Coxeter groups and Hopf algebras I

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Foreword

In the study of a mathematical system, algebraic structures allow for the discovery of more information. This is the motor behind the success of many areas of mathematics such as algebraic geometry, algebraic combinatorics, algebraic topology and others. This was certainly the motivation behind the observation of G.-C. Rota stating that various combinatorial objects possess natural product and coproduct structures. These structures give rise to a graded Hopf algebra, which is usually referred to as a combinatorial Hopf algebra. Typically, it is a graded vector space where the homogeneous components are spanned by finite sets of combinatorial objects of a given type and the algebraic structures are given by some constructions on those objects.

Recent foundational work has constructed many interesting combinatorial Hopf algebras and uncovered new connections between diverse subjects such as combinatorics, algebra, geometry, and theoretical physics. This has expanded the new and vibrant subject of combinatorial Hopf algebras. To give a few instances:

- Connes and Kreimer showed that a certain renormalization problem in quantum field theory can be encoded and solved using a Hopf algebra indexed by rooted trees.

- Loday and Ronco showed that a Hopf algebra indexed by planar binary trees is the free dendriform algebra on one generator. This is true for many types of algebras; the free algebra on one generator is a combinatorial Hopf algebra.

- In the context of polytope theory, some interesting enumerative combinatorial invariants induce a Hopf morphism from a Hopf algebra of posets to the Hopf algebra of quasi-symmetric functions.

- Krob and Thibon showed that the representation theory of the Hecke algebras at $q = 0$ is intimately related to the Hopf algebra structure of quasi-symmetric functions and non-commutative symmetric functions.

Some of the latest research in these areas has been the subject of a series of recent meetings, including an AMS/CMS meeting in Montré al in May 2002, a BIRS workshop in Banff in August 2004, and a CIRM workshop in Luminy in April 2005. It was suggested at the BIRS meeting that the draft text of M. Aguiar and S. Mahajan be expanded into the first monograph on the subject. Both are outstanding communicators. Their unified geometric approach using Coxeter complexes and projection maps allows us to construct many of the combinatorial Hopf algebras currently under study and further to understand their properties (freeness, cofreeness, etc.) and to describe morphisms among them.

The current monograph is the result of this great effort and it is for me a great pleasure to introduce it.

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Preface

This research monograph deals with the interaction between the theory of Coxeter groups on one hand and the relationships among several Hopf algebras of recent interest on the other hand. It is aimed at upper-level graduate students and researchers in these areas. The viewpoint is new and leads to a lot of simplification.

0.1 The first part: Chapters 1-3

The first part, barring Chapter 2, consists of standard material. The first two chapters are related to Coxeter theory, while the third chapter is related to Hopf algebras. We hope that they will make the second part more accessible.

Chapter 1 provides an introduction to some standard Coxeter theory written in language suitable for our purposes. The emphasis is on the gate property and the projection maps of Tits, which are crucial in almost everything that we do. The reader may be required to accept many facts on faith, since most proofs are omitted. This chapter is a prerequisite for Chapter 5.

Chapter 2 is completely self-contained. It begins with some standard material on left regular bands (LRBs). We then develop some new material on pointed faces, lunes and bilinear forms on LRBs, largely inspired from the descent theory of Coxeter groups (Chapter 5). We also introduce the concept of a projection poset which generalizes the concept of a LRB to take into account some nonassociative examples.

Chapter 3 provides a brief discussion on cofree coalgebras, the coradical filtration and the antipode, which are standard notions in the theory of Hopf algebras. We then briefly discuss three examples of Hopf algebras which have now become standard: namely, the Hopf algebras of symmetric functions Λ, noncommutative symmetric functions $\mathrm{N}\Lambda$ and quasi-symmetric functions $\mathrm{Q}\Lambda$.

0.2 The second part: Chapters 4-8

The second part consists of mostly original work. The well-prepared reader may start directly with this part and refer back to the first part as necessary. Chapter 4 provides a brief overview of this work, which is spread over the next four chapters. Chapter 5 is related to Coxeter theory, while Chapters 6, 7 and 8 are related to Hopf algebras. Each of them is kept as self-contained as possible; the reader may even read them as different papers. A more detailed overview is given in the introduction section of each of these four chapters. The results in the second part, which are stated without credit, are new to our knowledge.

0.3 Future work

At many points in this monograph we say, “This will be explained in a future work”. We plan to write a follow-up to this monograph, where these issues will be taken up. Our main motivation is not merely to prove new results or reprove existing results but rather to show that these ideas have a promising future.

0.4 Acknowledgements

We would like to acknowledge our debt to Jacques Tits, whose work provided the main foundation for this monograph. The work of Kenneth Brown on random walks and the literature on Hopf algebras, to which many mathematicians have contributed, provided us important guidelines. We would like to thank Nantel Bergeron for taking publishing initiative, Carl Riehm and Thomas Salisbury for publishing this volume in the Fields monograph series, the referees for their comments and V. Nandagopal for providing TeX assistance.

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0.5 Notation

$\mathbb{K}$ stands for a field of characteristic 0. For P a set, we write $\mathbb{K}P$ for the vector space over $\mathbb{K}$ with basis the elements of P and $\mathbb{K}P^*$ for its dual space. A word is written in italics if it is being defined at that place. While looking for a particular concept, the reader is advised to search both the notation and the subject index. The notation $[n]$ stands for the set $\{1, 2, \ldots, n\}$. The table below indicates the main letter conventions that we use.

subsets	S, T, U, V
compositions	α, β, γ
partitions	λ, μ, ρ
faces or set compositions	F, G, H, K, N, P, Q
chambers	C, D, E
pointed faces or fully nested set compositions	(F, D), (P, C)
flats or set partitions	X, Y
lunes or nested set partitions	L, M

We write Σ for the set of faces, and $\mathcal{C}$ for the set of chambers. Otherwise we use the roman script for the above sets. For example, Q is the set of pointed faces and L is the set of flats. For the coalgebras and algebras constructed from such sets, we use the calligraphic script $\mathcal{M}$, $\mathcal{N}$ and so on. There are some inevitable conflicts of notation; however, the context should keep things clear. For example, we also use the above letters F, M, K, H and S to denote various bases, V for a vector space, H for a Hopf algebra and S for an antipode.

Chapter 1

Coxeter groups

In this chapter, we review the necessary ideas on regular cell complexes, hyperplane arrangements and Coxeter groups. The material is for the most part standard; parts of it are taken from Brown [18].

1.1 Regular cell complexes and simplicial complexes

For some basic information on regular cell complexes, the reader may look at the book by Cooke and Finney [22]. Another reference is the book on oriented matroids by Björner, Las Vergnas, Sturmfels, White and Ziegler [14, Appendix 4.7].

Let Σ be a *pure* regular cell complex, that is, the maximal cells have the same dimension. In particular, Σ could be a pure simplicial complex. We will see some examples in the forthcoming sections. Elements of Σ are called *faces* and maximal faces are called *chambers*. Let $\mathcal{C}$ be the set of chambers.

We say two chambers are *adjacent* if they have a common codimension 1 face. A *gallery* is a sequence of chambers such that consecutive chambers are adjacent. We say that Σ is *gallery connected* if for any two chambers C and D, there is a gallery from C to D. For any $C, D \in \mathcal{C}$, we then define the *gallery distance* $\operatorname{dist}(C, D)$ to be the minimal length of a gallery connecting C and D. And any gallery which achieves this minimum is called a *minimum gallery* from C to D.

1.1.1 Gate property

An important concept related to the gallery metric is the gate property. It originated in the work of Tits on Coxeter complexes and buildings [99, Section 3.19.6]. The concept was first abstracted by Dress and Scharlau [86, 25]. The reader may also look at Abels [1], Mühlherr [66] and Mahajan [60] for some later work.

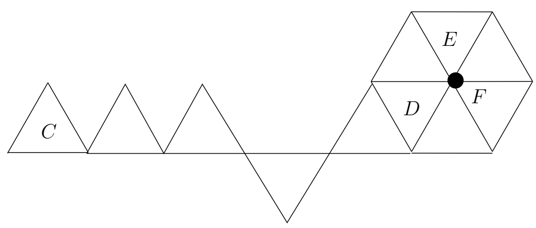


Figure 1.1: The gate property.

Gate property. For any face $F \in \Sigma$ and chamber $C \in \mathcal{C}$, there exists a chamber D containing F such that $\operatorname{dist}(C, D) \leq \operatorname{dist}(C, E)$, where E is any chamber containing F.

Furthermore, $\mathrm{dist}(C, E) = \mathrm{dist}(C, D) + \mathrm{dist}(D, E)$.

Figure 1.1 shows a part of a simplicial complex and illustrates the gate property. For F a face of Σ, let Σ_F consist of those faces which contain F. This is the *star* region of F, also denoted $\mathrm{star}(F)$. Let $\mathcal{C}_F$ be the set of chambers containing F. The gate property says that the star region $\mathrm{star}(F)$ when viewed from any chamber in the complex Σ appears to have a gate. In the above notation, the chamber D is the gate of $\mathrm{star}(F)$ when viewed from the chamber C.

A complex may or may not have the gate property. For instance, a polygon with an odd number of sides is a complex without the gate property. The gate property implies that Σ is *strongly connected*; that is, $\mathrm{star}(F)$ is gallery connected for all $F \in \Sigma$. In fact it implies that $\mathcal{C}_F$ is a *convex* subset of $\mathcal{C}$; that is, if D and E are any two chambers in $\mathcal{C}_F$ then any minimum gallery from D to E lies entirely in $\mathrm{star}(F)$ (hence in $\mathcal{C}_F$).

1.1.2 Link and join

We will need to deal with the concepts of link and join only for simplicial complexes. Hence for simplicity, we assume that Σ is a simplicial complex, but not necessarily pure.

We say that two faces of Σ are *joinable* if there is a third face containing both of them. The *link* of a face F, denoted $\mathrm{link}(F)$, is the subcomplex of Σ consisting of those faces which are disjoint from F but joinable to F. As a poset, $\mathrm{link}(F)$ is isomorphic to $\mathrm{star}(F)$. In Figure 1.1, for example, $\mathrm{star}(F)$ consists of the vertex F, and the six edges and six triangles which contain it. And $\mathrm{link}(F)$ is the outer hexagon, consisting of six vertices, six edges and the empty face.

Let Σ^1 and Σ^2 be simplicial complexes with vertex sets V_1 and V_2 respectively. Then the *join* of Σ^1 and Σ^2, denoted $\Sigma^1 * \Sigma^2$, is the simplicial complex with vertex set $V_1 \sqcup V_2$, and one face $F_1 \sqcup F_2$ for every $F_1 \in \Sigma^1$ and $F_2 \in \Sigma^2$. We denote $F_1 \sqcup F_2$ by $F_1 * F_2$, and call it the join of F_1 and F_2.

1.2 Hyperplane arrangements

A good reference for this section is Brown [18, Appendix A]. For more details, we recommend Brown [17, Chapter I]. The reader may also look at Orlik and Terao [71] or Ziegler [103]. The discussion below generalizes to oriented matroids [14]. A part of it (Sections 1.2.1 and 1.2.2) generalizes further to left regular bands (Section 2.2).

A *hyperplane arrangement* is a finite set of hyperplanes in a real vector space V. The arrangement is called *central* if all the hyperplanes pass through the origin, and *essential* if the intersection of all the hyperplanes is the zero subspace.

1.2.1 Faces

Let $\{\mathrm{H}_i\}_{i \in I}$ be an essential central hyperplane arrangement. For each i, let H_i^+ and H_i^- be the two open half-spaces defined by H_i. The choice of $+$ and $-$ is arbitrary but fixed. We say that H_i is the *supporting hyperplane* of H_i^+ and H_i^-. An open half-space together with its supporting hyperplane is a *closed* half-space. A *face* of the arrangement is a subset of V of the form

$$F = \bigcap_{i \in I} \mathrm{H}_i^{\epsilon_i},$$

where $\epsilon_i \in \{+, 0, -\}$ and $\mathrm{H}_i^0 = \mathrm{H}_i$. The totality Σ of all the faces is a poset under inclusion. The maximal faces are called *chambers*. A codimension one face of a chamber is called a *facet*. An arrangement is called *simplicial* if the chambers are simplicial cones.

Note that each face F can be defined by a sign sequence $(\epsilon_i(F))_{i \in I}$, where $\epsilon_i(F)$ is 0, $+$ or $-$, depending on whether F lies in H_i, H_i^+ or H_i^- respectively. It is clear that a chamber is a face F for which $\epsilon_i(F) \neq 0$ for each i. Each face F has an *opposite* face $\overline{F}$

obtained by replacing each $\epsilon_i(F)$ in the sign sequence defining F by its negative. We say that a hyperplane H_i *separates* faces F and K if $\epsilon_i(F)$ and $\epsilon_i(K)$ have opposite signs.

Less obviously, Σ is a semigroup. The product FK is the face with sign sequence

$$\epsilon_i(FK) = \begin{cases} \epsilon_i(F) & \text{if } \epsilon_i(F) \neq 0, \\ \epsilon_i(K) & \text{if } \epsilon_i(F) = 0. \end{cases} \tag{1.1}$$

We note some elementary but important properties of this product.

- The above product is associative. The zero subspace $\{0\}$ whose sign sequence is identically zero serves as the identity for this product.
- The set of chambers $\mathcal{C}$ is a two sided ideal in Σ.
- For a face F, we have $F\overline{F} = F$. And given faces F and P, if there exists a face G such that $FPG = F\overline{P}G$, then $FP = F\overline{P} = F$.

The product has a geometric meaning. Namely, if we move from a point of F to a point of K along a straight line then FK is the face that we are in after moving a small positive distance.

Remark A fairly complete study of the semigroup algebra associated to Σ can be found in recent work of Saliola [85].

1.2.2 Flats

Let L be the *intersection lattice* of the arrangement. It consists of those subspaces of V which can be obtained by intersecting some subset of hyperplanes in the arrangement. One may check that L is a poset under inclusion with a meet and join. In other words, L is a lattice, also referred to as the lattice of flats. We warn the reader that many authors order L by reverse inclusion, contrary to our convention.

Let $\mathrm{supp} : \Sigma \twoheadrightarrow \mathrm{L}$ be the map that sends a face F to its linear span. Equivalently, $\mathrm{supp}\, F$ is the intersection of the hyperplanes containing F. The support map satisfies the property

$$\mathrm{supp}\, FG = \mathrm{supp}\, F \vee \mathrm{supp}\, G. \tag{1.2}$$

Hence one may say that the support map is a semigroup homomorphism, with the product in L given by the join.

Let C be a chamber. The support of a codimension one face of C is called a *wall* of C. The set of walls of C is the unique minimal subset of hyperplanes which define C, see [17, Chapter 1, Section 4B, Proposition 1].

Remark There are various axiomatic approaches to oriented matroids, one of which uses covectors [14, Section 4.1.1]. In this approach, an oriented matroid is an appropriate collection of sign sequences which are closed under the product in (1.1). In this context, L is the underlying matroid obtained by forgetting the $+$ and $-$ signs, and Equation (1.2) holds. This is summarized in [14, Proposition 4.1.13], which is attributed to Edmonds and Mandel [63].

1.2.3 Spherical picture

The poset Σ has the structure of a regular cell complex homeomorphic to the sphere. This is obtained by cutting the hyperplane arrangement by the unit sphere, and identifying faces of the arrangement with cells on the sphere. The face $F = \{0\}$ is not visible in the spherical picture; it corresponds to the empty cell. In particular, the regular cell complex so obtained is pure. If the arrangement is simplicial then Σ becomes a pure simplicial complex. As far as notation goes, we do not distinguish between the linear and spherical models of Σ. The notions of Section 1.1 can now be applied to hyperplane arrangements, and in this case, we can say a lot more.

1.2.4 Gate property and other facts

The cell complex Σ of an hyperplane arrangement is gallery connected. The gallery distance $\operatorname{dist}(C, D)$ is equal to the number of hyperplanes which separate C and D. The maximum gallery distance is $\operatorname{dist}(C, \overline{C})$, which is independent of C and equal to the number of hyperplanes in the arrangement.

For chambers $E, D, C \in \mathcal{C}$, let the notation $E - \ldots - D - \ldots - C$ mean that there is a minimum gallery from E to C passing through D. Sometimes we use the more compact notation $E - D - C$. Then one can show that

$$E - D - C \iff \begin{array}{l} \text{If a hyperplane H separates } C \text{ and } D \\ \text{then it also separates } E \text{ and } C. \end{array} \tag{1.3}$$

This fact implies that a minimum gallery from C to D can always be extended to a minimum gallery $C - D - \overline{C}$.

Proposition 1.2.1 *The cell complex of faces of a central hyperplane arrangement satisfies the gate property.*

In fact, the gate of $\operatorname{star}(F)$ when viewed from C is the chamber FC, obtained by multiplying F and C using the product described in (1.1). This gives the combinatorial

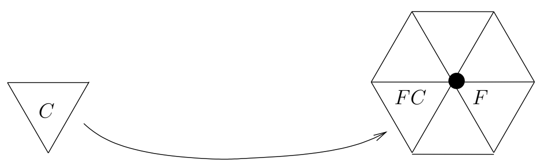


Figure 1.2: The projection map at work.

content of the geometry in the product on Σ. Namely, FC is the chamber closest to C in the gallery metric having F as a face. This is shown in Figure 1.2. We call FC the projection of C on F. The product in Σ can be recovered from the projection of chambers by

$$FP = \bigcap_{C:\ P \leq C} FC.$$

We call FP the projection of P on F.

1.3 Reflection arrangements

We review the basic facts that we need about a finite Coxeter group and its associated simplicial complex. The foundations of this theory were laid down by Tits [99]. Details can be found in Brown [17] and Mahajan [60]. The reader may also refer to Grove and Benson [39], Humphreys [47] or Bourbaki [16]. The example of type A_{n-1} is explained in the next section.

1.3.1 Finite reflection groups

A *finite reflection group* W on a real inner product space V is a finite group of orthogonal transformations of V generated by reflections s_{H} with respect to hyperplanes H through the origin. The set of hyperplanes H such that $s_{\mathrm{H}} \in W$ is the *reflection arrangement* associated with W. This arrangement is central but not necessarily essential. In the latter case, we can pass to an essential arrangement by taking the quotient of V by the subspace obtained by intersecting all the hyperplanes. The regular cell complex Σ of this essential arrangement is called the *Coxeter complex* of W. It turns out that the Coxeter complex Σ is always a simplicial complex. Furthermore, the action of W on V induces an action of W on Σ, and this action is simply transitive on the chambers. Thus the set $\mathcal{C}$ of chambers can be identified with W, once a "fundamental chamber" C_0 is chosen. We write wC_0 for the chamber corresponding to the element w of W.

The Coxeter complex has the structure of a semigroup given by (1.1), which commutes with the group action. In other words,

$$w(FK) = w(F)w(K)$$

for $w \in W$ and $F, K \in \Sigma$. This product appeared in the work of Tits on Coxeter complexes and buildings [99, Section 2.30]. He used the notation $\operatorname{proj}_F G$ instead of FG, since he viewed this operation as a geometric tool rather than as a product.

1.3.2 Types of faces

The number r of vertices of a chamber of Σ is called the *rank* of Σ (and of W); thus the dimension of Σ as a pure simplicial complex is $r-1$. It is known that one can color the vertices of Σ with r colors in such a way that vertices connected by an edge have distinct colors. The color of a vertex is also called its *label*, or its *type*, and we denote the set of all types by S. We can also define $\operatorname{type}(F)$ for any $F \in \Sigma$; it is the subset of S consisting of the types of the vertices of F. For example, every chamber has type S, while the empty face has type $\emptyset$. The action of W is type-preserving; moreover, two faces are in the same W-orbit if and only if they have the same type.

1.3.3 The Coxeter diagram

Choose a fundamental chamber C_0. It is known that the reflections s_i in the facets of C_0 generate W. In fact, W has a presentation of the form

$$\langle s_1, \dots, s_r \mid (s_i s_j)^{m_{ij}} \rangle \tag{1.4}$$

with $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$. A group with a presentation of this form is called a *Coxeter group*. The set of generators $\{s_1, \dots, s_r\}$ is usually denoted S and one says that the pair (W, S) is a *Coxeter system*. This terminology is due to Tits [99] and it recognizes the fact that the class of finite groups with a presentation as above were first studied by Coxeter [23]. With the condition of finiteness, it is the same as the class of finite reflection groups defined earlier.

The data in a Coxeter system is conveniently encoded in a picture called the *Coxeter diagram* of W. This diagram is a graph, with vertices and edges, defined as follows: There are r vertices, one for each generator s_i, and the vertices corresponding to s_i and s_j are connected by an edge if and only if $m_{ij} \geq 3$. If $m_{ij} \geq 4$ then we simply label the edge with the number m_{ij}. Figure 1.3 shows the Coxeter diagrams of type A_{n-1} and B_n.

It is customary to use the generators of W, or equivalently, the vertices of the Coxeter diagram to label the vertices of its Coxeter complex Σ. The rule is as follows.

A vertex of the fundamental chamber C_0 is labeled s_i if it is fixed by all the fundamental reflections except s_i. Since W acts transitively on $\mathcal{C}$ and the action is type-preserving, this determines the type of all the vertices of Σ.

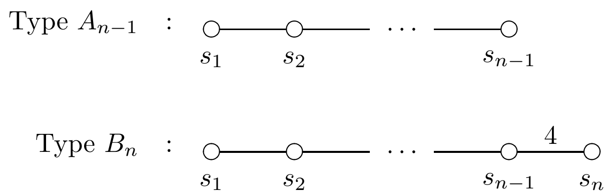


Figure 1.3: The Coxeter diagrams of type A_{n-1} and B_n.

1.3.4 The distance map

We write $l(w)$ for the minimum length of w expressed as a word using elements of S. Then the gallery metric is given by

$$\mathrm{dist}(uC_0, vC_0) = l(u^{-1}v).$$

This further suggests that we can define the W-valued gallery distance function

$$d : \mathcal{C} \times \mathcal{C} \to W$$

by the formula

$$d(uC_0, vC_0) = u^{-1}v.$$

It follows that this function is invariant under the diagonal action of W on $\mathcal{C} \times \mathcal{C}$. In other words,

$$d(C, D) = d(C', D') \iff \text{There exists a unique } w \text{ such that } wC = C',\ wD = D'.$$

Also it is clear that

$$d(E, C) = d(E, D)d(D, C). \tag{1.5}$$

The set $\mathcal{C} \times \mathcal{C}$ of pairs of chambers will play a central role in our theory.

1.3.5 The Bruhat order

We say that $u \le v$ in the *weak left Bruhat order* on W if there is a minimum gallery $E - D - C$ such that $d(D, C) = u$ and $d(E, C) = v$.

Alternatively,

$$\begin{aligned} u \le v \text{ in } W \quad &\iff \quad \text{There is a minimum gallery } v^{-1}C_0 - u^{-1}C_0 - C_0. \\ &\iff \quad \text{There is a minimum gallery } C_0 - vu^{-1}C_0 - vC_0. \end{aligned}$$

The first gallery condition is illustrated in Figure 1.4. The second gallery above is obtained from the first by multiplying by v.

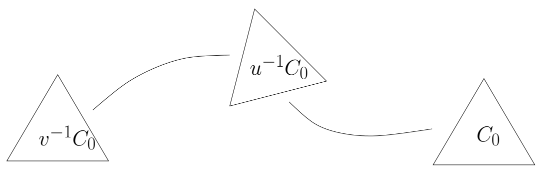


Figure 1.4: A minimum gallery that illustrates the partial order on W.

By letting $d(E, D) = w$ in the first definition above, one obtains a more combinatorial description of the weak left Bruhat order. Namely,

$$u \leq v \iff v = wu \text{ and } l(v) = l(w) + l(u).$$

The left in the notation refers to the fact that w appears to the left of u in the expression $v = wu$.

One can define the *weak right Bruhat order* on W, denoted $\leq_{rb}$, by the equation

$$u \leq_{rb} v \iff u^{-1} \leq v^{-1}.$$

The partial order $\leq$ will be used crucially in Chapters 5, 7 and 8, while the partial order $\leq_{rb}$ will only make a brief appearance in Chapter 7. Hence whenever we refer to the partial order on W, it always means the weak left Bruhat order.

1.3.6 The descent algebra: A geometric approach

For a Coxeter system (W, S), let

$$\overline{\mathrm{Q}} = \{T \mid T \leq S\}$$

be the poset of subsets of S ordered by inclusion. Let $\mathrm{des} : W \to \overline{\mathrm{Q}}$ be the descent map

$$\mathrm{des}(w) = \{s \in S \mid l(ws) < l(w)\}.$$

Let $\mathbb{K}W$ be the group algebra of W over the field $\mathbb{K}$. Solomon [92] showed that the elements

$$d_T = \sum_{\mathrm{des}(w) \leq T} w,$$

as T varies, give a basis for a subalgebra of $\mathbb{K}W$. This subalgebra is known as the descent algebra. Further Solomon also computed the radical of this algebra [92, Theorem 3]. A geometric version of his result is given in Lemma 2.6.6.

Let Σ be the Coxeter complex of W and $\mathbb{K}\Sigma$ be its semigroup algebra. Let $(\mathbb{K}\Sigma)^W$ be the algebra of invariants of the W-action on $\mathbb{K}\Sigma$. A basis for $(\mathbb{K}\Sigma)^W$ is given by

$$\sigma_T = \sum_{\mathrm{type}(F) = T} F,$$

as T ranges over all subsets of S. Bidigare [11] proved that the map

$$(\mathbb{K}\Sigma)^W \to \mathbb{K}W,$$

that sends σ_T to d_T is an algebra anti-homomorphism. It is easy to see that this map is injective and its image is precisely the descent algebra. Hence $(\mathbb{K}\Sigma)^W$ is anti-isomorphic to the descent algebra. The proof, which is conceptual and short, is also explained in Brown [18, Section 9.6].

1.3.7 Link and join

The relevance to us of the link and join operations on simplicial complexes is that Coxeter complexes are well behaved with respect to these operations. The facts written below will be crucially needed in Chapters 6, 7 and 8.

Link

Let $F \in \Sigma$ be a face of type $T \leq S$. And let $W_{S \setminus T}$ be the subgroup of W generated by $S \setminus T$. Then the link of F in Σ, denoted $\mathrm{link}(F)$, is again a Coxeter complex. The Coxeter group of $\mathrm{link}(F)$ can be viewed as a subgroup of W and it is a conjugate of $W_{S \setminus T}$. A subgroup of W of this form is known as a *parabolic* subgroup. The Coxeter diagram of $\mathrm{link}(F)$ is obtained from the Coxeter diagram of Σ by deleting all the vertices whose type is contained in T. The map

$$\Sigma \to \mathrm{link}(F),$$

that sends the face K to the face in $\mathrm{link}(F)$ which corresponds to FK, is a semigroup homomorphism. For convenience, we usually identify $\mathrm{link}(F)$ with $\mathrm{star}(F)$ and work with the map $\Sigma \to \mathrm{star}(F)$ that sends K to FK. This map also preserves opposites. Namely, if K and $\overline{K}$ are opposite faces in Σ then FK and $F\overline{K}$ are opposite faces in $\mathrm{star}(F)$.

Remark It is clear that if F and F' are faces of the same type then $\mathrm{link}(F) \cong \mathrm{link}(F')$.

Join

The join $\Sigma^1 * \Sigma^2$ of two Coxeter complexes is again a Coxeter complex, whose diagram is the disjoint union of the diagrams of Σ^1 and Σ^2. Its Coxeter group is the cartesian product of the two smaller Coxeter groups. Further, the join operation is compatible with the projection maps and the distance map, that is,

$$(H_1 * N_1)(H_2 * N_2) = (H_1 H_2 * N_1 N_2), \text{ where } H_i, N_i \in \Sigma^i.$$

$$d(C * C', D * D') = (d(C, D), d(C', D')).$$

In addition, a minimum gallery in $\Sigma^1 * \Sigma^2$ yields a minimum gallery in Σ^1 and a minimum gallery in Σ^2. And using the galleries in the two smaller complexes, one can reconstruct the original gallery. We refer to this fact as the compatibility of galleries with joins.

1.4 The Coxeter group of type A_{n-1}

The symmetric group S_n on n letters can be generated by $n-1$ transpositions s_1, s_2, $\ldots$, s_{n-1}, where s_i interchanges i and $i+1$ and fixes the other letters. These generators satisfy the relations

$$s_i^2 = 1,\ (s_i s_{i+1})^3 = 1,\ (s_i s_j)^2 = 1 \text{ if } i \text{ and } j \text{ differ by more than 1.}$$

This gives rise to a presentation for S_n, which is of the form written in (1.4). Hence S_n is a Coxeter group, which is also known as the Coxeter group of type A_{n-1}. Its Coxeter diagram is shown in Figure 1.3.

