements w such that $u \sqsubseteq w$ admits a simple description. Recall that for $J \subseteq I$, we denote by s_J the longest element of W_J. Proposition 3.6 suggests the following definition.

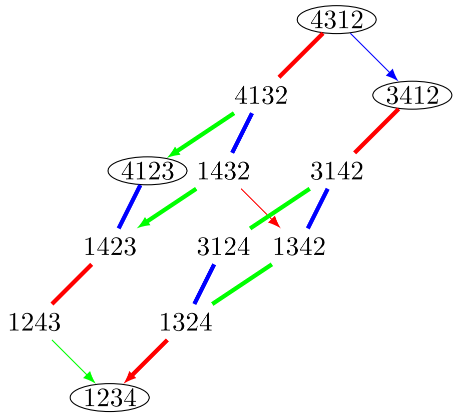
Definition 3.15. *Let $u \in W$. We call $k \in I$ a* short right nondescent *(resp. $j \in I$ a* short left nondescent*) of u if there exists $j \in I$ (resp. $k \in I$) such that*

$$s_j u = u s_k \,,$$

where the product is reduced (that is j and k are nondescents). An equivalent condition is that u maps the simple root α_k to a simple root (resp. the preimage of α_j is a simple root).

Set further

$$U_u := uW_K = [u, us_K]_R = W_J u = [u, s_J u]_L \,,$$



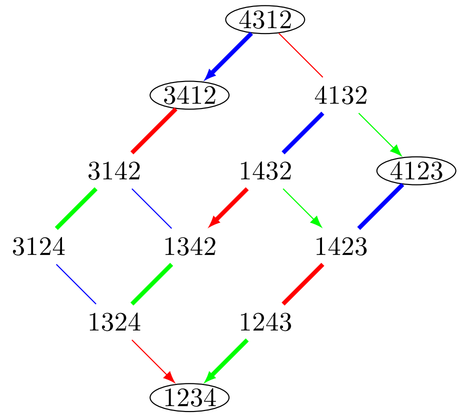


FIGURE 1. Two pictures of the interval $[1234, 4312]_R$ in right order in $\mathfrak{S}_4$ illustrating its proper tilings, for $J := \{3\}$ and $J := \{1, 2\}$, respectively. The thick edges highlight the tiling. The circled permutations are the cutting points, which are at the top of the tiling intervals. Blue, red, green lines correspond to s_1, s_2, s_3, respectively. See Section 6.4 for the definition of the orientation of the edges (this is $G^{(4312)}$); edges with no arrow tips point in both directions.

where $K := K(u)$ (resp. $J := J(u)$) is the set of short right (resp. left) nondescents of u.

Pictorially, one takes left and right order on W and associates to each vertex u the translate U_u above u of the parabolic subgroup generated by the short nondescents of u, which correspond to the simultaneous covers of u in both left and right order.

Example 3.16. In type A, i is short for $u \in \mathfrak{S}_n$ if $u(i+1) = u(i) + 1$, that is, there is a 2×2 identity block in columns $(i, i+1)$ of the permutation matrix of u. Furthermore U_u is obtained by looking at all identity blocks in u and replacing each by any permutation matrix.

The permutation 4312 of Example 3.11 has a single nondescent 3 which is short, and $U_{4312} = \{4312, 4321\}$.

Proposition 3.17. *U_u is the set of all w such that $u \sqsubseteq w$.*

In particular, it follows that:

- *If $u \leq_R v \leq_R w$ and $u \sqsubseteq w$, then $u \sqsubseteq v$.*
- *If $u \sqsubseteq w$ and $u \sqsubseteq w'$, then $u \sqsubseteq w \vee_R w'$.*

Proof. Note that w is in U_u if and only if there exists K such that $K \subseteq K(u)$ and $w^K = u$. By Proposition 3.6, this is equivalent to the existence of a block K such that $w^K = u$, that is $u \sqsubseteq w$. □

The following related lemma is used to prove that $(W, \sqsubseteq)$ is a poset.

Lemma 3.18. *If $u \sqsubseteq w$, then the set of short nondescents of w is a subset of the short nondescents of u, namely $K(w) \subseteq K(u)$.*

Proof. Let $k \in K(w)$, so that $ws_k = s_j w$ for some $j \in I$ and both sides are reduced. It follows from Lemma 3.4 that there exists $k' \in K(w)$ such that $s_j u = us_{k'}$ and both sides are reduced. Hence $k' \in K(u)$. Since the map $k \mapsto k'$ is injective it follows that $K(w) \subseteq K(u)$. □

Theorem 3.19. $(W, \sqsubseteq)$ *is a subposet of both left and right order.*

Proof. The relation $\sqsubseteq$ is reflexive since v is a cutting point of v with right block $\emptyset$; hence $v \sqsubseteq v$. Applying Proposition 3.6, it is a subrelation of left and right order: if $v \sqsubseteq w$ then $v = w^K \leq_R w$ for some K and $v = {}^J w \leq_L w$ for some J. Antisymmetry follows from the antisymmetry of left (or right) order.

For transitivity, let $v \sqsubseteq w$ and $w \sqsubseteq z$. Then $v = w^K$ and $w = z^{K'}$ for some right block K of w and K' of z. We claim that $v = z^{K \cup K'}$ with $K \cup K'$ a right block of z. Certainly $k \notin \mathrm{D}_R(v)$ for $k \in K$ since $v = w^K$. Since $w = z^{K'}$ with K' a block of z, all $k' \in K'$ are short nondescents of w and hence by Lemma 3.18 also short nondescents of v. This proves the claim. Therefore $v \sqsubseteq z$. □

Example 3.20. The cutting poset for $\mathfrak{S}_3$ and $\mathfrak{S}_4$ is given in Figure 2. As we can see on those figures, the cutting poset is not the intersection of the right and left order since w_0 is maximal for left and right order but not for cutting poset.

3.5. **Lattice properties of intervals.** In this section we show that the set of blocks and the set of cutting points $\{u \mid u \sqsubseteq w\}$ of a fixed $w \in W$ are endowed with the structure of distributive lattices (see Theorem 3.26).

We begin with a lemma which gives some properties of blocks that are contained in each other.

Lemma 3.21. *Fix $w \in W$. Let $K \subseteq K'$ be two right blocks of w and $J \subseteq J'$ be the corresponding left blocks, so that*

$$W_J w = w W_K, \quad W_{J'} w = w W_{K'}, \quad {}^J w = w^K \sqsubseteq w, \quad \text{and} \quad {}^{J'} w = w^{K'} \sqsubseteq w\,.$$

Then,

(i) $w^{K'} \leq_R w^K$ and $w^{K'} \leq_L w^K$;
(ii) K' is a right block of w^K and $w^{K'} \sqsubseteq w^K$;
(iii) K is a right block of ${}_{K'}w$ and ${}_{K'}w^K \sqsubseteq {}_{K'}w$.
Furthermore K is reduced for ${}_{K'}w$ if and only if it is reduced for w.

The same statements hold for left blocks.

Proof. (i) holds because $w^{K'} = (w^K)^{K'} \leq_R w^K \leq_R w$, and similarly on the left.

(ii) is a trivial consequence of (i) and Proposition 3.17.

For (iii), first note that $({}_{K'}w)^K = {}_{K'}(w^K)$, so that the notation ${}_{K'}w^K$ is unambiguous. Consider the bijection ϕ from K' to J' of Proposition 3.6, and note that $W_J w^{K'} = w^{K'} W_{\phi^{-1}(J)}$. Therefore,

$$w^{K'}\,{}_{K'}w W_K = w W_K = W_J\; w = W_J\; w^{K'}\,{}_{K'}w = w^{K'} W_{\phi^{-1}(J)}\;{}_{K'}w\,.$$

Simplifying by $w^{K'}$ on the left, one obtains that

$${}_{K'}w\; W_K = W_{\phi^{-1}(J)}\;{}_{K'}w\,,$$

proving that K is also a block of ${}_{K'}w$. The reduction statement is trivial. □

We saw in Remark 3.13 that the set of blocks is closed under unions and intersections in type A. This holds for general type.

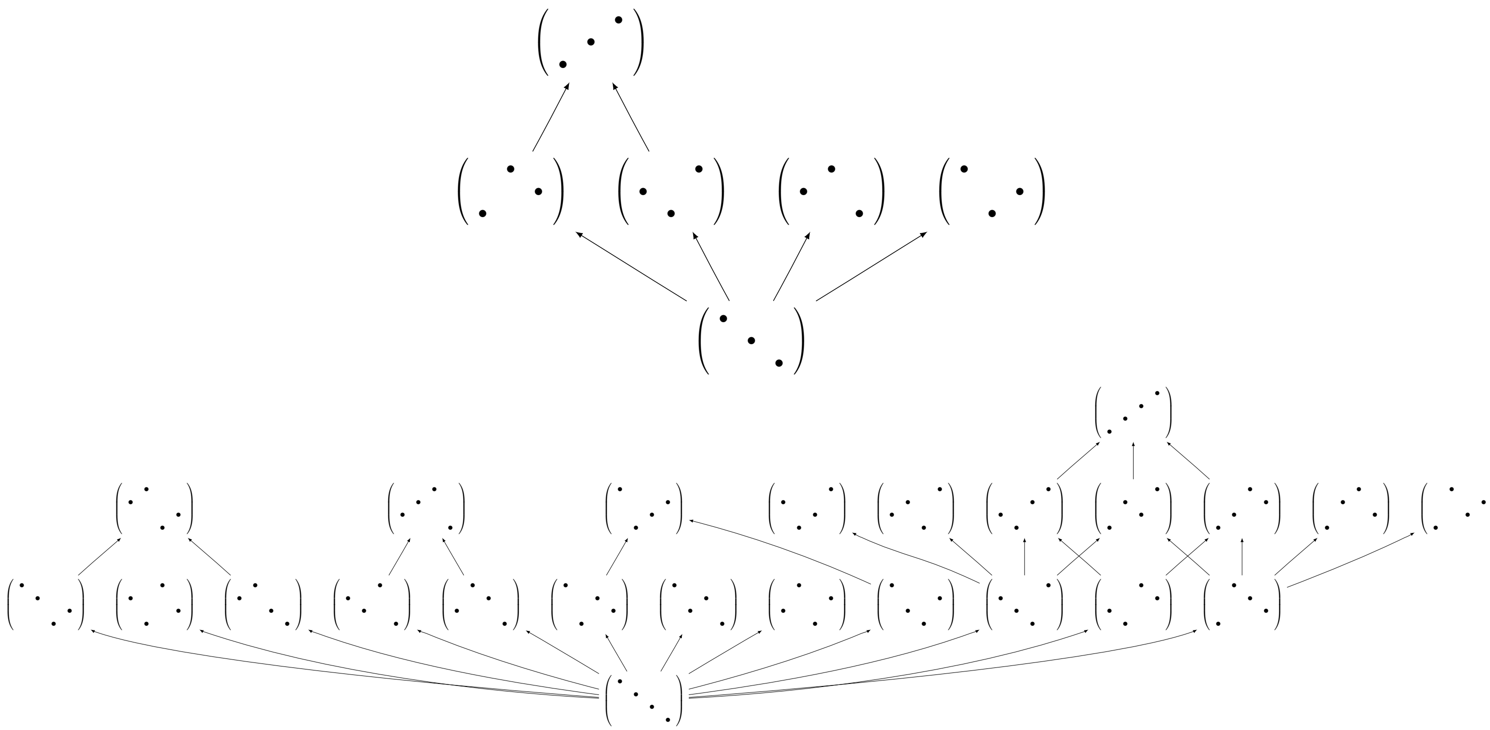


Figure 2. The cutting posets for the symmetric groups $\mathfrak{S}_3$ and $\mathfrak{S}_4$. Each permutation is represented by its permutation matrix, with the bullets marking the positions of the ones. Notice the Boolean sublattice appearing as the interval between the identity permutation at the bottom and the maximal permutation at the top; its elements are the minimal elements of the descent classes.

Proposition 3.22. *The set $\mathcal{B}_{\mathcal{R}}(w)$ (resp. $\mathcal{B}_{\mathcal{L}}(w)$) of right (resp. left) blocks is stable under union and intersection. Hence, it forms a distributive sublattice of the Boolean lattice $\mathcal{P}(I)$.*

Proof. Let K and K' be right blocks for $w \in W$, and J and J' be the corresponding left blocks, so that:

$$wW_K = W_J w \quad \text{and} \quad wW_{K'} = W_{J'} w.$$

Take $u \in W_{K\cap K'} = W_K \cap W_{K'}$. Then, wuw^{-1} is both in W_J and $W_{J'}$ and therefore in $W_J \cap W_{J'} = W_{J\cap J'}$. This implies $wW_{K\cap K'}w^{-1} \subseteq W_{J\cap J'}$. By symmetry, the inclusion $w^{-1}W_{J\cap J'}w \subseteq W_{K\cap K'}$ holds as well, and therefore $W_{J\cap J'}w = wW_{K\cap K'}$. In conclusion, $K \cap K'$ is a right block, with $J \cap J'$ as corresponding left block.

Now take $u \in W_{K\cup K'} = \langle W_K, W_{K'}\rangle$, and write u as a product $u_1u_1'u_2u_2'\cdots u_\ell u_\ell'$, where $u_i \in W_K$ and $u_i' \in W_{K'}$ for all $1 \le i \le \ell$. Then, for each i, $wu_iw^{-1} \in W_J$ and $wu_i'w^{-1} \in W_{J'}$. By composition, $wuw^{-1} \in W_J W_{J'} W_J W_{J'} \cdots W_J W_{J'} \subseteq W_{J\cup J'}$. Using symmetry as above, we conclude that $wW_{K\cup K'} = W_{J\cup J'}w$. In summary, $K \cup K'$ is a right block, with $J \cup J'$ as corresponding left block.

Finally, since blocks are stable under union and intersection, they form a sublattice of the Boolean lattice. Any sublattice of a distributive lattice is distributive. □

Next we relate the union and intersection operation on blocks with the meet and join operations in right and left order. We start with the following general statement which must be classical, though we have not found it in the literature.

Lemma 3.23. *Take $w \in W$ and $J, J', K, K' \subseteq I$. Then*

$$w^{K\cap K'} = w^K \vee_R w^{K'} \qquad \textit{and} \qquad {}^{J\cap J'}w = {}^{J}w \vee_L {}^{J'}w\,.$$

Proof. We include a proof for the sake of completeness. By Lemma 3.21 (i), $w^K, w^{K'} \le_R w^{K\cap K'}$, and therefore $v \le_R w^{K\cap K'}$, where $v = w^K \vee_R w^{K'}$. Suppose that v has a right descent $k \in K \cap K'$. Then vs_k is still bigger than w^K and $w^{K'}$ in right order, a contradiction to the definition of v. Hence $w^{K\cap K'} = w^K \vee_R w^{K'}$, as desired. The statement on the left follows by symmetry. □

Corollary 3.24. *Take $w \in W$. Let $K, K' \subseteq I$ be two right blocks of w and $J, J' \subseteq I$ the corresponding left blocks. Then, for the right block $K\cap K'$ (resp. left block $J\cap J'$)*

$$w^{K\cap K'} = {}^{J\cap J'}w = w^K \vee_R w^{K'} = {}^{J}w \vee_L {}^{J'}w\,.$$

The analogous statement of Lemma 3.23 for unions fails in general: take for example $w = 4231$ and $K = \{3\}$ and $K' = \{1, 2\}$, so that $w^K = 4213$ and $w^{K'} = 2341$; then $w^{K\cup K'} = 1234$, but $w^K \wedge_R w^{K'} = 2134$. However, it holds for blocks:

Lemma 3.25. *Take $w \in W$. Let $K, K' \subseteq I$ be two right blocks of w and $J, J' \subseteq I$ the corresponding left blocks. Then, for the right block $K \cup K'$ (resp. left block $J \cup J'$):*

$$w^{K\cup K'} = {}^{J\cup J'}w = w^K \wedge_R w^{K'} = {}^{J}w \wedge_L {}^{J'}w\,.$$

Furthermore, $K \cup K'$ is reduced whenever K and K' are reduced, and similarly for the left blocks.

Proof. By symmetry, it is enough to prove the statements for right blocks.

By Lemma 3.21 (i), $w^{K\cup K'} \le_R w^K, w^{K'}$, and therefore $w^{K\cup K'} \le_R w^K \wedge_R w^{K'}$.

Note that the interval $[w^{K\cup K'}, w]_R$ contains all the relevant points: w^K, $w^{K'}$, and $w^K \wedge_R w^{K'}$. Consider the translate of this interval obtained by dividing on the left by $w^{K\cup K'}$, or equivalently by using the map $u \mapsto {}_{K\cup K'}u$. By Lemma 3.21 (iii), K and K' are still blocks of ${}_{K\cup K'}w$. From now on, we may therefore assume without loss of generality that $w^{K\cup K'} = 1$. It follows at once that $[1, w]_R$ lies in the parabolic subgroup $W_{K\cup K'}$ and that $J \cup J' = K \cup K'$.

