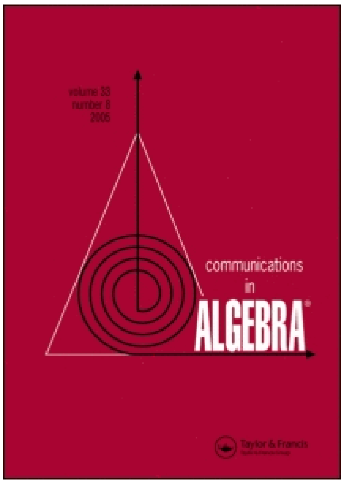

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A CLASS OF SEMIGROUPS OF FINITE REPRESENTATION TYPE

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In their study of the class of standardly stratified algebras with n irreducible representations, Ágoston et al. (to appear) have introduced two operators Σ and Ω acting on the class and satisfying the following relations:

$$\Sigma^2 = \Sigma, \qquad \Omega^2 = \Omega \qquad \textit{and} \qquad \Sigma(\Omega\Sigma)^{n-1} = (\Omega\Sigma)^{n-1}.$$

In this little note we are presenting a complete description of indecomposable linear representations of the respective semigroups

$$\bar{S}_n = \langle a, b \mid a^2 = a, b^2 = b, (ab)^{n-1}a = (ab)^{n-1}\rangle \dot{\cup} \{1\}$$

by constructing the graph semigroups T_n (see Dlab and Pospíchal, 2002) such that the semigroup algebras KT_n and $K\overline{S}_n$ are isomorphic. The number of indecomposable representations of $\bar{S}_n$ is $2(2n-1)$ for $n > 1$ (of which 4 are irreducible) and all indecomposable representations of $\overline{S}_n$ are uniserial.

Key Words: Finite representation type; Graph semigroups; Indecomposable representations; Linear representations of semigroups.

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STRUCTURE OF $K\overline{S}_n$

Given the semigroup

$$S_n = \langle a, b \mid a^2 = a, b^2 = b, (ab)^{n-1}a = (ab)^{n-1}\rangle,$$

attach to S_n the unity 1 and denote the resulting semigroup by

$$\overline{S}_n = S_n \dot{\cup} \{1\}.$$

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Thus, $\overline{S}_n$ has $2(2n-1)$ elements

$$a_{2k-1} = (ab)^{k-1}a, \qquad a_{2k} = (ab)^k \quad \text{for } 1 \le k \le n-1,$$
$$b_{2k-1} = (ba)^{k-1}b, b_{2k} = (ba)^k \qquad \text{for } 1 \le k \le n-1,$$
$$b_{2n-1} = (ba)^{n-1}b \text{ and } 1.$$

Note that $a_1 = a$ and $b_1 = b$.

One can easily check the following multiplication table for $\overline{S}_n$:

$$a_1 a_s = a_s \qquad \text{for all } 1 \le s \le 2n-2,$$
$$a_{2r}a_s = a_{2r+1}a_s = a_t, \qquad \text{where } t = \min(2r+s, 2n-2),$$
$$\text{for all } 1 \le r \le n-2 \text{ and } 1 \le s \le 2n-2,$$
$$a_{2n-2}a_s = a_{2n-2} \qquad \text{for all } 1 \le s \le 2n-2;$$
$$a_{2r-1}b_s = a_{2r}b_s = a_t, \qquad \text{where } t = \min(2r+s-1, 2n-2)$$
$$\text{for all } 1 \le r \le n-2 \text{ and } 1 \le s \le 2n-1,$$
$$b_{2r-1}a_s = b_{2r}a_s = b_t, \qquad \text{where } t = \min(2r+s-1, 2n-2)$$
$$\text{for all } 1 \le r \le n-1 \text{ and } 1 \le s \le 2n-2,$$
$$b_{2n-1}a_s = b_{2n-1} \qquad \text{for all } 1 \le s \le 2n-2;$$
$$b_1 b_s = b_s \qquad \text{for all } 1 \le s \le 2n-1$$

and

$$b_{2r}b_s = b_{2r+1}b_s = b_t, \qquad \text{where } t = \min(2r+s, 2n-1)$$
$$\text{for all } 1 \le r \le n-1 \text{ and } 1 \le s \le 2n-1.$$

Now, let K be a field and

$$A_n = K\overline{S}_n$$

the semigroup algebra of $\overline{S}_n$.

For $n = 1$, clearly, $A_1 \simeq K \oplus K = Kb \oplus K(1-b)$ is semisimple, and for $n = 2$, A_2 is hereditary with a K-basis consisting of

$$e_1 = b_1 - b_2, \qquad e_2 = a_1 - a_2, \qquad e_3 = 1 - b_1 + b_2 - a_1 + a_2 - b_3, \qquad e_4 = b_3,$$
$$g = b_2 - b_3 \qquad \text{and} \qquad h = a_2 - b_3.$$

Thus A_2 is the path algebra of the quiver

$$\overset{1}{\bullet} \xrightarrow{g} \overset{2}{\bullet} \;\dot{\cup}\; \overset{3}{\bullet} \xrightarrow{h} \overset{4}{\bullet}.$$

For $n \geq 3$, define the following elements of A_n:

$$e_1 = \sum_{i=1}^{2n-2} (-1)^{i+1} b_i,$$
$$e_2 = \sum_{i=1}^{2n-2} (-1)^{i+1} a_i,$$
$$e_3 = 1 - e_1 - e_2 - b_{2n-1},$$
$$e_4 = b_{2n-1},$$
$$g_1 = b_2 - b_3,$$
$$g_2 = a_2 - a_3 \quad \text{and}$$
$$g_3 = a_{2n-2} - b_{2n-1}.$$

It is a straightforward calculation to check the following multiplication table (Table 1).