1.4.1 The braid arrangement

The reflection arrangement in this case is the *braid arrangement* in $\mathbb{R}^n$. It is discussed in detail in [11, 12, 13, 19]. It consists of the $\binom{n}{2}$ hyperplanes H_{ij} defined by $x_i = x_j$, where $1 \leq i < j \leq n$. The intersection of all these hyperplanes is the line $x_1 = x_2 = \ldots = x_n$; so the arrangement is not essential. Each chamber is determined by an ordering of the coordinates, so it corresponds to a permutation. The faces of a chamber are obtained by changing to equalities some of the inequalities defining that chamber.

When $n = 4$, the arrangement consists of six planes in $\mathbb{R}^4$. By taking the quotient of $\mathbb{R}^4$ by the line $x_1 = x_2 = x_3 = x_4$, and cutting by the unit sphere, we obtain the spherical picture shown in Figure 1.5. It has been reproduced from Billera, Brown and Diaconis [13]. As an example, the permutation 2314 corresponds to the inequality

$$x_2 < x_3 < x_1 < x_4.$$

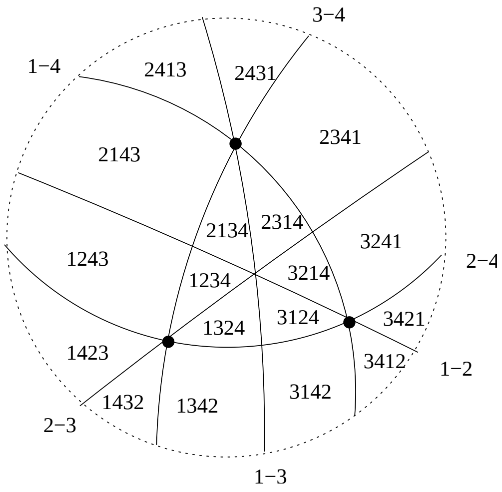


Figure 1.5: The braid arrangement when $n = 4$.

1.4.2 Types of faces

The symmetric group S_n acts on the braid arrangement by permuting the coordinates. We fix $x_1 < x_2 < \ldots < x_n$ to be the fundamental chamber C_0. The supports of the facets of C_0 are hyperplanes of the form $x_i = x_{i+1}$, where $1 \leq i \leq n-1$. The reflection in the hyperplane $x_i = x_{i+1}$ corresponds to the generator s_i of S_n that interchanges the coordinates x_i and x_{i+1}. The chamber C_0 has $n-1$ vertices, namely

$$\begin{aligned} s_1 \quad &: \quad x_1 < x_2 = \ldots = x_n, \\ s_2 \quad &: \quad x_1 = x_2 < x_3 = \ldots = x_n, \\ &\vdots \\ s_{n-1} \quad &: \quad x_1 = \ldots = x_{n-1} < x_n. \end{aligned}$$

The letters $s_1, s_2, \ldots, s_{n-1}$ on the left are labels assigned to each vertex by the rule mentioned in Section 1.3.3. Applying the action of W we see, for example, that

$$x_{\pi(1)} < x_{\pi(2)} = \ldots = x_{\pi(n)}$$

gives all vertices of type s_1 as π varies over the permutations of $[n]$.

1.4.3 Set compositions and partitions

A *composition* of the set $[n]$ is an ordered partition $F^1|\ldots|F^k$ of $[n]$. That is, $F^1, \ldots, F^k$ are disjoint nonempty sets whose union is $[n]$, and their order counts. We can encode the system of equalities and inequalities defining a face by a composition of $[n]$; the equalities are used to define the blocks and the inequalities to order them. For example, for $n = 4$,

$$x_1 = x_3 < x_2 = x_4 \qquad \longleftrightarrow \qquad 13|24.$$

Thus the faces of Σ are compositions of the set $[n]$. Observe that the vertices of type s_1 are two block compositions such that the first block is a singleton. Note that F is a face of H if and only if H consists of a composition of F^1 followed by a composition of F^2, and so on, that is, if and only if H is a refinement of F.

The product in Σ is also easy to describe in this language. We multiply two compositions by taking intersections and ordering them lexicographically; more precisely, if

$F = F^1|\dots|F^l$ and $H = H^1|\dots|H^m$, then

$$FH = (F^1 \cap H^1|\dots|F^1 \cap H^m|\dots|F^l \cap H^1|\dots|F^l \cap H^m)\hat{},$$

where the hat means "delete empty intersections". The 1-block composition is the identity for the product.

The lattice of flats L is the lattice of set partitions ordered by refinement. For example, for $n = 4$,

$$x_1 = x_3,\ x_2 = x_4 \qquad \longleftrightarrow \qquad \{13, 24\}.$$

The product or join of two set partitions is their smallest common refinement. More precisely, we multiply partitions by taking intersections of the parts and deleting empty intersections. The similarity between the product in Σ and L is explained by the support map. The support map $\Sigma \twoheadrightarrow \mathrm{L}$ forgets the ordering of the blocks. For example, for $n = 4$, the support map sends the face $13|24$ to $\{13, 24\}$.

Thus we see that set compositions and partitions emerge naturally in this example. In fact, one can explain this example in purely combinatorial terms without reference to hyperplane arrangements. More details are given in Section 5.4.

1.4.4 The Bruhat order

Let $\mathrm{Inv}(u)$ be the set of inversions of a permutation $u \in \mathrm{S}_n$, that is,

$$\mathrm{Inv}(u) \ := \ \{(i, j) \in [n] \times [n] \bigm| i < j \text{ and } u(i) > u(j)\}.$$

The inversion set determines the permutation. Given u and v in S_n, we write $u \leq v$ if $\mathrm{Inv}(u) \subseteq \mathrm{Inv}(v)$. This gives the weak left Bruhat order on S_n.

Note that $\mathrm{Inv}(u)$ can be identified with the set of hyperplanes which separate C_0 and $u^{-1}C_0$, by letting the pair (i, j) correspond to the hyperplane $x_i = x_j$. As an illustrative example, take $u = 3|4|2|1$. Then $(1, 3) \in \mathrm{Inv}(u)$. And note that the hyperplane $x_1 = x_3$ separates

$$x_1 < x_2 < x_3 < x_4 \quad \text{and} \quad x_4 < x_3 < x_1 < x_2,$$

which are the chambers C_0 and $u^{-1}C_0$ respectively. Now using (1.3), one sees that the above definition of the weak left Bruhat order is same as the gallery definition given earlier.

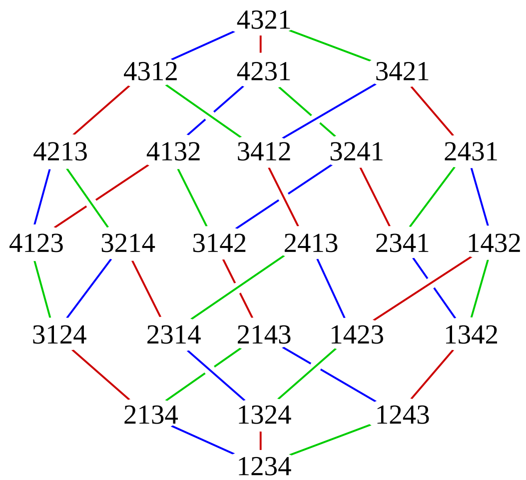


Figure 1.6: The weak left Bruhat order on S_4.

Figure 1.6, which is taken from [4], shows the partial order on S_4. It can also be drawn from Figure 1.5 by replacing each permutation by its inverse and drawing an edge between adjacent chambers.

Chapter 2

Left regular bands

Left regular bands, or LRBs for short, are semigroups that have been of recent interest in random walk theory. They are easy to define and work with and have a rich source of examples. For more details, see the seminal paper of Brown [18]. The main example is the poset of faces of a hyperplane arrangement defined in Section 1.2. The LRB terminology we use is motivated by this example. Coxeter complexes fall in this category as they arise from reflection arrangements. As a slightly more general example, we have the poset of covectors of an oriented matroid. More information about LRBs can be found in Grillet [38] and Petrich [76, 77]. The origin of LRBs can be traced to Schützenberger [90].

2.1 Why LRBs?

The main motivation for LRBs is that many of our results in Chapter 5 generalize to LRBs. Coxeter complexes, and more generally, the poset of faces of a hyperplane arrangement, to which most of the theory is applied, can be viewed as special cases. To effect this generalization, one is forced to develop the standard theory of LRBs further. We begin with the standard material in Section 2.2 and then present the new material in Sections 2.3-2.7.

In Section 2.3, we introduce the concept of a pointed face. This notion will allow us to properly formulate the adjointness properties of the descent map to be considered in Chapter 5. Similarly, Section 2.4 on sub and quotient LRBs is motivated by the Hopf algebra considerations in Chapter 6.

In Section 2.5, we define and study a bilinear form on any LRB Σ. This bilinear form controls the commutativity in diagram (5.8), which we will encounter in Chapter 5. We show that the radical of this form contains the radical of the semigroup algebra $\mathbb{K}\Sigma$, which was computed by Bidigare [11] and Brown [18]. Further we give a computable criterion for the equality of radicals to hold.

In Section 2.6, we specialize to the case when Σ is the Coxeter complex of a Coxeter group. In this situation, one can pass to invariants and induce a bilinear form on $(\mathbb{K}\Sigma)^W$, which can be identified with $\mathbb{K}\overline{\mathrm{Q}}$ defined in Section 1.3.6. We know that $(\mathbb{K}\Sigma)^W$ is anti-isomorphic to the descent algebra (Section 1.3.6). Following the method in Section 2.5, we show that the radical of the above form contains the radical of the descent algebra. Further we show that the two radicals are equal if the above mentioned criterion is satisfied.

Some of our results in the second part of Chapter 5 generalize further to projection posets, which is a notion that we introduce in Section 2.7. This allows us to consider nonassociative structures like buildings and modular lattices.

2.2 Faces and flats

The material in this section is taken from Brown [18, Appendix B]. In this section and the next, we define the basic objects related to LRBs. Towards the end of each section, we explain the examples of a hyperplane arrangement and the free LRB. The example of the braid arrangement is explicitly worked out in Sections 1.4 and 5.4. The reader may want to read these sections in parallel with the material below.

2.2.1 Faces

Let Σ be a *left-regular band*, or a LRB for short. It is a semigroup that satisfies the identities

$$x^2 = x \quad \text{and} \quad xyx = xy \tag{2.1}$$

for all $x, y \in \Sigma$. Early references to this concept occur in Klein-Barmen [50] and Schützenberger [90]. For simplicity, we assume that Σ is finite and has a unit. In this case, the first identity follows from the second.

The relation

$$x \leq y \iff xy = y$$

defines a partial order on Σ. Elements of Σ are called *faces* and for $x \leq y$, one says that x is a face of y.

- If $x \leq y$ then $zx \leq zy$ for any z; however, $xz \leq yz$ may not hold.
- If there is z such that $xz = y$ then $x \leq y$. In other words, x is always a face of xz.

The above properties follow from the definitions. A complete list of properties which we will need to use later is given in Section 2.7.2.

2.2.2 Flats

Define another relation $\preceq$ on Σ by $x \preceq y \iff yx = y$. This is transitive and reflexive, but not necessarily antisymmetric. We therefore obtain a poset L by identifying x and y if $x \preceq y$ and $y \preceq x$. We denote the quotient map by $\operatorname{supp}\colon \Sigma \twoheadrightarrow \mathrm{L}$. Then

$$yx = y \iff \operatorname{supp} x \leq \operatorname{supp} y$$

holds by definition. Elements of L are called *flats*. It follows that

$$xy = x \text{ and } yx = y \iff \operatorname{supp} x = \operatorname{supp} y. \tag{2.2}$$

The support map is order preserving. To see this, suppose that $x \leq y$, that is, $xy = y$. Premultiplying by y and using Equation (2.1), we conclude that $yx = y$ and hence $\operatorname{supp} x \leq \operatorname{supp} y$. Following [18, Appendix B], it can also be shown that L is a join semilattice and that

$$\operatorname{supp} xy = \operatorname{supp} x \vee \operatorname{supp} y. \tag{2.3}$$

In other words, the support map is a map of semigroups, with the product in L given by the join.

2.2.3 Chambers

We call an element $c \in \Sigma$ a *chamber* if $\operatorname{supp} c = \hat{1}$, where $\hat{1}$ is the largest element of L.

Proposition 2.2.1 *[18, Proposition 9] The following conditions on an element $c \in \Sigma$ are equivalent:*

1. *c is a chamber.*

2. *$cx = c$ for all $x \in \Sigma$.*

3. *c is maximal in the poset Σ.*

For a partial generalization, see Lemma 2.7.6. Thus the set $\mathcal{C}$ of chambers consists of the maximal elements of Σ and is a two sided ideal in Σ. Let $\mathcal{C}_x = \{c \in \mathcal{C} \mid x \leq c\}$. Observe the following.

Lemma 2.2.1 *If $xy = x$ and $yx = y$ then there is a bijection $\mathcal{C}_x \to \mathcal{C}_y$ given by $c \mapsto yc$ and with inverse $d \mapsto xd$.*

For a generalization of this result to projection posets, see Lemma 2.7.7.

2.2.4 Examples

Example The motivating example of a LRB is the poset of faces of a central hyperplane arrangement, with the product as given in (1.1). The notion of flats and the support map given by the LRB theory agree with those described in Section 1.2. The reader may compare Equations (2.3) and (1.2).

Example The free LRB on n letters consists of words with no letter repetitions. The product of u and v is the concatenation $(uv)\widehat{\ }$, where the hat means "delete the letters in v that have occurred in u". The empty word is the identity for this product. And $u \leq v$ if u is an initial subword of v. The chambers are the permutations of the n letters. The lattice of flats consists of subsets of the n letters, and the support map sends a word to the subset of letters it contains.

2.3 Pointed faces and lunes

There is an analogue of Sections 2.2.1 and 2.2.2 with faces and flats replaced by pointed faces and lunes respectively. Details are as below.

2.3.1 Pointed faces

Let $\mathrm{Q} = \{(x, c) \mid x \leq c\} \subseteq \Sigma \times \mathcal{C}$. Define a partial order on Q by

$$(x, c) \leq (y, d) \iff c = d \text{ and } x \leq y.$$

Elements of Q are called *pointed faces*.

2.3.2 Lunes

Define another relation $\preceq$ on Q by $(x, c) \preceq (y, d) \iff yx = y$ and $yc = d$. This is transitive and reflexive, but not necessarily antisymmetric. We therefore obtain a poset Z by identifying (x, c) and (y, d) if $(x, c) \preceq (y, d)$ and $(y, d) \preceq (x, c)$. We denote the quotient map by $\mathrm{lune}\colon \mathrm{Q} \twoheadrightarrow \mathrm{Z}$. Then

$$yx = y \text{ and } yc = d \iff \mathrm{lune}(x, c) \leq \mathrm{lune}(y, d)$$

holds by definition. Elements of Z are called *lunes* and $\mathrm{lune}(x, c)$ is called the lune of x and c. It follows that

$$xy = x,\ xd = c,\ yx = y \text{ and } yc = d \iff \mathrm{lune}(x, c) = \mathrm{lune}(y, d). \tag{2.4}$$

The lune map is order preserving. To see this, suppose that $(x, c) \leq (y, c)$, that is, $xy = y$. Argue as for the support map to conclude that $yx = y$ and hence $\mathrm{lune}(x, c) \leq \mathrm{lune}(y, c)$.

2.3.3 The relation of Q and Z with Σ and L

For a poset P, let $\mathbb{K}P$ be the vector space over $\mathbb{K}$ with basis the elements of P. Note that $\mathbb{K}\Sigma$ and $\mathbb{K}\mathrm{L}$ are semigroup algebras. The relation of Q and Z with Σ and L respectively can be seen as follows. Define the map base : $\mathrm{Q} \to \Sigma$ by $(x,c) \mapsto x$ and the map base* : $\mathbb{K}\Sigma \to \mathbb{K}\mathrm{Q}$ by $x \mapsto \sum_{c:\ x \le c} (x,c)$. These maps induce maps $\mathrm{Z} \to \mathrm{L}$ and $\mathbb{K}\mathrm{L} \to \mathbb{K}\mathrm{Z}$ so that the following diagrams commute.

$$
\begin{tikzcd}
\Sigma \arrow[d, twoheadrightarrow, "\text{supp}"'] & \mathrm{Q} \arrow[l, twoheadrightarrow, "\text{base}"'] \arrow[d, twoheadrightarrow, "\text{lune}"] \\
\mathrm{L} & \mathrm{Z} \arrow[l, twoheadrightarrow, "\text{base}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}\Sigma \arrow[r, hookrightarrow, "\text{base}^*"] \arrow[d, twoheadrightarrow, "\text{supp}"'] & \mathbb{K}\mathrm{Q} \arrow[d, twoheadrightarrow, "\text{lune}"] \\
\mathbb{K}\mathrm{L} \arrow[r, rightarrowtail, "\text{base}^*"'] & \mathbb{K}\mathrm{Z}
\end{tikzcd}
\tag{2.5}
$$

Proof For the first diagram, we need to show that

$$\text{lune}(x,c) = \text{lune}(y,d) \implies \text{supp}\, x = \text{supp}\, y.$$

This follows from (2.2) and (2.4).

For the second diagram, we need to show that

$$\text{supp}\, x = \text{supp}\, y \implies \sum_{c \in \mathcal{C}_x} \text{lune}(x,c) = \sum_{d \in \mathcal{C}_y} \text{lune}(y,d).$$

This follows from (2.2), (2.4) and Lemma 2.2.1.

□

2.3.4 Lunar regions

There is another approach one can take to lunes, which is closer to intuition and which justifies the terminology. Namely, define a map reg : $\mathrm{Q} \to \{R \mid R \subseteq \Sigma\}$ by

$$\text{reg}(x,c) = \{y \mid xy \le c\}. \tag{2.6}$$

The terminology R and $\text{reg}(x,c)$ indicate that these are "regions" in Σ. We say that $\text{reg}(x,c)$ is the lunar region of x and c in Σ. Let Z' be the image of the map reg. The sets Z' and Z are closely related; the precise relation between them is as follows.

Lemma 2.3.1 *There is a commutative diagram*

$$
\begin{tikzcd}
& \mathrm{Q} \arrow[dl, twoheadrightarrow, "\text{lune}"'] \arrow[dr, twoheadrightarrow, "\text{reg}"] & \\
\mathrm{Z} \arrow[rr, twoheadrightarrow, "\text{zone}"'] & & \mathrm{Z}'
\end{tikzcd}
$$

Equivalently, by (2.4), for $x \le c$ and $y \le d$, we have

$$xy = x,\ xd = c,\ yx = y \text{ and } yc = d \implies \text{reg}(x,c) = \text{reg}(y,d). \tag{2.7}$$

We call the induced map $\mathrm{Z} \to \mathrm{Z}'$ the zone map.

Proof Let x, y, c, d be as in the left hand side of (2.7). Now let $z \in \text{reg}(x,c)$, that is, $xzc = c$. Then

$$yzd = yxzyc = yxzc = yc = d.$$

For the first equality, we used $y = yx$ and $d = yc$. For the second equality, we used Equation (2.1). From the above equation, we conclude that $z \in \text{reg}(y,d)$. This shows that $\text{reg}(x,c) \subseteq \text{reg}(y,d)$ and the result follows by symmetry.

□

Open Question Identify the class of LRBs for which the zone map is a bijection; in other words, for which the reverse implication in (2.7) holds.

We give a partial answer to this question. The zone map is a bijection for the poset of faces of hyperplane arrangements, see Lemma 2.3.3. However, this fails for the free LRB, see Section 2.3.5. In the general case, one can say the following.

$$\operatorname{reg}(x,c)=\operatorname{reg}(y,d) \implies \operatorname{reg}(x,c)=\operatorname{reg}(xy,c)=\operatorname{reg}(y,d)=\operatorname{reg}(yx,d). \tag{2.8}$$

Note that $\operatorname{reg}(x,c)=\operatorname{reg}(y,d)$ implies that $y\in\operatorname{reg}(x,c)$, that is, $xy\leq c$. Hence the term $\operatorname{reg}(xy,c)$ written above makes sense.

Proof By symmetry, it is enough to show that $\operatorname{reg}(x,c)=\operatorname{reg}(xy,c)$. This follows by the following string of equivalences.

$$z\in\operatorname{reg}(xy,c) \iff xyz\leq c \iff yz\in\operatorname{reg}(x,c) \iff yz\in\operatorname{reg}(y,d)$$

$$yz\in\operatorname{reg}(y,d) \iff yyz\leq d \iff z\in\operatorname{reg}(y,d) \iff z\in\operatorname{reg}(x,c)$$

The third and last equivalence hold by the assumption $\operatorname{reg}(x,c)=\operatorname{reg}(y,d)$ and the rest hold by the definition of a lunar region given in (2.6).

□

2.3.5 Examples

Example We return to the example of hyperplane arrangements and first describe the set of lunar regions Z'. More material on lunes can be found in Billera, Brown and Diaconis [13] or Mahajan [60, Chapter 1]. Just as an element of L is an intersection of hyperplanes, an element of Z' is an intersection of a special set of closed half-spaces.

Lemma 2.3.2 *The lunar region of F and D, namely $\operatorname{reg}(F,D)$, is the intersection of those closed half-spaces which contain D and whose supporting hyperplane contains F. More precisely, $\operatorname{reg}(F,D)$ consists of those faces which lie in the above intersection.*

Proof Using (1.1), we obtain:

$$FK\leq D \iff \text{If a hyperplane H contains } F \text{ then it does not separate } K \text{ and } D.$$

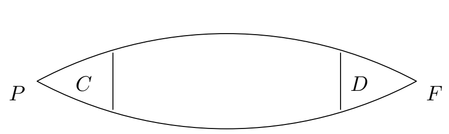
The lemma now follows from the definition of a lunar region given in (2.6).

Remark We note that $\operatorname{reg}(F,D)$ is a chamber in the subarrangement consisting of those hyperplanes which contain F. The walls of this chamber are same as the walls of D which contain F. This requires an extra argument which we leave to the reader. Hence $\operatorname{reg}(F,D)$ is in fact the intersection of those closed half-spaces which contain D and whose supporting hyperplane contains F and is a wall of D.

Remark In the lemma below, we will identify lunes and lunar regions. Hence we may say that the base of the lunar region $\operatorname{reg}(F,D)$ is $\operatorname{supp} F$.

In Figure 2.1, we have shown two schematic spherical pictures for lunar regions in a rank 3 arrangement. Note that hyperplanes in this case are great circles on the sphere. In the first picture, F is a vertex of the two dimensional chamber D; hence there are two supporting hyperplanes in question. The two great circles intersect at F and its opposite vertex P, dividing the sphere into four regions. The region containing D is the lunar region $\operatorname{reg}(F,D)$ and its base consists of the two vertices F and P.

In the second picture, F is an edge; hence there is only one supporting hyperplane in question. It divides the sphere into two regions. The region containing D is the lunar



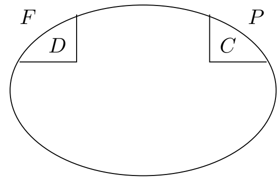


Figure 2.1: Two low dimensional pictures of the lunar regions $\mathrm{reg}(P, C) = \mathrm{reg}(F, D)$.

region $\mathrm{reg}(F, D)$ and its base is the hyperplane itself, which is shown as the ellipse passing through F and P. For a more concrete example of a lunar region, see the shaded region in Figure 7.2.

Lemma 2.3.3 *For the poset of faces of a central hyperplane arrangement, the* zone *map* $\mathrm{Z} \to \mathrm{Z}'$ *in Lemma 2.3.1 is a bijection.*

Proof By (2.7) and (2.8), it is enough to show that

$$\mathrm{reg}(F, D) = \mathrm{reg}(FP, D) \implies F = FP,$$

where F and P are arbitrary faces with $P \in \mathrm{reg}(F, D)$.

Let $\mathrm{reg}(F, D) = \mathrm{reg}(FP, D)$ and $\overline{P}$ be the opposite face to P. Since $P\overline{P} = P$, we have $FP\overline{P} = FP \leq D$. Hence by definition $\overline{P} \in \mathrm{reg}(FP, D)$, which by assumption implies $\overline{P} \in \mathrm{reg}(F, D)$. Therefore we obtain $FP, F\overline{P} \leq D$. By applying the third elementary property of the product listed in Section 1.2.1, we conclude that $FP = F\overline{P} = F$.

$\square$

Remark As one can see from the proof, the existence of an opposite makes central hyperplane arrangements special among LRBs.

Example We return to the example of the free LRB on n letters. From the definition, we have $\mathrm{lune}(x, c) = \mathrm{lune}(y, d)$ if x and y contain the same letters and the subword of c obtained by deleting the initial segment x is same as the subword of d obtained by deleting the initial segment y. Thus the set Z can be identified with the set of words in the n letters without repetition, which is the same as Σ. The lune map then sends the pointed face (x, c) to the subword of c obtained by deleting the initial segment x. In this notation, a word y is an element of $\mathrm{zone}(x)$ if the letters which are common to both x and y form an initial segment of x. In particular, the zone of a one letter word is the entire set Σ. This shows that the zone map is not injective. To give a concrete example, take $n = 3$ and the letters to be x, y and z. Then

$$\mathrm{reg}(xy, xyz) = \mathrm{reg}(yz, yzx) \quad \text{but} \quad \mathrm{lune}(xy, xyz) \neq \mathrm{lune}(yz, yzx),$$

which says that $\mathrm{zone}(z) = \mathrm{zone}(x)$.

2.4 Link and join of LRBs

In Chapter 6, we will construct Hopf algebras from the family of LRBs $\{\Sigma^n\}_{n \geq 0}$, where Σ^n is the Coxeter complex of S_n. In this section, we state two simple but useful lemmas in the construction. They are valid for any LRB.

2.4.1 SubLRB and quotient LRB

Let Σ be a LRB and Q, L and Z be as above. Let

$$\Sigma_x = \{y \in \Sigma \mid x \le y\}.$$

Then Σ_x is a LRB in its own right, which we may also call the link or star region of x in Σ. Denote its corresponding objects by Q_x, L_x and Z_x respectively. Explicitly, we have

$$\mathrm{Q}_x = \{(y,d) \in \mathrm{Q} \mid x \le y \le d\},$$

$$\mathrm{L}_x = \{X \in \mathrm{L} \mid \operatorname{supp} x \le X\}, \quad \text{and}$$

$$\mathrm{Z}_x = \{\operatorname{lune}(y,d) \in \mathrm{Z} \mid x \le y \le d\} = \{L \in \mathrm{Z} \mid \operatorname{supp} x \le \operatorname{base} L\}.$$

In addition to a subLRB, one can view Σ_x as a quotient LRB of Σ. The quotient map

$$x\cdot : \Sigma \twoheadrightarrow \Sigma_x$$

sends y to xy. This induces the map $x\cdot : \mathrm{Q} \twoheadrightarrow \mathrm{Q}_x$ which sends (y,d) to (xy, xd), the map $x\cdot : \mathrm{L} \twoheadrightarrow \mathrm{L}_x$ which sends X to $X \vee \operatorname{supp} x$, and the map $x\cdot : \mathrm{Z} \twoheadrightarrow \mathrm{Z}_x$ which sends $\operatorname{lune}(y,d)$ to $\operatorname{lune}(xy, xd)$.

Lemma 2.4.1 *The following diagrams commute.*

$$\begin{tikzcd}
\Sigma_x \arrow[r, hook] \arrow[d, "\operatorname{supp}"'] & \Sigma \arrow[r, two heads, "x\cdot"] \arrow[d, "\operatorname{supp}"] & \Sigma_x \arrow[d, "\operatorname{supp}"] \\
\mathrm{L}_x \arrow[r, hook] & \mathrm{L} \arrow[r, two heads, "x\cdot"'] & \mathrm{L}_x
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Q}_x \arrow[r, hook] \arrow[d, "\operatorname{lune}"'] & \mathrm{Q} \arrow[r, two heads, "x\cdot"] \arrow[d, "\operatorname{lune}"] & \mathrm{Q}_x \arrow[d, "\operatorname{lune}"] \\
\mathrm{Z}_x \arrow[r, hook] & \mathrm{Z} \arrow[r, two heads, "x\cdot"'] & \mathrm{Z}_x
\end{tikzcd}$$

The proof is a direct consequence of the definitions.

2.4.2 Product of LRBs

For $i = 1, 2$, let Σ^i be a LRB and Q^i, L^i and Z^i be the associated objects. Then the cartesian product $\Sigma = \Sigma^1 \times \Sigma^2$ is a LRB with componentwise multiplication; we may call Σ the join of Σ^1 and Σ^2.

Lemma 2.4.2 *The associated posets of* $\Sigma = \Sigma^1 \times \Sigma^2$ *are* $\mathrm{Q} = \mathrm{Q}^1 \times \mathrm{Q}^2$, $\mathrm{L} = \mathrm{L}^1 \times \mathrm{L}^2$ *and* $\mathrm{Z} = \mathrm{Z}^1 \times \mathrm{Z}^2$.