If $w^K \wedge_R w^{K'} = 1 = w^{K\cup K'}$, then we are done. Otherwise, let $i \in K\cup K' = J\cup J'$ be the first letter of some reduced word for $w^K \wedge_R w^{K'}$. Since $w^K \wedge_R w^{K'}$ is in the interval $[1, w^K]_R$, i cannot be in J; by symmetry i cannot be in J' either, a contradiction.

Assume further that K and K' are reduced. Then, any $k \in K$ appears in any reduced word for ${}_K w$, and therefore in any reduced word for ${}_{K\cup K'}w$ since ${}_K w \leq_L {}_{K\cup K'}w$. By symmetry, the same holds for $k' \in K'$. Hence $K \cup K'$ is reduced. □

Theorem 3.26. *The map $K \mapsto w^K$ (resp. $J \mapsto {}^J w$) defines a lattice antimorphism from the lattice $\mathcal{B}_{\mathcal{R}}(w)$ (resp. $\mathcal{B}_{\mathcal{L}}(w)$) of right (resp. left) blocks of w to both right and left order on W.*

The set of cutting points for w, which is the image set

$$\{w^K \mid K \in \mathcal{B}_{\mathcal{R}}(w)\} = \{{}^J w \mid J \in \mathcal{B}_{\mathcal{L}}(w)\}$$

of the previous map, is a distributive sublattice of right (resp. left) order.

Proof. The first statement is the combination of Lemmas 3.23 and 3.25. The second statement follows since the quotient of a distributive sublattice by a lattice morphism is a distributive lattice. □

Corollary 3.27. *Every interval of $(W, \sqsubseteq)$ is a distributive sublattice and an induced subposet of both left and right order.*

Proof. Take an interval in $(W, \sqsubseteq)$; without loss of generality, we may assume that it is of the form $[1, w]_{\sqsubseteq} = \{w^K \mid K \in \mathcal{RB}_{\mathcal{R}}(w)\}$. The interval $[1, w]_{\sqsubseteq}$ is not only a subposet of left (resp. right) order, but actually the induced subposet; indeed for K and K' right reduced blocks, and J and J' the corresponding left blocks,

$$w^K \leq_L w^{K'} \Leftrightarrow w^K \leq_R w^{K'} \Leftrightarrow J' \subseteq J \Leftrightarrow K' \subseteq K \Leftrightarrow w^K \leq_{\sqsubseteq} w^{K'} .$$

Therefore, using Theorem 3.26, it is a distributive sublattice of left (resp. right) order. □

Let us now consider the lower covers in the cutting poset for a fixed $w \in W$. They correspond to nontrivial blocks J which are minimal for inclusion, and in particular reduced.

Lemma 3.28. *Each minimal nontrivial (left) block J for $w \in W$ contains at least one element which is in no other minimal nontrivial block for w.*

Proof. Assume otherwise. Then, J is the union of its intersections with the other nontrivial blocks. Each such intersection is necessarily a trivial block, and a union of trivial blocks is a trivial block. Therefore, J is a trivial block, a contradiction. □

Corollary 3.29. *The semilattice of unions of minimal nontrivial blocks for a fixed $w \in W$ is free.*

Proof. This is a straightforward consequence of Lemma 3.28. Alternatively, this property is also a direct consequence of Corollary 3.27, since it holds in general for any distributive lattice. □

3.6. **Index sets for cutting points.** Recall that by Theorem 3.26 the cutting points of w form a distributive lattice. Hence, by Birkhoff's representation theorem, they can be indexed by some collection of subsets closed under unions and intersections. We therefore now aim at finding a suitable choice of indexing scheme for the cutting points of w. More precisely, for each w, we are looking for a pair $(\mathcal{K}^{(w)}, \phi^{(w)})$, where $\mathcal{K}^{(w)}$ is a subset of some Boolean lattice (typically $\mathcal{P}(I)$) such that $\mathcal{K}^{(w)}$ ordered by inclusion is a lattice, and

$$\phi^{(w)} \ : \ \mathcal{K}^{(w)} \longrightarrow [1, w]_{\sqsubseteq}$$

is an isomorphism (or antimorphism) of lattices.

Here are some of the desirable properties of this indexing:

(1) The indexing gives a Birkhoff's representation of the lattice of cutting points of w. Namely, $\mathcal{K}^{(w)}$ is a sublattice of the chosen Boolean lattice, and unions and intersections of indices correspond to joins and meets of cutting points.
(2) The isomorphism $\phi^{(w)}$ is given by the map $J \mapsto {}^{J}w$. In that case the choice amounts to defining a section of those maps.
(3) The indexing generalizes the usual combinatorics of descents.
(4) The indices are blocks: $\mathcal{K}^{(w)} \subset \mathcal{B}_{\mathcal{L}}(w)$.
(5) We may actually want to have two indexing sets $\mathcal{K}^{(w)}$ and $\mathcal{K}^{(w)}$, one on the left and one on the right, with a natural isomorphism between them.
(6) The index of u in $\mathcal{K}^{(w)}$ does not depend on w (as long as u is a cutting point of w). One may further ask for this index to not depend on W, so that the indexing does not change through embedding of parabolic subgroups.

Unfortunately, there does not seem to be an ideal choice satisfying all of these properties at once, and we therefore propose several imperfect alternatives.

3.6.1. *Indexing by reduced blocks.* The first natural choice is to take reduced blocks as indices; then, $\mathcal{K}^{(w)} = \mathcal{RB}_{\mathcal{R}}(w)$ (and similarly $\mathcal{J}^{(w)} = \mathcal{RB}_{\mathcal{L}}(w)$ on the left). This indexing scheme satisfies most of the desired properties, except that it does not provide a Birkhoff representation, and depends on w.

Remark 3.30. *By Lemma 3.25, if $K, K' \subseteq I$ are reduced right blocks for w, then $K \cup K'$ is also reduced. However, this is not necessarily the case for $K \cap K'$: consider for example the permutation $w = 4231$, $K = \{1,2\}$ and $K' = \{2,3\}$; then $K \cap K' = \{2\}$ is a block which is equivalent to the reduced block $\{\}$: $4231^{\{2\}} = 4231 = 4231^{\{\}}$.*

The union $K \cup K'$ of two blocks may be reduced even when the blocks are not both reduced. Consider for example the permutation $w = 4312$ as in Figure 1. Then $K = \{1,3\}$ and $K' = \{2,3\}$ are blocks and their union $K \cup K' = \{1,2,3\}$ is reduced, yet K is not reduced.

Proposition 3.31. *The poset $(\mathcal{RB}_{\mathcal{R}}(w), \subseteq)$ of reduced right blocks is a distributive lattice, with the meet and join operation given respectively by:*

$$K \vee K' = K \cup K' \qquad \text{and} \qquad K \wedge K' = \mathrm{red}(K \cap K')\ ,$$

where, for a block K, $\mathrm{red}(K)$ is the unique largest reduced block contained in K.

The map $\phi^{(w)} : K \mapsto w^K$ restricts to a lattice antiisomorphism from the lattice $\mathcal{B}_{\mathcal{R}}(w)$ of reduced right blocks of w to $[1, w]_{\sqsubseteq}$.

The same statements hold on the left.

Proof. By Proposition 3.22 and Lemma 3.25, $\mathcal{RB}_{\mathcal{R}}(w)$ is a dual Moore family of the Boolean lattice of I, or even of $\mathcal{B}_{\mathcal{R}}(w)$. Therefore, using Section 2.1, it is a lattice, with the given join and meet operations.

The lattice antiisomorphism of property follows from Lemma 3.25 and the coincidence of right order and $\sqsubseteq$ on $[1, w]_{\sqsubseteq}$ (Theorem 3.26). □

3.6.2. *Indexing by largest blocks.* The indexing by reduced blocks corresponds to the section of the lattice morphism $K \mapsto w^K$ by choosing the smallest block K in the fiber of a cutting point u. Instead, one could choose the largest block in the fiber of u, which is given by the set of short nondescents of u. This indexing scheme is independent of w. Also, by the same reasoning as above, the indexing sets $\mathcal{J}^{(w)}$ come endowed with a natural lattice structure. However, it does not give a Birkhoff representation: the meet is given by intersection, but the join is not given by union (take $w = 2143$; its cutting points are 1234, 1243, 2134, and 2143, indexed respectively by $\{1, 2, 3\}$, $\{1\}$, $\{3\}$, and $\{\}$).

3.6.3. *Birkhoff's representation using non-blocks.* We now relax the condition for the indices to be blocks. That is, we consider $K \mapsto w^K$ as a function from the full Boolean lattice $\mathcal{P}(I)$ to the minimal coset representatives of w. Beware that this map is no longer a lattice antimorphism; yet, the fiber of any u still admits a largest set $K = \overline{\mathrm{D}}_R(u) \subseteq I$, which is the complement of the right descent set of u. One can define a similar indexing on the left by $J = \overline{\mathrm{D}}_L(u)$. These indexings are independent of w and provide a Birkhoff representation for the lattice of cutting points (see Proposition 3.34). Define

$$\mathcal{DB}_{\mathcal{L}}(w) = \{\overline{\mathrm{D}}_L(u) \mid u \sqsubseteq w\} \qquad \text{and} \qquad \mathcal{DB}_{\mathcal{R}}(w) = \{\overline{\mathrm{D}}_R(u) \mid u \sqsubseteq w\}\ . \tag{3.1}$$

Remark 3.32. *Since $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ are not necessarily blocks anymore, the bijection between $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ is not induced anymore by a bijection at the level of descents: for example, for $u = 3142$, one has $\overline{\mathrm{D}}_L(u) = \{1, 3\}$ and $\overline{\mathrm{D}}_R(u) = \{2\}$.*

Remark 3.33. *Using $\mathrm{D}_R(u)$ instead of $\overline{\mathrm{D}}_R(u)$ would give an isomorphism instead of an antiisomorphism, and make the indexing further independent of W, at the price of slightly cluttering the notation w^K for cutting points.*

Proposition 3.34 (Birkhoff representation for the lattice of cutting points)**.** *The set $\mathcal{DB}_{\mathcal{R}}(w)$ of Equation (3.1) is a sublattice of the Boolean lattice, and the maps $K \mapsto w^K$ and $u \mapsto \overline{\mathrm{D}}_R(u)$ form a pair of reciprocal lattice antiisomorphisms with the lattice of cutting points of w. The same statement holds on the left.*

The proof of this Proposition uses the following property of left and right order (recall that $[1, w]_{\sqsubseteq}$ is a sublattice thereof).

Lemma 3.35 ([LCdPB94, Lemme 5])**.** *The maps*

$$\begin{cases} (W, \leq_L) & \to \mathcal{P}(I) \\ w & \mapsto \mathrm{D}_R(w) \end{cases} \qquad \begin{cases} (W, \leq_R) & \to \mathcal{P}(I) \\ w & \mapsto \mathrm{D}_L(w) \end{cases}$$

are surjective lattice morphisms.

Proof of Proposition 3.34. By construction, $\overline{\mathrm{D}}_L$ is a section of $K \mapsto w^K$, and these maps form a pair of reciprocal bijections between $\mathcal{DB}_{\mathcal{L}}(w)$ and the cutting points of w. Using Lemma 3.35, the map $\overline{\mathrm{D}}_L$ is a lattice antimorphism. Therefore its image set $\mathcal{DB}_{\mathcal{R}}(w)$ is a sublattice of the Boolean lattice. The argument on the left is the same. □

3.7. **A w-analogue of descent sets.** For each $w \in W$, we now provide a definition of a w-analogue on the interval $[1, w]_R$ of the usual combinatorics of (non)descents on W. From now on, we assume that we have chosen an indexation scheme so that the cutting points of w are given by $(w^K)_{K\in\mathcal{K}^{(w)}}$ or equivalently by $({}^J w)_{J\in\mathcal{J}^{(w)}}$.

Lemma 3.36. *Take a cutting point of w, and write it as $w^K = {}^J w$ for some $J, K \subseteq I$, which are not necessarily blocks. Then:*

(i) for $u \in [1, w]_R$, $u \in [1, {}^J w]_R$ if and only if $\mathrm{D}_L(u) \cap J = \emptyset$;

(ii) for $u \in [1, w]_L$, $u \in [1, w^K]_L$ if and only if $\mathrm{D}_R(u) \cap K = \emptyset$.

Proof. This is a straightforward corollary of Proposition 3.7: any element u of $[1, w]_R$ can be written uniquely as a product $u'v$ with $u' \in W_J$ and $v \in [1, {}^J w]_R$. So u is in $[1, {}^J w]_R$ if and only if $u' = 1$, which in turn is equivalent to v having no descents in J. This proves (i). The argument for (ii) is analogous. □

Example 3.37. For $w = w_0$, ${}^J w$ is the maximal element of a left descent class, and $[1, {}^J w]_R$ gives all elements of W whose left descent set is a subset of the left descent set of w.

Definition 3.38 (w-nondescent sets)**.** *For $u \in [1, w]_R$ define $J^{(w)}(u)$ to be the index $J \in \mathcal{J}^{(w)}$ of the lowest cutting point ${}^J w$ such that $u \in [1, {}^J w]_R$ (or the equivalent condition of Lemma 3.36). Define similarly $K^{(w)}(u)$ as the index in $\mathcal{K}^{(w)}$ of this cutting point.*

Example 3.39. When $w = w_0$, $J^{(w_0)}(u)$ and $K^{(w_0)}(u)$ are respectively the sets $\overline{\mathrm{D}}_L(u)$ and $\overline{\mathrm{D}}_R(u)$ of left and right nondescents of u.

Problem 3.40. *Given J, describe all the elements $w \in W$ such that J is a left block. This essentially only depends on ${}^J w$.*

3.8. **Properties of the cutting poset.** In this section we study the properties of the cutting poset $(W, \sqsubseteq)$ of Theorem 3.19 for the cutting relation $\sqsubseteq$ introduced in Definition 3.5 (see also Figure 2). The following theorem summarizes the results.

Theorem 3.41. *$(W, \sqsubseteq)$ is a distributive meet-semilattice with 1 as minimal element, and a subposet of both left and right order.*

Every interval of $(W, \sqsubseteq)$ is a distributive sublattice and a sublattice of both left and right order. The $\sqsubseteq$-lower covers of an element w correspond to the nontrivial blocks of w which are minimal for inclusion. The meet-semilattice L_w they generate (equivalently for $\wedge_L$, $\wedge_R$, or $\wedge_\sqsubseteq$) is free, and is in correspondence with the lattice of unions of these minimal nontrivial blocks.

The Möbius function is given by $\mu(u, w) = \pm 1$ if u is in L_w (with alternating sign according to the usual rule for the Boolean lattice), and 0 otherwise.

This Möbius function is used in Section 6.4 to compute the size of the simple modules of M.

Since $(W, \sqsubseteq)$ is almost a distributive lattice, Birkhoff's representation theorem suggests to embed it in the distributive lattice $O(I((W, \sqsubseteq)))$ of the lower sets of its join-irreducible elements.

Problem 3.42. *Describe the set $I(W, \sqsubseteq)$ of the join-irreducible elements of $(W, \sqsubseteq)$.*

Figure 2 seems to suggest that the join-irreducible elements of $(W, \sqsubseteq)$ form a tree, but this already fails for $n = 5$. We now briefly comment on the simplest join-irreducible elements, namely the immediate successors w of 1 in the cutting poset. Equivalent statements are that w admits exactly two reduced blocks $\{\}$ and B, possibly with $B = I$, or that the simple module S_w is of dimension $|[1, w]_R| - 1$. For a Coxeter group W, we denote by $S(W)$ the set of elements $w \neq 1$ having no proper reduced blocks, and $T(W)$ those having exactly two reduced blocks. Note that $T(W)$ is the disjoint union of the $S(W_J)$ for $J \subseteq I$.

Example 3.43. In type A, a permutation $w \in S(\mathfrak{S}_n)$ is uniquely obtained by taking a simple permutation, and inflating each 1 of its permutation matrix by an identity matrix. An element of $T(\mathfrak{S}_n)$ has a block diagonal matrix with one block in $S(\mathfrak{S}_m)$ for $m \leq n$, and $n - m$ 1×1 blocks. This gives an easy way to construct the generating series for $S(\mathfrak{S}_n)_{n\in\mathbb{N}}$ and for $T(\mathfrak{S}_n)_{n\in\mathbb{N}}$ from that of the simple permutations given in [AA05].

We now turn to the proof of Theorem 3.41, starting with some preliminary results.