Moreover, we can show that

$$(g_1 g_2)^{n-2} = b_{2n-3} - b_{2n-2} + (2n-5) b_{2n-1}.$$

Indeed, we can show that for any $1 \leq t \leq n-2$,

$$(g_1 g_2)^t = b_{2t+1} - b_{2t+2} + \sum_{p=t+1}^{n-2} \alpha_p (b_{2p+1} - b_{2p+2}) \qquad \text{for suitable integers } \alpha_p. \qquad (*)$$

This is true for $t = 1$, since

$$g_1 g_2 = b_3 - b_4 + (-1)(b_5 - b_6).$$

Observe that, in general,

$$(b_{2t+1} - b_{2t+2}) g_1 g_2 = b_{2t+3} - b_{2t+4} - 2\alpha(b_{2t+5} - b_{2t+6}) + \beta(b_{2t+7} - b_{2t+8}),$$

Table 1

$\cdot$	e_1	e_2	e_3	e_4	g_1	g_2	g_3
e_1	e_1	0	0	0	g_1	0	0
e_2	0	e_2	0	0	0	g_2	0
e_3	0	0	e_3	0	0	0	g_3
e_4	0	0	0	e_4	0	0	0
g_1	0	g_1	0	0	0	$g_1 g_2$	0
g_2	g_2	0	0	0	$g_2 g_1$	0	0
g_3	0	0	0	g_3	0	0	0

where α and β equal to 0 or 1, and thus, proceeding by induction, $(*)$ follows. In particular, for $t = n-2$, we have

$$(g_1g_2)^{n-2} = b_{2n-3} - b_{2n-2} + (2n-5)b_{2n-1}.$$

Furthermore, one can verify easily that

$$(g_1g_2)^{n-2}g_1 = b_{2n-2} - b_{2n-1},$$
$$(g_1g_2)^{n-1} = 0 \quad \text{and} \quad g_2(g_1g_2)^{n-2} = (g_2g_1)^{n-2}g_2 = 0.$$

Remark here that $(g_2g_1)^{n-2} = a_{2n-3} - a_{2n-2}$.

As a result, we can formulate the following theorem.

Theorem. *For every* $n \geq 3$,

$$A_n \simeq A_n^{(1)} \oplus A_n^{(2)},$$

where $A_n^{(1)} = KQ_1/\langle (g_2g_1)^{n-2}g_2 \rangle$ *is a factor algebra of the path algebra over the quiver*

$$Q_1 = \overset{1}{\bullet} \underset{g_2}{\overset{g_1}{\rightleftarrows}} \overset{2}{\bullet}$$

and $A_n^{(2)}$ *is the path algebra over the quiver*

$$Q_2 = \overset{3}{\bullet} \overset{g_3}{\longrightarrow} \overset{4}{\bullet}.$$

Thus, A_n *is of finite representation type: There are* $4n-5$ *indecomposable representations of* $A_n^{(1)}$ *and* 3 *indecomposable representations of* $A_n^{(2)}$*. All these representations are uniserial.*

Proof. It is easy to see that

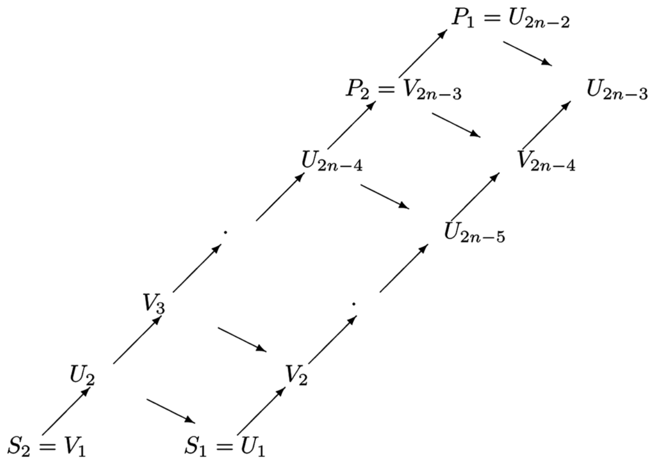
$$B_1 = \{e_1, g_1, g_1g_2, g_1g_2g_1, \ldots, (g_1g_2)^{n-2}g_1, e_2, g_2, g_2g_1, g_2g_1g_2, \ldots, (g_2g_1)^{n-2}\}$$

is a K-basis of $A_n^{(1)}$ and the set

$$B_2 = \{e_3, g_3, e_4\}$$

is a K-basis of $A_n^{(2)}$. Thus, $\dim_K A_n = \dim_K A_n^{(1)} + \dim_K A_n^{(2)}$ and the theorem follows.