2.5 Bilinear forms related to a LRB

In this section, we initiate a study of three bilinear forms related to a LRB. They are defined on $\mathbb{K}\mathrm{Q}$, $\mathbb{K}\Sigma$ and $\mathbb{K}\mathrm{L}$ respectively. The material in Sections 2.5.1-2.5.3, except Lemma 2.5.1 and Corollary 2.5.1, generalizes to projection posets, which are defined in Section 2.7.

2.5.1 The bilinear form on $\mathbb{K}\mathrm{Q}$

Define a symmetric bilinear form on $\mathbb{K}\mathrm{Q}$ by

$$\langle (x,c), (y,d) \rangle = \begin{cases} 1 & \text{if } yc = d \text{ and } xd = c, \text{ or equivalently,} \\ & \text{if } c \in \operatorname{reg}(y,d) \text{ and } d \in \operatorname{reg}(x,c), \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 2.2, we have shown the schematic picture of two intersecting lunar regions. It illustrates the case when $\langle (P,C),(F,D)\rangle = 1$ for the poset of faces of a hyperplane arrangement.

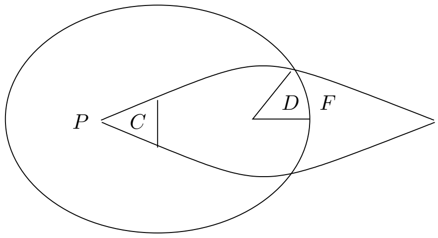


Figure 2.2: The pointed faces (P,C) and (F,D) lie in each other's lunar regions.

Open Question The above form is degenerate in general. Compute its radical.

As a partial answer, we give one source of degeneracy in the Coxeter case. In this case, by passing to invariants, we obtain an induced form on $(\mathbb{K}\mathrm{Q})^W \cong \mathbb{K}\overline{\mathrm{Q}}$, which we show to be degenerate in Section 2.6. This implies by general principles that the original form was also degenerate. For example, for $T,U \leq S$, if $T-U$ is in the radical of the induced form then

$$\sum_{D\in\mathcal{C}} (T_D, D) - (U_D, D) \tag{2.9}$$

belongs to the radical of the original form. Here T_D refers to the face of D which is of type T.

Example For type A, the elements of Σ and Q are set compositions and fully nested set compositions respectively, see Section 5.4. One can give an explicit combinatorial definition for the bilinear form on $\mathbb{K}\mathrm{Q}$. We illustrate it by the following example.

$$\langle (6|2|3\Big|5|1\Big|4|7),(4\Big|6|7\Big|2|5|1\Big|3)\rangle = 1.$$

This is because $6|2|3|5|1|4|7$ is a shuffle of $4, 6|7, 2|5|1, 3$ and $4|6|7|2|5|1|3$ is a shuffle of $6|2|3, 5|1, 4|7$. This should make the general definition clear. The reader can also play with Figure 1.5 and match the geometric and combinatorial definitions for $n = 4$.

To obtain an element in the radical of this form, one can take T and U to be two compositions, say $(1,2,1)$ and $(2,1,1)$, with the same underlying partition, and then use formula (2.9).

2.5.2 The pairing between $\mathbb{K}\mathrm{Q}$ and $\mathbb{K}\Sigma$

Consider the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathbb{K}\mathrm{Q}\times\mathbb{K}\Sigma \arrow[dr] & & \\
\mathbb{K}\Sigma\times\mathbb{K}\Sigma \arrow[ur] \arrow[dr] & & \mathbb{K}\mathrm{Q}\times\mathbb{K}\mathrm{Q} \arrow[r] & \mathbb{K}, \\
 & \mathbb{K}\Sigma\times\mathbb{K}\mathrm{Q} \arrow[ur] & &
\end{tikzcd} \tag{2.10}$$

induced by the map $\mathrm{base}^* : \mathbb{K}\Sigma \to \mathbb{K}\mathrm{Q}$ given by $x \mapsto \sum_{c:\ x\leq c} (x,c)$. The rightmost map is the bilinear form defined in Section 2.5.1.

Explicitly, the map $\mathbb{K}\mathrm{Q} \times \mathbb{K}\Sigma \to \mathbb{K}$ is given by

$$\langle (x, c), y \rangle = \begin{cases} 1 & \text{if } xyc = c, \text{ or equivalently, } y \in \operatorname{reg}(x, c), \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the map $\operatorname{reg} : \mathrm{Q} \to \mathrm{Z}'$, we have the following.

Lemma 2.5.1 *The kernel of the map* $\operatorname{reg} : \mathbb{K}\mathrm{Q} \to \mathbb{K}\mathrm{Z}'$ *lies in the left radical of the map* $\mathbb{K}\mathrm{Q} \times \mathbb{K}\Sigma \to \mathbb{K}$. *Hence there is a commutative diagram*

$$\begin{tikzcd} \mathbb{K}\mathrm{Q} \times \mathbb{K}\Sigma \arrow[d] \arrow[dr] & \\ & \mathbb{K} \\ \mathbb{K}\mathrm{Z}' \times \mathbb{K}\Sigma \arrow[ur] & \end{tikzcd}$$

2.5.3 The bilinear form on $\mathbb{K}\Sigma$

Note that diagram (2.10) defines a symmetric bilinear form on $\mathbb{K}\Sigma$. It is given by

$$\langle x, y \rangle_\Sigma = |\{(c, d) \mid xd = c,\ yc = d\}|. \tag{2.11}$$

There is an alternate way to define this bilinear form. For each $x \in \Sigma$, let $c_x = |\mathcal{C}_x|$ be the number of chambers $c \in \mathcal{C}$ such that $c \geq x$. Define a linear map $\zeta : \mathbb{K}\Sigma \to \mathbb{K}$ by

$$\zeta(x) = c_x. \tag{2.12}$$

Lemma 2.5.2 *We have* $\langle x, y \rangle_\Sigma = \zeta(xy)$.

Proof Since $\operatorname{supp}(xy) = \operatorname{supp}(yx)$, by Lemma 2.2.1, there is a bijection

$$\operatorname{bij} : \mathcal{C}_{xy} \to \mathcal{C}_{yx}$$

given by $c \mapsto yxc$ with inverse $d \mapsto xyd$.

Let $(\mathcal{C} \times \mathcal{C})_{x,y} = \{(c, d) \mid yc = d,\ xd = c\}$. Then

$$(c, d) \in (\mathcal{C} \times \mathcal{C})_{x,y} \iff c \in \mathcal{C}_{xy},\ d \in \mathcal{C}_{yx},\ \operatorname{bij}(c) = d.$$

To see the forward implication, note that $(c, d) \in (\mathcal{C} \times \mathcal{C})_{x,y}$ implies $y \leq d$ and $x \leq c$. Hence $yc = y(xc) = d$. This says that $yx \leq d$ and $\operatorname{bij}(c) = d$. Similarly $xy \leq c$. The backward implication is similar. This proves the lemma.

□

Corollary 2.5.1 *The form* $\langle\ ,\ \rangle_\Sigma$ *on* $\mathbb{K}\Sigma$ *is invariant. In other words,*

$$\langle x, yz \rangle_\Sigma = \langle xy, z \rangle_\Sigma.$$

2.5.4 The bilinear form on $\mathbb{K}\mathrm{L}$

The bilinear form on $\mathbb{K}\Sigma$ is far from being nondegenerate. We know from Lemma 2.2.1 that c_x depends only on $\operatorname{supp} x$. Hence for each $X \in \mathrm{L}$, let c_X be the number of chambers $c \in \mathcal{C}$ such that $c \geq x$, where x is any fixed element of Σ having support X. The map ζ factors through $\mathbb{K}\mathrm{L}$ giving a function $\zeta : \mathbb{K}\mathrm{L} \to \mathbb{K}$ with

$$\zeta(X) = c_X. \tag{2.13}$$

Now $\langle x, y \rangle_\Sigma = \zeta(xy) = \zeta(\operatorname{supp} xy) = \zeta(\operatorname{supp} x \vee \operatorname{supp} y)$. This shows the following.

Lemma 2.5.3 *The form* $\langle\ ,\ \rangle_\Sigma : \mathbb{K}\Sigma \times \mathbb{K}\Sigma \to \mathbb{K}$ *and the map* $\zeta : \mathbb{K}\Sigma \to \mathbb{K}$ *factor through* $\mathbb{K}\mathrm{L}$ *to give a form* $\langle\ ,\ \rangle_\mathrm{L} : \mathbb{K}\mathrm{L} \times \mathbb{K}\mathrm{L} \to \mathbb{K}$ *and a map* $\zeta : \mathbb{K}\mathrm{L} \to \mathbb{K}$ *satisfying*

$$\langle X, Y\rangle_\mathrm{L} = \zeta(X \vee Y).$$

In other words, there are two commutative diagrams

$$\begin{tikzcd} \mathbb{K}\Sigma \times \mathbb{K}\Sigma \arrow[d, "\mathrm{supp}\times\mathrm{supp}"'] \arrow[dr] & \\ \mathbb{K}\mathrm{L} \times \mathbb{K}\mathrm{L} \arrow[r] & \mathbb{K} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K}\Sigma \arrow[d, "\mathrm{supp}"'] \arrow[dr, "\zeta"] & \\ \mathbb{K}\mathrm{L} \arrow[r, "\zeta"'] & \mathbb{K} \end{tikzcd}$$

2.5.5 The nondegeneracy of the form on $\mathbb{K}\mathrm{L}$

Now we discuss conditions under which the induced form $\langle\ ,\ \rangle_\mathrm{L}$ on $\mathbb{K}\mathrm{L}$ is nondegenerate.

Definition 2.5.1 We define numbers n_X by the equation

$$\sum_{X \leq Y} n_Y = c_X,$$

for each $X \in \mathrm{L}$. Equivalently, $n_X = \sum_{X\leq Y} \mu(X, Y)c_Y$, where μ is the Möbius function of the lattice L.

The numbers n_X, in this generality, were defined by Brown. They are the generic multiplicities of certain random walks on the chambers of a LRB, see [18, Theorem 1]. For the special case of hyperplane arrangements, $n_X = |\mu(X, \hat{1})|$, where $\hat{1}$ is the maximum element of L and μ is its Möbius function. This follows from a formula of Zaslavsky [101]. The connection of these numbers to random walks was first made by Bidigare, Hanlon and Rockmore [11, 12].

Lemma 2.5.4 *The semigroup algebra* $\mathbb{K}\mathrm{L}$ *is split semisimple, that is, it is isomorphic to a product of copies of* $\mathbb{K}$*. Further, the form* $\langle\ ,\ \rangle_\mathrm{L} : \mathbb{K}\mathrm{L} \times \mathbb{K}\mathrm{L} \to \mathbb{K}$ *is nondegenerate* $\iff$ $n_X \neq 0$ *for each* $X \in \mathrm{L}$.

The first part is due to Solomon [91], see also Greene [37] and Stanley [93, Section 3.9]. It holds for any finite semilattice.

Proof Explicitly, for the first part, if $\mathbb{K}^\mathrm{L}$ denotes the algebra of functions from L to $\mathbb{K}$, then there is an algebra isomorphism $\mathbb{K}\mathrm{L} \xrightarrow{\cong} \mathbb{K}^\mathrm{L}$ given by $X \mapsto \sum_{X\leq Y} \delta_Y$, where δ_Y is defined to be 1 at Y and 0 elsewhere.

For the second part, let q_X be the element of $\mathbb{K}\mathrm{L}$, which corresponds to δ_X under this isomorphism. Then q_X are the orthogonal idempotents for the algebra $\mathbb{K}\mathrm{L}$. It follows from Definition 2.5.1 that $\zeta(q_X) = n_X$. We now compute the form $\langle\ ,\ \rangle_\mathrm{L}$ on the $\{q_X\}$ basis.

$$\langle q_X, q_Y\rangle_\mathrm{L} = \zeta(q_X q_Y) = \zeta(q_X)\delta_{X,Y} = n_X\delta_{X,Y}, \tag{2.14}$$

where $\delta_{X,Y}$ denotes the Kronecker delta. The result follows.

$\square$

Lemma 2.5.5 *The kernel of the support map* $\mathbb{K}\Sigma \to \mathbb{K}\mathrm{L}$ *is* $\mathrm{rad}(\mathbb{K}\Sigma)$*, where* $\mathrm{rad}(\mathbb{K}\Sigma)$ *stands for the Jacobson radical of* $\mathbb{K}\Sigma$.

This result is due to Bidigare and Brown.

Proof Bidigare [11] showed that for Σ arising from hyperplane arrangements, the kernel of the support map $\mathbb{K}\Sigma \to \mathbb{K}\mathrm{L}$ is nilpotent. And since $\mathbb{K}\mathrm{L}$ is semisimple, the result follows. The same proof was generalized to LRBs by Brown [18, Section 7.2].

$\square$

Corollary 2.5.2 *We have $n_X \neq 0$ for each $X \in \mathrm{L}$* $\iff$ $\mathrm{rad}\langle\ ,\ \rangle_\Sigma = \mathrm{rad}(\mathbb{K}\Sigma)$.

This follows from the previous two lemmas and gives a computable criterion to check equality of the radicals.

Open Question Identify the class of LRBs for which the above criterion holds.

It is known that the above criterion holds for Σ arising from hyperplane arrangements; see Orlik and Terao [71, Theorem 2.47]. We thank Victor Reiner for this reference. However one can check that the criterion fails for the free LRB. The reader who has come this far may be convinced that free LRBs are good for producing counterexamples.

Example We recall that for type A, the elements of Σ and L are set compositions and set partitions respectively.

If $F = F^1|F^2|\dots|F^k$ is a set composition then $c_F = f_1!f_2!\dots f_k!$ where $f_i = |F^i|$. Note that this depends only on the cardinalities of the F^i and not on their order. In other words, c_F only depends on $\operatorname{supp} F = \{F^1, F^2, \dots, F^k\}$, as expected.

Using the formula for the Möbius function of the poset of set partitions, we obtain:

$$n_X = |\mu(X, \hat{1})| = (x_1 - 1)!\dots(x_k - 1)!, \tag{2.15}$$

where $X = \{X^1, X^2, \dots, X^k\}$ and $x_i = |X^i|$. Note that $n_X \neq 0$ for each $X \in \mathrm{L}$, as already claimed above.

Remark In the Coxeter case, the numbers n_X are related to the invariant theory of the Coxeter group W. If X is the minimum element of L, then $n_X = |\mu(X, \hat{1})|$ is the product of the exponents of W, and hence in particular is nonzero. This is an old result of Orlik and Solomon [70], see also Orlik and Terao [71, Corollary 6.62]. For any other element X in the lattice, the interval $[X, \hat{1}]$ is isomorphic to the intersection lattice for another Coxeter group W', which is a parabolic subgroup of W; hence the above result applies.

2.6 Bilinear forms related to a Coxeter group

Let (W, S) be a Coxeter system and Σ be the Coxeter complex of W. Then the diagram

$$\begin{tikzcd}
\mathbb{K}\overline{\mathrm{Q}} \cong (\mathbb{K}\Sigma)^W \arrow[r, hook] \arrow[d, two heads, "\mathrm{supp}"'] & \mathbb{K}\Sigma \arrow[d, two heads, "\mathrm{supp}"] \\
\mathbb{K}\overline{\mathrm{L}} \cong (\mathbb{K}\mathrm{L})^W \arrow[r, hook] & \mathbb{K}\mathrm{L}
\end{tikzcd}$$

commutes. In Section 2.5, we studied bilinear forms on $\mathbb{K}\mathrm{Q}$, $\mathbb{K}\Sigma$ and $\mathbb{K}\mathrm{L}$ for any LRB Σ. In the setting of Coxeter groups, it is clear that these forms are invariant under the W-action, and hence induce symmetric bilinear forms on $\mathbb{K}\overline{\mathrm{Q}} \cong (\mathbb{K}\mathrm{Q})^W \cong (\mathbb{K}\Sigma)^W$ and $\mathbb{K}\overline{\mathrm{L}} \cong (\mathbb{K}\mathrm{L})^W$. In this section, we study these two induced bilinear forms.

We have encountered the object $\overline{\mathrm{Q}}$ in the context of the descent algebra (Section 1.3.6). The object $\overline{\mathrm{L}}$ is new; it is defined as the set of W-orbits in L. In fact, it has a partial order which it inherits from L.

2.6.1 The bilinear form on $(\mathbb{K}\Sigma)^W$

In this subsection, we give some interesting descriptions of the bilinear form on $(\mathbb{K}\Sigma)^W$. We recall that a basis for $(\mathbb{K}\Sigma)^W$ is given by

$$\sigma_T = \sum_{\mathrm{type}(F)=T} F,$$

as T ranges over all subsets of S.

Lemma 2.6.1 *We have*

$$\frac{1}{|W|}\langle \sigma_T, \sigma_U \rangle = |\{w \in W \mid \ \mathrm{des}(w) \leq T, \ \mathrm{des}(w^{-1}) \leq U\}|.$$

Compare this equation with [4, Proposition 9.4].

Proof Consider the set

$$(\mathcal{C} \times \mathcal{C})_{T,U} = \{(C, D) \mid T_C D = C \text{ and } U_D C = D\},$$

where T_C is the face of C of type T, and U_D is the face of D of type U. It is clear that this set is closed under the diagonal action of W on $\mathcal{C} \times \mathcal{C}$. From (2.11), we see that the left hand side of the above equation counts the number of W-orbits in this set. Further note that each W-orbit can be indexed by an element $w \in W$ using the rule $d(C, D) = w$. Hence one has to determine those $w \in W$, which occur as orbits in $(\mathcal{C} \times \mathcal{C})_{T,U}$. This is done by translating the condition on the projection maps into a condition on $\mathrm{des}(w)$ and $\mathrm{des}(w^{-1})$. This will be the content of Proposition 5.3.2. This gives the right hand side of the above equation.

□

Using Lemma 2.5.2, the bilinear form on $(\mathbb{K}\Sigma)^W$ can be written as

$$\langle \sigma_T, \sigma_U \rangle = \zeta(\sigma_T \sigma_U), \tag{2.16}$$

where ζ is given by (2.12). Now write

$$\sigma_T \sigma_U = \sum_{V \leq S} \alpha^V_{TU} \sigma_V.$$

Since $(\mathbb{K}\Sigma)^W$ is anti-isomorphic to the descent algebra, the constants α^V_{TU} may also be regarded as the structure constants of the descent algebra (Section 1.3.6). We now describe the bilinear form using these constants.

Lemma 2.6.2 *We have* $\langle \sigma_T, \sigma_U \rangle = |W| \sum_{V \leq S} \alpha^V_{TU}$.

Proof Using the above equations, we have $\langle \sigma_T, \sigma_U \rangle = \sum_{V \leq S} \alpha^V_{TU} \zeta(\sigma_V)$. It remains to show that

$$\zeta(\sigma_V) = |W| \ \text{ for all } \ V \leq S.$$

The left hand side counts pairs (F, D) with $F \leq D$ and $\mathrm{type}\, F = V$. Since every chamber has a unique face of type V, this is same as the number of chambers, which is $|W|$.

□

Lemma 2.6.3 *Let $F \leq H$ and* $\mathrm{type}\, F = T$ *and* $\mathrm{type}\, H = V$. *Then α^V_{TU} counts the number of faces N of type U such that $FN = H$.*

This lemma follows from the definition. Combining Lemmas 2.6.2 and 2.6.3 gives us the following description.

Lemma 2.6.4 *Let $F \leq D$ and* $\operatorname{type} F = T$. *Then*

$$\frac{1}{|W|}\langle \sigma_T, \sigma_U \rangle = |\{N \in \operatorname{reg}(F, D) \mid \operatorname{type} N = U\}|.$$

Note that the symmetry of the bilinear form is not at all clear from the right hand side. Hence we can use the symmetry along with this lemma to derive a nontrivial result about lunar regions as follows.

Corollary 2.6.1 *The number of faces of type U in Σ, say f_U, is equal to the number of chambers in the lunar region* $\operatorname{reg}(F, D)$*, where F is any face of type U and $F \leq D$.*

This result is also implied by Mahajan [60, Lemma 5].

Proof Recall that S is the set of reflections and any chamber D is of type S. By the symmetry of the bilinear form on $(\mathbb{K}\Sigma)^W$, we have

$$\frac{1}{|W|}\langle \sigma_S, \sigma_U \rangle = \frac{1}{|W|}\langle \sigma_U, \sigma_S \rangle.$$

Now use Lemma 2.6.4 on both sides. Note that $\operatorname{reg}(D, D) = \Sigma$. Hence the left hand side is the number of faces of type U in Σ. And the right hand side is the number of chambers in the lunar region $\operatorname{reg}(F, D)$, where F is any face of type U and $F \leq D$.

$\square$

2.6.2 The bilinear form on $(\mathbb{K}\mathrm{L})^W$ and its nondegeneracy

It is clear that the bilinear form on $(\mathbb{K}\Sigma)^W$ factors through the map $(\mathbb{K}\Sigma)^W \to (\mathbb{K}\mathrm{L})^W$ to give a bilinear form $\langle\ ,\ \rangle : (\mathbb{K}\mathrm{L})^W \times (\mathbb{K}\mathrm{L})^W \to \mathbb{K}$. In analogy with Lemma 2.5.4, one can show:

Lemma 2.6.5 *The algebra $(\mathbb{K}\mathrm{L})^W$ is split semisimple. Further, the form $\langle\ ,\ \rangle : (\mathbb{K}\mathrm{L})^W \times (\mathbb{K}\mathrm{L})^W \to \mathbb{K}$ is nondegenerate $\iff$ $n_X \neq 0$ for each $X \in \mathrm{L}$.*

The proof is straightforward and is given in Lemma 5.7.1. The main step is to construct a basis for $(\mathbb{K}\mathrm{L})^W$ such that each basis element when viewed as an element of $\mathbb{K}\mathrm{L}$ is a sum of primitive idempotents.

Lemma 2.6.6 *The kernel of the support map $(\mathbb{K}\Sigma)^W \to (\mathbb{K}\mathrm{L})^W$ is* $\operatorname{rad}((\mathbb{K}\Sigma)^W)$.

Proof The kernel is nilpotent because it sits inside $\operatorname{rad}(\mathbb{K}\Sigma)$ which is nilpotent. And since $(\mathbb{K}\mathrm{L})^W$ is semisimple, the result follows.

$\square$

The above result (phrased in a different language) was obtained by Solomon [92, Theorem 3]. For other proofs and additional related results, see Atkinson [6], Garsia and Reutenauer [32] and Krob, Leclerc and Thibon [51, Corollary 3.11]. A lift of the the primitive idempotents of $(\mathbb{K}\mathrm{L})^W$ to $(\mathbb{K}\Sigma)^W$ is given by Bergeron, Bergeron, Howlett and Taylor [7]. For a recent survey on the descent algebra, see Schocker [89].

The previous two lemmas give us the following corollary relating the radical of the descent algebra to the radical of the bilinear form on it.

Corollary 2.6.2 *We have $n_X \neq 0$ for each $X \in \mathrm{L}$ $\iff$ $\operatorname{rad}\langle\ ,\ \rangle_{(\mathbb{K}\Sigma)^W} = \operatorname{rad}((\mathbb{K}\Sigma)^W)$.*

Note that the criterion for the equality of the radicals in the coinvariant case is the same as the criterion obtained in Corollary 2.5.2.

2.7 Projection posets

In this section, we introduce projection posets, which are more general than LRBs. The main motivation is that some of our results hold in this generality. However, the constructions of the objects L and Z are specific to LRBs; they do not generalize to projection posets.

2.7.1 Definition and examples

Definition 2.7.1 A projection poset Σ is a poset with a (not necessarily associative) product $\Sigma \times \Sigma \to \Sigma$ that satisfies:

(1) The product $x_1 x_2 \ldots x_n$ is well-defined if there exist y, z such that for every $1 \leq i \leq n$, either $x_i \leq y$ or $x_i \leq z$.

(2) $y^2 = y$ and $yzy = yz$.

(3) $yz = z \iff y \leq z$.

(4) The set of chambers (maximal elements) $\mathcal{C}$ is a left ideal in Σ.

Proposition 2.7.1 *Every LRB is a projection poset.*

Proof Since the product in a LRB is associative, $x_1 x_2 \ldots x_n$ is always well-defined; so (1) holds. Property (2) holds by definition. The next two properties are a part of LRB foundations, which were discussed in Section 2.2. Note that the definition of a LRB does not involve any poset. The point is to first define a relation on a LRB using property (3) and then to prove that it is a partial order.

□

A projection poset is not an empty generalization of a LRB. The generalization does indeed give new examples.

Proposition 2.7.2 *The poset of faces of a building and the order complex of a modular lattice are projection posets.*

Proof A building is a simplicial complex obtained by gluing together Coxeter complexes. These Coxeter complexes are called the apartments of the building. For a building Δ, given $y, z \in \Delta$, one can always choose an apartment containing y and z. Details can be found in Brown [17]. Tits defined a product (not necessarily associative) on the poset of faces of a building [99, Section 3.19]. One way to describe the product yz is to first choose an apartment containing y and z, and then take the product in that apartment.

Readers may now readily see the origin of property (1). Namely, one can choose an apartment containing the elements $x_1, x_2, \ldots, x_n$. The product $x_1 x_2 \ldots x_n$ is then well-defined because the product within an apartment is associative. Similarly, buildings also satisfy properties (2) – (4). All of them involve at most two distinct elements, hence one can always choose an apartment containing them and argue as above.

In [1], Abels showed that the order complex of any modular lattice behaves like the building of type A. The role of apartments is played by distributive lattices. The order complex of a distributive lattice corresponds to a convex set of chambers in the braid arrangement [1, Proposition 2.5]. Hence it is an example of a LRB; in particular, its product is associative. The same argument as for buildings then shows that the order complex of a modular lattice is a projection poset.

□

The examples in the proposition above were referred to as nonassociative LRBs in Mahajan [60, Chapter 1], where related material can be found.

2.7.2 Elementary facts

We now generalize some known facts about LRBs to projection posets. These facts, though elementary, are important. They will allow us to use projection posets as the basis for an axiomatic treatment of Hopf algebras in Chapter 6.

Lemma 2.7.1 *We have $y \leq yz$.*

Proof By property (1), we know that yyz is well-defined. By property (2), we have $y^2 = y$; hence $y(yz) = (yy)z = yz$. Now, applying property (3), we obtain $y \leq yz$.

□

Lemma 2.7.2 *If $y \leq z$ then $zy = z$.*

Proof Let $y \leq z$. Then by property (3), we have $yz = z$. Premultiplying by z, we get $zyz = z^2$. Property (3) now implies that $zy = z$.

□

Lemma 2.7.3 *If $x \leq y \leq z$ and $xw = z$ then $yw = z$.*

Proof This follows from the sequence of equalities $yw = yxw = yz = z$. The first equality uses Lemma 2.7.2.

□

Lemma 2.7.4 *An element $\emptyset$ is the identity in Σ if and only if $\emptyset$ is the unique minimal element in Σ.*

Proof From property (3), we have

$$\emptyset x = x \text{ for all } x \iff \emptyset \leq x \text{ for all } x.$$

Also by Lemma 2.7.2, if $\emptyset \leq x$ for all x then $x\emptyset = x$ for all x.

□

Lemma 2.7.5 *If $y \leq z$ then $xy \leq xz$.*

Proof Let $y \leq z$. Then $xyxz$ is well-defined by property (1). By properties (2) and (3), we have $xyxz = xyz = xz$. Therefore, we get $xy(xz) = xz$, which by property (3) says that $xy \leq xz$.

□

The set of chambers $\mathcal{C}$ in Σ can also be characterized using the product in Σ as below, also see Proposition 2.2.1.

Lemma 2.7.6 *We have that c is a chamber in $\Sigma \iff cx = c$ for all $x \in \Sigma$.*

Proof For the forward implication, note by Lemma 2.7.1 that $c \leq cx$ for all $x, c \in \Sigma$; so if c is maximal then $c = cx$ for all $x \in \Sigma$. Conversely, if $cx = c$ for all $x \in \Sigma$ then c is maximal because $c \leq x \implies cx = x \implies c = x$.

□

Let $\mathcal{C}_x = \{c \in \mathcal{C} \mid x \leq c\}$. The following is a generalization of Lemma 2.2.1.

Lemma 2.7.7 *If $xy = x$ and $yx = y$ then there is a bijection $\mathcal{C}_x \to \mathcal{C}_y$ given by $c \mapsto yc$ and with inverse $d \mapsto xd$.*

Proof By property (4), we know that the maps in the lemma are well-defined. We want to show that they are inverse to each other. By symmetry, it is enough to show that for $x \leq c$, we have $x(yc) = c$. By property (1), we know that xyc is well-defined. Further, from assumption and property (3), we get $xyc = xc = c$.

$\square$

Corollary 2.7.1 *There is a bijection $\mathcal{C}_{xy} \to \mathcal{C}_{yx}$ given by $c \mapsto yxc$ with inverse $d \mapsto xyd$.*

Proof By property (1), we know that $yxxy$ is well-defined and by property (2), we have $yxxy = yx$. Similarly, we have $xyyx = xy$. The assertion now follows from the previous lemma.