Lemma 3.44. *$(W, \sqsubseteq)$ is a partial join-semilattice. Namely, when the join exists, it is unique and given by the join in left and in right order:*

$$v \vee_{\sqsubseteq} v' = v \vee_L v' = v \vee_R v' .$$

Proof. Take v and v' with at least one common successor. Applying Corollary 3.27 to the interval $[1, w]_{\sqsubseteq}$ for any such common successor w, one obtains $v, v' \sqsubseteq v \vee_R v' = v \vee_L v' \sqsubseteq w$. Therefore, $v \vee_R v' = v \vee_L v'$ is the join of v and v' in the cutting order. □

Lemma 3.45. *$(W, \sqsubseteq)$ is a meet-semilattice. Namely, for $v, v' \in W$*

$$v \wedge_{\sqsubseteq} v' = \bigvee_{u \sqsubseteq v, v'} u ,$$

where $\bigvee$ is the join for the cutting order (or equivalently for left or right order). If further v and v' have a common successor, then

$$v \wedge_{\sqsubseteq} v' = v \wedge_R v' = v \wedge_L v' .$$

Proof. The first part follows from a general result. Namely, for any poset, the following statements are equivalent (see e.g. [Pou10, Proposition 6.3]):

(i) Any bounded nonempty part has an upper bound.
(ii) Any bounded nonempty part has a lower bound.

Here we reprove this fact for the sake of self-containment. Take u and u' two common cutting points for v and v'. Then, using Lemma 3.44, their join exists and $u \vee_{\sqsubseteq} u' = u \vee_R u' = u \vee_L u'$ is also a cutting point for v and v'. The first statement follows by repeated iteration over all common cutting points.

Now assume that v and v' have a common successor w. Then applying Corollary 3.27 to the interval $[1, w]_{\sqsubseteq}$, we find that $v \wedge_R v' = v \wedge_L v'$ is the meet of v and v' in the cutting order. □

Proof of Theorem 3.41. $(W, \sqsubseteq)$ is a distributive meet-semilattice by Lemma 3.45, together with Corollary 3.29 for the distributivity. The argument is in fact general: any poset with a minimal element 1 such that all intervals $[1, x]$ are distributive lattices and such that any two elements admit either a join or no common successor

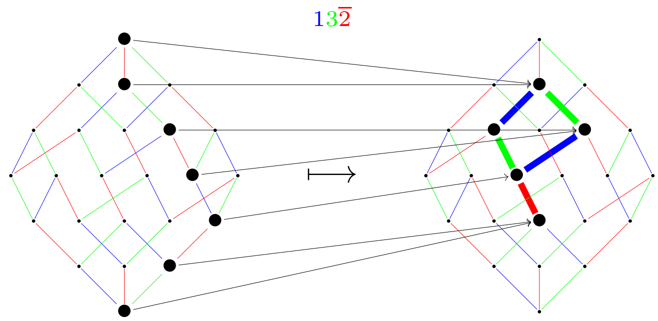


Figure 3. A partial picture of the graph of the element $f := \pi_1\pi_3\overline{\pi}_2$ of the monoid $M(\mathfrak{S}_4)$. On both sides, the underlying poset is left order of $\mathfrak{S}_4$ (with 1 at the bottom, and the same color code as in Figure 1); on the right, the bold dots depict the image set of f. The arrows from the left to the right describe the image of each point along some chain from 1 to w_0.

is a distributive meet-semilattice (see [Ede86] for literature on such). The end of the first statement is Theorem 3.19.

The first statement about the interval $[1, w]_\sqsubseteq$ is Corollary 3.27. The top part of this interval $[1, w]_\sqsubseteq$ is further described in Corollary 3.29. The value of $\mu(u, w)$ depends only on this interval, and we conclude the remaining statements using Rota's Crosscut Theorem [Rot64] on Möbius functions for lattices (see also [BS97, Theorem 1.3]). □

Problem 3.46. *Blocks are join-irreducible if there is only one minimal nontrivial block below it. Determine the distributive lattice associated with the cutting poset via Birkhoff's theory from the join-irreducibles.*

4. The combinatorics of $M(W)$

In this section we study the combinatorics of the biHecke monoid $M(W)$ of a finite Coxeter group W. In particular, we prove in Sections 4.1 and 4.2 that its elements preserve left order and Bruhat order, and derive in Section 4.3 properties of their image sets and fibers. In Sections 4.4 and 4.5, we prove the key combinatorial ingredients for the enumeration of the simple modules of $M(W)$ in Section 7: $M(W)$ is aperiodic and admits $|W|$ conjugacy classes of idempotents. Finally, in Section 4.6 we study the Green relations as introduced in Section 2.5 and involutions on $M(W)$ in Section 4.7.

4.1. Preservation of left order.

Recall that $M(W)$ is defined by its right action on elements in W by (1.5) and (1.6). The following key proposition, illustrated in Figure 3, states that it therefore preserves properties on the left.

Proposition 4.1. *Take $f \in M(W)$, $w \in W$, and $j \in I$. Then, $(s_j w).f$ is either $w.f$ or $s_j(w.f)$.*

The proof of Proposition 4.1 is a consequence of the associativity of the 0-Hecke monoid and relies on the following lemma, which is a nice algebraic (partial) formulation of the Exchange Property [BB05, Section 1.5].

Lemma 4.2. *Let $w \in W$ and $i, j \in I$ such that $j \notin \mathrm{D}_L(w)$. Then*

$$(s_j w).\pi_i = \begin{cases} w.\pi_i & \text{if } j \in \mathrm{D}_L(w.\pi_i), \\ s_j(w.\pi_i) & \text{otherwise.} \end{cases}$$

The same result holds with π_i replaced by $\overline{\pi}_i$.

Proof. Recall that for any $w, v \in W$, $w.\pi_v = 1.(\pi_w \pi_v)$. Set $w' = w.\pi_i$. Then

$$\begin{aligned}(s_j w).\pi_i = 1.(\pi_{s_j w}\pi_i) = 1.((\pi_j \pi_w)\pi_i) = 1.(\pi_j(\pi_w \pi_i)) = 1.(\pi_j \pi_{w'}) \\ = \begin{cases} 1.\pi_{w'} = w' & \text{if } j \in \mathrm{D}_L(w'), \\ 1.\pi_{s_j w'} = s_j w' & \text{otherwise.} \end{cases}\end{aligned}$$

The result for $\overline{\pi}_i$ follows from Remark 1.1 and the fact that $w_0 s_j = s_{j'} w_0$ for some $j' \in I$ by Example 3.9 and Lemma 3.3 with $w = w_0$ and $K = \{j\}$. □

Proof of Proposition 4.1. Any element $f \in M(W)$ can be written as a product of π_i and $\overline{\pi}_i$. Lemma 4.2 describes the action of π_i and $\overline{\pi}_i$ on the Hasse diagram of left order. By applying induction, each π_i and $\overline{\pi}_i$ in the expansion of f satisfies all desired properties, and hence so does f (the statement holds trivially for the identity). □

Proposition 4.3. *For $f \in M(W)$, the following holds:*

(i) f preserves left order:

$$w \leq_L w' \quad \Rightarrow \quad w.f \leq_L w'.f \quad \text{for } w, w' \in W.$$

(ii) Take $w \leq_L w'$ in W, and consider a maximal chain

$$w.f = v_1 \xrightarrow{i_1} v_2 \xrightarrow{i_2} \cdots \xrightarrow{i_{k-1}} v_k = w'.f\,.$$

Then, there is a maximal chain:

$$\begin{aligned} w = u_{1,1} \to \cdots \to u_{1,\ell_1} \xrightarrow{i_1} u_{2,1} \to \cdots \to u_{2,\ell_2} \xrightarrow{i_2} \quad \ldots \\ \ldots \quad \xrightarrow{i_{k-1}} u_{k,1} \to \cdots \to u_{k,\ell_k} = w'\,, \end{aligned} \tag{4.1}$$

such that $u_{j,l}.f = v_j$ for all $1 \leq j \leq k$ and $1 \leq l \leq \ell_j$.

(iii) f is length contracting; namely, for $w \leq_L w'$:

$$\ell(w'.f) - \ell(w.f) \leq \ell(w') - \ell(w).$$

Furthermore, when equality holds, $(w'.f)(w.f)^{-1} = w'w^{-1}$.

(iv) Let $J = [a, b]_L$ be an interval in left order. Then the image of J under f denoted by $J.f$ has $a.f$ and $b.f$ as minimal and maximal element, respectively. Furthermore, $J.f$ is connected. If $\ell(b.f) - \ell(a.f) = \ell(b) - \ell(a)$, then $J.f$ is isomorphic to J, that is $x.f = (xa^{-1})(a.f)$ for $x \in J$.

Proof. (i) and (ii) are direct consequences of Proposition 4.1, using induction.
(iii) follows from (ii).
(iv) follows from (i), (ii), and (iii) applied to $a \leq_L x$ for all $x \in [a, b]_L$. □

4.2. **Preservation of Bruhat order.** Recall the following well-known property of Bruhat order of Coxeter groups.

Proposition 4.4 (Lifting Property [BB05, p.35])**.** *Suppose $u <_B v$ and $i \in \mathrm{D}_R(v)$ but $i \notin \mathrm{D}_R(u)$. Then, $u \leq_B vs_i$ and $us_i \leq_B v$.*

The next proposition is a consequence of the Lifting Property.

Proposition 4.5. *The elements f of $M(W)$ preserve Bruhat order. That is for $u, v \in W$*

$$u \leq_B v \quad \Longrightarrow \quad u.f \leq_B v.f.$$

Proof. It suffices to show the property for π_i and $\overline{\pi}_i$ since they generate $M(W)$. For these, the claim of the proposition is trivial if i is a right descent of u, or i is not a right descent of v. Otherwise, we can apply the Lifting Property:

$$\begin{aligned} u.\pi_i = us_i \leq_B v = v.\pi_i, \\ u.\overline{\pi}_i = u \leq_B vs_i = v.\overline{\pi}_i. \end{aligned}$$ □

Remark 4.6. *By Lemma 2.3, the preimage of a point is a convex set, but need not be an interval. For example, the preimage of $s_1s_3 \in \mathfrak{S}_4$ (or 2143 in one-line notation) of $f = \overline{\pi}_1\pi_2\pi_1\pi_3\overline{\pi}_2\overline{\pi}_3\overline{\pi}_1\overline{\pi}_2$ is*

$$\{2413, 2341, 4213, 3412, 3241, 2431, 4312, 4231, 3421, 4321\},$$

which in Bruhat order has two maximal elements 2413 and 2341 and hence is not an interval.

The next result is a corollary of Proposition 4.3.

Corollary 4.7. *Let $f \in M$.*

(i) If $1.f = 1$, then f is regressive for Bruhat order: $w.f \leq_B w$ for all $w \in W$.
(ii) If $w_0.f = w_0$, then f is extensive for Bruhat order: $w.f \geq_B w$ for all $w \in W$.

Proof. First suppose that $1.f = 1$. Let $w.f = s_{i_k} \cdots s_{i_1}$ be a reduced decomposition of $w.f$. This defines a maximal chain

$$1.f = 1 = v_0 \stackrel{i_1}{\to} \cdots \stackrel{i_{k-2}}{\to} v_{k-2} \stackrel{i_{k-1}}{\to} v_{k-1} \stackrel{i_k}{\to} v_k = w.f$$

in left order. By Proposition 4.3 (ii) there is a larger chain from 1 to w so that there is a reduced word for w which contains $s_{i_k} \cdots s_{i_1}$ as a subword. Hence by the subword property of Bruhat order $w.f \leq_B w$. This proves (i).

Now let $w_0.f = w_0$. By similar arguments as above, constructing a maximal chain from $w.f$ to $w_0.f$ in left order, one finds that $w_0(w.f)^{-1} \leq_B w_0w^{-1}$. By [BB05, Proposition 2.3.4], the map $v \mapsto w_0v$ is a Bruhat antiautomorphism and by the subword property $v \mapsto v^{-1}$ is a Bruhat automorphism. This implies $w \leq_B w.f$ as desired for (ii). □

4.3. **Fibers and image sets.** Viewing elements of the biHecke monoid $M(W)$ as functions on W, we now study properties of their fibers and image sets.

Proposition 4.8.

(i) The image set $\mathrm{im}(f)$ for any $f \in M(W)$ is connected (see Definition 2.1) with a unique minimal and maximal element in left order.
(ii) The image set of an idempotent in $M(W)$ is an interval in left order.

Proof. The first statement follows immediately from Proposition 4.3 (iv) with $J = [1, w_0]_L$.

For the second statement, let $e \in M(W)$ be an idempotent with image set $\mathrm{im}(e)$. By Proposition 4.3 (iv) with $J = [1, w_0]_L$, we have that $1.e$ (resp. $w_0.e$) is the minimal (resp. maximal) element of $\mathrm{im}(e)$. Then by Proposition 4.3 (ii), for every maximal chain in left order between $1.e$ and $w_0.e$, there is a maximal chain in left order of preimage points. Since e is an idempotent, there must be such a chain which contains the original chain. Hence all chains in left order between $1.e$ and $w_0.e$ are in $\mathrm{im}(e)$, proving that $\mathrm{im}(e)$ is an interval. □

Note that the above proof, in particular Proposition 4.3 (ii), heavily uses the fact that the edges in left order are colored.

Definition 4.9. *For any $f \in M(W)$, we call the set of fibers of f, denoted by* $\mathrm{fibers}(f)$*, the (unordered) set-partition of W associated by the equivalence relation $w \equiv w'$ if $w.f = w'.f$.*

Proposition 4.10. *Take $f \in M(W)$, and consider the Hasse diagram of left order contracted with respect to the fibers of f. Then, this graph is isomorphic to left order restricted on the image set.*

Proof. See Appendix A on colored graphs. □

Proposition 4.11. *Any element $f \in M(W)$ is characterized by its set of fibers and $1.f$.*

Proof. Fix a choice of fibers. Contract the left order with respect to the fibers. By Proposition 4.10 this graph has to be isomorphic to the left order on the image set.

Once the lowest element in the image set $1.f$ is fixed, this isomorphism is forced, since by Proposition 4.8 (i) the graphs are (weakly) connected, have a unique minimal element, and there is at most one arrow of a given color leaving each node. □

Proposition 4.11 makes it possible to visualize nontrivial elements of the monoid (see Figure 4).

Recall that a set-partition $\Lambda = \{\Lambda_i\}$ is said to be finer than the set-partition $\Lambda' = \{\Lambda'_i\}$ if for all i there exists a j such that $\Lambda_i \subseteq \Lambda'_j$. This is denoted by $\Lambda \preceq \Lambda'$. The refinement relation is a partial order.

For $f \in M$, define the *type* of f by

$$\mathrm{type}(f) := \mathrm{type}([1.f, w_0.f]_L) = (w_0.f)(1.f)^{-1}\,. \tag{4.2}$$

The *rank* of $f \in M$ is the cardinality of the image set $\mathrm{im}(f)$.

Lemma 4.12. *Fix $f \in M$. For $h = fg \in fM$, one has:*

(1) $\mathrm{fibers}(f) \preceq \mathrm{fibers}(h)$
(2) $\mathrm{type}(h) \leq_B \mathrm{type}(f)$
(3) $\mathrm{rank}(h) \leq \mathrm{rank}(f)$.

Furthermore, the following are equivalent

(i) $\mathrm{fibers}(h) = \mathrm{fibers}(f)$
(ii) $\mathrm{rank}(h) = \mathrm{rank}(f)$
(iii) $\mathrm{type}(h) = \mathrm{type}(f)$
(iv) $\ell(w_0.h) - \ell(1.h) = \ell(w_0.f) - \ell(1.f)$.