Remark 1. Denoting by S_1 and S_2 the simple representations of $A_n^{(1)}$ corresponding to the idempotents e_1 and e_2, respectively, we can display the Auslander-Reiten quiver as follows:



where $U_t, 1 \leq t \leq 2n-2$, are the uniserial modules with the composition series $S_1, S_2, S_1, S_2, \ldots$ of length t and $V_t, 1 \leq t \leq 2n-3$, are the uniserial modules with the composition series $S_2, S_1, S_2, S_1, \ldots$ of length t. Here, P_1 and P_2 are the indecomposable projective representations.

Remark 2. Note that A_n is isomorphic to the semigroup algebra of the graph semigroup T_n defined by the set $E = \{e_1, e_2, e_3, e_4\}$ of the orthogonal idempotents and the set $G = \{g_1, g_2, g_3\}$ of the generators in the terminology of Dlab and Pospíchal (2002).

Remark 3. Let us describe the (unique) expressions for the generators a and b of S_n in terms of the basis $B_1 \cup B_2$:

$$a = e_2 + g_2 + \kappa_1 g_2 g_1 g_2 + \kappa_2 (g_2 g_1)^2 g_2 + \cdots + \kappa_{n-3} (g_2 g_1)^{n-3} g_2 + g_3 + e_4$$

with suitable integers $\kappa_t, 1 \leq t \leq n-3$, and

$$b = e_1 + g_1 + \lambda_1 g_1 g_2 g_1 + \lambda_2 (g_1 g_2)^2 g_1 + \cdots + \lambda_{n-3} (g_1 g_2)^{n-3} g_1 + (g_1 g_2)^{n-2} g_1 + e_4$$

with suitable integers $\lambda_t, 1 \leq t \leq n-3$.

The expressions follow immediately using the relations

$$g_2 (g_1 g_2)^t = a_{2t+2} - a_{2t+3} + \sum_{p=t+1}^{n-3} \kappa_p (a_{2p+2} - a_{2p+3})$$

and

$$(g_1 g_2)^t g_1 = b_{2t+2} - b_{2t+3} + \sum_{p=t+1}^{n-3} \lambda_p (b_{2p+2} - b_{2p+3})$$

that can be easily derived from $(*)$.

Table 2

$\cdot$	a_{2r-1}	a_{2r}	b_{2r-1}	b_{2r}
a_{2s-1}	a_{h-3}	a_{h-2}	a_{h-2}	a_{h-1}
a_{2s}	a_{h-1}	a_h	a_{h-2}	a_{h-1}
b_{2s-1}	b_{h-2}	b_{h-1}	b_{h-3}	b_{h-2}
b_{2s}	b_{h-2}	b_{h-1}	b_{h-1}	b_h

FINAL REMARKS

Defining formally a_p to be the product $abab\ldots$ with p factors (resulting in the equalities $a_p = a_{2n-2}$ for all $p \geq 2n-2$) and b_q to be the product $baba\ldots$ with q factors (and thus having $b_q = b_{2n-1}$ for all $q \geq 2n-1$), we obtain the following simple multiplication table (Table 2) with $h = 2(r+s)$.

From Table 2, one can get immediately the following formulae (for all $t \geq 1$):

$$(g_1g_2)^t = \sum_{i=0}^{2t-1}(-1)^i \binom{2t-1}{i}(b_{2(t+i)+1} - b_{2(t+i)+2}),$$

$$(g_1g_2)^t g_1 = \sum_{i=0}^{2t}(-1)^i \binom{2t}{i}(b_{2(t+i)+2} - b_{2(t+i)+3}),$$

$$(g_2g_1)^t = \sum_{i=0}^{2t-1}(-1)^i \binom{2t-1}{i}(a_{2(t+i)+1} - a_{2(t+i)+2}),$$

and

$$(g_2g_1)^t g_2 = \sum_{i=0}^{2t}(-1)^i \binom{2t}{i}(a_{2(t+i)+2} - a_{2(t+i)+3}).$$

It is then a matter of simple computations to write down the formulae for a and b in terms of $e_1, e_2, e_3, e_4, g_1, g_2$, and g_3 explicitly. The reader may find easily that $\kappa_p = \lambda_p$ for all $p \geq 1$ with $\kappa_1 = \lambda_1 = 1$ and that, writing $q = p - [\frac{p-1}{3}]$,

$$\kappa_p = 1 + \sum_{i=1}^{q-1}(-1)^{q+i+1}\binom{2(p-q+i)}{q-i}\kappa_{p-q+i} \qquad \text{for all } p \geq 2.$$

Thus, for instance, $\kappa_5 = 273$ and $\kappa_{10} = 1430715$.

Added December 13, 2007: In a recent letter, Prof. József Pelikán of Eötvös Loránd University in Budapest informs us that

$$\kappa_p = \frac{1}{2p+1}\binom{3p}{p}.$$

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