$\square$

Lemma 2.7.8 *Let $y \leq z$. Then*

$$xz \leq c \iff xy \leq c \text{ and } yxz \leq yc.$$

This result will be crucially used in Chapter 6, see the corollary to Proposition 6.4.5. If z is a chamber, say d, then we may say

$$xd = c \iff xy \leq c \text{ and } yxd = yc.$$

This result will be used in Proposition 6.5.6.

Proof Since $y \leq z$, by Lemma 2.7.5, we get $xy \leq xz$. The forward implication is easy. Let $xz \leq c$. Then using the above inequality, $xy \leq xz \leq c$. This proves one part. And by Lemma 2.7.5, we have $yxz \leq yc$, which proves the second part.

Conversely, let $xy \leq c$ and $yxz \leq yc$. From Lemma 2.7.5 and the second assumption, we have $xyxz \leq xyc$. Note that since $y \leq z$ and $x \leq c$, both sides are well-defined by property (1). Further by properties (2) and (3), we have $xyxz = xyz = xz$. And by the first assumption and property (3), we have $xyc = c$. Hence $xz \leq c$.

$\square$

Chapter 3

Hopf algebras

For the definition of a Hopf algebra and basic examples, the reader may refer to Kassel [49], Montgomery [65] or Sweedler [95]. Roughly a Hopf algebra is a vector space with a product and a unit, a coproduct and a counit and an antipode; the structures being compatible in an appropriate sense.

Our Hopf algebras will have linear bases indexed by combinatorial objects of various kinds. The study of Hopf algebras of this type was initiated by Joni and Rota [48] and continued by Schmitt [87, 88], Ehrenborg [29] and others.

3.1 Hopf algebras

In this section, we review the notions of cofree graded coalgebras, coradical filtration and antipode, which are relevant to us. The material is directly taken from [4].

3.1.1 Cofree graded coalgebras

Let V be a vector space over $\mathbb{K}$ and set

$$Q(V) \ := \ \bigoplus_{k\geq 0} V^{\otimes k} \, .$$

The space $Q(V)$, graded by k, becomes a graded coalgebra with the *deconcatenation* coproduct

$$\Delta(v_1 \otimes \ldots \otimes v_k) \ = \ \sum_{i=0}^{k} \, (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k) \, , \tag{3.1}$$

and counit $\epsilon(v_1 \otimes \cdots \otimes v_k) = 0$ for $k \geq 1$. The coalgebra $Q(V)$ is *connected*, in the sense that the component of degree 0 is identified with the base field via ϵ.

We call $Q(V)$ the *cofree graded coalgebra* cogenerated by V. The canonical projection $\pi : Q(V) \to V$ satisfies the following universal property. Given a graded coalgebra $C = \oplus_{k\geq 0} C^k$ and a linear map $\varphi : C \to V$ where $\varphi(C^k) = 0$ when $k \neq 1$, there is a unique morphism of graded coalgebras $\hat{\varphi} : C \to Q(V)$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, dashed, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & & Q(V) \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Explicitly, $\hat{\varphi}$ is defined by

$$\hat{\varphi}_{|_{C^k}} = \varphi^{\otimes k} \Delta^{(k-1)} \, ,$$

where $\Delta^{(k-1)}$ is the iterated coproduct explained in Section 3.1.3. In particular,

$$\hat{\varphi}_{|_{C^0}} = \epsilon,\ \hat{\varphi}_{|_{C^1}} = \varphi, \quad \text{and} \quad \hat{\varphi}_{|_{C^2}} = (\varphi \otimes \varphi)\Delta.$$

For more explanation, see Sweedler [95, Lemma 12.2.7]. For a more general result, see Quillen [79, Appendix B] and Loday and Ronco [55].

3.1.2 The coradical filtration

Let C be a graded connected coalgebra. The *coradical* $C^{(0)}$ of C is the 1-dimensional component in degree 0 (identified with the base field via the counit). The *primitive elements* of C are

$$P(C) \ := \ \{x \in C \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}\,.$$

Set $C^{(1)} := C^{(0)} \oplus P(C)$, the first level of the coradical filtration. More generally, the k-th level of the *coradical filtration* is

$$C^{(k)} \ := \ \big(\Delta^{(k)}\big)^{-1}\Big(\sum_{i+j=k} C^{\otimes i} \otimes C^{(0)} \otimes C^{\otimes j}\Big)\,.$$

We have $C^{(0)} \subseteq C^{(1)} \subseteq C^{(2)} \subseteq \cdots \subseteq C = \bigcup_{k\geq 0} C^{(k)}$, and

$$\Delta(C^{(k)}) \ \subseteq \ \sum_{i+j=k} C^{(i)} \otimes C^{(j)}\,.$$

Thus, the coradical filtration measures the complexity of iterated coproducts.

Suppose now that C is the cofree graded coalgebra $Q(V)$. Then the space of primitive elements is just V, and the k-th level of the coradical filtration is $\oplus_{i=0}^{k} V^{\otimes i}$. These are straightforward consequences of the definition of the deconcatenation coproduct.

3.1.3 Antipode

There is a general formula for the antipode of a graded connected Hopf algebra H, due to Takeuchi [96, Lemma 14] (see also Milnor and Moore [64]). Let H be an arbitrary bialgebra with structure maps: multiplication $m\colon H \otimes H \to H$, unit $u\colon \mathbb{K} \to H$, comultiplication $\Delta\colon H \to H \otimes H$, and counit $\epsilon\colon H \to \mathbb{K}$. Set $m^{(1)} = m$, $\Delta^{(1)} = \Delta$, and for any $k \geq 2$,

$$\begin{aligned} m^{(k)} \ &= \ m(m^{(k-1)} \otimes \mathrm{id}) \ : \ H^{\otimes k+1} \to H, \qquad \text{and} \\ \Delta^{(k)} \ &= \ (\Delta^{(k-1)} \otimes \mathrm{id})\Delta \ : \ H \to H^{\otimes k+1}\,. \end{aligned}$$

These are the higher or iterated products and coproducts. We also set

$$\begin{aligned} m^{(-1)} \ &= \ u\colon \mathbb{K} \to H, \\ \Delta^{(-1)} \ &= \ \epsilon\colon H \to \mathbb{K}, \qquad \text{and} \\ m^{(0)} \ &= \ \Delta^{(0)} \ = \ \mathrm{id}\colon H \to H\,. \end{aligned}$$

If $f : H \to H$ is any linear map, the convolution powers of f are, for any $k \geq 0$,

$$f^{*k} \ = \ m^{(k-1)} f^{\otimes k} \Delta^{(k-1)}\,.$$

In particular, $f^{*0} = u\epsilon$ and $f^{*1} = f$.

We set $\pi := \mathrm{id} - u\epsilon$. If π is locally nilpotent with respect to convolution, then $\mathrm{id} = u\epsilon + \pi$ is invertible with respect to convolution, with inverse

$$S \ = \ \sum_{k\geq 0}(-\pi)^{*k} \ = \ \sum_{k\geq 0}(-1)^k m^{(k-1)} \pi^{\otimes k} \Delta^{(k-1)}\,. \tag{3.2}$$

This is certainly the case if H is a graded connected bialgebra, in which case π annihilates the component of degree 0 (and hence π^{*k} annihilates components of degree $< k$). Thus Equation (3.2) is a general formula for the antipode of a graded connected Hopf algebra.

In general, the interest is in finding an explicit formula for the structure constants of the antipode, which formula (3.2) does not always give because many cancellations often take place. This problem won't be pursued in the examples we will consider.

3.2 Hopf algebras: Examples

We are interested in the following diagram of connected graded Hopf algebras that relates the Hopf algebra of permutations $\mathrm{S}\Lambda$ and the Hopf algebra of symmetric functions Λ, noncommutative symmetric functions $\mathrm{N}\Lambda$ and quasi-symmetric functions $\mathrm{Q}\Lambda$.

$$\begin{tikzcd} \mathrm{N}\Lambda \arrow[r, hook] \arrow[d] & \mathrm{S}\Lambda \arrow[d] & & \mathbb{K}\overline{\mathrm{Q}}^n \arrow[r, hook] \arrow[d] & \mathbb{K}\mathrm{S}_n \arrow[d] \\ \Lambda \arrow[r, hook] & \mathrm{Q}\Lambda & & \mathbb{K}\overline{\mathrm{L}}^n \arrow[r, hook] & \mathbb{K}(\overline{\mathrm{Q}}^n)^* \end{tikzcd} \tag{3.3}$$

The diagram on the right shows the nth graded pieces of these Hopf algebras. For P a set, we write $\mathbb{K}P$ for the vector space over $\mathbb{K}$ with basis the elements of P, and $\mathbb{K}P^*$ for its dual space. The three sets, namely $\overline{\mathrm{L}}^n$, $\overline{\mathrm{Q}}^n$ and S_n, and also the structure of these Hopf algebras, are summarized in Table 3.1.

Table 3.1: Hopf algebras, their indexing sets and structure.

Hopf algebra	Indexing set	Comm.	Cocomm.	Structure
Λ	$\overline{\mathrm{L}}^n$ = partitions of n	Yes	Yes	Self-dual, free comm. and cofree cocomm.
$\mathrm{Q}\Lambda$	$\overline{\mathrm{Q}}^n$ = compositions of n	Yes	No	Free comm. and cofree
$\mathrm{N}\Lambda$	$\overline{\mathrm{Q}}^n$ = compositions of n	No	Yes	Free and cofree cocomm.
$\mathrm{S}\Lambda$	S_n = permutations of n	No	No	Self-dual, free and cofree

The Hopf algebra $\mathrm{S}\Lambda$ defined by Malvenuto [61] will be adequately dealt in Chapter 7. In this section, we explain the remaining three Hopf algebras. For a recent survey on related topics, see Hazewinkel [42, 43]. For an extensive theory and applications to representation theory of the symmetric group, see Blessenohl and Schocker [15].

3.2.1 The Hopf algebra Λ

The Hopf algebra Λ of symmetric functions is most often viewed as a subalgebra of the algebra of formal power series in countably many variables $x_1, x_2, \ldots$. Details can be found in Fulton [31], Macdonald [59], Sagan [84] and Stanley [94]. However, we treat Λ as an intrinsic object.

Definition 3.2.1 A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a nonincreasing finite sequence of positive integers which add up to n. We write $\mathrm{parts}(\lambda)$ for the number of parts in λ. We denote partitions by the letters λ, μ and ρ.

We say that $\lambda \leq \mu$ if μ refines λ, that is, μ is obtained by refining each part in λ and then rearranging the parts in descending order. This defines a partial order on $\overline{\mathrm{L}}^n$, the set of partitions of n. We warn the reader that this is different from the standard containment or dominance partial orders on partitions.

A *false-shuffle* of partitions λ and μ is a shuffle of the components of λ written is some order and the components of μ written is some order. For example, $(2, \mathbf{1}, 3, 1, \mathbf{3}, \mathbf{1})$ is a false-shuffle of $(3, 2, 1)$ and $(\mathbf{3}, \mathbf{1}, \mathbf{1})$.

A *shuffle* of partitions λ and μ is a false-shuffle of λ and μ, whose entries are nonincreasing. For example, $(3, \mathbf{3}, 2, 1, \mathbf{1}, \mathbf{1})$ and $(\mathbf{3}, 3, 2, \mathbf{1}, 1, \mathbf{1})$ are distinct shuffles of $(3, 2, 1)$ and $(\mathbf{3}, \mathbf{1}, \mathbf{1})$.

A *quasi-shuffle* of partitions λ and μ is a false-shuffle of λ and μ, where in addition we may replace any number of pairs of consecutive components (λ_i, μ_j) in the false-shuffle by $\lambda_i + \mu_j$, and the resulting entries are nonincreasing. For example, $(\mathbf{3}, 3, 1 + \mathbf{1}, 2, \mathbf{1})$ is a quasi-shuffle of $(3, 2, 1)$ and $(\mathbf{3}, \mathbf{1}, \mathbf{1})$.

The space Λ is equipped with a variety of basis, all indexed by partitions. We will mainly deal with the monomial m basis, the homogeneous h basis and the power sum p basis. The change of bases matrices can be found in the references mentioned above. The product in the h and p basis is given by

$$h_\lambda h_\mu = h_{\lambda \sqcup \mu} \quad \text{and} \quad p_\lambda p_\mu = p_{\lambda \sqcup \mu},$$

where $\sqcup$ denotes the union as multisets. This follows because the h and p basis are defined by

$$h_\lambda = h_{(\lambda_1)} h_{(\lambda_2)} \dots h_{(\lambda_k)} \quad \text{and} \quad p_\lambda = p_{(\lambda_1)} p_{(\lambda_2)} \dots p_{(\lambda_k)}$$

for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. It is true that $p_{(i)} = m_{(i)}$ for all i; however m_λ is defined differently from p_λ. The product in the m basis is given by

$$m_\lambda m_\mu = \sum_{\rho:\ \rho \text{ a quasi-shuffle of } \lambda \text{ and } \mu} m_\rho.$$

Remark The definition of a quasi-shuffle is concocted so that the above formula holds. Note that the definition is a little complicated and involves ordering the components of the partitions, which is not so natural.

There are two ways in which one can make things easier. One way is to work with $\mathrm{Q}\Lambda$, the Hopf algebra of ordered partitions or compositions (Section 3.2.2). The definition of a quasi-shuffle for compositions is much simpler. Another way is to work with Π, the Hopf algebra of set partitions (Section 6.2.10). The definition of a quasi-shuffle for set partitions is also simple, and more importantly, can be given without ordering the components of the partitions.

The coalgebra structure of Λ was first pointed out by Geissinger [34] who also showed that Λ is a self-dual Hopf algebra. The isomorphism of Λ with its dual is via the standard inner product on Λ with the m and h as dual bases. The Hopf algebra viewpoint on symmetric functions can be found in Zelevinsky [102]. As an algebra, Λ is free commutative on $h_{(1)}, h_{(2)}, \dots$. The formula

$$\Delta(h_{(n)}) = \sum_{i=0}^{n} h_{(i)} \otimes h_{(n-i)}, \tag{3.4}$$

extended as an algebra map defines the coproduct on Λ. The reader may write down the general expression for $\Delta(h_\lambda)$. It is clear that Λ is both commutative and cocommutative. Hence Λ is the cofree graded connected cocommutative coalgebra cogenerated by $P(\Lambda)$ by the Milnor-Moore theorem, see [64, Theorem 5.18] or Quillen [79, Theorem 4.5].

Remark The cofreeness of Λ is in the category of cocommutative coalgebras, and hence a little different from the setup in Section 3.1.1.

On the m basis, the coproduct is as follows.

$$\Delta(m_\lambda) = \sum m_\mu \otimes m_\rho,$$

where the sum is over all ordered pairs (μ, ρ) such that $\mu \sqcup \rho = \lambda$ as multisets. It is clear that $m_{(1)}$, $m_{(2)}$, ... forms a basis for the space of primitive elements of Λ. Since $p_{(i)} = m_{(i)}$ for all i, the $p_{(i)}$'s are primitive. One can now check that the coproduct on the p basis has the same expression as on the m basis.

The antipode of Λ is given by the formulas

$$S(p_\lambda) \;=\; (-1)^{\text{parts}(\lambda)}\; p_\lambda, \qquad\qquad S(m_\lambda) \;=\; (-1)^{\text{parts}(\lambda)} \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu\,,$$

where parts(λ) is the number of parts of λ, the partial order on partitions is as given in Definition 3.2.1, and $c_{\lambda\mu}$ is the number of compositions with underlying partition λ, which refine μ.

3.2.2 The Hopf algebra $\mathrm{Q}\Lambda$

Quasi-symmetric functions $\mathrm{Q}\Lambda$ were introduced by Gessel [36] as a subalgebra of the algebra of formal power series in countably many variables $x_1, x_2, \ldots$ (although with hindsight one can recognize them in work of Cartier [20]). A discussion can be found in Stanley [94, Section 7.19], Reutenauer [82, Section 9.4] and Bertet, Krob, Morvan, Novelli, Phan and Thibon [10]. The Hopf algebra structure of $\mathrm{Q}\Lambda$ was introduced by Malvenuto [61, Section 4.1]. The description of the product in some form or another can be found in Cartier [20, Formula (7)], Hoffman [46], Hazewinkel [41] and Ehrenborg [29, Lemma 3.3]. We recall some standard notions.

Definition 3.2.2 A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n is a finite sequence of positive integers which add up to n. If we replace positive by nonnegative, then we get a *weak composition*. We denote compositions by the letters α, β and γ.

We say that $\alpha \leq \beta$ if β refines α. This defines a partial order on $\overline{\mathrm{Q}}^n$, the set of compositions of n.

A *shuffle* of compositions α and β is a shuffle of the components of α and β. For example, $(3, \mathbf{4}, \mathbf{2}, 5, \mathbf{7}, 2, 8)$ is a shuffle of $(3, 5, 2, 8)$ and $(\mathbf{4}, \mathbf{2}, \mathbf{7})$.

A *quasi-shuffle* of compositions α and β is a shuffle of the components of α and β, where in addition we may replace any number of pairs of consecutive components (α_i, β_j) in the shuffle by $\alpha_i + \beta_j$. For example, $(1 + \mathbf{4}, \mathbf{1}, 3 + \mathbf{1}, 2, \mathbf{2})$ is a quasi-shuffle of $(1, 3, 2)$ and $(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{2})$.

For a composition α, we let supp(α) denote the underlying partition of α.

The Hopf algebra $\mathrm{Q}\Lambda$ has two well-known basis, the monomial basis M_α and the fundamental basis F_α, both indexed by compositions. They are related by the equation

$$F_\alpha \;=\; \sum_{\alpha \leq \beta} M_\beta.$$

The inclusion map $\Lambda \to \mathrm{Q}\Lambda$ sends

$$m_\lambda \mapsto \sum_{\alpha\colon\ \mathrm{supp}(\alpha) = \lambda} M_\alpha \tag{3.5}$$

The Hopf algebra structure of $\mathrm{Q}\Lambda$ on the M basis is defined as below. The formulas on the F basis are discussed in Section 8.4.

$$\Delta(M_{(\alpha_1, \alpha_2, \ldots, \alpha_k)}) = \sum_{i=0}^{k} M_{(\alpha_1, \ldots, \alpha_i)} \otimes M_{(\alpha_{i+1}, \ldots, \alpha_k)}$$

Comparing the coproduct with Equation (3.1) shows that QΛ is a cofree graded coalgebra, with M_α graded by the number of parts in α.

$$M_\alpha * M_\beta = \sum_{\gamma:\ \gamma \text{ a quasi-shuffle of } \alpha \text{ and } \beta} M_\gamma.$$

We would like to clarify a small point about the definition of this product. The terms $(1+\mathbf{4}, \mathbf{1}, 3+\mathbf{1}, 2, \mathbf{2})$ and $(1+\mathbf{4}, \mathbf{1}, 3+\mathbf{1}, \mathbf{2}, 2)$ are both distinct quasi-shuffles of $(1,3,2)$ and $(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{2})$, though they have the same underlying composition. Hence $M_{(5,1,4,2,2,)}$ appears with a coefficient of 2 in the product $M_{(1,3,2)} * M_{(4,1,1,2)}$.

One may now check directly that Λ is a subHopf algebra of QΛ. The antipode of QΛ is given by the formula

$$S(M_\alpha) \;=\; (-1)^{\text{parts}(\alpha)} \sum_{\beta \leq \alpha} M_{\overline{\beta}},$$

where parts(α) is the number of parts of α, and if $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ then $\overline{\beta}$ is β written in reverse order $(\beta_k, \ldots, \beta_2, \beta_1)$.

3.2.3 The Hopf algebra NΛ

The Hopf algebra NΛ was introduced as a noncommutative analog of symmetric functions by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [35]. Accordingly, it can be viewed as a subalgebra of the algebra of formal power series in countably many noncommutative variables. This object has been the center of intense activity; the lecture notes by Thibon [98] may be a good place to start. There is a series of papers on this topic [51, 28, 53, 54, 26, 27], apart from numerous other references.

In analogy with Λ, one can define the complete H basis for NΛ, again indexed by compositions. The product in NΛ in the H basis is given by

$$H_\alpha * H_\beta = H_{(\alpha,\beta)}.$$

It is then clear that NΛ is free on $H_{(1)}$, $H_{(2)}$, ... The coproduct is again defined by Equation (3.4), with H instead of h, and extended as an algebra map. Explicitly,

$$\Delta(H_\gamma) = \sum H_\alpha \otimes H_\beta,$$

the sum being over pairs $(\hat{\alpha}, \hat{\beta})$ of weak compositions such that $\hat{\alpha}$ and $\hat{\beta}$ do not have a 0 in the same place, $\gamma = \hat{\alpha} + \hat{\beta}$, and deleting the 0 entries gives α and β.

One may check that NΛ is a graded Hopf algebra with the above product and coproduct. The cofreeness of NΛ as a graded connected cocommutative coalgebra follows from the Milnor-Moore theorem again. The primitive elements are described in [35, Proposition 3.10] and [62]. In the latter paper, Malvenuto and Reutenauer identify the primitive elements with a free Lie algebra. To mention another fact, the Hopf algebra NΛ has an internal product which is isomorphic to the product in the descent algebra of type A [36, 62, 35].

3.2.4 The duality between QΛ and NΛ

One may observe that NΛ is the graded Hopf algebra dual of QΛ, with the H being the dual of the M basis, see [35, 62]. Hence by duality, QΛ is free as a commutative algebra over $\mathbb{K}$. Hazewinkel has shown that in fact QΛ is free over the integers [40]. It is also clear that the map NΛ → Λ which sends H_α to $H_{\text{supp}(\alpha)}$ is a map of Hopf algebras and is dual to the one in (3.5). It is natural to consider the dual of the F basis of QΛ; we call this the K basis of NΛ. The product and coproduct formulas in the K basis can be written by dualizing the formulas in the F basis which are given in Section 8.4.

Chapter 4

A brief overview

We briefly summarize the contents of the second part of the monograph.

4.1 Abstract: Chapter 5

In Chapter 5, we initiate a systematic study of the descent theory for Coxeter groups. This brings to the fore two posets, namely $\mathcal{C} \times \mathcal{C}$ and Q, which are defined in terms of the Coxeter complex Σ associated to a Coxeter system (W, S). We have already encountered the poset Q in Section 2.2 in the more general context of LRBs. The poset $\mathcal{C} \times \mathcal{C}$ consisting of pairs of chambers is more specific to the Coxeter case. Its partial order mimics the weak left Bruhat order on W.

The descent theory consists of three order preserving maps denoted Road, GRoad and Θ that relate the two posets $\mathcal{C} \times \mathcal{C}$ and Q. The maps themselves are related to one another in an interesting way, namely, Road and GRoad are the left and right adjoints to Θ respectively. The arguments in the proofs involve repeated use of the gate property satisfied by Σ (Proposition 1.2.1).

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[rr, bend left, "\mathrm{Road}"] \arrow[rr, bend right, "\mathrm{GRoad}"'] & & \mathrm{Q} \arrow[ll, "\Theta"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
W \arrow[rr, bend left, "\mathrm{des}"] \arrow[rr, bend right, "\mathrm{gdes}"'] & & \overline{\mathrm{Q}} \arrow[ll, "\theta"']
\end{tikzcd}$$

Figure 4.1: The descent theory.

The Coxeter group W acts on $\mathcal{C} \times \mathcal{C}$ and Q and also on the maps relating them. By taking W-coinvariants, we obtain maps denoted des, gdes and θ that relate the posets W under the weak left Bruhat order and $\overline{\mathrm{Q}} = \{T \mid T \leq S\}$ under subset inclusion. These are shown in Figure 4.1. This gives us the notion of descent and global descent of an element of W. The map des, already defined in Section 1.3.6, is standard while the map gdes is new. It generalizes the notion of global descent of a permutation [4].

Let $\mathbb{K}\Sigma$ be the semigroup algebra over $\mathbb{K}$ with basis the elements of Σ. The radical of this algebra is well understood (Lemma 2.5.5). Its semisimple quotient is $\mathbb{K}\mathrm{L}$, where L is the poset of flats of Σ. In Section 2.5, we defined a symmetric and invariant bilinear form on $\mathbb{K}\Sigma$. In the second part of Chapter 5, we will see how this form emerges naturally from descent theory. The relation between the radical of the form and the radical of the algebra was given in Corollary 2.5.2.

$$\begin{tikzcd}
\mathbb{K}\Sigma \arrow[r, hook] \arrow[dd, two heads] & \mathbb{K}\mathrm{Q} \arrow[r, hook] & \mathbb{K}(\mathcal{C}\times\mathcal{C})^* \arrow[d, two heads] \\
 & & \mathbb{K}\mathrm{Q}^* \arrow[d, two heads] \\
\mathbb{K}\mathrm{L} \arrow[rr, hook] & & \mathbb{K}\Sigma^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}\overline{\mathrm{Q}} \arrow[r, hook] \arrow[d, two heads] & \mathbb{K}W \arrow[d, two heads] \\
\mathbb{K}\overline{\mathrm{L}} \arrow[r, hook] & \mathbb{K}(\overline{\mathrm{Q}})^*
\end{tikzcd}$$

Figure 4.2: The commutative diagrams.

The bilinear form on $\mathbb{K}\Sigma$ allows us to define a commutative diagram that relates $\mathbb{K}\Sigma$, $\mathbb{K}\mathrm{Q}$, $\mathbb{K}(\mathcal{C}\times\mathcal{C})$, $\mathbb{K}\mathrm{L}$ and their duals. A part of this diagram is shown in Figure 4.2 on the left. The composite map from $\mathbb{K}\Sigma$ to $\mathbb{K}\Sigma^*$ is the one induced by the bilinear form on $\mathbb{K}\Sigma$. By taking W-coinvariants, we obtain a quotient diagram, which is shown on the right. For type A, this coincides with the right part of diagram (3.3).

The abstract setting enables us to generalize some of the results by replacing Σ by a left regular band (Chapter 2). In order to incorporate nonassociative examples like buildings and modular lattices, we generalize even further to projection posets, which is a notion that we introduced in Section 2.7.

4.2 Abstract: Chapter 6

In Chapter 6, we study how Hopf algebras enter in this context. We start with the family of symmetric groups S_n, for $n \geq 0$. For each n, we have the corresponding objects denoted $\mathbb{K}\Sigma^n$, $\mathbb{K}\mathrm{Q}^n$, $\mathbb{K}(\mathcal{C}^n\times\mathcal{C}^n)$ and so on and the commutative diagram relating them, shown in Figure 4.2. By taking the direct sum over all n, we obtain a diagram of graded vector spaces, a part of which is shown on the left below in Figure 4.3.

$$\begin{tikzcd}
\mathrm{M}\Pi \arrow[r, hook] \arrow[dd, two heads] & \mathrm{N}\Pi \arrow[r, hook] & \mathrm{S}\Pi \arrow[d, two heads] \\
 & & \mathrm{Q}\Pi \arrow[d, two heads] \\
\Pi \arrow[rr, hook] & & \mathrm{P}\Pi
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{N}\Lambda \arrow[r, hook] \arrow[d, two heads] & \mathrm{S}\Lambda \arrow[d, two heads] \\
\Lambda \arrow[r, hook] & \mathrm{Q}\Lambda
\end{tikzcd}$$

Figure 4.3: The external commutative diagrams.

We put a graded Hopf algebra structure on each object in the diagram on the left and then show that all the maps are in fact morphisms of Hopf algebras. The semigroup structure of Σ plays an important role in the definition of these Hopf algebras, as well as in the proofs. The combinatorial structures that arise in this theory are set partitions, set compositions, pairs of permutations and so on. We have seen some of this in Section 1.4. To achieve more clarity, we write down abstract algebra and coalgebra axioms for a family of projection posets and show that the above example is then a special case.

Following the philosophy of Chapter 5, we take the coinvariant quotient of the diagram on the left. This yields the diagram on the right, which we have met before (Diagram (3.3)).

4.3 Abstract: Chapters 7 and 8

In Chapter 7, we give an application of our methods. Namely, we study the Hopf algebra

$$\mathrm{S}\Pi = \bigoplus_{n \geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*,$$

indexed by pairs of permutations that is introduced in Chapter 6. In particular, we show that it is free and cofree. The method of proof is similar to the one used in [4] to study the Hopf algebra $\mathrm{S}\Lambda$ of permutations. Namely, we compute the product and coproduct of $\mathrm{S}\Pi$ in different bases. A notable difference is our extensive use of the semigroup structure of Σ. The above two Hopf algebras are related. Namely, there is a quotient map from $\mathrm{S}\Pi$ to $\mathrm{S}\Lambda$. Using this map, one obtains some known as well as new results on $\mathrm{S}\Lambda$.

In Chapter 8, in a similar way, we study and relate the Hopf algebra $\mathrm{Q}\Pi$, also introduced in Chapter 6, and the Hopf algebra $\mathrm{Q}\Lambda$ of quasi-symmetric functions.