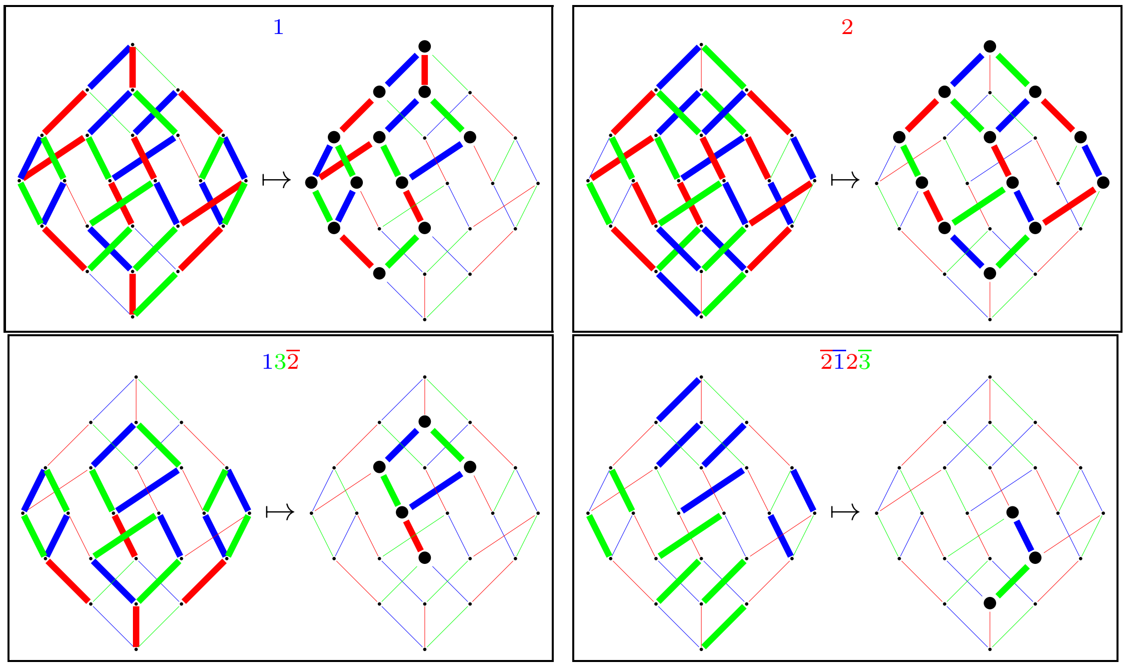


FIGURE 4. The elements $f = \pi_1$, π_2, $\pi_1\pi_3\overline{\pi}_2$ and $\overline{\pi}_2\overline{\pi}_1\pi_2\overline{\pi}_3$ of $M(\mathfrak{S}_4)$. As in Figure 3, the underlying poset on both sides is left order on $\mathfrak{S}_4$, and the bold dots on the right sides depict the image set of f. On the left side, an edge between two elements of W is thick if they are not in the same fiber. This information completely describes f; indeed $u = 1$ on the left is mapped to the lowest element of the image set on the right; each time one moves u up along a thick edge on the left, its image $u.f$ is moved up along the edge of the same color on the right.

If any, and therefore all, of the above hold, then h is completely determined (within fM) by $1.h$.

Proof. For $f, g \in M$, the statement $\operatorname{fibers}(f) \preceq \operatorname{fibers}(fg)$ is obvious.

By Proposition 4.3 (iii) and (iv), we know that for $f, g \in M$ either $\operatorname{type}(fg) = \operatorname{type}(f)$, or $\ell(w_0.(fg)) - \ell(1.(fg)) < \ell(w_0.f) - \ell(1.f)$. In the latter case by Proposition 4.5 $\operatorname{type}(fg) <_B \operatorname{type}(f)$. The second case occurs precisely when $\operatorname{fibers}(f)$ is strictly finer than $\operatorname{fibers}(fg)$, or equivalently $\operatorname{rank}(fg) < \operatorname{rank}(f)$.

The last statement, that if $\operatorname{fibers}(h) = \operatorname{fibers}(f)$ then h is determined by $1.h$, follows from Proposition 4.11. $\square$

4.4. **Aperiodicity.** Recall from Section 2.5 that a monoid M is called *aperiodic* if for any $f \in M$, there exists $k > 0$ such that $f^{k+1} = f^k$. Note that, in this case, $f^\omega := f^k = f^{k+1} = \dots$ is an idempotent.

Proposition 4.13. *The biHecke monoid $M(W)$ is* aperiodic.

Proof. From Proposition 4.3 (iv), we know that $\operatorname{im}(f^k)$ has a minimal element $a_k = 1.f^k$ and a maximal element $b_k = w_0.f^k$ in left order. Since $\operatorname{im}(f^{k+1}) \subseteq \operatorname{im}(f^k)$, we have $a_{k+1} \geq_L a_k$ and $b_{k+1} \leq_L b_k$. Therefore, both sequences a_k and b_k must ultimately be constant.

This implies that, for N big enough, a_N and b_N are fixed points. Applying Proposition 4.3 (iii) yields that all elements in $[a_N, b_N]_L$ are fixed points under f. It follows successively that $\mathrm{im}(f^N) = [a_N, b_N]_L$, $f^N = f^{N+1} = \cdots$, and $\mathrm{fix}(f) = [a_N, b_N]_L$. □

Corollary 4.14. *The set of fixed points of an element $f \in M(W)$ is an interval in left order.*

Proof. The set of fixed point of f is the image set of f^ω, which by Proposition 4.8 (ii) is an interval in left order. □

4.5. **Idempotents.** We now study properties of idempotents in $M(W)$.

Proposition 4.15.

(i) *For $w \in W$*

$$e_w := \pi_{w^{-1}w_0} \overline{\pi}_{w_0 w}$$

is the unique idempotent such that $1.e_w = 1$ and $w_0.e_w = w$. Its image set is $[1, w]_L$, and it satisfies:

$$u.e_w = \max_{\leq_B} \big([1, u]_B \cap [1, w]_L \big) \ .$$

(ii) *Similarly, for $w \in W$,*

$$\tilde{e}_w := \overline{\pi}_{w^{-1}} \pi_w$$

is the unique idempotent with image set $[w, w_0]_L$, and it satisfies a dual formula.

(iii) *Furthermore,*

$$e_{a,b} := \overline{\pi}_{a^{-1}} e_{ba^{-1}} \pi_a$$

is an idempotent with image set $[a, b]_L$.

Proof. (i): Clearly, the image of e_w is a subset of $[1, w]_L$. Applying Remark 2.7 shows that $[1, w]_L$ is successively mapped bijectively to $[w^{-1}w_0, w_0]_L$ and back to $[1, w]_L$. So e_w is an idempotent with image set $[1, w]_L$. Reciprocally, let f be an idempotent such that $1.f = 1$ and $w_0.f = w$. Then, by Proposition 4.5 f preserves Bruhat order and by Corollary 4.7 (i) $u.f \leq_B u$ for all $u \in W$. Furthermore, by Proposition 4.8, the image set of f is the interval $[1, w]_L$. Using Proposition 2.4, uniqueness and the given formula follow.

Statement (ii) is dual to (i) and is proved similarly.

(iii): The image set of $e_{ba^{-1}}$ is $[1, ba^{-1}]_L$; hence the image set of $e_{a,b}$ is a subset of $[a, b]_L$. We conclude by checking that $[a, b]_L$ is mapped bijectively at each step $\overline{\pi}_{a^{-1}}$, $e_{ba^{-1}}$ and π_a (see also Remark 2.7), and therefore consists of fixed points. □

Remark 4.16. *For $f \in M$, $fe_v = fe_{u.e_v}$, where $u = w_0.f$.*

Proof. Use the formula of Proposition 4.15 (i). □

Corollary 4.17. *For $u, w \in W$, the intersection $[1, u]_B \cap [1, w]_L$ is a $\leq_L$-lower set with a unique maximal element v in Bruhat order. The maximum is given by $v = u.e_w$.*

4.6. **Green relations.** We have now gathered enough information about the combinatorics of $M(W)$ to give a partial description of its Green relations which will be used in the study of the representation theory of $M(W)$. As an example, Figure 5 completely describes the Green relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{H}$ for $M(\mathfrak{S}_3)$. In the sequel, we describe $\mathcal{R}$-classes for general elements, as well as $\mathcal{J}$-order on regular elements. In particular, we obtain that the conjugacy classes of idempotents are indexed by the elements of W, and that $\mathcal{J}$-order on regular classes is given by left-right order $<_{LR}$ on W. Note that the latter is *not* a lattice, unlike for the variety $\mathcal{DA}$ (which consists of all aperiodic monoids all of whose simple modules are dimension 1; see e.g. [GMS09]).

Proposition 4.18. *Two elements $f, g \in M(W)$ are in the same $\mathcal{R}$-class if and only if they have the same fibers. In particular, the $\mathcal{R}$-class of f is given by:*

$$\mathcal{R}(f) = \{h \in fM \mid \operatorname{rank}(h) = \operatorname{rank}(f)\} = \{f_u \mid u \in [1, \operatorname{type}(f)^{-1} w_0]_R\}\,, \tag{4.3}$$

where f_u is the unique element of M such that $\operatorname{fibers}(f_u) = \operatorname{fibers}(f)$ *and* $1.f_u = u$.

Proof. It is a general easy fact about monoids of functions that elements in the same $\mathcal{R}$-class have the same fibers (see also Lemma 4.12). Reciprocally, if g has the same fibers as f, then one can use Remark 2.7 to define $g' = g\overline{\pi}_{(1.g)^{-1}}\pi_{1.f}$ such that $\operatorname{fibers}(g') = \operatorname{fibers}(f)$ and $1.g' = 1.f$. Also by Proposition 4.11, $f = g' \in gM$, and similarly, $g \in fM$.

Equation (4.3) follows using Lemma 4.12 and Remark 2.7. □

Lemma 4.19. *Let e and f be idempotents of M with respective image sets $[a, b]_L$ and $[c, d]_L$. Then, $f \leq_{\mathcal{J}} e$ if and only if $dc^{-1} \leq_{LR} ba^{-1}$.*

In particular, two idempotents e and f are conjugates if and only if the intervals $[a, b]_L$ and $[c, d]_L$ are of the same type: $dc^{-1} = ba^{-1}$.

The above properties extend to any two regular elements (elements whose $\mathcal{J}$-class contains an idempotent).

Proof. First note that an interval $[c, d]_L$ is isomorphic to a subinterval of $[a, b]_L$ if and only $dc^{-1} \leq_{LR} ba^{-1}$. This follows from Proposition 2.5 and the fact that $[c, d]_L$ is a subinterval of $[a, b]_L$ if and only if $[ca^{-1}, da^{-1}]_L$ is a subinterval of $[1, ba^{-1}]_L$. But then dc^{-1} is a subfactor of ba^{-1}.

Assume first that $dc^{-1} \leq_{LR} ba^{-1}$, and let $[c', d']_L$ be a subinterval of $[a, b]_L$ isomorphic to $[c, d]_L$. Using Proposition 2.5, take $u, v \in M$ which induce reciprocal bijections between $[c, d]_L$ and $[c', d']_L$. Then, $f = fuev$ is a conjugate of e.

Reciprocally, assume that $f = uev$ with $u, v \in M$. Without loss of generality, we may assume that $u = ue$ so that $\operatorname{im}(u) \subseteq [a, b]_L$. Set $c' = c.u$ and $d' = d.u$. Since $f = ff = fuv$, and using Proposition 4.3, the functions u and v must induce reciprocal isomorphisms between $[c, d]_L$ and $[c', d']_L$, the latter being a subinterval of $[a, b]_L$. Therefore, $dc^{-1} \leq_{LR} ba^{-1}$.

To conclude, note that a regular element has the same type as any idempotent in its $\mathcal{J}$-class. □

Corollary 4.20. *The idempotents $(e_w)_{w \in W}$ form a complete set of representatives of regular $\mathcal{J}$-classes in M.*

Example 4.21. For $w \in W$, the idempotents e_w and $\tilde{e}_{w^{-1}w_0}$ are in the same $\mathcal{J}$-class. This follows immediately from Lemma 4.19, or by direct computation using

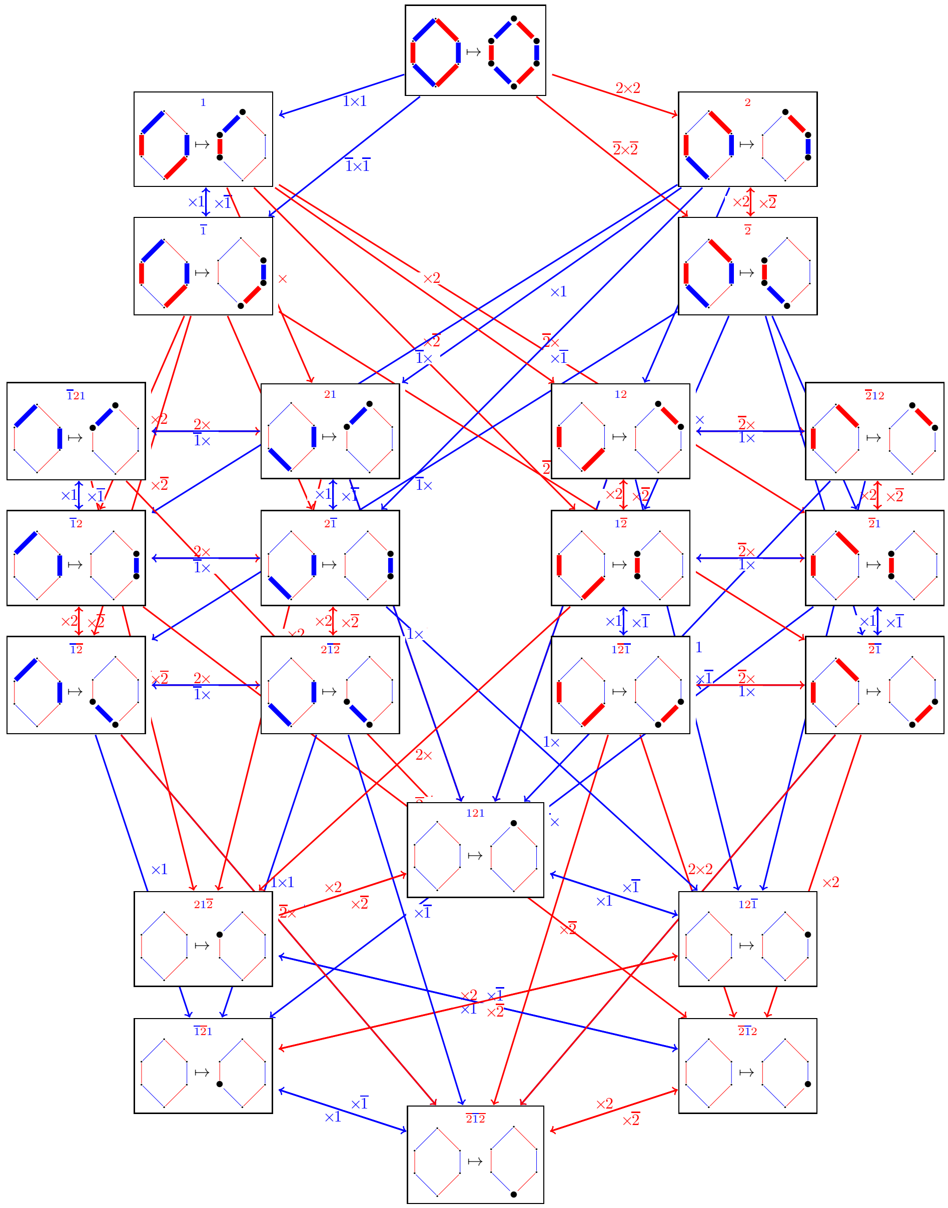


FIGURE 5. The graph of $\mathcal{J}$-order for $M(\mathfrak{S}_3)$. The vertices are the 19 elements of $M(\mathfrak{S}_3)$, each drawn as in Figure 4. The edges give both the left and right Cayley graph of $M(\mathfrak{S}_3)$; for example, there is an arrow $f \xrightarrow{\times\pi_1} g$ if $g = f\pi_1$, and an arrow $f \xrightarrow{\pi_1\times\pi_1} g$ if $g = f\pi_1 = \pi_1 f$. The picture also highlights the $\mathcal{J}$-classes of $M(\mathfrak{S}_3)$, and the corresponding eggbox pictures (i.e. the decomposition of the $\mathcal{J}$-classes into $\mathcal{L}$ and $\mathcal{R}$-classes); namely, from top to bottom, there is one $\mathcal{J}$-class of size $1 = 1\times 1$, two $\mathcal{J}$-classes of size $2 = 1\times 2$, two $\mathcal{J}$-classes of size $6 = 2\times 3$, and one $\mathcal{J}$-class of size $6 = 1\times 6$, where $n\times m$ gives the dimension of the eggbox picture. In other words the $\mathcal{J}$-class splits into n $\mathcal{R}$-classes of size m and also into m $\mathcal{L}$-classes of size n. This example is specific in that all $\mathcal{J}$-classes are regular.

the explicit expressions for e_w and $\tilde{e}_{w^{-1}w_0}$ in Proposition 4.15:

$$e_w = e_w^2 = \pi_{w^{-1}w_0}\overline{\pi}_{w_0w}\pi_{w^{-1}w_0}\overline{\pi}_{w_0w} = \pi_{w^{-1}w_0}\tilde{e}_{w^{-1}w_0}\overline{\pi}_{w_0w}\,,$$
$$\tilde{e}_{w^{-1}w_0} = \tilde{e}^2_{w^{-1}w_0} = \overline{\pi}_{w_0w}\pi_{w^{-1}w_0}\overline{\pi}_{w_0w}\pi_{w^{-1}w_0} = \overline{\pi}_{w_0w}e_w\pi_{w^{-1}w_0}\,.$$

Corollary 4.22. *The image of a regular element is an interval in left order.*

Proof. A regular element has the same type, and same size of image set as any idempotent in its $\mathcal{J}$-class. □

Remark 4.23. *The reciprocal is false: In type B_3, the element $\overline{\pi}_1\overline{\pi}_3\overline{\pi}_2\pi_1\overline{\pi}_3\overline{\pi}_2\overline{\pi}_1$ has the interval $[1, s_2s_3s_2]_L$ as image set, but it is not regular. The same holds in type A_4 with the element $\pi_2\pi_1\overline{\pi}_4\pi_3\overline{\pi}_2\overline{\pi}_1\overline{\pi}_3\pi_4\overline{\pi}_2\overline{\pi}_3\overline{\pi}_4$.*

Problem 4.24. *Describe $\mathcal{L}$-classes in general, and $\mathcal{L}$-order, $\mathcal{R}$-order, as well as $\mathcal{J}$-order on nonregular elements.*

4.7. **Involutions and consequences.** Define an involution $*$ on W by

$$w \mapsto w^* := w_0w,$$

where w_0 is the maximal element of W. Moreover, define the *bar* map $- : M(W) \to M(W)$ by requiring that for a given $f \in M(W)$

$$w.\overline{f} := (w^*.f)^* \qquad \text{for all } w \in W.$$

Proposition 4.25. *The bar involution is a monoid endomorphism of $M(W)$ which exchanges π_i and $\overline{\pi}_i$.*

Proof. For any $w \in W$ and $f, g \in M(W)$, one has

$$w.\overline{fg} = (w^*.f.g)^* = (((w^*.f)^*)^*.g)^*$$

because $w \mapsto w^*$ is an involution. Therefore, $w.\overline{fg} = w.\overline{f}.\overline{g}$, so that bar is a monoid endomorphism from the monoid of functions from W to itself. Moreover, it is easy to see that bar exchanges π_i and $\overline{\pi}_i$, so that it fixes $M(W)$. □

The previous proposition has some interesting consequences when applied to idempotents: For any $w \in W$, the bar involution is a bijection from e_wM to $\overline{e}_wM$. But $\overline{e}_w$ fixes w_0 and sends $1 = w_0^*$ to w^*, so that $\overline{e}_w = e_{w^*,w_0} = \tilde{e}_{w_0w}$. The latter is in turn conjugate to $e_{w_0w^{-1}w_0}$ by Example 4.21. This implies the following result.

Corollary 4.26. *The ideals e_wM and $e_{w_0w^{-1}w_0}M$ are in bijection.*

5. The Borel submonoid $M_1(W)$ and its representation theory

In the previous section, we outlined the importance of the idempotents $(e_w)_{w\in W}$. A crucial feature is that they live in a "Borel" submonoid $M_1 \subseteq M$ of elements of the biHecke monoid which fix the identity:

$$M_1 = \{f \in M \mid 1.f = 1\}\,.$$

In this section we study this monoid and its representation theory, as an intermediate step toward the representation theory of M (see Section 6). For this application, it is actually more convenient to work with the submonoid fixing w_0 instead of 1:

$$M_{w_0} = \{f \in M \mid w_0.f = w_0\}\,.$$

Since, both monoids are isomorphic under the involution of Section 4.7, and since the interaction of M_{w_0} with Bruhat order is notationally simpler, we focus on M_1 in this section.

From the definition it is clear that M_1 is indeed a submonoid which contains the idempotents $(e_w)_{w\in W}$. Furthermore, by Proposition 4.5 and Corollary 4.7 its elements are both order-preserving and regressive for Bruhat order. As a consequence, the products of M_1 enjoy some triangularity property:

Corollary 5.1. *For $f, g \in M_1$, define the relation $f \le g$ if $w.f \le_B w.g$ for all $w \in W$. Then $\le$ defines a partial order on M_1 such that $fg \le f$ and $fg \le g$ for all $f, g \in M_1$.*

Remark 5.2. *For $w \in W$, $w.M_1$ is the interval $[1, w]_B$ in Bruhat order.*

Proof. By Corollary 4.7, for $f \in M_1$, we have $w.f \le_B w$. Take reciprocally $v \in [1, w]_B$. Then, using Proposition 4.15, $w.e_v = v$. □

By Proposition 2.10, a monoid with such an order is $\mathcal{J}$-trivial [Pin10], and the description of its representation theory is the topic of a separate paper [DHST10].

5.1. **Representation theory.** In this subsection, we summarize the main results of [DHST10] in the setting of the Borel submonoid M_1 (see this reference for proofs). The results apply without modification to M_{w_0}.

For each $w \in W$ define S^1_w (written $S^{w_0}_w$ for M_{w_0}) to be the one-dimensional vector space with basis $\{\epsilon_w\}$ together with the right operation of any $f \in M_1$ given by

$$\epsilon_w.f := \begin{cases} \epsilon_w & \text{if } w.f = w, \\ 0 & \text{otherwise.} \end{cases}$$

The basic features of the representation theory of M_1 can be stated as follows:

Theorem 5.3. *The radical of $\mathbb{K}[M_1]$ is the ideal with basis $(f^\omega - f)_f$ for $f \in M_1$ non-idempotent. The quotient of $\mathbb{K}[M_1]$ by its radical is commutative. Therefore, all simple M_1-modules are one-dimensional. In fact, the family $\{S^1_w\}_{w\in W}$ forms a complete system of representatives of the simple M_1-modules.*

Recall from Section 2.5 that $\mathcal{J}$-order is the partial order $\le_{\mathcal{J}}$ defined by $f \le_{\mathcal{J}} g$ if there exists $x, y \in M_1$ such that $f = xgy$. The restriction of the $\mathcal{J}$-order to idempotents has a very simple description:

Proposition 5.4. *For $u, v \in W$, the following are equivalent:*

- $e_u e_v = e_u$
- $u \le_L v$
- $e_v e_u = e_u$
- $e_u \le_{\mathcal{J}} e_v$.

Moreover, $(e_u e_v)^\omega = e_{u\wedge_L v}$, where $u \wedge_L v$ is the meet (or greatest lower bound) of u and v in left order.

As a consequence the following definition makes sense.

Definition 5.5. *For any element $x \in M_1$, define*

$$\operatorname{lfix}(x) := \min_{\le_L}\{u \in W \mid e_u x = x\} \quad \textit{and} \quad \operatorname{rfix}(x) := \min_{\le_L}\{u \in W \mid x e_u = x\} = w_0.f$$

the min *being taken for the left order.*

Then, the projective modules and Cartan invariants can be described as follows:

Theorem 5.6. *There is an explicit basis $(b_x)_{x\in M_1}$ of $\mathbb{K}[M_1]$ such that, for all $w \in W$,*

- *the family $\{b_x \mid x \in M_1 \text{ with } \mathrm{lfix}(x) = w\}$ is a basis for the right projective module P^1_w associated to S^1_w;*
- *the family $\{b_x \mid \mathrm{rfix}(x) = w\}_{x\in M_1}$ is a basis for the left projective module associated to S^1_w.*

Moreover, the Cartan invariant of $\mathbb{K}[M_1]$ defined by $c_{u,v} := \dim(e_u\mathbb{K}[M_1]e_v)$ for $u, v \in W$ is given by $c_{u,v} = |C_{u,v}|$, where

$$C_{u,v} := \{f \in M_1 \mid \mathrm{lfix}(f) = u \text{ and } \mathrm{rfix}(f) = v\}\,.$$

Remark 5.7. *In terms of characters, the previous theorem can be restated as*

$$[P^1_u] = \sum_{f\in M_1,\mathrm{lfix}(f)=u} [S^1_{w_0.f}]\,, \tag{5.1}$$

which gives the following character for the right regular representation

$$[\mathbb{K}.M_1] = \sum_{f\in M_1} [S^1_{w_0.f}]\,. \tag{5.2}$$

Problem 5.8. *Specialize the generic description of the Cartan matrix and projective modules (and of the quiver, see* [DHST10]*) to M_1, if at all possible to express them in terms of the combinatorics of the Coxeter group W.*

Recall that the 0-Hecke monoid $H_0(W)$ is a submonoid of $M_{w_0}(W)$. As a consequence any $M_{w_0}(W)$-module is a $H_0(W)$-module and one can consider the decomposition map $G_0(M_{w_0}(W)) \to G_0(H_0(W))$. It is given by the following formula:

Proposition 5.9. *For $w \in W$, let $S^{w_0}_w$ be the simple $M_{w_0}(W)$-module defined by*

$$\epsilon_w.f := \begin{cases} \epsilon_w & \text{if } w.f = w,\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, for $J \subseteq I$, let $S^{H_0}_J$ be the simple $H_0(W)$-module defined by

$$\mu_J.\pi_i := \begin{cases} \mu_I & \text{if } i \in J,\\ 0 & \text{otherwise.}\end{cases}$$

Then, the restriction of $S^{w_0}_w$ to $H_0(W)$ is isomorphic to $S^{H_0}_{Des(w)}$. The decomposition map $G_0(M_{w_0}(W)) \to G_0(H_0(W))$ is therefore given by $[S^{w_0}_w] \mapsto [S^{H_0}_{Des(w)}]$.

Proof. By definition of the action, $w.\pi_i = w$ if and only if $i \in \mathrm{D}_R(W)$. □

Problem 5.10. *The monoids $M_1(\mathfrak{S}_n)$, for $n \in \mathbb{N}$, form a tower of monoids with a natural embedding $M_1(\mathfrak{S}_n) \times M_1(\mathfrak{S}_m) \hookrightarrow M_1(\mathfrak{S}_{m+n})$. As outlined in the introduction, it would hence be interesting to understand the induction and restriction functors in this setting, and in particular to describe the bialgebra obtained from the associated Grothendieck groups. This would give a representation theoretic interpretation of some bases of* FQSym.

In this context, Proposition 5.9 provides an interpretation of the surjective coalgebra morphism FQSym $\twoheadrightarrow$ QSym, through the restriction along the following commutative diagram of monoid inclusions (see [DHT02] for more details):

$$\begin{tikzcd} H_0(\mathfrak{S}_n) \times H_0(\mathfrak{S}_m) \arrow[r, hook] \arrow[d, hook] & M_1(\mathfrak{S}_n) \times M_1(\mathfrak{S}_m) \arrow[d, hook] \\ H_0(\mathfrak{S}_{n+m}) \arrow[r, hook] & M_1(\mathfrak{S}_{n+m})\,. \end{tikzcd}$$

5.2. **Minimal generating set.** Interestingly, M_1 can be defined as the submonoid of M generated by the idempotents $(e_w)_{w\in W}$, and in fact the subset of those idempotents indexed by Grassmannian elements (an element $w \in W$ is *Grassmannian* if it has at most one descent).

Theorem 5.11. *M_1 has a unique minimal generating set which consists of the idempotents e_w where $w^{-1}w_0$ is right Grassmanian.*

In type A_{n-1} this minimal generating set is of size $2^n - n$ (which is the number of Grassmannian elements in this case [Man01]).

Proof. Define the length $\ell(f)$ of an element $f \in M$ as the length of a minimal expression of f as a product of the generators π_i's and $\overline{\pi}_i$'s. We now prove by induction on the length that M_1 is generated by $\{e_w \mid w \in W\}$.

Take an element $f \in M_1$ of length l. If $l = 0$ we are done. Otherwise, since $1.f = 1$, an expression of f as a product of π_i's and $\overline{\pi}_i$ contains at least one $\overline{\pi}_i$. Write $f = gh$ where $g = \pi_w\overline{\pi}_i$ for some $w \in W$ and $h \in M$ so that $\ell(w) + 1 + \ell(h) = l$.

Claim: $f = e_{w_0(ws_i)^{-1}}\pi_w h$ and $\pi_w h \in M_1$.

It follows from the claim that $\ell(\pi_w h) < l$, and hence since $\pi_w h \in M_1$ we can apply induction to conclude that M_1 is generated by $\{e_w \mid w \in W\}$.

Let us prove the claim. By minimality of l, i is not a descent of w (otherwise, we would obtain a shorter expression for f: $f = \pi_w\overline{\pi}_i h = \pi_{w'}\pi_i\overline{\pi}_i h = \pi_{w'}\overline{\pi}_i h$ where $\ell(w') < \ell(w)$). Therefore, $1.g = 1.(\pi_w\overline{\pi}_i) = w$. Since $f \in M_1$ it follows that $w.h = 1$ and therefore $\pi_w h \in M_1$. It further follows that $\overline{\pi}_{w^{-1}}\pi_w$ acts trivially on the image set $[w, w_0]_L$ of g, and therefore $f = g\overline{\pi}_{w^{-1}}\pi_w h$. Note that $g\overline{\pi}_{w^{-1}} = \pi_w\overline{\pi}_i\overline{\pi}_{w^{-1}} = \pi_w\pi_i\overline{\pi}_i\overline{\pi}_{w^{-1}} = e_{w_0(ws_i)^{-1}}$.

By Proposition 5.4, the idempotents of M_1 are generated by the meet-irreducible idempotents e_w in $\mathcal{J}$ order. Here x is meet-irreducible if and only if for any $a, b \in M_1$ such that $x = a \wedge b$ implies that $x = a$ or $x = b$. These meet-irreducible elements are indexed by meet-irreducible elements w in left order, that is those $w \in W$ having at most one left nondescent, or equivalent such that $w_0 w^{-1}$ is (right) Grassmanian.

The uniqueness of the minimal generating set is true for any $\mathcal{J}$-trivial monoid with a minimal generating set [Doy84, Theorem 2] [Doy91, Theorem 1]. □

Actually one can be much more precise:

Proposition 5.12. *Any element $f \in M_1$ can be written as a product $e_{w_1} \cdots e_{w_k}$, where:*

- *$w_1 >_B \cdots >_B w_k$ is a chain in Bruhat order such that any two consecutive terms w_i and w_{i+1} are incomparable in left order;*
- *$w_i = \mathrm{rfix}(e_{w_1} \cdots e_{w_i}) = \mathrm{lfix}(e_{w_i} \cdots e_{w_k})$.*

Proof. Start from any expression $e_{w_1} \cdots e_{w_k}$ for f. We show that if any of the conditions of the proposition are not satisfied, the expression can be reduced to a strictly smaller (in length, or in Bruhat, term by term) expression, so that induction can be applied.

- If $u \not\geq_B v$, then by Remark 4.16 $e_u e_v = e_u e_{u.e_v}$ with $u.e_v <_B v$.
- If $u <_L v$, then $e_u e_v = e_u$, and similarly on the right.
- If the left symbol e_u for $e_{w_i} \cdots e_{w_k}$ is not e_{w_i}, then $u <_L w_i$ and

$$e_{w_i} \cdots e_{w_k} = e_u e_{w_i} \cdots e_{w_k} = e_u e_{w_{i+1}} \cdots e_{w_k}.$$

Similarly on the right. □

Corollary 5.13. *The Cartan matrix of M_1 is upper-unitriangular with respect to Bruhat order. Namely, for $f \in M_1$, $\mathrm{lfix}(f) \geq_B \mathrm{rfix}(f)$, with equality if and only if f is an idempotent.*

Lemma 5.14. *If $v \leq_B u$ in Bruhat order and $u' = \mathrm{lfix}(e_u e_v)$, then*

$$v \leq_B u' \quad \textit{and} \quad u' \leq_L u.$$

Proof. By Definition 5.5, $u' \leq_L u$ since $e_u(e_u e_v) = e_u e_v$ and for M_1 the minimum is measured in left order. Also by Proposition 4.15

$$v = w_0.e_u e_v = w_0.e_{u'} e_u e_v \leq_B u'.$$ □

Lemma 5.15. *If u covers v in Bruhat order and $u' = \mathrm{lfix}(e_u e_v)$, then either $u' = u$, or $u' = v$ and $e_u e_v = e_v e_u$.*

Proof. By Lemma 5.14, we have that either $u' = u$ or $u' = v$, since u covers v in Bruhat order. When $u' = v$, we have again by Lemma 5.14 that $v \leq_L u$. Hence $e_u e_v = e_v = e_v e_u$. □

6. Translation modules and w-biHecke algebras

The main purpose of this section is to pave the ground for the construction of the simple modules S_w of the biHecke monoid $M = M(W)$ in Section 7.1.

As for any aperiodic monoid, each such simple module is associated with some regular $\mathcal{J}$-class D of the monoid, and can be constructed as a quotient of the span $\mathbb{K}.\mathcal{R}(f)$ of the $\mathcal{R}$-class of any idempotent f in D, endowed with its natural right M-module structure (see Section 2.7).

In Section 6.1, we endow the interval $[1, w]_R$ with a natural structure of a combinatorial M-module T_w, called *translation module*, and show that, for any $f \in M$, regular or not, the right M-module $\mathbb{K}.\mathcal{R}(f)$ is always isomorphic to some T_w.

The translation modules will play an ubiquitous role for the representation theory of M in Section 7: indeed T_w can be obtained by induction from the simple modules S_w of M, and the right regular representation of M admits a composition series in terms of the T_w which mimics the composition series of the right regular representation of M_{w_0} in terms of its simple modules S_w. Reciprocally T_w, and therefore the right regular representation of M, restricts naturally to M_{w_0}. Finally, T_w is closely related to the projective module P_w of M (Corollary 7.3).