Chapter 5

The descent theory for Coxeter groups

5.1 Introduction

In this chapter, we initiate a systematic study of a generalized descent theory for finite Coxeter groups. We construct a commutative diagram involving interesting objects related to Coxeter groups, which is central to the study of some Hopf algebras of recent interest. The connection with Hopf algebras will be treated in detail in the next chapters. The current chapter is divided into two parts, whose contents we summarize briefly.

5.1.1 The first part: Sections 5.2-5.5

Standard material

Let S_n be the symmetric group on n letters and regard it as a poset under the weak left Bruhat order. Let $\overline{\mathrm{Q}}^n$ denote the poset of subsets of $[n-1]$ under inclusion. It is the same as the poset of compositions of length n under refinement. In [4], Aguiar and Sottile considered two order preserving maps $\mathrm{des}, \mathrm{gdes} : \mathrm{S}_n \twoheadrightarrow \overline{\mathrm{Q}}^n$ that map a permutation to its descent and global descent set respectively. Further they considered a third order preserving map $\theta : \overline{\mathrm{Q}}^n \hookrightarrow \mathrm{S}_n$ and showed that the maps des and gdes are the left and right adjoints respectively to θ. They used these ideas to study $\mathrm{S}\Lambda$, the graded Hopf algebra of permutations introduced by Malvenuto [61].

In the first part of this chapter, we generalize the above results to any finite Coxeter group W. The papers by Reading [81, 80] generalize these ideas in a different direction from the one we take.

New material

The picture so far should be regarded as existing on the W-invariant-coinvariant level, and which can be derived from a more fundamental picture. Accordingly, in Section 5.2, we define three maps Road, GRoad and Θ and develop a descent theory. In particular, we show that the maps Road and GRoad are the left and right adjoints respectively to Θ. We refer to this as the lifted picture. The motivation for the "road" terminology is also given in Section 5.2.

The maps Road, GRoad and Θ commute with the action of the Coxeter group W, and by moding out the action, one recovers the original maps, namely, des, gdes and θ. This philosophy is summarized in Proposition 5.2.1. The descriptions of the maps des, gdes and θ are given in Section 5.3. The notion of shuffles for arbitrary Coxeter groups is well-known (minimal coset representatives of parabolic subgroups), see Humphreys [47, Section 1.10]. In Section 5.3, we discuss this notion, but from a geometric viewpoint. In Sections 5.4 and 5.5, we discuss the examples of type A_{n-1} and $A_1^{\times(n-1)}$ in explicit combinatorial terms.

The advantage of working with the lifted picture is both greater generality and conceptual clarity. For example, for the most part, the Coxeter group W is not necessary for the lifted picture and one may work with any central hyperplane arrangement. However there are problems if one wants to generalize further to left regular bands, see the open question at the end of Section 5.2.

5.1.2 The second part: Sections 5.6-5.7

Standard material

The Hopf algebra of permutations $\mathrm{S}\Lambda$ of Malvenuto [61] is related to three other graded Hopf algebras by a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{N}\Lambda \arrow[r, hook] \arrow[d, two heads] & \mathrm{S}\Lambda \arrow[d, two heads] \\ \Lambda \arrow[r, hook] & \mathrm{Q}\Lambda \end{tikzcd} \tag{5.1}$$

where Λ, $\mathrm{Q}\Lambda$ and $\mathrm{N}\Lambda$ are the Hopf algebras of symmetric functions [34, 102], quasi-symmetric functions [36, 61, 46] and noncommutative symmetric functions [35] respectively. For more details, see Section 3.2.

For P a poset, let $\mathbb{K}P$ be the vector space over the field $\mathbb{K}$ with basis the elements of P, and let $\mathbb{K}P^*$ be its dual space. The maps des, gdes and θ play a key role in the definition of the maps in diagram (5.1). The vector spaces in degree n of these graded Hopf algebras are as follows.

$$\begin{tikzcd} \mathbb{K}\overline{\mathrm{Q}}^n \arrow[r, hook] \arrow[d, two heads] & \mathbb{K}\mathrm{S}_n \arrow[d, two heads] \\ \mathbb{K}\overline{\mathrm{L}}^n \arrow[r, hook] & \mathbb{K}(\overline{\mathrm{Q}}^n)^* \end{tikzcd} \tag{5.2}$$

Here $\overline{\mathrm{L}}^n$ is the poset of partitions of n, with partial order as given in Definition 3.2.1.

New material

Let (W, S) be a Coxeter system and let Σ be its Coxeter complex. In the second part of this chapter, we construct a commutative diagram as above for any Coxeter group W, see diagram (5.17) in Section 5.7. It is a slightly expanded form of diagram (5.2). We replace S_n by W, and $\overline{\mathrm{Q}}^n$ by

$$\overline{\mathrm{Q}} = \{T \mid T \le S\},$$

and $\overline{\mathrm{L}}^n$ by something appropriate, see Section 5.2.3. The main tool in the construction is a bilinear form on $\mathbb{K}\overline{\mathrm{Q}}$. However, we mention that the construction of the Hopf algebras in diagram (5.1) is special to the example of type A, the symmetric groups.

In the summary of the first part of the chapter above, we mentioned that the maps des, gdes and θ can be derived from more fundamental maps, namely Road, GRoad and Θ respectively. We continue with this philosophy in the second part of the chapter. Namely, in Section 5.6, we define the commutative diagram (5.8) to go along with these lifted maps. In this case, the commutativity in the lifted diagram (5.8) is controlled by a bilinear form on $\mathbb{K}\Sigma$. Note that the W-orbits in Σ can be identified with $\overline{\mathrm{Q}}$. By moding out the action of W, we recover diagram (5.17). In particular, for the example of type A_{n-1}, one can view diagram (5.2) as a quotient of a more fundamental diagram.

As in the first part, the lifted picture works for any central hyperplane arrangement. However, in contrast to the first part, the second part generalizes to LRBs in a satisfactory way. And a part of it generalizes further to projection posets.

5.2 The descent theory for Coxeter groups

In this section and the next, we present the descent theory for any Coxeter group W, as was outlined briefly in the introduction.

5.2.1 Preliminaries

We recall some definitions and facts from Chapter 1. Let (W, S) be a Coxeter system and Σ be the Coxeter complex of W. Then the Coxeter complex Σ is a *chamber complex*, that is, a gallery connected pure simplicial complex. It follows that Σ is a meet semilattice with the partial order given by face inclusion and the meet of F and G given by their intersection $F \cap G$. Further Σ has the structure of a semigroup. For $F, G \in \Sigma$, we call the product FG as the projection of G on F. Let $\mathcal{C}$ be the set of chambers in Σ. Then $\mathcal{C}$ is a two sided ideal in Σ. We use the letters C, D and E to denote chambers. Associated to Σ is a lattice L and an order preserving surjective map $\operatorname{supp} : \Sigma \twoheadrightarrow \mathrm{L}$, called the support map, such that

$$\operatorname{supp} FG = \operatorname{supp} F \vee \operatorname{supp} G.$$

Let

$$\mathrm{Q} = \{(F, D) \mid F \leq D\} \subseteq \Sigma \times \mathcal{C} \quad \text{and} \quad \overline{\mathrm{Q}} = \{T \mid T \leq S\}.$$

The Coxeter group W acts on Σ and hence on $\mathcal{C}$, the set of chambers in Σ. Further the action on $\mathcal{C}$ is simply transitive. The action of W induces diagonal actions on $\mathcal{C} \times \mathcal{C}$ and Q. Let $d : \mathcal{C} \times \mathcal{C} \twoheadrightarrow W$ and $\text{type} : \mathrm{Q} \twoheadrightarrow \overline{\mathrm{Q}}$, where d is the distance map and type maps (F, D) to the type of the face F. These maps are invariant under the W-action on $\mathcal{C} \times \mathcal{C}$ and Σ respectively. In fact, we get induced isomorphisms

$$(\mathcal{C} \times \mathcal{C})_W \xrightarrow{\cong} W \quad \text{and} \quad \Sigma_W \xrightarrow{\cong} \overline{\mathrm{Q}}$$

from the respective spaces of orbits. The notation $—_W$ refers to the set of W-orbits in —.

5.2.2 Summary

We now summarize the content of this section and the next section.

Proposition 5.2.1 *The following diagrams commute.*

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, two heads, "d"] \arrow[d, "\text{Road}"'] & W \arrow[d, "\text{des}"] \\ \mathrm{Q} \arrow[r, two heads, "\text{type}"'] & \overline{\mathrm{Q}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, two heads, "d"] \arrow[d, "\text{GRoad}"'] & W \arrow[d, "\text{gdes}"] \\ \mathrm{Q} \arrow[r, two heads, "\text{type}"'] & \overline{\mathrm{Q}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, two heads, "d"] & W \\ \mathrm{Q} \arrow[r, two heads, "\text{type}"'] \arrow[u, "\Theta"] & \overline{\mathrm{Q}} \arrow[u, "\theta"'] \end{tikzcd} \tag{5.3}$$

All objects are posets and all maps are order preserving.

Note on the proof In Section 5.2.4, we define three partial orders each on $\mathcal{C} \times \mathcal{C}$ and Q. The ones relevant to this proposition are $\leq$ and $\preceq$. The partial order on W is the weak left Bruhat order and on $\overline{\mathrm{Q}}$ is subset inclusion.

In Sections 5.2.5-5.2.7, we define the maps Road, GRoad and Θ respectively and show that they are order preserving for $\leq$ and $\preceq$. It is clear from the definitions that they commute with the action of the Coxeter group W. Hence we get induced order preserving maps on the orbit spaces des, gdes : $(\mathcal{C}\times\mathcal{C})_W \to \mathrm{Q}_W$ and $\theta : \mathrm{Q}_W \to (\mathcal{C}\times\mathcal{C})_W$. In Lemmas 5.2.1 and 5.2.2, we show respectively that the maps d and type are order preserving, and that the identifications

$$(\mathcal{C}\times\mathcal{C})_W \xrightarrow{\cong} W \quad \text{and} \quad \Sigma_W \xrightarrow{\cong} \overline{\mathrm{Q}}$$

are in fact poset isomorphisms. This yields the right hand columns of diagram (5.3). They are described in detail in Section 5.3.

□

In Section 5.2.3, we discuss the objects Z and $\overline{\mathrm{L}}$ that are used in the second part of the chapter. In Sections 5.2.4-5.2.7, we discuss the various aspects in the proof of Proposition 5.2.1 as explained above. In Section 5.2.8, we show that the maps Road and GRoad are the left and right adjoints respectively to Θ (Proposition 5.2.5). We also consider the maps Des and GDes, which are obtained from Road and GRoad by composing with the projection $\mathrm{Q} \twoheadrightarrow \Sigma$ on the first coordinate. By construction, it follows that the maps des and gdes are the left and right adjoints respectively to θ (Proposition 5.3.5). As mentioned in the introduction, for the symmetric group S_n, this result was obtained in [4].

Remark The reader may omit the discussion pertaining to the partial orders $\leq'$ and $\preceq$ on a first reading of this section.

5.2.3 The posets Z and $\overline{\mathrm{L}}$

Associated to Q is a poset Z and an order preserving surjective map lune : $\mathrm{Q} \twoheadrightarrow \mathrm{Z}$. The partial order $\leq$ on Q is the one relevant to this statement. This is explained in the general setting of a LRB in Section 2.3. The construction of Z from Q is analogous to the construction of the poset of flats L from Σ. Further the map base : $\mathrm{Q} \to \Sigma$ that sends (F, D) to F induces a map base : $\mathrm{Z} \to \mathrm{L}$.

$$\begin{tikzcd}
\Sigma \arrow[dr, two heads, "\mathrm{type}"'] \arrow[dd, two heads, "\mathrm{supp}"'] & & \mathrm{Q} \arrow[ll, two heads, "\mathrm{base}"'] \arrow[dl, two heads, "\mathrm{type}"] \arrow[dd, two heads, "\mathrm{lune}"] \\
& \overline{\mathrm{Q}} \arrow[dd, two heads] & \\
\mathrm{L} \arrow[dr, two heads] & & \mathrm{Z} \arrow[ll, two heads] \arrow[dl, two heads] \\
& \overline{\mathrm{L}} &
\end{tikzcd} \tag{5.4}$$

Note that $\mathrm{Q}_W \cong \Sigma_W \cong \overline{\mathrm{Q}}$ as posets. Similarly, it is true that $\mathrm{Z}_W \cong \mathrm{L}_W$. We call this quotient poset $\overline{\mathrm{L}}$. The map $\overline{\mathrm{Q}} \to \overline{\mathrm{L}}$ in diagram (5.4) is the induced map on the orbit spaces from both the supp and lune maps.

5.2.4 The partial orders on $\mathcal{C}\times\mathcal{C}$ and Q

Recall that the Coxeter group W acts simply transitively on the set of chambers $\mathcal{C}$. Fix a fundamental chamber C_0 in Σ and use it as a reference point to identify $\mathcal{C}$ with W. We use the notation discussed in Section 1.3 that wC_0 is the chamber that corresponds to $w \in W$. Note that $d(C_0, wC_0) = w$.

Definition 5.2.1 Let $\leq$ be the weak left Bruhat order on W. Define a partial order on the set $\mathcal{C}$ by

$$uC_0 \leq_b vC_0 \quad \text{in} \quad \mathcal{C} \quad \Longleftrightarrow \quad u \leq v \quad \text{in} \quad W.$$

The subscript "b" stands for Bruhat.

Notation As mentioned in Chapter 1, the notation $C_2 - C_1 - D$ indicates a minimum gallery $C_2 - \ldots - C_1 - \ldots - D$ from C_2 to D passing through C_1.

Definition 5.2.2 We define three partial orders on $\mathcal{C} \times \mathcal{C}$.

$$\begin{aligned}
(C_1, D_1) \leq (C_2, D_2) &\iff D_1 = D_2 = D \text{ and } C_2 - C_1 - D. \\
&\iff D_1 = D_2 = D \text{ and } d(C_1, D_1) \leq d(C_2, D_2). \\
(C_1, D_1) \leq' (C_2, D_2) &\iff D_1 \leq_b D_2 \text{ and } d(C_1, D_1) = d(C_2, D_2). \\
(C_1, D_1) \preceq (C_2, D_2) &\iff \exists E \ni (C_1, D_1) \leq (E, D_1) \text{ and } (E, D_1) \leq' (C_2, D_2).
\end{aligned}$$

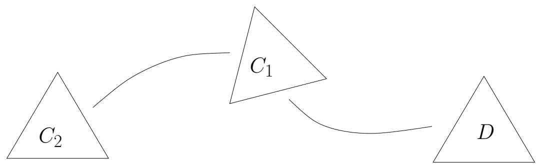


Figure 5.1: A minimum gallery that illustrates the partial order $\leq$ on $\mathcal{C} \times \mathcal{C}$.

We make some elementary observations.

- In the definition of $\preceq$, only one E can satisfy the required condition; namely the one that satisfies $d(E, D_1) = d(C_2, D_2)$.
- Unlike $\leq$, the partial orders $\leq'$ and $\preceq$ depend on the choice of the fundamental chamber C_0, since they involve the partial order $\leq_b$.
- It is clear that $(C_1, D_1) \leq (C_2, D_2)$ implies $(C_1, D_1) \preceq (C_2, D_2)$. The converse is of course not true. However if $(C_1, D_1) \preceq (C_2, D_2)$ then there exists a unique element, say (E, D_1), in the same W-orbit as (C_2, D_2) for which $(C_1, D_1) \leq (E, D_1)$. We conclude that $\leq$ and $\preceq$ induce the same partial order on $(\mathcal{C} \times \mathcal{C})_W$.

Lemma 5.2.1 *For the partial orders $\leq$, $\leq'$ and $\preceq$ on $\mathcal{C} \times \mathcal{C}$, the map $d : \mathcal{C} \times \mathcal{C} \to W$ is order preserving. This induces a poset map $(\mathcal{C} \times \mathcal{C})_W \to W$, which is a set isomorphism. And for $\leq$ and $\preceq$, the induced map is, in fact, a poset isomorphism.*

Proof For $\leq$ and $\leq'$, it is clear that d is order preserving; compare Figures 1.4 and 5.1. And these two facts imply that d is order preserving for $\preceq$ as well.

In view of the third observation made above, it is enough to prove the second claim for $\leq$. Note that, as a poset, $\mathcal{C} \times \mathcal{C}$ is made of $|W|$ disjoint components. Each component is obtained by fixing the second coordinate and varying the first. Further W acts simply transitively on the set of components, and d maps each component isomorphically onto W as a poset. This proves the second claim for $\leq$.

$\square$

Remark In Chapter 7, we will use the partial orders $\leq$ and $\preceq$ to study the structure of the Hopf algebra SΠ of pairs of permutations. This provides a separate motivation.

Definition 5.2.3 Let $\mathrm{Q} = \{(F, D) \mid F \le D\} \subseteq \Sigma \times \mathcal{C}$ be the set of pointed faces. Define three partial orders on Q as follows.

$$(F_1, D_1) \le (F_2, D_2) \iff D_1 = D_2 \text{ and } F_1 \le F_2.$$

$$(F_1, D_1) \le' (F_2, D_2) \iff D_1 \le_b D_2 \text{ and } \operatorname{type} F_1 = \operatorname{type} F_2.$$

$$(F_1, D_1) \preceq (F_2, D_2) \iff \exists H \ni (F_1, D_1) \le (H, D_1) \text{ and } (H, D_1) \le' (F_2, D_2).$$

In analogy with $\mathcal{C} \times \mathcal{C}$, we make the following observations.

- In the definition of $\preceq$, only one H can satisfy the required condition; namely the face of D_1 whose type is the same as that of F_2.
- Unlike $\le$, the partial orders $\le'$ and $\preceq$ depend on the choice of the fundamental chamber C_0.
- It is clear that $(F_1, D_1) \le (F_2, D_2)$ implies $(F_1, D_1) \preceq (F_2, D_2)$. The converse is not true. However if $(F_1, D_1) \preceq (F_2, D_2)$ then there exists a unique element, say (H, D_1), in the same W-orbit as (F_2, D_2) for which $(F_1, D_1) \le (H, D_1)$. We conclude that $\le$ and $\preceq$ induce the same partial order on Q_W.

Let type : $\mathrm{Q} \to \overline{\mathrm{Q}}$ map (F, D) to the type of the face F. It is invariant under the W-action on Q and there is an induced isomorphism $\mathrm{Q}_W \xrightarrow{\cong} \overline{\mathrm{Q}}$ of sets.

Lemma 5.2.2 *For the partial orders $\le$, $\le'$ and $\preceq$ on Q, the map* type : $\mathrm{Q} \to \overline{\mathrm{Q}}$ *is order preserving. This induces a poset map $\mathrm{Q}_W \to \overline{\mathrm{Q}}$, which is a set isomorphism. And for $\le$ and $\preceq$, the induced map is, in fact, a poset isomorphism.*

The proof is along the same lines as that for Lemma 5.2.1.

5.2.5 The map Road

We begin by explaining the notion of a descent.

Definition 5.2.4 The chamber D has a descent with respect to the chamber C at a vertex v of D if there is a minimal gallery from C to D that passes through the facet $D \setminus v$ in the final step, that is, if the support of $D \setminus v$ separates C and D. This is illustrated in Figure 5.2.

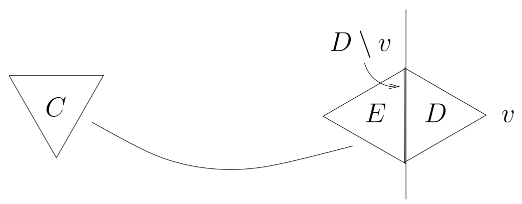


Figure 5.2: The chamber D has a descent with respect to the chamber C at v.

And $\operatorname{Des}(C, D)$ is the face of D spanned by the vertices v of D at which D has a descent with respect to C. This defines a map $\operatorname{Des} : \mathcal{C} \times \mathcal{C} \to \Sigma$ and a map $\operatorname{Road} : \mathcal{C} \times \mathcal{C} \to \mathrm{Q}$ by

$$\operatorname{Road}(C, D) = (\operatorname{Des}(C, D), D).$$

Remark One may say that $\operatorname{Asc}(C, D) = D \setminus \operatorname{Des}(C, D)$ consists of those vertices v at which D has an ascent with respect to C. In this sense, the map Road keeps both the descent and the ascent information; hence the name.

Descent sets can be characterized in terms of the semigroup structure on Σ as follows.

Proposition 5.2.2 *Given chambers C, D and a face $F \leq D$, we have*

$$FC = D \iff \mathrm{Des}(C, D) \leq F.$$

Thus $\mathrm{Des}(C, D)$ *is the smallest face $F \leq D$ such that $FC = D$.*

The above observation is due to Brown [18, Proposition 4].

Lemma 5.2.3 *For $\leq$ and $\preceq$, the map* $\mathrm{Road} : \mathcal{C} \times \mathcal{C} \to \mathrm{Q}$ *is order preserving.*

Proof We prove the lemma only for the partial order $\leq$. Let $(C_1, D) \leq (C_2, D)$ and $v \leq \mathrm{Des}(C_1, D)$. To prove the lemma, we need to show that $v \leq \mathrm{Des}(C_2, D)$.

The first assumption gives a minimum gallery $C_2 - \ldots - C_1 - \ldots - D$ and the second gives a minimum gallery $C_1 - \ldots - E \overset{D\setminus v}{\text{---}} D$. Replacing the second part of the first gallery by the second gallery, one obtains a minimum gallery $C_2 - \ldots - C_1 - \ldots - E \overset{D\setminus v}{\text{---}} D$. This shows that $v \leq \mathrm{Des}(C_2, D)$.

□

5.2.6 The map GRoad

Next we introduce global descents. This notion is meaningful only when W is finite. Recall that in this case, there exists the notion of opposite faces. Let $^{-} : \Sigma \to \Sigma$ be the opposite map that sends a face F to its opposite face $\overline{F}$.

Definition 5.2.5 The chamber D has a global descent with respect to the chamber C at the vertex v of D if, for every chamber $E \in \mathrm{star}(v)$, the star region of v, there is a minimal gallery from C to E that passes through the facet $E \setminus v$ in the final step.

And $\mathrm{GDes}(C, D)$ is the face of D spanned by the vertices v of D at which D has a global descent with respect to C. This defines a map $\mathrm{GDes} : \mathcal{C} \times \mathcal{C} \to \Sigma$ and a map $\mathrm{GRoad} : \mathcal{C} \times \mathcal{C} \to \mathrm{Q}$ by

$$\mathrm{GRoad}(C, D) = (\mathrm{GDes}(C, D), D).$$

Note that by definition $\mathrm{GDes}(C, D) \leq \mathrm{Des}(C, D)$.

Figure 5.3: The chamber D has a global descent with respect to the chamber C at v.

Proposition 5.2.3 *We have* $\mathrm{GDes}(C, D) = D \cap \overline{C}$.

Proof Suppose that D has a global descent with respect to C at v, see Figure 5.3. The definition implies that all chambers in $\mathrm{star}(v)$ have a descent (and global descent) with respect to C at v. The fact mentioned after Equation (1.3) now implies that the opposite chamber $\overline{C} \in \mathrm{star}(v)$.

Conversely, if $\overline{C} \in \mathrm{star}(v)$ then the convexity of $\mathrm{star}(v)$ implies that all chambers in $\mathrm{star}(v)$ have a descent with respect to C at v.

Lemma 5.2.4 *For $\leq$ and $\preceq$, the map* $\mathrm{GRoad} : \mathcal{C} \times \mathcal{C} \to \mathrm{Q}$ *is order preserving.*

Proof We only prove the lemma for the partial order $\leq$. Let $(C_1, D) \leq (C_2, D)$ and $v \leq \mathrm{GDes}(C_1, D) = D \cap \overline{C}_1$, that is, $\overline{D}, C_1 \in \mathrm{star}(\overline{v})$. To prove the lemma, we need to show that $C_2 \in \mathrm{star}(\overline{v})$.

The first assumption gives a minimum gallery $C_2 - \ldots - C_1 - \ldots - D$, which we can extend to $\overline{D} - \ldots - C_2 - \ldots - C_1 - \ldots - D$. Hence by restriction, we have a minimum gallery $\overline{D} - \ldots - C_2 - \ldots - C_1$ such that $\overline{D}, C_1 \in \mathrm{star}(\overline{v})$. The convexity of a star region now implies that $C_2 \in \mathrm{star}(\overline{v})$.

$\square$

5.2.7 The map Θ

Next we introduce the map Θ. We continue to assume that W is finite. We first prove a preliminary fact.

Fact 5.2.1 Let $F \leq D$ and C be any chamber. Then

$$\overline{F}D - C - D \iff FC = D \iff C \in \mathrm{reg}(F, D).$$

The last equivalence is just the definition of $\mathrm{reg}(F, D)$, which is the lunar region of F and D, as given in (2.6).

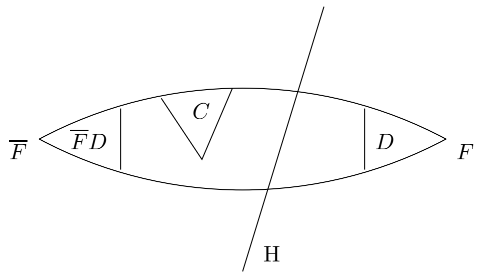


Figure 5.4: A chamber C in the lunar region $\mathrm{reg}(F, D)$.

Proof The first equivalence is a consequence of the following three statements.

$FC = D$	$\iff$	If a hyperplane H separates C and D then it does not pass through F.
H separates $\overline{F}D$ and D	$\iff$	H does not pass through F.
$\overline{F}D - C - D$	$\iff$	If a hyperplane H separates C and D then it also separates $\overline{F}D$ and D.

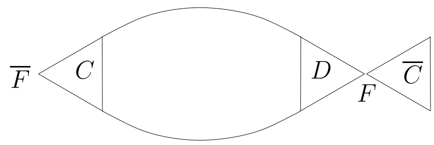
Figure 5.4 shows the lunar region $\mathrm{reg}(F, D)$ and illustrates the situation in all the statements above. For the first two statements, one can argue with sign sequences using (1.1). The third statement is same as (1.3).

$\square$

Definition 5.2.6 Let $\Theta : \mathrm{Q} \to \mathcal{C} \times \mathcal{C}$, where for F, D fixed, $\Theta(F, D)$ is the maximal element of the poset $\{(C, D) \mid FC = D\}$ in the partial order $\leq$ on $\mathcal{C} \times \mathcal{C}$.

For Θ to be well-defined, we need to show that there is a unique maximum. First note that $F\overline{F}D = FD = D$. Hence $(\overline{F}D, D) \in \{(C, D) \mid FC = D\}$. Now Fact 5.2.1 implies that $(\overline{F}D, D)$ is the maximal element in $\{(C, D) \mid FC = D\}$ in the partial order $\leq$ on $\mathcal{C} \times \mathcal{C}$. We have shown the following.

Proposition 5.2.4 *We have* $\Theta(F, D) = (\overline{F}D, D)$.



We also have $(C, D) = \Theta(F, D) \iff (D, C) = \Theta(\overline{F}, C)$.

It follows from the gate property and the above proposition that:

Lemma 5.2.5 *For $\leq$ and $\preceq$, the map $\Theta : \mathrm{Q} \to \mathcal{C} \times \mathcal{C}$ is order preserving.*

5.2.8 Connection among the three maps

Proposition 5.2.5 *For $\leq$ and $\preceq$, the maps* Road *and* GRoad *are the left and right adjoints respectively to Θ. In other words,*

(i) $\mathrm{Road}(C, D) \leq (F, D) \iff (C, D) \leq \Theta(F, D)$.

(ii) $(F, D) \leq \mathrm{GRoad}(C, D) \iff \Theta(F, D) \leq (C, D)$.

And the same statement with $\leq$ replaced by $\preceq$.

Proof We prove the proposition only for the partial order $\leq$. Note that the result for $\preceq$ can be deduced using the result for $\leq$. To see (i), note that

$$\mathrm{Road}(C, D) \leq (F, D) \iff FC = D \iff (C, D) \leq \Theta(F, D). \tag{5.5}$$

The first equivalence follows from Proposition 5.2.2 and the second from Fact 5.2.1 and Proposition 5.2.4.

To see (ii), note that

$$\begin{aligned} (F, D) \leq \mathrm{GRoad}(C, D) &\iff F \leq \overline{C} &&\iff \overline{F} \leq C \\ &\iff \overline{D} - C - \overline{F}D - D &&\iff C - \overline{F}D - D \\ &\iff \Theta(F, D) \leq (C, D). \end{aligned}$$

The first equivalence is by Proposition 5.2.3. For the third equivalence, we use the gate property in one direction and the convexity of $\mathrm{star}(\overline{F})$ and $\overline{F} \leq \overline{D}$ in the other direction. For the fourth equivalence, we use the fact mentioned after Equation (1.3). And for the last equivalence, we use Proposition 5.2.4.