By taking the quotient of $\mathbb{K}[M]$ through its representation on T_w, we obtain a w-analogue $\mathcal{H}W^{(w)}$ of the biHecke algebra $\mathcal{H}W$. This algebra turns out to be interesting in its own right, and we proceed by generalizing most of the results of [HT09] on the representation theory of $\mathcal{H}W$.

As a first step, we introduce in Section 6.2 a collection of submodules $P_J^{(w)}$ of T_w, which are analogues of the projective modules of $\mathcal{H}W$. Unlike for $\mathcal{H}W$, not any subset J of I yields such a submodule, and this is where the combinatorics of the blocks of w as introduced in Section 3 enters the game. In a second step, we derive in Section 6.3 a lower bound on the dimension of $\mathcal{H}W^{(w)}$; this requires a (fairly involved) combinatorial construction of a family of functions on $[1,w]_R$ which is triangular with respect to Bruhat order. In Section 6.4 we combine these results to derive the dimension and representation theory of $\mathcal{H}W^{(w)}$: projective and simple modules, Cartan matrix, quiver, etc (see Theorem 6.19).

6.1. **Translation modules and w-biHecke algebras.** Let us begin by defining the right class modules for a general monoid A.

Definition 6.1. *Let A be a monoid. For $a \in A$, let $\mathbb{K}\mathcal{R}_<(a) = \mathbb{K}\{b \in aA \mid b <_{\mathcal{R}} a\}$. Then the right class module is defined as*

$$\mathbb{K}\mathcal{R}(a) := \mathbb{K}aA/\mathbb{K}\mathcal{R}_<(a).$$

Definition 6.2. *Fix $f \in M$. Then*

$$\mathbb{K}\mathcal{R}(f) := \mathbb{K}fM/\mathbb{K}\mathcal{R}_<(f)$$

is called the translation module *associated with f.*

The basis of $\mathbb{K}\mathcal{R}(f)$ is the right class $\mathcal{R}(f)$ which is described in Proposition 4.18.

Proposition 6.3. *Set $w = \operatorname{type}(f)^{-1}w_0$. Then $(f_u)_{u\in[1,w]_R}$ forms a basis of $\mathbb{K}\mathcal{R}(f)$ such that:*

$$\begin{aligned}
f_u.\pi_i &= \begin{cases} f_u & \text{if } i \in \mathrm{D}_R(u) \\ f_{us_i} & \text{if } i \notin \mathrm{D}_R(u) \text{ and } us_i \in [1,w]_R \\ 0 & \text{otherwise;} \end{cases} \\
f_u.\overline{\pi}_i &= \begin{cases} f_{us_i} & \text{if } i \in \mathrm{D}_R(u) \\ f_u & \text{if } i \notin \mathrm{D}_R(u) \text{ and } us_i \in [1,w]_R \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{6.1}$$

In particular, the action of a general element $g \in M$ on the basis f_u of the translation module either annihilates f_u or agrees with the usual action of W: $f_u.g = f_{u.g}$.

Proof. By Definition 6.2 and Proposition 4.18, $(f_u)_{u\in[1,w]_R}$ form a basis of $\mathbb{K}\mathcal{R}(f)$.

The action of π_i agrees with right multiplication, except when the index v of the new f_v is no longer in $[1,w]_R$, in which case the element is annihilated. The action of $\overline{\pi}_i$ also agrees with right multiplication. However, due to the relations $\pi_i\overline{\pi}_i = \overline{\pi}_i$ and $\overline{\pi}_i\pi_i = \pi_i$, we need that $\overline{\pi}_i$ annihilates f_u if $i \notin \mathrm{D}_R(u)$ and $us_i \notin [1,w]_R$.

The last statement follows by induction writing $f \in M$ in terms of the generators π_i and $\overline{\pi}_i$ and using (6.1). □

Proposition 6.3 gives a combinatorial model for translation modules. It is clear that two functions with the same type yield isomorphic translation modules. The converse also holds:

Proposition 6.4. *For any $f, f' \in M$, the translation modules $\mathbb{K}\mathcal{R}(f)$ and $\mathbb{K}\mathcal{R}(f')$ are isomorphic if and only if $\operatorname{type}(f) = \operatorname{type}(f')$.*

Proof. By Proposition 6.3, it is clear that if $\mathrm{type}(f) = \mathrm{type}(f')$, then $\mathbb{K}\mathcal{R}(f) \cong \mathbb{K}\mathcal{R}(f')$.

Conversely, suppose $\mathrm{type}(f) \neq \mathrm{type}(f')$. Then we also have $w \neq w'$, where $w = \mathrm{type}(f)^{-1}w_0$ and $w' = \mathrm{type}(f')^{-1}w_0$. Without loss of generality, we may assume that $\ell(w) \geq \ell(w')$. Using the combinatorial model of Proposition 6.3, we then have

$$f_1.\pi_w = f_w \neq 0 \qquad \text{and} \qquad f'_1.\pi_w = 0\ ,$$

so that $\mathbb{K}\mathcal{R}(f) \not\cong \mathbb{K}\mathcal{R}(f')$. □

Note that it is not obvious from the combinatorial action of π_i and $\overline{\pi}_i$ of Proposition 6.3 that the result indeed gives a module. However, since it agrees with the right action on the quotient space as in Definition 6.2, this is true. By Proposition 6.4, we may choose a canonical representative for translation modules.

Definition 6.5. *We define $T_w := \mathbb{K}\mathcal{R}(e_{w,w_0})$ for all $w \in W$, and identify its basis with $[1,w]_R$ via $u \mapsto f_u$, where $f = e_{w,w_0}$.*

For the remainder of this section for $f \in M$ and $u \in [1,w]_R$, unless otherwise specified, $u.f$ means the action of f on u in the translation module T_w.

Definition 6.6. *The w-biHecke algebra $\mathcal{H}W^{(w)}$ is the natural quotient of $\mathbb{K}[M(W)]$ through its representation on T_w. In other words, it is the subalgebra of $\mathrm{End}(T_w)$ generated by the operators π_i and $\overline{\pi}_i$ of Proposition 6.3.*

6.2. **Left antisymmetric submodules.** By analogy with the simple reflections in the Hecke group algebra, we define for each $i \in I$ the operator $s_i := \pi_i + \overline{\pi}_i - 1$. For $u \in [1,w]_R$, the action on the translation module T_w is given by

$$u.s_i = \begin{cases} us_i & \text{if } us_i \in [1,w]_R, \\ -u & \text{otherwise.} \end{cases} \tag{6.2}$$

These operators are still involutions, but do not always satisfy the braid relations.

Example 6.7. Take W of type A_2 and $w = s_1$. The translation module T_w has two basis elements $B = (1, s_1)$ and the matrices for s_1 and s_2 on this basis are given by

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

It is not hard to check that then $s_1s_2s_1 \neq s_2s_1s_2$.

Similarly, one can define operators $\overleftarrow{s}_i$ acting on the left on the translation module T_w:

$$\overleftarrow{s}_i.u = \begin{cases} s_iu & \text{if } s_iu \in [1,w]_R, \\ -u & \text{otherwise.} \end{cases} \tag{6.3}$$

Definition 6.8. *For $J \subseteq I$, set $P_J^{(w)} := \{v \in T_w \mid \overleftarrow{s}_i.v = -v \quad \forall i \in J\}$.*

For $w = w_0$, these are the projective modules P_J of the biHecke algebra [HT09].

Proposition 6.9. *Take $w \in W$ and $J \subseteq I$. Then, the following are equivalent:*

(i) ${}^Jw$ is a cutting point of w;

(ii) $P_J^{(w)}$ is an M-submodule of T_w.

Furthermore, when any, and therefore all, of the above hold, $P_J^{(w)}$ *is isomorphic to* $T_{^J w}$*, and its basis is indexed by* $[1, {}^J w]_R$*, that is, assuming* $J \in \mathcal{J}^{(w)}$, $\{v \in [1,w]_R, J \subset J^{(w)}(v)\}$.

Proof. (ii) $\Rightarrow$ (i): Set

$$v_J^w := \sum_{u \in [1,w_J]_R} (-1)^{\ell(u)-\ell(w_J)} u.$$

Up to a scalar factor, this is the unique vector in $P_J^{(w)}$ with support contained in $[1, w_J]_R$. Then,

$$v_J^w.\pi_{^J w} = \sum_{\substack{u\in[1,w_J]_R \\ \text{s.t. } u^J w \in [1,w]_R}} (-1)^{\ell(uv)-\ell(w_J v)} u^J w,$$

$$v_J^w.\pi_v \overline{\pi}_{v^{-1}} = \sum_{\substack{u\in[1,w_J]_R \\ \text{s.t. } u^J w \in [1,w]_R}} (-1)^{\ell(u)-\ell(w_J)} u.$$

Therefore, if ${}^J w \not\leq_R w$, then $v_J^w.\pi_{^J w}\overline{\pi}_{^J w^{-1}}$ is a nonzero vector with support strictly included in $[1, w_J]_R$ and therefore not in $P_J^{(w)}$. By Proposition 3.8 this proves (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): If (i) holds, then the action of π_i (resp. $\overline{\pi}_i$) on $v_J^w.\pi_v$ either leaves it unmodified, kills it (if $vs_i = s_j v$ for some j) or maps it to $v_J^w.\pi_{vs_i}$. The vectors $(v_J^w.\pi_v)_{v\in[1,{}^J w]_R}$ form a basis of $P_J^{(w)}$ which is stable by M.

The last statement follows straightforwardly. □

It is clear from the definition that for $J_1, J_2 \subseteq I$, $P_{J_1\cup J_2}^{(w)} = P_{J_1}^{(w)} \cap P_{J_2}^{(w)}$. Since the set $\mathcal{RB}_{\mathcal{L}}(w)$ of left blocks of w is stable under union, the set of M-modules $(P_J^{(w)})_{J\in\mathcal{RB}_{\mathcal{L}}(w)}$ is stable under intersection. On the other hand, unless J_1 and J_2 are comparable, $P_{J_1\cup J_2}^{(w)}$ is a strict subspace of $P_{J_1}^{(w)} + P_{J_2}^{(w)}$. This motivates the following definition.

Definition 6.10. *For* $J \in \mathcal{J}^{(w)}$*, we define the module*

$$S_J^{(w)} := P_J^{(w)} / \sum_{J' \supsetneq J, J' \in \mathcal{RB}_{\mathcal{L}}(w)} P_{J'}^{(w)}, \tag{6.4}$$

Remark 6.11. *By the last statement of Proposition 6.9, and the triangularity of the natural basis of the modules* $P_{J'}^{(w)}$*, the basis of* $S_J^{(w)}$ *is given by*

$$[1, {}^J w]_R \setminus \bigcup_{v \sqsubset {}^J w} [1, v]_R = \{v \in [1,w]_R, J \subset J^{(w)}(v)\}. \tag{6.5}$$

6.3. **A (maximal) Bruhat-triangular family of the w-biHecke algebra.** Consider the submonoid F in $\mathcal{H}W^{(w)}$ generated by the operators π_i, $\overline{\pi}_i$, and s_i, for $i \in I$. For $f \in F$ and $u \in [1,w]_R$, we have $u.f = \pm v$ for some $v \in [1,w]_R$. For our purposes, the signs can be ignored and f be considered as a function from $[1,w]_R$ to $[1,w]_R$.

Definition 6.12. *For* $u, v \in [1,w]_R$*, a function* $f \in F$ *is called* (u,v)-triangular *(for Bruhat order) if* v *is the unique minimal element of* $\mathrm{im}(f)$ *and* u *is the unique maximal element of* $f^{-1}(v)$ *(all minimal and maximal elements in this context are with respect to Bruhat order).*

Recall the notion of maximal reduced right block $K^{(w)}(u)$ of Definition 3.38.

Proposition 6.13. *Take $u, v \in [1, w]_R$ such $K^{(w)}(u) \subseteq K^{(w)}(v)$. Then, there exists a (u, v)-triangular function $f_{u,v}$ in F.*

For example, for $w = 4312$ in $\mathfrak{S}_4$, the condition on u and v is equivalent to the existence of a path from u to v in the digraph $G^{(4312)}$ (see Figure 1 and Section 6.4).

The proof of Proposition 6.13 relies on several remarks and lemmas that are given in the sequel of this section. The construction of $f_{u,v}$ is explicit, and the triangularity derives from $f_{u,v}$ being either in M, or close enough to be bounded below by an element of M. It follows from the upcoming Theorem 6.19 that the condition on u and v is not only sufficient but also necessary.

Remark 6.14. *If f is (u, v)-triangular and g is (v, v')-triangular, then fg is (u, v')-triangular.*

Remark 6.15. *Take $x \in [1, w]_R$ and let $i \in I$. Then, $x.\overline{\pi}_i \leq_R x.s_i$.*

By repeated application, for $S \subset I$, and $i_1, \ldots, i_k \in S$, $x.\overline{\pi}_S \leq_R x.s_{i_1} \cdots s_{i_k}$, where recall that $\overline{\pi}_S$ is the longest element in the generators $\{\overline{\pi}_j \mid j \in S\}$.

Lemma 6.16. *Take $u \in [1, w]_R$, and define $f_{u,u} := e_{u,w_0} = \overline{\pi}_{u^{-1}}\pi_u$. Then:*

(i) $f_{u,u}$ *is (u, u)-triangular.*
(ii) *For $v \in [1, w]_R$, either $v.f_{u,u} = 0$ or $v.f_{u,u} \geq_B v$.*
(iii) $\operatorname{im}(f_{u,u}) = [u, w_0]_L \cap [1, w]_R$.

Proof. First consider the case $w = w_0$. Then, (ii) and (iii) hold by Lemma 4.15.

Now take any $w \in W$. By Proposition 6.3 the action of $f \in M$ on the translation module T_w either agrees with the action on W or yields 0. Hence in particular Proposition 4.5 still applies, which yields (ii). This also implies the inclusion $\operatorname{im}(f_{u,u})\backslash\{0\} \subset [u, w_0]_L \cap [1, w]_R$. The reverse inclusion is straightforward: if $u' = xu$, then $u'.f_{u,u} = xu.\overline{\pi}_{u^{-1}}\pi_u = x\pi_u = xu = u'$. Therefore (iii) holds as well.

Finally, (iii) implies that u is the unique minimal element of $\operatorname{im}(f_{u,u})$, and (ii) implies that u is the unique maximal element in $f_{u,u}^{-1}(u)$; therefore (i) holds. □

Lemma 6.17. *If $u >_R v$, then $f_{u,v} := f_{u,u}\overline{\pi}_{u^{-1}v}$ is (u, v)-triangular.*

Proof. By Lemma 6.16 (iii), the image set of $f_{u,u}$ is a subset of $[u, w_0]_L$. Therefore, by Remark 2.7, $\overline{\pi}_{u^{-1}v}$ translates it isomorphically to the interval $[v, w_0u^{-1}v]_L$. In particular, the fibers are preserved: $f_{u,v}^{-1}(v) = f_{u,u}^{-1}(u)$, and the triangularity of $f_{u,v}$ follows. □

Lemma 6.18. *Take $u \in [1, w]_R$. Then, either u is a cutting point of w, or there exists a (u, v)-triangular function $f_{u,v}$ in F with $u <_R v \leq_R w$.*

Proof. Let J be the set of short nondescents i of u, and set $V := U_u \cap [1, w]_R$ (recall from Definition 3.15 that $U_u := uW_J$). By Proposition 3.17, V is the set of $w' \in [1, w]_R$ such that $u \sqsubseteq w'$. Furthermore, V is a lattice (it is the intersection of the two lattices $(uW_J, <_R)$ and $[1, w]_R$) with u as unique minimal element; in particular, $V \subset [u, w]_R$.

If $w \in V$ (which includes the case $u = w$ and $J = \{\}$), then u is a cutting point for w and we are done.

Otherwise, consider a shortest sequence $i_1, \ldots, i_k$ such that $\{i_1, \ldots, i_k\} \cap \mathrm{D}_R(u) = \emptyset$ and $v' = us_{i_1} \cdots s_{i_k} \notin V$. Such a sequence must exist since $w \notin V$. Set $S :=$

$\{i_1, \ldots, i_k\}$. Note that $i_1, \ldots, i_{k-1}$ are in J but i_k is not. Furthermore, $u \not\sqsubseteq v'$ while $u = v'^S$ because $v' \in uW_S$ and $S \cap \mathrm{D}_R(u) = \emptyset$.

Case 1: $v' \in \mathrm{im}(f_{u,u})$. Then, $u <_L v'$. Combining this with $u = v'^S$ yields that $u \sqsubseteq v'$, a contradiction.

Case 2: $v' \notin \mathrm{im}(f_{u,u})$. Set $v := us_{i_1}$, and define $f_{u,v} := f_{u,u}\sigma\pi_{i_1}$, where

$$\sigma := s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2} \,. \tag{6.6}$$

Note that for $k = 1$, we have $\sigma = 1$. We now prove that $f_{u,v}$ is (u, v)-triangular.