$\square$

Proposition 5.2.6 *The map Θ is a section to both* Road *and* GRoad. *In particular, it implies that Θ is injective and* Road *and* GRoad *are surjective.*

Proof Let $(C, D) = \Theta(F, D)$. Then by Proposition 5.2.5,

$$\mathrm{Road}(C, D) \leq (F, D) \leq \mathrm{GRoad}(C, D).$$

However by definition, $\mathrm{GRoad}(C, D) \leq \mathrm{Road}(C, D)$. Hence

$$\mathrm{Road}(\Theta(F, D)) = (F, D) = \mathrm{GRoad}(\Theta(F, D)),$$

which proves the proposition.

□

Remark It is clear that many results in this section hold for hyperplane arrangements. The objects W, $\overline{\mathrm{Q}}$, and $\overline{\mathrm{L}}$ and the maps involving them, for example d and type, are special to the Coxeter case. The remaining part, with the exception of the partial orders $\leq'$ and $\preceq$, is valid for any central hyperplane arrangement.

Open Question It is not clear how to generalize the results in this section to LRBs. The problem starts right with defining the descent map. For that, one needs to assume that a LRB satisfies the projection axiom $(P2)$. These projection axioms were defined in Mahajan [60, Chapter 1, page 15]. They are not to be confused with the axioms that we will define in Chapter 6. Formulate the correct abstract framework to define descents, global descents and so forth.

5.3 The coinvariant descent theory for Coxeter groups

Let $l : W \to \mathbb{Z}$ be the length function. In the previous section, we proved the existence of the maps $\mathrm{des}, \mathrm{gdes} : W \to \overline{\mathrm{Q}}$ and $\theta : \overline{\mathrm{Q}} \to W$. In Sections 5.3.1-5.3.3, we explicitly describe these maps. As expected, they have very similar descriptions to the ones for the maps $\mathrm{Des}, \mathrm{GDes}$ and Θ. In Section 5.3.4, we deduce that the maps des and gdes are the left and right adjoints respectively to θ. In the rest of the section, we look at some additional results. In Section 5.3.5, we discuss shuffles, and in Section 5.3.6, we consider some sets that show up in the product in the M basis of the Hopf algebra of permutations $\mathrm{S}\Lambda$. The later part will be relevant to Chapter 7.

Notation For $T \leq S$ and $D \in \mathcal{C}$, it is convenient to write T_D for the face of D of type T.

5.3.1 The map des

Let $C, D \in \mathcal{C}$ be such that $d(C, D) = w$. Then by definition, $\mathrm{des}(w)$ is the type of the face $\mathrm{Des}(C, D)$. Let $E \in \mathcal{C}$ be such that $d(D, E) = s$. Hence from Equation (1.5), we have $d(C, E) = ws$. Then in analogy with Definition 5.2.4, one can say:

The element $w \in W$ has a descent at $s \in S$ if there is a minimum gallery $C - E - D$, or equivalently, if $l(ws) < l(w)$. In other words:

Proposition 5.3.1 *We have* $\mathrm{des}(w) = \{s \in S \mid l(ws) < l(w)\}$.

Thus, we recover the familiar notion of descent. Alternatively, from Proposition 5.2.2:

Proposition 5.3.2 *Let $C, D \in \mathcal{C}$ and $T \leq S$. Further let $d(C, D) = w$ and T_D be the face of D of type T. Then*

$$T_D C = D \iff \mathrm{des}(w) \leq T.$$

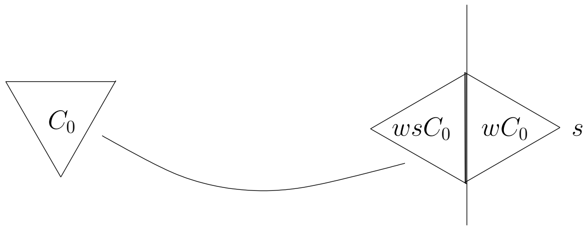


Figure 5.5: A descent at s for an element $w \in W$.

Since $d(C_0, wC_0) = w$, we can take for example $C = C_0$ and $D = wC_0$ and say that $\mathrm{des}(w)$ is the type of the face $\mathrm{Des}(C_0, wC_0)$. In analogy with Definition 5.2.4, one says:

The element $w \in W$ has a descent at $s \in S$ if there is a minimum gallery from C_0 to wC_0 passing through wsC_0. This is illustrated in Figure 5.5.

Alternatively, from the above proposition, $\mathrm{des}(w)$ is the type of the smallest face F such that $FC_0 = wC_0$.

Remark Following our earlier terminology, one should denote the map des by the term "road". However in this particular case, such a distinction is not necessary because by taking complement in the set S, the descent and ascent sets determine each other.

5.3.2 The map gdes

Let $C, D \in \mathcal{C}$ be such that $d(C, D) = w$. Then by definition, $\mathrm{gdes}(w)$ is the type of the face $\mathrm{GDes}(C, D)$. Since $d(C_0, wC_0) = w$, we can take $C = C_0$ and $D = wC_0$ and say that $\mathrm{gdes}(w)$ is the type of the face $\mathrm{GDes}(C_0, wC_0)$. Applying Definition 5.2.5 gives us the following.

Let v be the vertex of wC_0 of type s. Then the element $w \in W$ has a global descent at $s \in S$ if $s \in \mathrm{des}(u)$ for every $uC_0 \in \mathrm{star}(v)$.

Figure 5.6: A global descent at s for an element $w \in W$.

The chambers in $\mathrm{star}(v)$ are characterized by the set $\{wzC_0 \mid z \text{ is any word written using generators other than } s\}$. Hence $w \in W$ has a global descent at $s \in S$ if $l(wzs) < l(wz)$, for any word z written using generators other than s.

Proposition 5.3.3 *We have* $\mathrm{gdes}(w) = \{s \in S \mid l(wzs) < l(wz)$ *for any word* z *written using generators other than* $s\}$.

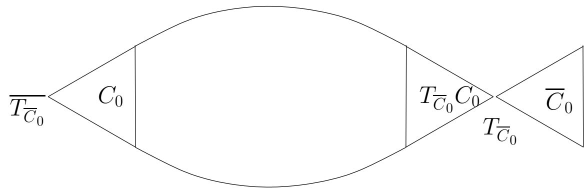
5.3.3 The map θ

Let $\theta : \overline{\mathrm{Q}} \to W$ be the map induced from $\Theta : \mathrm{Q} \to \mathcal{C} \times \mathcal{C}$. The analogy with Definition 5.2.6 is given later in Proposition 5.3.7. In analogy with Proposition 5.3.4, one can say:

Proposition 5.3.4 *For* $T \in \overline{\mathrm{Q}}$, *let* T_D *be the face of type* T *of* D. *Then* $\theta : \overline{\mathrm{Q}} \to W$ *is given by* $T \mapsto \theta(T)$ *where* $\theta(T) = d(\overline{T_D}D, D)$.

Corollary 5.3.1 *For $T \in \overline{\mathrm{Q}}$, let $T_{\overline{C}_0}$ be the face of type T of $\overline{C}_0$. Then $\theta : \overline{\mathrm{Q}} \to W$ is given by $T \mapsto \theta(T)$ where*

$$\theta(T) = d(C_0, T_{\overline{C}_0}C_0), \quad \textit{or equivalently}, \quad \theta(T)C_0 = T_{\overline{C}_0}C_0.$$



Proof Take $D = T_{\overline{C}_0}C_0$ in the previous proposition. The face T_D of type T of D is $T_{\overline{C}_0}$. Hence $\overline{T_{\overline{C}_0}}D = \overline{T_{\overline{C}_0}}T_{\overline{C}_0}C_0 = \overline{T_{\overline{C}_0}}C_0 = C_0$.

□

Remark It may not be true that $\overline{T_{\overline{C}_0}} = T_{C_0}$, where T_{C_0} is the face of type T of C_0. This is because the opposite map does not preserve types in general.

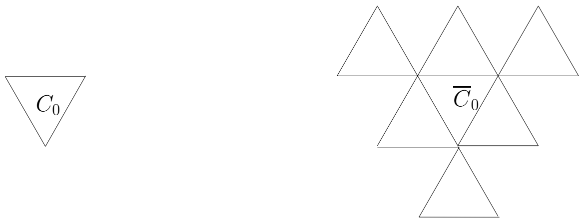


Figure 5.7: The image of the θ map in rank 3.

The corollary says that except for C_0, the image of the map θ viewed as distance from C_0, is concentrated around $\overline{C}_0$. Alternatively, it isolates a set of "long" elements of W. Consider the rank 3 case when W has 3 generators. The image of the θ map then consists of $2^3 = 8$ chambers. A schematic picture for that is shown in Figure 5.7.

5.3.4 Connection among the three maps

The map $d : \mathcal{C} \times \mathcal{C} \to W$ has a section given by $w \mapsto (w^{-1}C_0, C_0)$. Similarly, the map type : $\mathrm{Q} \to \overline{\mathrm{Q}}$ has a section given by $T \mapsto (T_{C_0}, C_0)$, where T_{C_0} is the face of type T of C_0. Further, both the sections are order preserving for each of the partial orders $\leq$ and $\preceq$.

Note that the second coordinate in the image of both maps is always C_0. Further, the maps Road, GRoad and Θ preserve the second coordinate. Since the action of W is simply transitive on $\mathcal{C}$, and W and $\overline{\mathrm{Q}}$ are the orbit spaces of $\mathcal{C} \times \mathcal{C}$ and Q respectively, we obtain three commutative diagrams.

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[d, "\mathrm{Road}"'] & W \arrow[l, hook'] \arrow[d, "\mathrm{des}"] \\
\mathrm{Q} & \overline{\mathrm{Q}} \arrow[l, hook']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[d, "\mathrm{GRoad}"'] & W \arrow[l, hook'] \arrow[d, "\mathrm{gdes}"] \\
\mathrm{Q} & \overline{\mathrm{Q}} \arrow[l, hook']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} & W \arrow[l, hook'] \\
\mathrm{Q} \arrow[u, "\Theta"] & \overline{\mathrm{Q}} \arrow[l, hook'] \arrow[u, "\theta"']
\end{tikzcd}
\tag{5.6}$$

Using diagrams (5.3) and (5.6), now Propositions 5.2.5 and 5.2.6 yield the following corollaries respectively.

Proposition 5.3.5 *The maps* des *and* gdes *are the left and right adjoints respectively to* θ. *In other words,*

(i) $\mathrm{des}(w) \leq T \iff w \leq \theta(T)$.

(ii) $T \leq \mathrm{gdes}(w) \iff \theta(T) \leq w$.

Proposition 5.3.6 *The map* θ *is a section to both* des *and* gdes.

5.3.5 Shuffles

This subsection is optional. By using the map des, we define the notion of shuffles for a Coxeter group. We then describe it by more geometric objects and explain how the map θ fits into the picture. The motivation for the terminology comes from the example of type A_{n-1} given in Section 5.4.6.

Definition 5.3.1 For $T \leq S$, define the set of T-shuffles by

$$\mathrm{Sh}_T = \{\sigma \in W \mid \mathrm{des}\,\sigma \leq T\}.$$

Let Σ_T be the set of faces in Σ of type T. For a fixed $T \leq S$, one can write

$$\mathcal{C} = \bigsqcup_{F \in \Sigma_T} \mathcal{C}_F, \quad \text{where } \mathcal{C}_F = \{D \mid F \leq D\}.$$

This is because every chamber $D \in \mathcal{C}$ has a unique face of type T.

Recall that we have fixed a fundamental chamber C_0 in $\mathcal{C}$. In each star region $\mathcal{C}_F$, by the gate property, there is a special chamber, namely FC_0, closest to C_0 in the gallery metric. Under the identification of $\mathcal{C}$ with W, it corresponds to a T-shuffle. More precisely:

Lemma 5.3.1 *There is a bijection*

$$\pi : \Sigma_T \to \mathrm{Sh}_T$$

given by $F \mapsto \sigma$, *where* $FC_0 = \sigma C_0$, *or equivalently,* $\sigma = d(C_0, FC_0)$. *Under this bijection,* $T_{\overline{C}_0}$, *the face of type* T *of* $\overline{C}_0$ *maps to* $\theta(T)$.

In the identity $FC_0 = \sigma C_0$, the left hand side is the product of F and C_0 while the right hand side is the action of σ on C_0.

Proof The first fact is a simple consequence of Proposition 5.3.2. The second fact follows from Corollary 5.3.1.

□

Figure 5.8 shows two bold dots, which are faces of type T, along with their star regions. The element $\sigma = d(C_0, FC_0)$, which corresponds to F, has been shown as a vector pointing from C_0 to FC_0.

Lemma 5.3.2 *Let* $T \leq S$ *and* $D \in \mathcal{C}$ *be fixed, and let* T_D *be the face of* D *of type* T. *Then the distance map*

$$d : \{(C, D) \mid C \in \mathrm{reg}(T_D, D)\} \longrightarrow \mathrm{Sh}_T$$

is a bijection. Under this bijection, $(\overline{T_D}D, D)$ *maps to* $\theta(T)$. *And further*

$$\mathrm{Sh}_T = \{\sigma \in W \mid \sigma^{-1}C_0 \in \mathrm{reg}(T_{C_0}, C_0)\}.$$

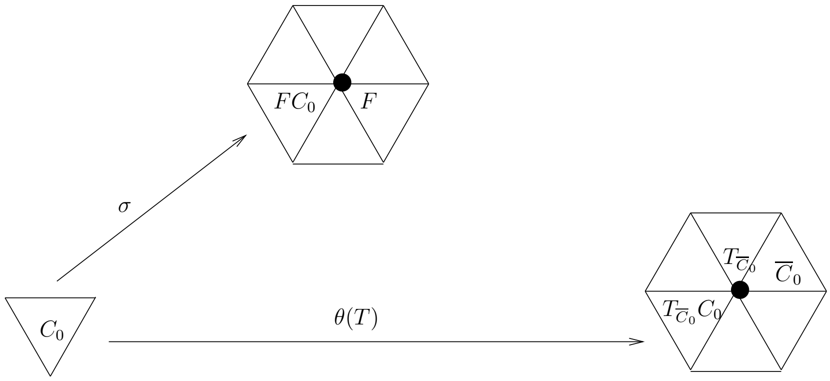


Figure 5.8: T-shuffles correspond to faces of type T.

Note that, in the left hand side in the above bijection, the second coordinate is always fixed to be D. This bijection is dual to the one in Lemma 5.3.1, where the first coordinate is fixed to be C_0. For more on this duality, see Mahajan [60, Chapter 1, pages 32-36].

Proof Observe that by definition,

$$C \in \operatorname{reg}(T_D, D) \iff T_D C = D.$$

The first claim now follows from Proposition 5.3.2. The second claim follows from Proposition 5.3.4. Setting $D = C_0$ yields the third claim.

□

Remark Note that by the above lemma, the set

$$\{\sigma \in W \mid \operatorname{des}(\sigma^{-1}) \leq T\},$$

closely related to Sh_T, corresponds to chambers in the lunar region of T_{C_0} and C_0. This makes the connection between shuffles and lunar regions precise. The reader may look at the definition of the product in $\mathrm{S}\Lambda$ in this regard (Definition 7.2.13).

Proposition 5.3.7 *The element $\theta(T)$ is the maximal element in Sh_T in the weak left Bruhat order on W. Equivalently,*

$$FC_0 \leq_b T_{\overline{C_0}} C_0$$

for any face F of type T.

Proof We know from Definition 5.2.6 and Proposition 5.2.4 that $(\overline{T_D D}, D)$ is the maximal element in $\{(C, D) \mid C \in \operatorname{reg}(T_D, D)\}$. Hence the result follows from Lemma 5.3.2 and the fact that d is order preserving.

□

For $W = \mathrm{S}_n$, the above result is written in [4, Lemma 2.8]. In Chapter 7, we will generalize this result further. For that purpose, the following direct proof of the above proposition will be useful.

Alternative proof Let $FC_0 = uC_0$ and $T_{\overline{C_0}}C_0 = vC_0$. We want to show that there is a minimum gallery $C_0 - vu^{-1}C_0 - vC_0$. For that, observe that vu^{-1} takes FC_0 to $T_{\overline{C_0}}C_0$ and hence F to $T_{\overline{C_0}}$. Therefore

$$T_{\overline{C_0}}(vu^{-1}C_0) = vu^{-1}(FC_0) = T_{\overline{C_0}}C_0.$$

This says that $T_{\overline{C}_0}C_0$ is the gate of $\mathrm{star}(T_{\overline{C}_0})$ when viewed from $vu^{-1}C_0$. Hence by the gate property, we have,

$$C_0 - vu^{-1}C_0 - T_{\overline{C}_0}C_0 - \overline{C}_0.$$

By restriction, we obtain the required minimum gallery.

□

5.3.6 Sets related to the product in the M basis of $\mathrm{S}\Lambda$

This subsection is optional for this chapter. Let $T \leq S$ and $W_{S\setminus T}$ be the parabolic subgroup of W generated by $S \setminus T$. Let $w \in W$ and $x \in W_{S\setminus T}$. Further let $G \leq C_0$ be the face of type T of the fundamental chamber.

Lemma 5.3.3 *The sets $S^0_w(x)$, $S^+_w(x)$ and $S^-_w(x)$, defined below, are in bijection with one another.*

The definition of the set $S^0_w(x)$ has a geometric flavor, while definitions of the other two sets have a combinatorial flavor. These sets show up when one considers the product in the M basis of the Hopf algebra of permutations $\mathrm{S}\Lambda$. We will see this in Chapter 7. Though this motivation is specific to type A_{n-1}, the discussion below is general. We suppress the dependence on T in the notation.

Definition 5.3.2

$$S^0_w(x) = \left\{ (C,D) \in \mathcal{C} \times \mathcal{C}, d(C,D) = w, GD = C_0 \;\middle|\; \begin{array}{l} (i)\ C - GC - GD - D \\ (ii)\ GC = x^{-1}C_0. \end{array} \right\}$$

Note that this definition depends on the choice of C_0. From the gate property, there are two shorter ways to rewrite condition (i), namely, $C - GC - D$, or $C - GD - D$.

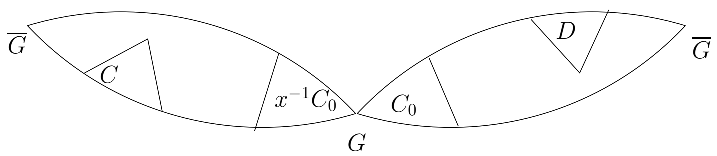


Figure 5.9: The set $S^0_w(x)$.

To get a general idea of what is going on, the reader may refer to Figure 5.9. In the figure, the face G is shown as a vertex. The two lunar regions, containing the chambers C and D respectively, lie on a sphere and meet at $\overline{G}$, the vertex opposite to G. For simplicity, this is not shown in the figure, where there are two vertices labeled $\overline{G}$.

We have defined $S^0_w(x)$ using the Coxeter complex Σ. Since $d(C,D) = w$, knowing D determines C and vice versa. So one can also define $S^0_w(x)$ using the Coxeter group W by getting rid of one of the coordinates. This leads to two more sets $S^+_w(x)$ and $S^-_w(x)$ as below.

Let $d(D, C_0) = \sigma$. Then by Proposition 5.3.2,

$$\sigma \in \mathrm{Sh}_T \iff GD = C_0.$$

So $S^0_w(x)$ can be rewritten as

$$\left\{ \sigma \in \mathrm{Sh}_T \;\middle|\; \begin{array}{l} (i)\ C - GC - GD - D \\ (ii)\ GC = x^{-1}C_0. \end{array} \right\},$$

where D and C are defined by $d(D, C_0) = \sigma$ and $d(C, D) = w$.

Note that a gallery condition can be rephrased using the partial order $\leq$ on $\mathcal{C} \times \mathcal{C}$. And using the order preserving map d, one can then write it using the weak left order on W. Hence condition (i) can be rewritten in two ways as follows.

$$d(GC, D) \leq d(C, D) \iff x\sigma^{-1} \leq w, \quad \text{or} \quad d(GD, D) \leq d(C, D) \iff \sigma^{-1} \leq w.$$

Further note that, if $x\sigma^{-1} \leq w$ then

$$GC = x^{-1}C_0 \iff \begin{array}{l} \text{For } E \in \operatorname{star}(G), \text{ a minimum gallery } C - E - D \text{ implies} \\ \text{a minimum gallery } x^{-1}C_0 - E - C_0. \end{array}$$

This follows from the gate property. Now let $d(E, C_0) = y \in W_{S\setminus T}$, that is, $E = y^{-1}C_0$. Note that y and E determine each other. Now the right hand side in the above equation can be rewritten as:

$$\text{For } x, y \in W_{S\setminus T}, \text{ if } y\sigma^{-1} \leq w, \text{ then } y \leq x.$$

This motivates the following definition.

Definition 5.3.3

$$S_w^+(x) = \left\{ \sigma \in \mathrm{Sh}_T \;\middle|\; \begin{array}{l} (i)\ x\sigma^{-1} \leq w. \\ (ii) \text{ For } y \in W_{S\setminus T}, \text{ if } y\sigma^{-1} \leq w, \text{ then } y \leq x. \end{array} \right\}$$

For type A_{n-1}, the above set is considered in [4, Equations (4.2) and (4.4)].

Analogous to the analysis above, instead of $d(D, C_0) = \sigma$, we can start with the equation $d(C, x^{-1}C_0) = \sigma$. Then by Proposition 5.3.2,

$$\sigma \in \mathrm{Sh}_T \iff GC = x^{-1}C_0.$$

This leads us to the following.

Definition 5.3.4

$$S_w^-(x) = \left\{ \sigma \in \mathrm{Sh}_T \;\middle|\; \begin{array}{l} (i)\ x^{-1}\sigma^{-1} \leq w^{-1}. \\ (ii) \text{ For } y \in W_{S\setminus T}, \text{ if } y^{-1}\sigma^{-1} \leq w^{-1}, \text{ then } y^{-1} \leq x^{-1}. \end{array} \right\}$$

In the above discussion, along with the definitions, we also indicated the proof of Lemma 5.3.3. The details can be filled in by the reader.

5.4 The example of type A_{n-1}

This example arises from $W = \mathrm{S}_n$, the symmetric group on n letters (Section 1.4). We now explain this example in direct combinatorial terms without reference to hyperplane arrangements or the general theory of Coxeter groups. We use the superscript n to indicate the dependence on n.

5.4.1 The posets Σ^n and L^n

We quickly recall the content of Section 1.4.3. The Coxeter complex Σ^n can be identified with the poset of compositions $F = F^1|\dots|F^l$ of the set $[n]$. For example, $347|16|258$ is an element of Σ^8. We multiply two set compositions by taking intersections and ordering them lexicographically; more precisely, if $F = F^1|\dots|F^l$ and $H = H^1|\dots|H^m$, then

$$FH = (F^1 \cap H^1|\dots|F^1 \cap H^m|\dots|F^l \cap H^1|\dots|F^l \cap H^m)\widehat{\ },$$

where the hat means "delete empty intersections". For example,

$$(347|16|258)(6|157|28|34) = (7|34|6|1|5|28).$$

For the partial order of Σ^n, we say $F \leq K$ if K is a refinement of F. The set of chambers $\mathcal{C}^n$ in Σ^n consists of set compositions with singleton blocks, so they correspond to permutations of $[n]$. The opposite $\overline{F}$ of a face $F = F^1|\dots|F^l$ is obtained by reversing the order of the blocks, that is, $\overline{F} = F^l|\dots|F^1$.

The lattice L^n is the poset of set partitions again under refinement and the support map $\mathrm{supp} : \Sigma^n \twoheadrightarrow \mathrm{L}^n$ forgets the ordering of the blocks. For example,

$$\mathrm{supp}(347|16|258) = \{347, 16, 25\}.$$

5.4.2 The posets Q^n and Z^n

By definition, Q^n consists of pairs (F, D), where F is a set composition and D is a set composition with singleton blocks that refines F. For example,

$$(236|15|4, 6|2|3|5|1|4) \ \in \ \mathrm{Q}^6.$$

It is convenient to write this element as $(6|2|3\Big|5|1\Big|4)$, using small and big bars. We call this a fully nested set composition of [6]. Similarly, the poset Z^n associated to Q^n can be described using fully nested set partitions.

Table 5.1: Combinatorial notions for type A_{n-1}.

Σ^n = set compositions	Q^n = fully nested set compositions
L^n = set partitions	Z^n = fully nested set partitions

Definition 5.4.1 A nested set composition is a sequence $F = F^1\Big|F^2\Big|\dots\Big|F^l$, in which each F^i is a set composition of A^i, and $A^1|\dots|A^l$ is a set composition of $[n]$. For example,

$$(3|15\Big|7|48|29\Big|6)$$

is a nested composition of [9]. A fully nested set composition is a nested set composition with singleton blocks.

Equivalently, a nested set composition is a pair (F, H) of set compositions such that $F \leq H$. In the above example, the pair is $(135|24789|6, 3|15|7|48|29|6)$. And a fully nested set composition is a pair (F, D) with $F \leq D$ and D a chamber.

Definition 5.4.2 A nested set partition is a set $L = \{L^1, \dots, L^l\}$, in which each L^i is a set composition of A^i, and $\{A^1, \dots, A^l\}$ is a set partition of $[n]$. For example,

$$\{3|56, 2|17, 4\}$$

is a nested partition of [7]. A fully nested set partition is a nested set partition with singleton blocks.

We now describe the partial orders on Q^n and Z^n in this language. We say $x \leq y$ in Q^n if y is obtained from x by replacing zero or more occurrences of a small bar by a big bar. Similarly, we say $x \leq y$ in Z^n if y is obtained from x by deleting zero or more occurrences of a bar. For example,

$$(3|1\Big|5|4|2\Big|6) \leq (3\Big|1\Big|5|4\Big|2\Big|6) \text{ in } \mathrm{Q}^6 \qquad \{3|5,2|1,4\} \leq \{3,5,2|1,4\} \text{ in } \mathrm{Z}^5.$$

Next we describe the maps related to these two objects. The map $\mathrm{Q}^n \to \Sigma^n$ forgets the small bars, and replaces the big bars by small bars. For example,

$$(6|2|3\Big|5|1\Big|4) \mapsto (236|15|4).$$

The map lune : $\mathrm{Q}^n \to \mathrm{Z}^n$ forgets the big bars. For example,

$$\mathrm{lune}(6|2|3\Big|5|1\Big|4) = \{6|2|3,5|1,4\}.$$

The map $\mathrm{Z}^n \to \mathrm{L}^n$ forgets the small bars. For example,

$$\{6|2|3,5|1,4\} \mapsto \{236,15,4\}.$$

As already mentioned, the map supp : $\Sigma^n \to \mathrm{L}^n$ forgets the ordering among the blocks, or equivalently, forgets the small bars.

5.4.3 The quotient posets $\overline{\mathrm{Q}}^n$ and $\overline{\mathrm{L}}^n$

The symmetric group S_n acts on Σ^n and Q^n. The set of S_n-orbits in both cases can be identified with the set of compositions of n, which we denote $\overline{\mathrm{Q}}^n$. The induced partial order on $\overline{\mathrm{Q}}^n$ is given by refinement of compositions. The quotient map type : $\Sigma^n \to \overline{\mathrm{Q}}^n$ (resp. $\mathrm{Q}^n \to \overline{\mathrm{Q}}^n$) sends a set composition (resp. fully nested set composition) to its underlying composition. For example,

$$\mathrm{type}(347|16|258) = (3,2,3).$$

The symmetric group S_n acts on L^n and Z^n. The set of S_n-orbits in both cases can be identified with the set of partitions of n, which we denote $\overline{\mathrm{L}}^n$. The induced partial order on $\overline{\mathrm{L}}^n$ is as given in Definition 3.2.1. The map $\mathrm{L}^n \to \overline{\mathrm{L}}^n$ (resp. $\mathrm{Z}^n \to \overline{\mathrm{L}}^n$) sends a set partition (resp. fully nested set partition) to its underlying partition. For example,

$$\{347,16,258\} \mapsto (3,3,2).$$

The maps supp : $\Sigma^n \to \mathrm{L}^n$ and lune : $\mathrm{Q}^n \to \mathrm{Z}^n$ induce the map $\overline{\mathrm{Q}}^n \to \overline{\mathrm{L}}^n$ which sends a composition to its underlying partition.

Remark We have explained combinatorially all the objects and maps that occur in diagram (5.4) for this particular example.

5.4.4 The maps Road, GRoad and Θ

The projection and opposite map for type A_{n-1} were explicitly described in Section 5.4.1. The maps Des, GDes and Θ can be written by applying Propositions 5.2.2, 5.2.3 and 5.3.4 to these descriptions.

Definition 5.4.3 The descent map Des : $\mathcal{C}^n \times \mathcal{C}^n \to \Sigma^n$ is defined as follows.

- $\mathrm{Des}(C, D) = F$ is the face of $D = D^1|D^2|\ldots|D^n$ such that D^i and D^{i+1} lie in different blocks of F iff D^i appears after D^{i+1} in C.