First, we consider the fiber $f_{u,v}^{-1}(v)$. By minimality of k, and up to sign, s_{i_k} fixes all the elements of V at distance at most $k-2$ of u. Hence, $\sigma^{-1}(u) = u$. Simultaneously,

$$v.\sigma^{-1} = v.s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2} = v'.s_{i_{k-1}} \cdots s_{i_2} \in v'W_J \,. \tag{6.7}$$

Hence, $v.\sigma^{-1} \notin \mathrm{im}(f_{u,u})$ because $v' \notin \mathrm{im}(f_{u,u})$ and $\mathrm{im}(f_{u,u})$ is stable under right multiplication by s_j for $j \in J$. Putting everything together:

$$f_{u,v}^{-1}(v) = f_{u,u}^{-1}(\sigma^{-1}(\pi_{i_1}^{-1}(v))) = f_{u,u}^{-1}(\sigma^{-1}(\{u, v\})) = f_{u,u}^{-1}(\{u\}) = [1, u]_B \cap [1, w]_R \,. \tag{6.8}$$

Therefore, u is the unique length maximal element of $f_{u,v}^{-1}(v)$, as desired.

We take now $x \in \mathrm{im}(f_{u,u})$, and apply Proposition 4.5 repeatedly. To start with:

$$u = 1.f_{u,u} \leq_B x.f_{u,u} \,. \tag{6.9}$$

Using Remark 6.15:

$$u = u.\overline{\pi}_S \leq_B (x.f_{u,u}).\overline{\pi}_S \leq_B (x.f_{u,u}).\sigma = x.f_{u,u}.\sigma \,. \tag{6.10}$$

It follows that:

$$v = u.\pi_{i_1} \leq_B (x.f_{u,u}.\sigma).\pi_{i_1} = x.f_{u,v} \,. \tag{6.11}$$

In particular, v is the unique Bruhat minimal element of $\mathrm{im}(f_{u,v})$, as desired. □

Proof of Proposition 6.13. Since W is finite, repeated application of Lemma 6.18 yields a finite sequence of triangular functions

$$f_{u,u_1}, \ldots, f_{u_{k-1},u_k} \,, \quad \text{where} \quad u <_R u_1 <_R \cdots <_R u_k$$

and u_k is a cutting point w^J of w. Since $u <_R w^J$, one has $J \subset K^{(w)}(u) \subset K^{(w)}(v)$, and therefore $u_k = w^J >_R v$. Then, applying Lemma 6.17 one can construct a (u_k, v)-triangular function $f_{u_k,v}$. Finally, by Remark 6.14, composing all these triangular functions gives a (u, v)-triangular function $f_{u,u_1} \cdots f_{u_{k-1},u_k} f_{u_k,v}$. □

6.4. **Representation theory of the w-biHecke algebra.** Consider the digraph $G^{(w)}$ on $[1, w]_R$ with an edge $u \mapsto v$ if $u = vs_i$ for some i and $J^{(w)}(u) \subseteq J^{(w)}(v)$. Up to orientation, this is the Hasse diagram of right order (see for example Figure 1). The following theorem is a generalization of [HT09, Section 3.3].

Theorem 6.19. *$\mathcal{H}W^{(w)}$ is the maximal algebra stabilizing all modules $P_J^{(w)}$ for $J \in \mathcal{RB}_{\mathcal{L}}(w)$*

$$\mathcal{H}W^{(w)} = \{f \in \mathrm{End}(T_w) \mid f(P_J^{(w)}) \subseteq P_J^{(w)}\}$$

The elements $f_{u,v}$ of Proposition 6.13 form a basis $\mathcal{H}W^{(w)}$; in particular,

$$\dim \mathcal{H}W^{(w)} = |\{(u, v) \mid J^{(w)}(u) \subseteq J^{(w)}(v)\}| \,. \tag{6.12}$$

$\mathcal{H}W^{(w)}$ is the digraph algebra of the graph $G^{(w)}$.

The family $(P_J^{(w)})_{J\in\mathcal{RB}_{\mathcal{L}}(w)}$ *forms a complete system of representatives of the indecomposable projective modules of* $\mathcal{H}W^{(w)}$.

The family $(S_J^{(w)})_{J\in\mathcal{RB}_{\mathcal{L}}(w)}$ *forms a complete system of representatives of the simple modules of* $\mathcal{H}W^{(w)}$. *The dimension of* $S_J^{(w)}$ *is the size of the corresponding* w*-nondescent class.*

$\mathcal{H}W^{(w)}$ *is Morita equivalent to the poset algebra of the lattice* $[1,w]_{\sqsubseteq}$. *In particular, its Cartan matrix is the incidence matrix and its quiver the Hasse diagram of this lattice.*

Proof. From Proposition 6.13, one derives by triangularity that $\dim\mathcal{H}W^{(w)} \geq \{(u,v) \mid K^{(w)}(u) \subseteq K^{(w)}(v)\}$. The stability of all the subspaces $P_J^{(w)}$ imposes the converse equality. Hence, $\mathcal{H}W^{(w)}$ is exactly the subalgebra of $\mathrm{End}(T_w)$ stabilizing each $P_J^{(w)}$. The remaining statements follow straightforwardly, as in [HT09, Section 3.3]. See also e.g. [DHST10, Section 3.7.4] for the Cartan matrix and quiver of a poset algebra. □

7. The representation theory of $M(W)$

In this section, we gather all results of the preceding sections in order to describe the representation theory of $M = M(W)$. The main result is Theorem 7.1 which gives the simple modules of M. We further relate the representation theory of M to the representation theory of M_{w_0}. In particular, we prove that the translation modules are exactly the modules induced by the simple modules of M_{w_0}. We then conclude by computing some characters and the decomposition map from M to M_{w_0}.

7.1. **Simple modules.** We now study the simple modules of the biHecke monoid M and also show that the translation modules are indecomposable.

Theorem 7.1.

(i) *The biHecke monoid* M *admits* $|W|$ *non-isomorphic simple modules* $(S_w)_{w\in W}$ *(resp. projective indecomposable modules* $(P_w)_{w\in W}$*).*

(ii) *The simple module* S_w *is isomorphic to the top simple module*

$$S_{\{\}}^{(w)} = T_w / \sum_{v \sqsubset w} T_v$$

of the translation module T_w. *Its dimension is given by*

$$\dim S_w = \left| [1,w]_R \setminus \bigcup_{v\sqsubset w} [1,v]_R \right| .$$

In general, the simple quotient module $S_J^{(w)}$ *of* T_w *is isomorphic to* S_{J_w} *of* M.

Proof. (i) Since M is aperiodic (Proposition 4.13), it has only trivial subgroups. Applying a theorem of Clifford, Munn, and Ponizovskiĭ(see Section 2.6) it follows that the simple modules are in correspondence with the $\mathcal{J}$-classes (or conjugacy classes) of idempotents. By Corollary 4.20, there are $|W|$ of them. Finally, for any finite-dimensional algebra, the simple and indecomposable projective modules share the same indexing set (see [CR06, Corollary 54.14]).

The statement in (ii) is a straightforward application of Theorem 6.19. Note that the fact that S_w is the top of T_w is a special case of the construction of the simple modules of a finite monoid in [GMS09, Theorem 7] (see Section 2.6); indeed

Cartan type	$\lvert M(W)\rvert$	$(\dim S_w)_w$	$\sum_w \dim S_w$	$\lvert W\rvert$
A_1	3	1^2	2	2
$A_2 = I_2(3)$	23	$1^2 1^2 2^2$	8	6
A_3	477	$1^8 2^4 3^4 4^6 5^2$	62	24
A_4	31103		770	120
$B_2 = I_2(4)$	49	$1^2 1^2 2^2 3^2$	14	8
B_3	5455	$1^8 2^4 3^4 4^6 5^7 6^4 7^4 8^4 9^1 10^2 11^2 12^2$	246	48
$G_2 = I_2(6)$	153	$1^2 1^2 2^2 3^2 4^2 5^2$	32	
$A_1 \times A_1$	9	$1^2 1^2$	4	4
$I_2(p)$		$1^2 1^2 \cdots (p-1)^2$	$2 + p(p-1)$	$2p$

TABLE 1. Statistics on the biHecke monoids $M(W)$ for the small Coxeter groups. In column three, $1^8 2^4 \cdots 5^2$ means that there are 8 simple modules of dimension 1, 4 of dimension 2, and so on.

T_w is isomorphic to $\mathbb{K}e_w M_{\geq J}$, where J is the $\mathcal{J}$-class of e_w. The dimension formula follows from the construction of S_w (see Remark 6.11). □

Example 7.2. The simple module S_{4312} is of dimension 3, with basis indexed by $\{4312, 4132, 1432\}$ (see Figure 1). The other simple modules S_{3412}, S_{4123}, and S_{1234} are of dimension 5, 3, and 1, respectively. See also Table 1.

In general, the two extreme cases are, on the one hand, when w is the maximal element of a parabolic subgroup, in which case the simple module is of dimension 1 and, on the other hand, when w is an immediate successor of 1 in the cutting poset (see Example 3.43), in which case the simple module is of dimension $|T_w|-1$. In the other cases, one can use Theorem 3.41 to calculate the dimension of S_w by inclusion-exclusion from the sizes of the intervals $[1, {}^J w]_R$ where ${}^J w$ runs through the free sublattice at the top of the interval $[1,w]_{\sqsubseteq}$ of the cutting poset. Note that the sizes of the intervals in W can also be computed by a similar inclusion-exclusion (the Möbius function for right order is given by $\mu(u,w) = (-1)^k$ if the interval $[u,w]_R$ is isomorphic to some W_J with $|J| = k$, and 0 otherwise). This may open the door for some generating series manipulations to derive statistics like the sum of the dimension of the simple modules.

Corollary 7.3. *The translation module T_w is an indecomposable M-module, quotient of the projective module P_w of M.*

Proof. The top module of T_w is isomorphic to S_w. The result follows by a general theorem for finite-dimensional algebras (see [CR06, Corollary 54.14]). □

7.2. **From $M_{w_0}(W)$ to $M(W)$.** In this section, we use our knowledge on M_{w_0} to learn more about M.

Proposition 7.4. *The translation module T_w is isomorphic to the induction to M of the simple module $S_w^{w_0}$ of M_{w_0}.*

The proof of this proposition follows immediately from the next two results.

Lemma 7.5. *Let A and B be two finite monoids with $B \subseteq A$ and assume that:*

(i) $\mathcal{R}$-order on B is induced by $\mathcal{R}$-order on A, that is, for all $x, y \in B$

$$x <_{\mathcal{R}^A} y \qquad \Longleftrightarrow \qquad x <_{\mathcal{R}^B} y\,,$$

where $<_{\mathcal{R}^A}$ and $<_{\mathcal{R}^B}$ denote $\mathcal{R}$-order on A and B, respectively.

(ii) Any $\mathcal{R}$-class of A intersects B.

Then, for any $f \in B$, the right class module $\mathbb{K}\mathcal{R}^A(f)$ is isomorphic to the induction from B to A of the right class module $\mathbb{K}\mathcal{R}^B(f)$:

$$\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f)\!\uparrow_B^A.$$

Furthermore, any right class module of A is of this form.

Proof. Recall that for a B-module Y, the module $Y\!\uparrow_B^A$ induced by Y from B to A is given by $Y\!\uparrow_B^A := Y \otimes_{\mathbb{K}B} \mathbb{K}A$.

By construction of the right class modules (see Definition 6.1), we have the following exact sequences:

$$0 \to \mathbb{K}\mathcal{R}_<^B(f) \to \mathbb{K}fB \to \mathbb{K}\mathcal{R}^B(f) \to 0\,, \tag{7.1}$$

$$0 \to \mathbb{K}\mathcal{R}_<^A(f) \to \mathbb{K}fA \to \mathbb{K}\mathcal{R}^A(f) \to 0\,. \tag{7.2}$$

Consider now the sequence obtained by tensoring (7.1) by $\mathbb{K}A$:

$$0 \to \mathbb{K}\mathcal{R}_<^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A \to \mathbb{K}fB \otimes_{\mathbb{K}B} \mathbb{K}A \to \mathbb{K}\mathcal{R}^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A \to 0\,. \tag{7.3}$$

We want to prove that it is isomorphic to (7.2).

First note that, since $\mathbb{K}B$ is a subalgebra of $\mathbb{K}A$, we have $b \otimes a = 1 \otimes ba$ for $b \in B$ and $a \in A$. Therefore the product map

$$\begin{cases} \mathbb{K}fB \otimes_{\mathbb{K}B} \mathbb{K}A & \longrightarrow \mathbb{K}fA \\ fb \otimes a & \longmapsto fba \end{cases}$$

is an isomorphism of A-modules.

Next consider the restriction μ of this map to $\mathbb{K}\mathcal{R}_<^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A$. Its image set is $\mathbb{K}\mathcal{R}_<^B(f)A$, and we claim that $\mathbb{K}\mathcal{R}_<^B(f)A = \mathbb{K}\mathcal{R}_<^A(f)$:

- Inclusion $\subseteq$: If $b <_{\mathcal{R}^B} f$ for $b \in B$, then using (i) we have $ba \leq_{\mathcal{R}^A} b <_{\mathcal{R}^A} f$ for any $a \in A$.
- Inclusion $\supseteq$: If $a <_{\mathcal{R}^A} f$ for $a \in A$, then using (ii) there exists an element $b \in B$ such that $b\ \mathcal{R}^A\ a$. Then $b <_{\mathcal{R}^A} f$ and therefore by (i), $b \in \mathbb{K}\mathcal{R}_<^B(f)$ and it follows that $a \in \mathbb{K}\mathcal{R}_<^B(f)A$.

Therefore, μ restricts to an A-module isomorphism from $\mathbb{K}\mathcal{R}_<^B(f) \otimes_B A$ to $\mathbb{K}\mathcal{R}_<^A(f)$.

We are left to show that the third and fourth entries of the sequences (7.2) and (7.3) are isomorphic. However, it is a well-known fact that the functor $? \otimes_{\mathbb{K}B} \mathbb{K}A$ is right exact, so that the sequence

$$0 \to \mathbb{K}\mathcal{R}_<^A(f) \to \mathbb{K}fA \to \mathbb{K}\mathcal{R}^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A \to 0$$

is exact. Comparing with (7.2), we obtain that

$$\mathbb{K}\mathcal{R}^A(f) \cong \mathbb{K}\mathcal{R}^B(f) \otimes_{\mathbb{K}B} \mathbb{K}A\,,$$

where the latter is isomorphic to $\mathbb{K}\mathcal{R}^B(f)\!\uparrow_B^A$ by definition. □

Lemma 7.6. *The biHecke monoid and its Borel submonoid $M_{w_0}(W) \subseteq M(W)$ satisfy conditions* (i) *and* (ii) *of Lemma 7.5.*

Proof. By Proposition 4.18, for any $f \in M$ there exists a unique element $f_1 \in \mathcal{R}(f) \cap M_1$. Using the bar involution of Section 4.7, one finds a unique $\overline{f}_1 \in \mathcal{R}(f) \cap M_{w_0}$. This proves condition (ii) of Lemma 7.5.

Now let $f, g \in M_{w_0}$. It is obvious that $f <_{\mathcal{R}^{M_{w_0}}} g$ implies that $f <_{\mathcal{R}^M} g$ since M_{w_0} is a submonoid of M. Suppose that $f <_{\mathcal{R}^M} g$, so that $f = gx$ for $x \in M$. Note that $w_0.f = w_0.g = w_0$, which implies that $w_0.x = w_0$ as well and hence $x \in M_{w_0}$. Therefore, condition (i) of Lemma 7.5 holds. ☐

Proof of Proposition 7.4. Let $g_w := e_{w,w_0}$. By definition, the translation module is the quotient $T_w = \mathbb{K}g_wM/\mathbb{K}\mathcal{R}_<(g_w)$, whereas $S_w^{w_0} = \mathbb{K}g_wM_{w_0}/\mathbb{K}\mathcal{R}_<^{w_0}(g_w)$. Since $M_{w_0} \subseteq M$ satisfies the conditions of Lemma 7.5 by Lemma 7.6, Proposition 7.4 follows. ☐

Theorem 7.7. *The right regular representation of M admits a composition series with factors all isomorphic to translation modules, and its character is given by*

$$[\mathbb{K}M] = \sum_{f \in M_{w_0}} [T_{1.f}]\,. \tag{7.4}$$

Proof. As any monoid algebra, $\mathbb{K}M$ admits a composition series where each composition factor is given by (the linear span of) a $\mathcal{R}$-class of M. By Proposition 6.4, each such composition factor is isomorphic to the translation module $T_{1.f}$, where f is the unique element of the $\mathcal{R}$-class which lies in M_{w_0}. The character formula follows. Alternatively, it can be obtained using Proposition 7.4 and the character formula for the right regular representation of M_{w_0} (see Remark 5.7):

$$[\mathbb{K}M_{w_0}]_{M_{w_0}} = \sum_{f \in M_{w_0}} [S_{1.f}^{w_0}]_{M_{w_0}}\,. \tag{7.5}$$ ☐

Proposition 7.8. *For any $w \in W$ the translation module T_w is multiplicity-free as an M_{w_0}-module and its character is given by*

$$[T_w]_{M_{w_0}} = \sum_{u \in [1,w]_R} [S_u^{w_0}]_{M_{w_0}}\,. \tag{7.6}$$

Proof. Let f be an element in M which yields the translation module T_w, and define f_u as in Proposition 4.18.