For example, $\mathrm{Des}(2|4|1|3, 1|4|3|2) = (1|34|2)$.

Definition 5.4.4 The global descent map $\mathrm{GDes} : \mathcal{C}^n \times \mathcal{C}^n \to \Sigma^n$ is defined as follows.

- $\mathrm{GDes}(C, D) = F$ is the face of $D = D^1|D^2|\dots|D^n$ such that D^i and D^{i+1} lie in different blocks of F iff D^j appears after D^k in C for all $j \le i$ and $i+1 \le k$.

For example, $\mathrm{GDes}(2|4|1|3, 1|4|3|2) = (134|2)$.

Definition 5.4.5 The map $\Theta : \mathrm{Q}^n \to \mathcal{C}^n \times \mathcal{C}^n$ is defined as follows.

- $\Theta(F, D) = (C, D)$, where C is obtained by reversing the order on the blocks of F and ordering the elements within each block using the order of D.

For example, $\Theta(2|5\Big|1\Big|6|3|7\Big|4) = (4|6|3|7|1|2|5, 2|5|1|6|3|7|4)$.

5.4.5 The maps des, gdes and θ

The maps $\mathrm{Des}, \mathrm{GDes} : \mathcal{C}^n \times \mathcal{C}^n \to \Sigma^n$ and $\Theta : \mathrm{Q}^n \to \mathcal{C}^n \times \mathcal{C}^n$ are maps of S_n-sets. Hence they induce maps $\mathrm{des}, \mathrm{gdes} : \mathrm{S}_n \to \overline{\mathrm{Q}}^n$ and $\theta : \overline{\mathrm{Q}}^n \to \mathrm{S}_n$ on the S_n-orbits. The map des gives the usual notion of descent of a permutation and the maps gdes and θ coincide with the definitions given in [4]. We now describe them.

Write a permutation $w \in \mathrm{S}_n$ using the one-line notation as $w^1w^2\dots w^n$. Recall that the generating set S is $\{s_1, \dots, s_{n-1}\}$, where s_i is the transposition $(i, i+1)$. Note that s_iw interchanges the letters i and $i+1$, while ws_i interchanges the letters in positions i and $i+1$. In other words, $ws_i = w^1w^2\dots w^{i+1}w^i\dots w^n$. Note that

$$l(ws_i) < l(w) \iff w^i > w^{i+1}.$$

Hence by Proposition 5.3.1, one obtains:

Definition 5.4.6 The element w has a descent at s_i (or position i) if $w^i > w^{i+1}$.

Now we explain the map gdes. Observe that the set

$$\{z \mid z \text{ can be written using generators other than } s_i\}$$

consists of precisely those permutations which can be written as a permutation of the first i letters followed by a permutation of the remaining $n-i$ letters. For such a z, we have $wz = \underbrace{w^1\dots w^i}_{\text{permuted}}\underbrace{w^{i+1}\dots w^n}_{\text{permuted}}$. Combining with the above definition of des and the definition of gdes given by Proposition 5.3.3, one obtains:

Definition 5.4.7 The element w has a global descent at s_i (or position i) if $w^j > w^k$ for all $j \le i$ and $k \ge i+1$.

Now we explain the map θ. The fundamental chamber C_0 is the set composition $(1|2|\dots|n)$ and its opposite chamber $\overline{C}_0$ is $(n|n-1|\dots|2|1)$. Let $T = \{t_1 < \dots < t_k\}$ be any subset of $[n-1]$. The face of type T of $\overline{C}_0$ is then given by the k block set composition

$$(n{-}t_1{+}1\dots n|\ n{-}t_2{+}1\dots n{-}t_1|\ \dots\ |1\dots n{-}t_k).$$

Multiplying on the right by C_0 and using Proposition 5.3.4, we obtain:

Definition 5.4.8 For $T \le S$, we have

$$\theta(T) = (n{-}t_1{+}1|\dots|n|\ n{-}t_2{+}1|\dots|n{-}t_1|\ \dots\ |1|\dots|n{-}t_k).$$

5.4.6 Shuffles

Let $T = \{t_1 < \cdots < t_k\}$ be any subset of $[n-1]$. It follows from the definitions that

$$\mathrm{Sh}_T = \{\sigma \in \mathrm{S}_n \mid \sigma_1 < \sigma_2 < \cdots < \sigma_{t_1}, \sigma_{t_1+1} < \cdots < \sigma_{t_2}, \ldots, \sigma_{t_{k-1}+1} < \cdots < \sigma_{t_k}\}.$$

As an example, let $n = 4$ and $T = \{2\}$. Then in the composition notation, $T = (2, 2)$ and

$$\mathrm{Sh}_T = \{1|2|3|4, 1|3|2|4, 1|4|2|3, 2|3|1|4, 2|4|1|3, 3|4|1|2\}.$$

The set of inverses of the six elements obtained by shuffling $1|2$ and $3|4$ is precisely the set above. This explains the shuffle terminology used in Section 5.3.5.

5.5 The toy example of type $A_1^{\times(n-1)}$

In this section, we give a toy example to illustrate the theory. It deals with the coordinate hyperplane arrangement arising from the Coxeter group $\mathbb{Z}_2^{n-1}$, that is, the Coxeter group of type $A_1^{\times(n-1)}$. One reason we stress this example is that it belongs to an infinite family and hence is relevant to external structures. As in the previous example, we use the superscript n to indicate the dependence on n.

First note that $\mathbb{Z}_2$ or equivalently S_2 is the Coxeter group of type A_1 generated by a single element of order 2. By taking direct product of $n-1$ copies of $\mathbb{Z}_2$, we obtain the Coxeter group $\mathbb{Z}_2^{n-1}$ with generating set $S = \{s_1, \ldots, s_{n-1}\}$ and presentation as below.

$$s_i^2 = 1, \ (s_i s_j)^2 = 1 \text{ if } i \neq j.$$

The reflection arrangement in this case is the *coordinate arrangement* in $\mathbb{R}^{n-1}$. It consists of the hyperplanes defined by $x_i = 0$, where $1 \leq i \leq n-1$. The generator s_i acts on the arrangement by changing the ith coordinate to its negative and keeping the other coordinates unchanged.

We now explain the relevant objects for this example in direct combinatorial terms without reference to hyperplane arrangements.

5.5.1 The posets Σ^n and L^n

Let Σ^n be the poset

$$\underbrace{\begin{array}{c} + \quad - \\ \diagdown \; \diagup \\ 0 \end{array} \times \begin{array}{c} + \quad - \\ \diagdown \; \diagup \\ 0 \end{array} \times \cdots \times \begin{array}{c} + \quad - \\ \diagdown \; \diagup \\ 0 \end{array}}_{n-1}$$

Namely, elements of Σ^n are sequences $F = F^1 F^2 \ldots F^{n-1}$ of length $n-1$ in the alphabet $\{+, 0, -\}$. And $F \leq K$ in Σ^n if K is obtained by replacing some of the zeroes in F by either a $+$ or a $-$. The product FK is the face with sign sequence

$$(FK)^i = \begin{cases} F^i & \text{if } F^i \neq 0, \\ K^i & \text{if } F^i = 0. \end{cases} \tag{5.7}$$

This is the usual product rule in an oriented matroid as written in (1.1). Chambers in Σ^n are sequences of length $n-1$ in the alphabet $\{+, -\}$. The opposite $\overline{F}$ of a face F is obtained by changing a $+$ to a $-$ and viceversa.

The lattice L^n can be identified with subsets of $[n-1]$, that is, $\overline{\mathrm{Q}}^n$, but under reverse inclusion. The map $\mathrm{supp} : \Sigma^n \to \mathrm{L}^n$ indicates the positions of the zero elements of a sign sequence.

5.5.2 The posets Q^n and Z^n

The elements of Q^n can be regarded as stacked signed sequences. For example,

$$\begin{pmatrix} \overset{+}{0} - \overset{-}{0} + - \overset{+}{0} \end{pmatrix} \in \mathrm{Q}^n$$

and stands for the pair $(0-0+-0, +--+-+)$. For the partial order on Q^n, we say that $x \leq y$ if y is obtained from x by replacing zero or more occurrences of a stack by the sign on top of the stack. For example,

$$\begin{pmatrix} \overset{+}{0} - \overset{-}{0} + - \overset{+}{0} \end{pmatrix} \leq \begin{pmatrix} \overset{+}{0} - - + - \overset{+}{0} \end{pmatrix}.$$

The elements of Z^n are signed subsets of $[n-1]$. For example, $\{+1, -3, +6\}$ is an element of Z^7. The partial order on Z^n is reverse inclusion. For example,

$$\{+1, -3, +6\} \leq \{+1, +6\}.$$

The map $\mathrm{Q}^n \to \Sigma^n$ forgets the signs above the zeroes. For example,

$$\begin{pmatrix} \overset{+}{0} - \overset{-}{0} + - \overset{+}{0} \end{pmatrix} \mapsto 0-0+-0.$$

The map $\mathrm{lune} : \mathrm{Q}^n \to \mathrm{Z}^n$ picks the positions of the zero elements and assigns them the signs on top. For example,

$$\begin{pmatrix} \overset{+}{0} - \overset{-}{0} + - \overset{+}{0} \end{pmatrix} \mapsto \{+1, -3, +6\}.$$

The map $\mathrm{Z}^n \to \mathrm{L}^n = \overline{\mathrm{Q}}^n$ just forgets the signs of the signed set.

5.5.3 The quotient posets $\overline{\mathrm{Q}}^n$ and $\overline{\mathrm{L}}^n$

The group $\mathbb{Z}_2$ acts on $\{+,-\}$ where the nontrivial element of $\mathbb{Z}_2$ flips signs. This induces an action of the group $\mathbb{Z}_2^{n-1}$ on Σ^n and Q^n. The set of orbits in both cases can be identified with the set of compositions of n, which we denote $\overline{\mathrm{Q}}^n$. The induced partial order on $\overline{\mathrm{Q}}^n$ is given by refinement of compositions. The quotient map $\mathrm{type} : \Sigma^n \to \overline{\mathrm{Q}}^n$ indicates the positions of the nonzero elements of the sign sequence. The quotient map $\mathrm{type} : \mathrm{Q}^n \to \overline{\mathrm{Q}}^n$ indicates the positions of the nonzero elements of the stacked sign sequence. For example,

$$\mathrm{type} \begin{pmatrix} \overset{+}{0} - \overset{-}{0} + - \overset{+}{0} \end{pmatrix} = \{2, 4, 5\}.$$

Similarly the group $\mathbb{Z}_2^{n-1}$ acts on Z^n and L^n. The set of orbits denoted $\overline{\mathrm{L}}^n$ is the same in both cases. Every element of L^n is fixed by $\mathbb{Z}_2^{n-1}$. Hence the set of orbits $\overline{\mathrm{L}}^n \cong \mathrm{L}^n$. The quotient map $\mathrm{Z}^n \to \overline{\mathrm{L}}^n$ just forgets the signs of the signed set.

The maps $\mathrm{supp} : \Sigma^n \to \mathrm{L}^n$ and $\mathrm{lune} : \mathrm{Q}^n \to \mathrm{Z}^n$ induce the map $\overline{\mathrm{Q}}^n \to \overline{\mathrm{L}}^n$ which is an isomorphism sending a subset to its complement.

Remark We have explained combinatorially all the objects and maps that occur in diagram (5.4) for this particular example.

Remark The distance map $d : \mathcal{C}^n \times \mathcal{C}^n \to \mathbb{Z}_2^{n-1}$ sends (C, D) to the product of those s_i's for which $C^i \neq D^i$. The order of the s_i's is not important since $\mathbb{Z}_2^{n-1}$ is abelian. In fact, the family $\mathbb{Z}_2^n$, as n varies, are the only abelian Coxeter groups.

5.5.4 The maps Des, GDes and Θ

Definition 5.5.1 The notion of descent and global descent coincides for this example. We have

$$\mathrm{Des}(C,D) = \mathrm{GDes}(C,D) = F, \text{ where } F^i = \begin{cases} 0 & \text{if } C^i = D^i, \\ D^i & \text{otherwise.} \end{cases}$$

For example, $\mathrm{Des}(+--+,-+-+) = -+00$.

Definition 5.5.2 For the map Θ, we have

$$\Theta(F,D) = (C,D), \text{ where } \begin{cases} C^i = D^i & \text{if } F^i = 0, \\ C^i \neq D^i & \text{if } F^i \neq 0. \end{cases}$$

For example, $\Theta(+00-,++--) = (-+-+,++--)$.

5.5.5 The maps des, gdes and θ

Note that as a set, $\mathbb{Z}_2^{n-1}$ can be indentified with the set of words of length $n-1$ in the alphabet $\{0,1\}$. For example, for $n=6$,

$$s_2 s_4 \longleftrightarrow (0,1,0,1,0).$$

Passing to orbits, we obtain the map $\mathrm{des}, \mathrm{gdes} : \mathbb{Z}_2^{n-1} \xrightarrow{\cong} \overline{\mathrm{Q}}^n$ that sends a word to the subset that indicates the positions of the 1's. It is an isomorphism in this case. The induced map $\theta : \overline{\mathrm{Q}}^n \xrightarrow{\cong} \mathbb{Z}_2^{n-1}$ is the inverse to the above map.

5.6 The commutative diagram (5.8)

The goal of this section is to prove the following theorem.

Theorem 5.6.1 *For Σ a left regular band (LRB), the following diagram of vector spaces commutes.*

$$\begin{tikzcd}
\mathbb{K}\Sigma \arrow[r, "b^*"] \arrow[dd, two heads, "\mathrm{supp}"'] \arrow[dddrr, "\Upsilon^*"] & \mathbb{K}\mathrm{Q} \arrow[rr, "\Theta"] \arrow[dd, two heads, "\mathrm{lune}"] \arrow[dddrrr, "\Psi"] & & \mathbb{K}(\mathcal{C}\times\mathcal{C}) \arrow[dr, "s"] & \\
& & & & \mathbb{K}(\mathcal{C}\times\mathcal{C})^* \arrow[dd, "\mathrm{Road}"] \\
\mathbb{K}\mathrm{L} \arrow[r, "b^*"] \arrow[ddrr, "\Phi"'] & \mathbb{K}\mathrm{Z} \arrow[ddrrr, "\Upsilon"] & & & \\
& & \mathbb{K}\mathrm{Z}^* \arrow[rr, hook, "\mathrm{lune}^*"] \arrow[d, "b"] & & \mathbb{K}\mathrm{Q}^* \arrow[d, "b"] \\
& & \mathbb{K}\mathrm{L}^* \arrow[rr, hook, "\mathrm{supp}^*"'] & & \mathbb{K}\Sigma^*
\end{tikzcd} \tag{5.8}$$

For simplicity, we have abbreviated the base map to the letter b. If Σ is the Coxeter complex of a Coxeter group W then the above is a commutative diagram of W-modules.

Note on the proof The proof is given in two parts. Cutting along the map Ψ, the above diagram splits into two halves. The commutativity of each half is proved separately (Propositions 5.6.1 and 5.6.7). The organization is as below.

Let Σ be a Coxeter complex. Recall from Sections 5.2.3 and 5.2.4 that given Σ, one can construct posets $\mathrm{L}, \mathrm{Q}, \mathrm{Z}$ and $\mathcal{C}\times\mathcal{C}$. As remarked at the end of Section 5.2, apart from the absence of the partial orders $\leq'$ and $\preceq$, everything works the same for the poset of faces of any central hyperplane arrangement. In Section 5.6.1, we explain how the ten objects in diagram (5.8) are constructed from these posets. In Sections 5.6.2 and 5.6.3, we define the maps s, Θ, Road and Ψ and in Section 5.6.4 show that they fit in a commutative diagram (Proposition 5.6.1). The maps Θ and Road are defined using the corresponding maps in Section 5.2. We then explain how to adapt the proof to the case when Σ is a LRB, or a projection poset. This is the top half of diagram (5.8).

For the bottom half of diagram (5.8), we directly work with a LRB. We use the results from Sections 2.2 and 2.5, where most of the work is done. The strategy is to break the diagram into five smaller diagrams as below. In Section 5.6.5, we define the maps supp, lune and base* and show that they fit in a commutative diagram. In Section 5.6.6, we dualize to get four more maps and the dual commutative diagram. In Section 5.6.7, we explain the maps Φ and Υ. In Section 5.6.8, we draw three more commutative diagrams, which then imply the result (Proposition 5.6.7).

□

Remark Observe that scaling the five maps, namely, s, Ψ, Φ, Υ and Υ^* by the same factor does not affect the commutativity of the diagram.

5.6.1 The objects in diagram (5.8)

Let Σ be a Coxeter complex. The ten objects we consider are the vector spaces $\mathbb{K}\Sigma$, $\mathbb{K}\mathrm{L}$, $\mathbb{K}\mathrm{Q}$, $\mathbb{K}\mathrm{Z}$, $\mathbb{K}(\mathcal{C}\times\mathcal{C})$ over $\mathbb{K}$ and their duals. The superscript $*$ refers to the dual space or the dual map. Since each space is constructed by linearizing a poset, it has a canonical basis. However, in certain cases, it is better to use the partial order to define more bases, see Table 5.2.

Table 5.2: Vector spaces associated to Σ and their bases.

Vector space	Bases	Dual space	Dual bases
$\mathbb{K}\Sigma$	H	$\mathbb{K}\Sigma^*$	M
$\mathbb{K}\mathrm{Q}$	R, H, K	$\mathbb{K}\mathrm{Q}^*$	S, M, F
$\mathbb{K}(\mathcal{C}\times\mathcal{C})$	R, H, K	$\mathbb{K}(\mathcal{C}\times\mathcal{C})^*$	S, M, F
$\mathbb{K}\mathrm{L}$	h, q	$\mathbb{K}\mathrm{L}^*$	m, p
$\mathbb{K}\mathrm{Z}$	h	$\mathbb{K}\mathrm{Z}^*$	m

For example, for both $\mathbb{K}\mathrm{Q}$ and $\mathbb{K}(\mathcal{C}\times\mathcal{C})$, we let K be the canonical basis. We then define the R and H bases by the first two formulas below.

$$H_A = \sum_{B\leq A} K_B, \qquad R_A = \sum_{B\preceq A} K_B, \qquad R_A = \sum_{B\leq' A} H_B, \tag{5.9}$$

where A and B are elements of the underlying poset, and $\leq$, $\preceq$ and $\leq'$ are the partial orders given in Definitions 5.2.2 and 5.2.3. The last equality above follows from definitions of the partial orders. For example,

$$H_{(E,C)} = \sum_{(D,C)\leq(E,C)} K_{(D,C)} \text{ in } \mathbb{K}(\mathcal{C}\times\mathcal{C}) \text{ and } H_{(P,C)} = \sum_{Q\leq P} K_{(Q,C)} \text{ in } \mathbb{K}\mathrm{Q},$$

and so on. The M (resp. F) basis is defined as dual to the H (resp. K) basis. It follows from (5.9) that

$$F_B = \sum_{B \leq A} M_A, \quad F_B = \sum_{B \preceq A} S_A, \quad M_B = \sum_{B \leq' A} S_A. \tag{5.10}$$

The case of $\mathbb{K}\mathrm{L}$ is a little different. Instead of the H and K bases, we have the h and q bases related by

$$h_X = \sum_{X \leq Y} q_Y \quad \text{in} \quad \mathbb{K}\mathrm{L}. \tag{5.11}$$

Dually, $\mathbb{K}\mathrm{L}^*$ has the m and p bases related by

$$p_Y = \sum_{X \leq Y} m_X \quad \text{in} \quad \mathbb{K}\mathrm{L}^*. \tag{5.12}$$

The motivation for the q and p bases comes from the algebra structure of $\mathbb{K}\mathrm{L}$, see Lemma 5.6.3.

Remark The R and S bases are specific to the Coxeter case, since they involve the partial orders $\leq'$ and $\preceq$. Apart from that, the above discussion and what follows works for the poset of faces of any central hyperplane arrangement.

Example Recall from Section 5.4 that for the symmetric group S_n, the poset L, which we denoted L^n, is indexed by partitions of the set $[n]$. In this case, the graded vector space

$$\Pi_{\mathrm{L}^*} = \bigoplus_{n \geq 0} \mathbb{K}(\mathrm{L}^n)^*$$

is called the space of symmetric functions in noncommuting variables. It has a theory similar to that of symmetric functions. The definitions of the p and m bases above coincide with those considered by Gebhard, Rosas and Sagan [33, 83]; namely, p and m are the power and monomial basis respectively. The map Φ is an isomorphism in this case. If we push the h basis from $\mathbb{K}\mathrm{L}$ to $\mathbb{K}\mathrm{L}^*$ using Φ then it gives a basis of Π_{L^*}, denoted by the same letter in [83]. The map Φ is induced by a bilinear form on $\mathbb{K}\mathrm{L}$, see Definition 5.6.4. Pushing this form to $\mathbb{K}\mathrm{L}^*$ via Φ gives an inner product on Π_{L^*}, with the h and m bases being dual. The p basis is a lift of the power sum basis of symmetric functions. For more explanation, see Section 5.7.1 and Definition 5.7.8.

Remark The spaces Π (short for Π_{L^*}) and $\mathrm{N}\Lambda$ provide different noncommutative analogues to Λ, the space of symmetric functions (Section 3.2). Our treatment brings clarity to this distinction.

5.6.2 The maps s, Θ and Road

Definition 5.6.1 We define the switch map $s : \mathbb{K}(\mathcal{C} \times \mathcal{C}) \to \mathbb{K}(\mathcal{C} \times \mathcal{C})^*$ by

$$K_{(D,C)} \mapsto F_{(C,D)}.$$

Lemma 5.6.1 *Let $\Theta : \mathrm{Q} \to \mathcal{C} \times \mathcal{C}$ be the map defined in Section 5.2. The following are three equivalent definitions of the map $\Theta : \mathbb{K}\mathrm{Q} \to \mathbb{K}(\mathcal{C} \times \mathcal{C})$.*

$$\begin{aligned}
\Theta(K_{(P,C)}) &= \sum_{\mathrm{Des}(D,C)=P} K_{(D,C)} && (K \text{ basis}).\\
\Theta(H_{(P,C)}) &= H_{\Theta(P,C)} && (H \text{ basis}).\\
\Theta(R_{(P,C)}) &= R_{\Theta(P,C)} && (R \text{ basis}).
\end{aligned}$$

Proof The equivalence of the first two definitions can be seen by a simple computation. We use the formula in the K basis to derive the formula in the H basis.

$$\begin{aligned}\Theta(H_{(P,C)}) = \sum_{(Q,C)\leq(P,C)} \Theta(K_{(Q,C)}) &= \sum_{\mathrm{Road}(D,C)\leq(P,C)} K_{(D,C)} \\ &= \sum_{(D,C)\leq\Theta(P,C)} K_{(D,C)} \qquad = H_{\Theta(P,C)}.\end{aligned}$$

For the third equality, we used Proposition 5.2.5, part (i). The first and last equalities follow from (5.9), which is the relation between the H and K bases.

Replacing H by R and $\leq$ by $\preceq$ in the computation, and using Proposition 5.2.5, part (i), one gets the equivalence between the first and third definitions.

$\square$

The map Road : $\mathbb{K}(\mathcal{C}\times\mathcal{C})^* \to \mathbb{K}\mathrm{Q}^*$ is defined as the dual to the map $\Theta : \mathbb{K}\mathrm{Q} \to \mathbb{K}(\mathcal{C}\times\mathcal{C})$. By duality, Lemma 5.6.1 gives:

Lemma 5.6.2 *The following are three equivalent definitions of the map* Road : $\mathbb{K}(\mathcal{C}\times\mathcal{C})^* \to \mathbb{K}\mathrm{Q}^*$.

$$\begin{aligned}\mathrm{Road}(F_{(C,D)}) &= F_{\mathrm{Road}(C,D)} && (F\ \textit{basis}). \\ \mathrm{Road}(M_{(C,D)}) &= \begin{cases} M_{\mathrm{Road}\,(C,D)} & \textit{if } (C,D) = \Theta(\mathrm{Road}(C,D)) \\ 0 & \textit{otherwise.}\end{cases} && (M\ \textit{basis}).\end{aligned}$$

By replacing M by S, one gets the expression on the S basis.

A particularly useful expression for the map Θ is obtained when we start in the H basis and end in the K basis. It is given by the equation

$$\Theta(H_{(P,C)}) = \sum_{D:\ PD=C} K_{(D,C)}. \tag{5.13}$$

This can be seen by using the first part of the computation in Lemma 5.6.1 and Equation (5.5). Similarly, an useful expression for Road is obtained by going from the F basis to the M basis. It is given by

$$\mathrm{Road}(F_{(C,D)}) = \sum_{F:\ FC=D} M_{(F,D)}. \tag{5.14}$$

An important feature of the above two formulas is that they are defined solely in terms of projection maps, and thus make sense not only for hyperplane arrangements or LRBs, but also for projection posets. We will exploit this feature in Chapter 6.

5.6.3 The bilinear form on $\mathbb{K}\mathrm{Q}$

Define a symmetric bilinear form on $\mathbb{K}\mathrm{Q}$ by

$$\langle H_{(P,C)}, H_{(F,D)}\rangle = \begin{cases} 1 & \text{if } FC = D \text{ and } PD = C, \\ 0 & \text{otherwise.}\end{cases}$$

The above bilinear form induces a map $\Psi : \mathbb{K}\mathrm{Q} \mapsto \mathbb{K}\mathrm{Q}^*$ given by

$$\Psi(H_{(P,C)}) = \sum_{(F,D)} \langle (P,C),(F,D)\rangle\ M_{(F,D)}. \tag{5.15}$$

A detailed discussion on this bilinear form is given in Section 2.5.

5.6.4 The top half of diagram (5.8)

Proposition 5.6.1 *For Σ a LRB, or more generally, a projection poset, the top half of diagram* (5.8) *commutes.*

$$\begin{tikzcd} \mathbb{K}\mathrm{Q} \arrow[r, "\Theta"] \arrow[d, "\Psi"'] & \mathbb{K}(\mathcal{C}\times\mathcal{C}) \arrow[d, "s"] \\ \mathbb{K}\mathrm{Q}^* & \mathbb{K}(\mathcal{C}\times\mathcal{C})^* \arrow[l, "\mathrm{Road}"] \end{tikzcd}$$

Proof First, let Σ be a central hyperplane arrangement. We want to show that $\Psi = \mathrm{Road} \circ s \circ \Theta$. This is a direct computation. From Equation (5.13) and Definition 5.6.1, we obtain

$$s \circ \Theta(H_{(P,C)}) = \sum_{D:\ PD=C} F_{(C,D)}.$$

Now by Equation (5.14),

$$\mathrm{Road} \circ s \circ \Theta(H_{(P,C)}) = \sum_{D:\ PD=C} \ \sum_{F:\ FC=D} M_{(F,D)} = \sum_{(F,D)} \langle (P,C),(F,D)\rangle \ M_{(F,D)}.$$

The proposition now follows from the definition of Ψ.

Now let Σ be any projection poset. Given Σ, one can define the objects Q and $\mathcal{C}\times\mathcal{C}$, as in Section 2.2. Note that, in this generality, the partial order on $\mathcal{C}\times\mathcal{C}$ is not at all clear. Hence, we only consider the K basis on $\mathbb{K}(\mathcal{C}\times\mathcal{C})$ and dually the F basis on $\mathbb{K}(\mathcal{C}\times\mathcal{C})^*$, and define the maps Θ and Road by Equations (5.13) and (5.14) respectively. The advantage of these definitions is that they only involve projection maps, and hence make sense for any projection poset. Similarly, the bilinear form on $\mathbb{K}\mathrm{Q}$ and hence the map $\Psi : \mathbb{K}\mathrm{Q} \to \mathbb{K}\mathrm{Q}^*$ can be defined for any projection poset, as in Section 2.5. The same computation as above then proves Proposition 5.6.1 for projection posets.

$\square$

Remark The map Road in the generality of projection posets may not be surjective. For example, take Σ to be a commutative LRB. Then it has a single chamber and surjectivity is not possible.

5.6.5 The maps supp, lune and base*

Definition 5.6.2 We define the maps supp, lune and base* in diagram (5.8). For that, we use the maps supp, lune and base of Section 5.2.3.

- supp : $\mathbb{K}\Sigma \to \mathbb{K}\mathrm{L}$ given by $H_P \mapsto h_{\mathrm{supp}\, P}$.
- lune : $\mathbb{K}\mathrm{Q} \to \mathbb{K}\mathrm{Z}$ given by $H_{(P,C)} \mapsto h_{\mathrm{lune}(P,C)}$.
- base* : $\mathbb{K}\Sigma \to \mathbb{K}\mathrm{Q}$ given by $H_P \mapsto \sum_{C:\ P\leq C} H_{(P,C)}$.

  In other words, we sum over all pointed faces whose base is P.
- base* : $\mathbb{K}\mathrm{L} \to \mathbb{K}\mathrm{Z}$ given by $h_X \mapsto \sum_{L:\ \mathrm{base}\, L = X} h_L$.