Take some sequence $u_1, \dots, u_m$ (for $m = |[1,w]_R|$) of the elements of $[1,w]_R$ which is length increasing, and define the corresponding sequence of subspaces by $X_i := \mathbb{K}\{u_1, \dots, u_i\}$. Using Lemma 6.16, each such subspace is stable by M_{w_0}, and therefore $X_0 \subset \cdots \subset X_m$ forms an M_{w_0} composition series of T_w. Consider now a composition factor X_i/X_{i-1}, which is of dimension 1. Again, by Lemma 6.16, e_{v,w_0} fixes u_i if and only if $v \leq_L u_i$ (that is if the image set $[u_i, w^{-1}w_0u_i]_L$ of f_{u_i} is contained in the image set $[v, w_0]_L$ of e_{v,w_0}), and kills it otherwise. Hence, X_i/X_{i-1} is isomorphic to $S_{u_i}^{w_0}$. ☐

Theorem 7.9. *The decomposition map of M over M_{w_0} is lower uni-triangular for right order, with $0, 1$ entries. More explicitly,*

$$[S_w]_{M_{w_0}} = \sum_{u \in [1,w]_R \setminus \bigcup_{v \sqsubset w} [1,v]_R} [S_u^{w_0}]_{M_{w_0}}\,. \tag{7.7}$$

Proof. Since S_w is a quotient of T_w, its composition factors form a subset of the composition factors for T_w. Hence, using Proposition 7.8, the decomposition matrix of M over M_{w_0} is lower triangular for right order, with $0, 1$ entries. Furthermore, by construction (see Remark 6.11 and Theorem 7.1 (ii)), $S_w = T_w / \sum_{v \sqsubset w} T_v$; using Proposition 7.8 the sum on the right hand side contains at least one composition factor isomorphic to $S_u^{w_0}$ for each u in $[1, v]_R$ with $v \sqsubset w$; therefore S_w has no such composition factor. We conclude using the dimension formula of Theorem 7.1 (ii). □

Example 7.10. Following up on Example 7.2, the decomposition of the M-simple module S_{4312} over M_{w_0} is given by $[S_{4312}]_{M_{w_0}} = [S^{w_0}_{4312}] + [S^{w_0}_{4132}] + [S^{w_0}_{1432}]$. See also Figure 1 and the decomposition matrices given in Appendix B.2.

8. Research in progress

Our guiding problem is the search for a formula for the cardinality of the biHecke monoid. Using a standard result of the representation theory of finite-dimensional algebras together with the results of this paper, we can now write

$$|M(W)| = \sum_{w \in W} |\dim S_w| |\dim P_w| ,$$

where $\dim S_w$ is given by an inclusion-exclusion formula. It remains to determine the dimensions of the projective modules P_w.

While studying the representation theory of the Borel submonoid M_1 as an intermediate step, the authors realized that many of the combinatorial ingredients that arose were well-known in the semigroup community (for example the Green relations and related classes, automorphism groups, etc.), and hence the representation theory of M_1 is naturally expressed in the context of $\mathcal{J}$-trivial monoids (see [DHST10]). This sparked the investigation of the representation theory of more general classes of monoids, and in particular aperiodic monoids.

At the current stage, it appears that the Cartan matrix of an aperiodic monoid (and therefore the composition series of its projective modules, and by consequence their dimensions) is completely determined by the knowledge of the composition series for both left and right class modules. In other words, the study in this paper of right class modules (i.e. translation modules), whose original purpose was to construct the simple modules using [GMS09, Theorem 7], turns out to complete half of this program. The remaining half, in progress, is the decomposition of left class modules.

At the combinatorial level, this requires to control $\mathcal{L}$-order. Loosely speaking, $\mathcal{L}$-order is essentially given by left and right order in W; however, within $\mathcal{L}$-classes the structure seems more elusive, in particular because fibers are more difficult to describe than image sets. Another difficulty is that, unlike for $\mathcal{R}$-class modules, $\mathcal{L}$-class modules are not all isomorphic to regular ones (i.e. classes containing idempotents).

Yet, the general theory gives that the decomposition matrix should be upper triangular for left-right order for regular classes, and upper triangular for Bruhat order for nonregular ones, with no left-right "arrow" for left-right order. It further suggests the existence of an analogue of the cutting poset on the left.

We conclude by illustrating this for $W = \mathfrak{S}_4$ in Figure 6. The blue arrows are the covering relations of the cutting poset, which encode the composition series of

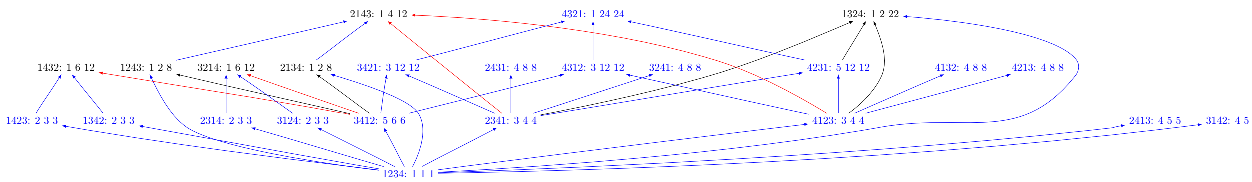


Figure 6. Graph encoding the characters of left and right class modules, and therefore the Cartan invariant matrix for $M(\mathfrak{S}_4)$. See the text for details.

the translation modules (i.e. right class modules). Namely, the character of T_w is given by the sum of $q^k[S_u]$ for u below w in the cutting poset, with k the distance from u to w in that poset. For example:

$$\begin{aligned}
[T_{2143}] &= [S_{2143}] + q[S_{1243}] + q[S_{2134}] + q^2[S_{1234}] \\
[T_{2341}] &= [S_{2341}] + q[S_{1234}] \\
[T_{4123}] &= [S_{4123}] + q[S_{4123}].
\end{aligned}$$

Similarly the black (resp. red) arrows encode the composition series of regular (resp. nonregular) left classes. In this simple example, the q-character of a right projective module P_w is then given by

$$[P_w] = [T_w] + \sum_u q[T_u],$$

where (u, w) is a black or red arrow in the graph. For example,

$$\begin{aligned}
[P_{2143}] &= [T_{2143}] + q[T_{2341}] + q[T_{4123}] \\
&= [S_{2143}] + q[S_{1243}] + q[S_{2134}] + q[S_{2341}] + q[S_{4123}] + 3q^2[S_{1234}]\,.
\end{aligned}$$

Appendix A. Monoid of edge surjective morphism of a colored graph

Let C be a set whose elements are called colors. We consider colored simple digraphs without loops. More precisely, a C-colored graph is a triple $G = (V, E, c)$, where V is the set of vertices of G, $E \subset V \times V / \{(x, x) \mid x \in V\}$ is the set of (oriented) edges of G, and $c : E \to C$ is the coloring map.

Definition A.1. *Let $G = (V, E, c)$ and $G' = (V', E', c')$ be two colored graphs. An* edge surjective *morphism (or ES-morphism) from G to G' is a map $f : V \to V'$ such that*

- *For any edge $(a, b) \in E$, either $f(a) = f(b)$, or $(f(a), f(b)) \in E'$ and $c(a, b) = c'(f(a), f(b))$.*
- *For any edge $(a', b') \in E'$ with a' and b' in the image set of f there exists an edge $(a, b) \in E$ such that $f(a) = a'$ and $f(b) = b'$.*

Note that by analogy to categories, instead of ES-morphism, we can speak about full morphisms.

The following proposition shows that colored graphs together with edge surjective morphisms form a category.

Proposition A.2. *For any colored graphs G, G_1, G_2, G_3:*

- *The identity* id $: G \to G$ *is an ES-morphism;*

- *For any ES-morphism $f : G_1 \to G_2$ and $g : G_2 \to G_3$ the composed function $g \circ f : G_1 \to G_3$ is an ES-morphism.*

Corollary A.3. *For any colored graph G, the set of ES-morphisms from G to G is a submonoid of the monoid of the functions from G to G.*

Here are some general properties of ES-morphisms:

Proposition A.4. *Let G_1 and G_2 be two colored graphs and f an ES-morphism from G_1 to G_2. Then the image of any path in G_1 is a path in G_2.*

In our particular case, we have some more properties:

(i) The graph is acyclic, with unique source and sink. In particular, it is (weakly) connected.
(ii) The graph is ranked by the integers, and edges occur only between two consecutive ranks.
(iii) The graph is C-regular, which means that for any vertex v and any color c, there is exactly one edge entering or leaving v with color c.

Remarks A.5. *Proposition 4.1 gives that our monoid is a submonoid of the $M(G)$ monoid for left order.*

Propositions 4.3 and 4.11 are generic, and would apply to any $M(G)$. For the latter, we just need that G is C-regular.

A natural source of colored graphs are crystal graphs. A question that arises is how the G-monoid of a crystal looks like.

Appendix B. Tables

B.1. **q-Cartan invariant matrices.** We give the Cartan invariant matrix for M_{w_0} and M in types A_1, A_2 and A_3. The q-parameter records the layer in the radical filtration. The extra rows and columns entitled "Simpl." and "Proj." give the dimension of the simple and projective modules, on the right for right modules and below for left modules. When all simple modules are one-dimensional, the column is omitted.

q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_2)$ (type A_1):

	12	21	Proj.
12	1	.	1
21	.	1	1
Proj.	1	1	

q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_3)$ (type A_2):

	123	132	213	231	312	321	Proj.
123	1	.	.	.	.	.	1
132	.	1	.	.	q	.	2
213	.	.	1	q	.	.	2
231	.	.	.	1	.	.	1
312	.	.	.	.	1	.	1
321	.	.	.	.	.	1	1
Proj.	1	1	1	2	2	1	

q-Cartan invariant matrix of $M_{w_0}(\mathfrak{S}_4)$ (type A_3):

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Proj.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1243	.	1	.	.	q	.	.	.	.	.	q	q^2	.	.	.	.	q	.	q^2	.	.	.	.	.	6
1324	.	.	1	q	.	.	.	.	.	q	.	.	q	q^2	.	q^2	q^3	.	q	q^2	.	q	.	.	10
1342	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	q^2	.	.	q	.	.	.	.	4
1423	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	2
1432	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	q	.	.	q	.	.	q^2	.	4
2134	.	.	.	.	.	.	1	.	q	q^2	q	.	.	.	.	.	q	.	.	.	q^2	.	.	.	6
2143	.	.	.	.	.	.	.	1	.	q	q	q^2	.	.	.	.	.	.	q	.	q^2	q^3	.	.	7
2314	.	.	.	.	.	.	.	.	1	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
2341	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2413	.	.	.	.	.	.	.	.	.	.	1	q	.	.	.	.	.	.	.	.	q	q^2	.	.	4
2431	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	2
3124	.	.	.	.	.	.	.	.	.	.	.	.	1	q	.	q	q^2	.	.	.	.	.	.	.	4
3142	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	q	.	.	.	.	.	.	.	2
3214	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	q	q	q^2	.	.	.	.	.	.	4
3241	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	q	.	.	.	.	.	.	2
3412	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1
3421	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1
4123	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1
4132	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	q	.	2
4213	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	q	.	.	2
4231	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
4312	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1
4321	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
Proj.	1	1	1	2	2	1	1	1	2	5	4	4	2	4	1	4	9	3	5	4	4	6	3	1	

q-Cartan invariant matrix of $M(\mathfrak{S}_2)$ (type A_1):

	12	21	Simpl.	Proj.
12	1	.	1	1
21	q	1	1	2
Simpl.	1	1		
Proj.	2	1		

q-Cartan invariant matrix of $M(\mathfrak{S}_3)$ (type A_2):

	123	132	213	231	312	321	Simpl.	Proj.
123	1	.	.	.	.		1	1
132	q	1	.	.	.		1	2
213	q	.	1	.	.		1	2
231	q	.	.	1	.		2	3
312	q	.	.	.	1		2	3
321	q^2	.	.	q	q	1	1	6
Simpl.	1	1	1	2	2	1		
Proj.	8	1	1	3	3	1		

q-Cartan invariant matrix of $M(\mathfrak{S}_4)$ (type A_3):

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Simpl.	Proj.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
1243	q^2+q	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	8
1324	q^3+2q^2+q	.	1	.	.	.	.	.	.	q^2+q	.	.	.	.	.	.	.	.	q^2+q	.	.	q	.	.	1	22
1342	q	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
1423	q	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
1432	$2q^2$	.	.	q	q	1	.	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	12
2134	q^2+q	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	1	8
2143	$3q^2$	q	.	.	.	.	q	1	.	q	.	.	.	.	.	.	.	.	q	.	.	.	.	.	1	12
2314	q	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	3
2341	q	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	4
2413	q	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	4	5
2431	q^2	.	.	.	.	.	.	.	.	q	.	1	.	.	.	.	.	.	.	.	.	.	.	.	4	8
3124	q	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2	3
3142	q	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	4	5
3214	$2q^2$	.	.	.	.	.	.	.	q	.	.	.	q	.	1	.	q	.	.	.	.	.	.	.	1	12
3241	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	1	.	.	.	.	.	.	.	.	4	8
3412	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	5	6
3421	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	q	1	.	.	.	.	.	.	3	12
4123	q	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	3	4
4132	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	1	.	.	.	.	4	8
4213	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	1	.	.	.	4	8
4231	q^2	.	.	.	.	.	.	.	.	q	.	.	.	.	.	.	.	.	q	.	.	1	.	.	5	12
4312	q^2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	q	.	q	.	.	.	1	.	3	12
4321	q^3	.	.	.	.	.	.	.	.	q^2	.	.	.	.	.	.	q^2	q	q^2	.	.	q	q	1	1	24
Simpl.	1	1	1	2	2	1	1	1	2	3	4	4	2	4	1	4	5	3	3	4	4	5	3	1		
Proj.	71	2	1	3	3	1	2	1	3	23	4	4	3	4	1	4	16	4	23	4	4	7	4	1		

B.2. **Decomposition matrices.** Since M_{w_0} is a submonoid of M, any simple M-module is also a simple M_{w_0}-module. The following matrices give the (generalized) M_{w_0} character of the simple M-module. The table reads as follows: for any two permutations σ, τ, the coefficient $m_{\sigma,\tau}$ gives the Jordan-Hölder multiplicity of the M_{w_0}-module $S_\tau^{w_0}$ in the M-module S_σ. In particular, since the simple M_{w_0}-modules are of dimension 1, summing each line one recovers the dimension of the simple M-modules, as shown.

Decomposition matrix of $M(\mathfrak{S}_2)$ on $M_{w_0}(\mathfrak{S}_2)$ (type A_1):

	12	21	Simpl.
12	1	.	1
21	.	1	1

Decomposition matrix of $M(\mathfrak{S}_3)$ on $M_{w_0}(\mathfrak{S}_3)$ (type A_2):

	123	132	213	231	312	321	Simpl.
123	1	.	.	.	.	.	1
132	.	1	.	.	.	.	1
213	.	.	1	.	.	.	1
231	.	.	1	1	.	.	2
312	.	1	.	.	1	.	2
321	.	.	.	.	.	1	1

Decomposition matrix of $M(\mathfrak{S}_4)$ on $M_{w_0}(\mathfrak{S}_4)$ (type A_3):

	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321	Simpl.
1234	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1243	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1324	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
1342	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
1423	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
1432	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2134	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2143	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
2314	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
2341	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3
2413	.	1	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	4
2431	.	1	.	.	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	4
3124	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2
3142	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	4
3214	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1
3241	.	.	1	.	.	.	.	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	4
3412	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	5
3421	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.	3
4123	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	3
4132	.	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	4
4213	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	4
4231	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	1	1	.	.	5
4312	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	3
4321	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1

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LITIS (EA 4108), Université de Rouen, Avenue de l'Université BP12 76801 Saint-Etienne du Rouvray, France and Institut Gaspard Monge (UMR 8049), France

E-mail address: florent.hivert@univ-rouen.fr

Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616, U.S.A.

E-mail address: anne@math.ucdavis.edu

Univ Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay, F-91405; CNRS, Orsay, F-91405, France

E-mail address: Nicolas.Thiery@u-psud.fr