Proposition 5.6.2 *For Σ a LRB, the following diagram commutes.*

$$\begin{tikzcd} \mathbb{K}\Sigma \arrow[r, "\mathrm{base}^*"] \arrow[d, "\mathrm{supp}"'] & \mathbb{K}\mathrm{Q} \arrow[d, "\mathrm{lune}"] \\ \mathbb{K}\mathrm{L} \arrow[r, "\mathrm{base}^*"'] & \mathbb{K}\mathrm{Z} \end{tikzcd}$$

The content of this proposition is that the map base* on the bottom is well defined. This is explained in diagram (2.5).

5.6.6 The dual maps supp*, lune* and base

Definition 5.6.3 By dualizing the maps in Definition 5.6.2, we obtain the maps supp*, lune* and base.

- supp* : $\mathbb{K}\mathrm{L}^* \to \mathbb{K}\Sigma^*$ given by $m_Y \mapsto \sum_{F:\ \mathrm{supp}\, F=Y} M_F$.
- lune* : $\mathbb{K}\mathrm{Z}^* \to \mathbb{K}\mathrm{Q}^*$ given by $m_L \mapsto \sum_{(F,D):\ \mathrm{lune}(F,D)=L} M_{(F,D)}$.
- base : $\mathbb{K}\mathrm{Q}^* \to \mathbb{K}\Sigma^*$ given by $M_{(F,D)} \mapsto M_F$.

  We recall that F is the base of the pointed face (F,D).
- base : $\mathbb{K}\mathrm{Z}^* \to \mathbb{K}\mathrm{L}^*$ given by $m_L \mapsto m_{\mathrm{base}\, L}$.

By duality, we obtain:

Proposition 5.6.3 *For Σ a LRB, the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{K}\mathrm{Z}^* \arrow[r, "\mathrm{lune}^*"] \arrow[d, "\mathrm{base}"'] & \mathbb{K}\mathrm{Q}^* \arrow[d, "\mathrm{base}"] \\
\mathbb{K}\mathrm{L}^* \arrow[r, "\mathrm{supp}^*"'] & \mathbb{K}\Sigma^*
\end{tikzcd}$$

5.6.7 The maps Φ and Υ

In Section 2.3.4, we proposed a second approach to lunes. Namely, we defined an object Z$'$ closely related to Z and maps reg : $\mathrm{Q} \to \mathrm{Z}'$ and zone : $\mathrm{Z} \to \mathrm{Z}'$.

Definition 5.6.4 We now define the map Φ and the maps related to the space $\mathbb{K}\mathrm{Z}'$.

- reg : $\mathbb{K}\mathrm{Q} \to \mathbb{K}\mathrm{Z}'$ given by $H_{(P,C)} \to h_{\mathrm{reg}(P,C)}$.
- zone : $\mathbb{K}\mathrm{Z} \to \mathbb{K}\mathrm{Z}'$ given by $h_{\mathrm{lune}(P,C)} \to h_{\mathrm{reg}(P,C)}$.
- $\Upsilon : \mathbb{K}\mathrm{Z}' \to \mathbb{K}\Sigma^*$ given by $h_L \mapsto \sum_{F \in L} M_F$.
- $\Phi : \mathbb{K}\mathrm{L} \to \mathbb{K}\mathrm{L}^*$ given by $\Phi(h_X) = \sum_Y \langle X, Y \rangle_{\mathrm{L}}\ m_Y$, where $\langle\ ,\ \rangle_{\mathrm{L}}$ is the bilinear form given in Section 2.5.4.

5.6.8 The bottom half of diagram (5.8)

By combining Lemmas 2.3.1 and 2.5.1, we obtain:

Proposition 5.6.4 *The following diagram commutes.*

$$\begin{tikzcd}
 & \mathbb{K}\mathrm{Q} \arrow[dl, "\mathrm{lune}"'] \arrow[d, "\mathrm{reg}"] \arrow[dr, "\mathrm{base}\circ\Psi"] & \\
\mathbb{K}\mathrm{Z} \arrow[r, "\mathrm{zone}"'] & \mathbb{K}\mathrm{Z}' \arrow[r, "\Upsilon"'] & \mathbb{K}\Sigma^*
\end{tikzcd}$$

The map $\mathrm{base} \circ \Psi : \mathbb{K}\mathrm{Q} \to \mathbb{K}\Sigma^*$ is given by

$$H_{(P,C)} \mapsto \sum_{F \in \mathrm{reg}(P,C)} M_F.$$

For the poset of faces of central hyperplane arrangements, by Lemma 2.3.3, there is no difference between $\mathbb{K}\mathrm{Z}'$ and $\mathbb{K}\mathrm{Z}$. Hence, to avoid unnecessary complications, we have omitted $\mathbb{K}\mathrm{Z}'$ from diagram (5.8), and referred to the composite $\Upsilon \circ \mathrm{zone}$ as simply Υ. By duality, we obtain:

Proposition 5.6.5 *The following diagram commutes.*

$$\begin{tikzcd}
\mathbb{K}\Sigma \arrow[d, "\Upsilon^*"'] \arrow[dr, "\Psi \circ \mathrm{base}^*"] & \\
\mathbb{K}\mathrm{Z}'^* \arrow[r, "\mathrm{reg}^*"'] \arrow[d, "\mathrm{zone}^*"'] & \mathbb{K}\mathrm{Q}^* \\
\mathbb{K}\mathrm{Z}^* \arrow[ur, "\mathrm{lune}^*"'] &
\end{tikzcd}$$

By Lemma 2.5.3, we obtain:

Proposition 5.6.6 *The following diagram commutes.*

$$\begin{tikzcd}
\mathbb{K}\Sigma \arrow[r, "\mathrm{base} \circ \Psi \circ \mathrm{base}^*"] \arrow[d, two heads, "\mathrm{supp}"'] & \mathbb{K}\Sigma^* \\
\mathbb{K}\mathrm{L} \arrow[r, "\Phi"'] & \mathbb{K}\mathrm{L}^* \arrow[u, hook, "\mathrm{supp}^*"']
\end{tikzcd}$$

The map $\mathrm{base} \circ \Psi \circ \mathrm{base}^* : \mathbb{K}\Sigma \to \mathbb{K}\Sigma^*$ is given by

$$\mathrm{base} \circ \Psi \circ \mathrm{base}^*(H_P) \;=\; \sum_F \langle P, F\rangle_\Sigma \; M_F,$$

where $\langle P, F\rangle_\Sigma$ is the bilinear form given in Section 2.5.3.

Proposition 5.6.7 *The bottom half of diagram* (5.8) *commutes.*

Proof This follows from Propositions 5.6.2-5.6.6.

□

5.6.9 The algebra $\mathbb{K}\mathrm{L}$

The spaces $\mathbb{K}\Sigma$ and $\mathbb{K}\mathrm{L}$ are algebras with the products

$$H_F H_K = H_{FK} \qquad \text{and} \qquad h_X h_Y = h_{X \vee Y}$$

respectively. The content of Equation (2.3) is that the map $\mathrm{supp} : \mathbb{K}\Sigma \to \mathbb{K}\mathrm{L}$ in Definition 5.6.2 is a map of algebras. The structure of the algebra $\mathbb{K}\mathrm{L}$ is explained in Section 2.5.5. Here we restate Lemma 2.5.4, making explicit the significance of the q basis of $\mathbb{K}\mathrm{L}$.

Lemma 5.6.3 *Elements of the q basis of* $\mathbb{K}\mathrm{L}$ *are the primitive idempotents for the split semisimple algebra* $\mathbb{K}\mathrm{L}$*. Further, the q basis is an orthogonal basis for* $\mathbb{K}\mathrm{L}$ *with respect to the bilinear form on* $\mathbb{K}\mathrm{L}$*. More precisely,*

$$\langle q_X, q_Y\rangle_{\mathrm{L}} = n_X \delta_{X,Y}, \tag{5.16}$$

with n_X *as in Definition 2.5.1.*

Proof Following Section 2.5.5, there is an algebra isomorphism $\mathbb{K}\mathrm{L} \xrightarrow{\cong} \mathbb{K}^{\mathrm{L}}$ given by $h_X \mapsto \sum_{X\leq Y} \delta_Y$. Comparing with Equation (5.11), we see that $q_X \mapsto \delta_X$ under this isomorphism. This proves the first part of the lemma. The second part is same as Equation (2.14).

□

5.7 The coinvariant commutative diagram (5.17)

In this section, we prove the following theorem. It is specific to the Coxeter case.

Theorem 5.7.1 *For* W *a Coxeter group, the following diagram of vector spaces commutes.*

$$\begin{tikzcd}
\mathbb{K}\overline{\mathrm{Q}} \arrow[r, hook, "\theta"] \arrow[d, "\mathrm{supp}"'] \arrow[rrdd, "\psi"] & \mathbb{K}W \arrow[dr, "s"] & \\
& & \mathbb{K}W^* \arrow[d, "\mathrm{des}"] \\
\mathbb{K}\overline{\mathrm{L}} \arrow[r, "\phi"'] & \mathbb{K}\overline{\mathrm{L}}^* \arrow[r, hook, "\mathrm{supp}^*"'] & \mathbb{K}\overline{\mathrm{Q}}^*
\end{tikzcd} \tag{5.17}$$

Recall that the partial orders on W and $\overline{\mathrm{Q}}$ are given by the weak left Bruhat order and subset inclusion respectively. And $\overline{\mathrm{L}}$ is the poset defined in Section 5.2.3. The maps in the diagram above are given in Definitions 5.7.9-5.7.15. The maps θ and des are written in terms of the corresponding maps of Section 5.3.

Note on the proof Recall a simple fact. For G a group and V a G-module, let V^* be the dual G-module and V^G and V_G be the space of invariants and coinvariants respectively. Then there is a canonical identification $(V^G)^* \cong (V^*)_G$.

Recall from Section 5.2 that $\Sigma_W \cong \mathrm{Q}_W \cong \overline{\mathrm{Q}}$, and $(\mathcal{C}\times\mathcal{C})_W \cong W$ and $\mathrm{L}_W \cong \mathrm{Z}_W \cong \overline{\mathrm{L}}$. By linearizing, we obtain $(\mathbb{K}\Sigma)_W \cong (\mathbb{K}\mathrm{Q})_W \cong \mathbb{K}\overline{\mathrm{Q}}$, and so on, where $-_W$ refers to the space of coinvariants. We have similar statements for the space of invariants. Now the Coxeter group W acts on the commutative diagram (5.8). If we take W-invariants for the spaces $\mathbb{K}\Sigma$, $\mathbb{K}\mathrm{L}$, $\mathbb{K}\mathrm{Q}$, $\mathbb{K}\mathrm{Z}$ and $\mathbb{K}(\mathcal{C}\times\mathcal{C})$ and W-coinvariants for their duals then diagram (5.8) immediately induces a diagram of the above form. All that remains to be done is to make the above isomorphisms and the induced maps explicit. For this, the reader should consult Propositions 5.7.1-5.7.4.

□

Remark For G a finite group and V a G-module, the composite map

$$V^G \hookrightarrow V \twoheadrightarrow V_G$$

is an isomorphism in characteristic 0. In other words, invariants and coinvariants can be identified in this case. Hence by removing invariants from the picture, one may also view diagram (5.17) as a coinvariant quotient of diagram (5.8). This different viewpoint is explained in Section 5.7.6.

The identification of W-invariants and W-coinvariants in our examples involves a factor of $|W|$, see the proof of Proposition 5.7.5. This is the reason why such a normalization factor is required in Section 5.7.4.

Notation We know that $\Sigma_W \cong \overline{\mathrm{Q}}$ and $\mathrm{L}_W \cong \overline{\mathrm{L}}$. For $T \in \overline{\mathrm{Q}}$ and $\lambda \in \overline{\mathrm{L}}$, it is convenient to let $\mathcal{O}_T$ and $\mathcal{O}_\lambda$ denote the orbits in Σ and L corresponding to T in $\overline{\mathrm{Q}}$ and λ in $\overline{\mathrm{L}}$ respectively.

5.7.1 The objects in diagram (5.17)

Let W, $\overline{\mathrm{Q}}$ and $\overline{\mathrm{L}}$ be the posets in Section 5.2 and $\mathbb{K}$ be a field of characteristic zero. The six objects we consider are the vector spaces $\mathbb{K}W$, $\mathbb{K}\overline{\mathrm{Q}}$ and $\mathbb{K}\overline{\mathrm{L}}$, and their duals. For each of them, we define two bases as shown in Table 5.3.

Table 5.3: Vector spaces associated to W and their bases.

Vector space	Bases	Dual objects	Dual bases
$\mathbb{K}\overline{\mathrm{Q}}$	H, K	$\mathbb{K}\overline{\mathrm{Q}}^*$	M, F
$\mathbb{K}W$	H, K	$\mathbb{K}W^*$	M, F
$\mathbb{K}\overline{\mathrm{L}}$	h, q	$\mathbb{K}\overline{\mathrm{L}}^*$	m, p

The H and K basis are related by

$$H_u = \sum_{v \le u} K_v \text{ in } \mathbb{K}W \quad \text{and} \quad H_T = \sum_{U \le T} K_U \text{ in } \mathbb{K}\overline{\mathrm{Q}}.$$

Dually, the M and F basis are related by

$$F_v = \sum_{v \le u} M_u \text{ in } \mathbb{K}W^* \quad \text{and} \quad F_U = \sum_{U \le T} M_T \text{ in } \mathbb{K}\overline{\mathrm{Q}}^*.$$

Analogous to the situation for L, the poset $\overline{\mathrm{L}}$ is a little different. The change of basis formulas are

$$h_\mu = \sum_{\mu \le \lambda} R_{\lambda\mu} q_\lambda \text{ in } \mathbb{K}\overline{\mathrm{L}} \quad \text{and} \quad p_\lambda = \sum_{\mu \le \lambda} R_{\lambda\mu} m_\mu \text{ in } \mathbb{K}\overline{\mathrm{L}}^*.$$

The coefficients $R_{\lambda\mu}$ are defined using Definition 5.7.8 and Equation (5.12). This means that

$$p_\lambda = \sum_{X : X \le Y} |\mu| \; m_\mu, \tag{5.18}$$

where $Y \in \mathcal{O}_\lambda$ is a fixed set partition and μ is defined by $X \in \mathcal{O}_\mu$. Note that $X \le Y$ implies that $\mu \le \lambda$, with the partial order on partitions given in Definition 3.2.1. It is then clear that the $R_{\lambda\mu}$ are nonnegative. We do not make them any more explicit in this generality; for the type A case, see Fact 5.7.2. The q and p basis are relevant to the algebra structure of $\mathbb{K}\overline{\mathrm{L}}$, see Lemma 5.7.1.

Example Recall from Section 5.4 that for the symmetric group S_n, the poset $\overline{\mathrm{L}}$, which we denoted $\overline{\mathrm{L}}^n$, is indexed by partitions of n. In this case, the graded vector space

$$\Lambda_{\mathrm{L}^*} = \bigoplus_{n \ge 0} \mathbb{K}(\overline{\mathrm{L}}^n)^*$$

is the familiar space of symmetric functions (Section 3.2.1). And the p and m bases defined above are the power and monomial basis respectively of Λ_{L^*}. The map ϕ is an isomorphism in this case. If we push the h basis of $\mathbb{K}\overline{\mathrm{L}}$ to $\mathbb{K}\overline{\mathrm{L}}^*$ using ϕ then it gives the homogeneous basis of Λ_{L^*}. The map ϕ is induced by a nondegenerate bilinear form on $\mathbb{K}\overline{\mathrm{L}}$, see Definition 5.7.15. If we push this form to $\mathbb{K}\overline{\mathrm{L}}^*$ via ϕ then we recover the usual inner product on symmetric functions, with the h and m bases being dual.

5.7.2 The maps from invariants

Proposition 5.7.1 *The following diagram of vector spaces commutes.*

$$\begin{tikzcd}
& \mathbb{K}\Sigma \arrow[r, "\mathrm{base}^*"] \arrow[dd] & \mathbb{K}\mathrm{Q} \arrow[rr, hook, "\Theta"] \arrow[dd] & & \mathbb{K}(\mathcal{C}\times\mathcal{C}) \\
\mathbb{K}\overline{\mathrm{Q}} \arrow[ur, hook] \arrow[rrr, hook] \arrow[dd] & & & \mathbb{K}W \arrow[ur, hook] & \\
& \mathbb{K}\mathrm{L} \arrow[r, "\mathrm{base}^*"'] & \mathbb{K}\mathrm{Z} & & \\
\mathbb{K}\overline{\mathrm{L}} \arrow[ur, hook] & & & &
\end{tikzcd} \tag{5.19}$$

For the maps from the invariants, we make specific choices of normalization factors. They are given in Definitions 5.7.1-5.7.4 below. The induced maps $\mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}W$ and $\mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}\overline{\mathrm{L}}$ are given in Definitions 5.7.10 and 5.7.12 below.

The proof is a straightforward check and is omitted.

Definition 5.7.1 The map $\mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}\Sigma$ is given by

$$H_T \mapsto \sum_{P\in\Sigma_T} H_P,$$

where Σ_T is the set of all faces of type T. Note that $\Sigma_T = \mathcal{O}_T$.

Definition 5.7.2 By composing with the map $\mathrm{base}^* : \mathbb{K}\Sigma \hookrightarrow \mathbb{K}\mathrm{Q}$, we get:
On the H basis, the map $\mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}\mathrm{Q}$ is given by

$$H_T \mapsto \sum_{P\leq C, P\in\Sigma_T} H_{(P,C)}.$$

Observe that the map has the same expression on the K basis as well.

Definition 5.7.3 On the K basis, the map $\mathbb{K}W \to \mathbb{K}(\mathcal{C}\times\mathcal{C})$ is given by

$$K_w \mapsto \sum_{d(D,C)=w} K_{(D,C)}.$$

The map has the same expression on the H basis also. On the R basis, one has

$$H_w \mapsto R_{(D,\overline{C}_0)},$$

with D such that $d(D,\overline{C}_0) = w$.

Definition 5.7.4 On the h basis, the map $\mathbb{K}\overline{\mathrm{L}} \to \mathbb{K}\mathrm{L}$ is given by

$$h_\lambda \mapsto |\lambda| \sum_{X\in\mathcal{O}_\lambda} h_X,$$

where $|\lambda| = \frac{|\mathcal{O}_T|}{|\mathcal{O}_\lambda|}$ for any $T \in \overline{\mathrm{Q}}$ such that $\operatorname{supp} T = \lambda$. As the notation suggests, this number only depends on λ and not on the specific choice of T. The reason for this is that if $w \in W$ fixes a face $P \in \Sigma$ then it fixes $\operatorname{supp} P$ pointwise.

On the q basis, the map $\mathbb{K}\overline{\mathrm{L}} \to \mathbb{K}\mathrm{L}$ is given by

$$q_\lambda \mapsto \sum_{X \in \mathcal{O}_\lambda} q_X.$$

Example Recall from Section 5.4 that for the symmetric group S_n, the poset $\overline{\mathrm{L}}$ is indexed by partitions of n and L is indexed by partitions of the set $[n]$. Say, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. Another common way to write λ is $1^{r_1}2^{r_2}\dots m^{r_m}$, which says that 1 occurs r_1 times, 2 occurs r_2 times, and so on. Then

$$|\mathcal{O}_\lambda| = \frac{n!}{\lambda_1!\lambda_2!\dots\lambda_k! r_1!\dots r_m!}. \tag{5.20}$$

An element $T \in \overline{\mathrm{Q}}$ such that $\operatorname{supp} T = \lambda$ is any composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with underlying partition λ, that is, the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ are some permutation of $\lambda_1, \lambda_2, \dots, \lambda_k$. First observe that

$$|\mathcal{O}_\alpha| = \frac{n!}{\alpha_1!\alpha_2!\dots\alpha_k!}.$$

This shows that $|\lambda| = r_1!\dots r_m!$.

Fact 5.7.1 For $\mu = (\mu_1, \dots, \mu_k)$ and any $X \in \mathcal{O}_\mu$, we have that $|\mu|$ is the number of compositions $F = F^1|\cdots|F^k$ with $\operatorname{supp} F = X$ and $|F^i| = \mu_i$.

Proof A composition F as above is obtained by arranging the parts of X in decreasing order of size. There are precisely $|\mu|$ choices for doing this since the parts of X of the same size can be arranged amongst themselves in any order.

□

For completeness, we derive the description of the coefficients $R_{\lambda\mu}$ given in Stanley [94, Proposition 7.7.1].

Fact 5.7.2 Let $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_k)$. Then $R_{\lambda\mu}$ is the number of compositions $B = B^1|\cdots|B^k$ of the set $[l]$ such that

$$\mu_j = \sum_{i \in B^j} \lambda_i, \qquad 1 \leq j \leq k. \tag{5.21}$$

We note that the above count is nonzero only if $\mu \leq \lambda$, with the partial order on partitions given in Definition 3.2.1. This is consistent with what we expect.

Proof Let $Y = \{Y^1, \dots, Y^l\}$ be a fixed set partition with $|Y^i| = \lambda_i$. From Equation (5.18) and the previous fact, we have

$$R_{\lambda\mu} = \{F = F^1|\cdots|F^k \mid \operatorname{supp} F \leq Y \text{ and } |F^i| = \mu_i\}.$$

The set compositions F such that $\operatorname{supp} F \leq Y$ can be identified with compositions of the set Y, or equivalently, compositions of the set $[l]$. It is clear that under this correspondence, the condition $|F^i| = \mu_i$ corresponds to Equation (5.21).

□

5.7.3 The maps to coinvariants

Proposition 5.7.2 *The following diagram of vector spaces commutes.*

$$
\begin{tikzcd}
 & & & \mathbb{K}(\mathcal{C}\times\mathcal{C})^* \arrow[dl, two heads] \arrow[dd, "\text{Road}"] \\
 & & \mathbb{K}W^* \arrow[ddd] & \\
 & \mathbb{K}\mathrm{Z}^* \arrow[rr, hook] \arrow[d, "\text{base}"'] & & \mathbb{K}\mathrm{Q}^* \arrow[d, "\text{base}"] \\
 & \mathbb{K}\mathrm{L}^* \arrow[rr, hook] \arrow[dl, two heads] & & \mathbb{K}\Sigma^* \arrow[dl, two heads] \\
\mathbb{K}\overline{\mathrm{L}}^* \arrow[rr, hook] & & \mathbb{K}\overline{\mathrm{Q}}^* &
\end{tikzcd}
\tag{5.22}
$$

The diagram above is obtained by dualizing diagram (5.19). Hence no proof is necessary. The explicit maps to coinvariants are written down in Definitions 5.7.5-5.7.8 below for completeness. The maps $\mathbb{K}W^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ and $\mathbb{K}\overline{\mathrm{L}}^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ are given by Definitions 5.7.11 and 5.7.13 below.

Definition 5.7.5 The map $\mathbb{K}\Sigma^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$M_F \mapsto M_{\mathrm{type}\, F}.$$

Definition 5.7.6 On the M basis, the map $\mathbb{K}\mathrm{Q}^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$M_{(F,D)} \mapsto M_{\mathrm{type}\, F}.$$

Observe that the map has the same expression on the F basis as well.

Definition 5.7.7 On the F basis, the map $\mathbb{K}(\mathcal{C}\times\mathcal{C})^* \to \mathbb{K}W^*$ is given by

$$F_{(C,D)} \mapsto F_{d(C,D)}.$$

The map has the same expression on the M basis also. On the S basis, one has

$$S_{(C,D)} \mapsto \begin{cases} M_{d(C,D)} & \text{if } D = \overline{C}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.7.8 On the m basis, the map $\mathbb{K}\mathrm{L}^* \to \mathbb{K}\overline{\mathrm{L}}^*$ is given by

$$m_X \mapsto |\lambda|\; m_\lambda,$$

where λ is defined by $X \in \mathcal{O}_\lambda$, and $|\lambda|$ is as in Definition 5.7.4.

On the p basis, the map $\mathbb{K}\mathrm{L}^* \to \mathbb{K}\overline{\mathrm{L}}^*$ is given by

$$p_X \mapsto p_\lambda,$$

with λ as above.

For type A_{n-1}, these two maps have been considered by Rosas and Sagan [83]. This idea of studying partitions by lifting them to set partitions goes back to Doubilet [24].

5.7.4 The maps in diagram (5.17)

In this section, we normalize the maps s, Ψ, Φ, Υ and Υ^* in diagram (5.8) by the factor $\frac{1}{|\mathcal{C}|} = \frac{1}{|W|}$ and indicate the difference with an overline, see the remark before Section 5.6.1. In analogy with Section 5.6, the maps θ and des in diagram (5.17) are defined in both the F and M bases. The equivalence between the two definitions can be derived using Proposition 5.3.5.

Definition 5.7.9 The switch map $s : \mathbb{K}W \to \mathbb{K}W^*$ is given by

$$K_w \mapsto F_{w^{-1}}.$$

Proposition 5.7.3 *The following diagram commutes.*

$$\begin{tikzcd} \mathbb{K}(\mathcal{C} \times \mathcal{C}) \arrow[r, "\overline{s}"] & \mathbb{K}(\mathcal{C} \times \mathcal{C})^* \arrow[d] \\ \mathbb{K}W \arrow[u, hook] \arrow[r, "s"'] & \mathbb{K}W^* \end{tikzcd}$$

The proof is a simple check.

Definition 5.7.10 On the H basis, the map $\theta : \mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}W$ is given by

$$H_T \mapsto H_{\theta(T)}.$$

On the K basis, the map $\theta : \mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}W$ is given by

$$K_T \mapsto \sum_{\mathrm{des}(w)=T} K_w.$$

Definition 5.7.11 On the F basis, the map des : $\mathbb{K}W^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$F_w \mapsto F_{\mathrm{des}(w)}.$$

For type A_{n-1}, this map was defined by Malvenuto in her thesis [61].

On the M basis, the map des : $\mathbb{K}W^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$\mathrm{des}(M_w) = \begin{cases} M_{\mathrm{des}(w)} & \text{if } w = \theta(\mathrm{des}(w)), \\ 0 & \text{otherwise.} \end{cases}$$

For type A_{n-1}, this formula was obtained in [4].

Definition 5.7.12 The map supp : $\mathbb{K}\overline{\mathrm{Q}} \to \mathbb{K}\overline{\mathrm{L}}$ is given by

$$H_T \mapsto h_{\mathrm{supp}\, T}.$$

Definition 5.7.13 The map $\mathrm{supp}^* : \mathbb{K}\overline{\mathrm{L}}^* \to \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$m_\lambda \mapsto \sum_{\mathrm{supp}\, T=\lambda} M_T.$$

Definition 5.7.14 The map $\psi : \mathbb{K}\overline{\mathrm{Q}} \mapsto \mathbb{K}\overline{\mathrm{Q}}^*$ is given by

$$\psi(H_T) = \sum_U \langle T, U\rangle\ M_U.$$

The bilinear form on $\mathbb{K}\Sigma$ in Section 2.5.3 via Definition 5.7.1 induces a bilinear form on $\mathbb{K}\overline{\mathrm{Q}}$. The notation $\langle\ ,\ \rangle$ above refers to this form, divided by the factor of $|W|$. In other words,

$$\langle T, U\rangle = \frac{1}{|W|}\langle \sigma_T, \sigma_U\rangle,$$

where $\langle \sigma_T, \sigma_U\rangle$ is defined by Equation (2.16).

Definition 5.7.15 The map $\phi : \mathbb{K}\overline{\mathrm{L}} \mapsto \mathbb{K}\overline{\mathrm{L}}^*$ is given by

$$\phi(h_\lambda) = \sum_\mu \langle \lambda, \mu\rangle\ m_\mu.$$

The bilinear form on $\mathbb{K}\mathrm{L}$ in Section 2.5.4 via Definition 5.7.4 induces a bilinear form on $\mathbb{K}\overline{\mathrm{L}}$. The notation $\langle\ ,\ \rangle$ above refers to this form, divided by the factor of $|W|$.

From the above two definitions, we obtain:

Proposition 5.7.4 *The following diagrams commute.*

$$\begin{tikzcd} \mathbb{K}\Sigma \arrow[r, "\mathrm{base}\circ\overline{\Psi}\circ\mathrm{base}^*"] & \mathbb{K}\Sigma^* \arrow[d, two heads] \\ \mathbb{K}\overline{\mathrm{Q}} \arrow[u, hook] \arrow[r, "\psi"'] & \mathbb{K}\overline{\mathrm{Q}}^* \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K}\mathrm{L} \arrow[r, "\overline{\Phi}"] & \mathbb{K}\mathrm{L}^* \arrow[d, two heads] \\ \mathbb{K}\overline{\mathrm{L}} \arrow[u, hook] \arrow[r, "\phi"'] & \mathbb{K}\overline{\mathrm{L}}^* \end{tikzcd}$$

5.7.5 The algebra $\mathbb{K}\overline{\mathrm{L}}$

Recall that $\mathbb{K}\overline{\mathrm{L}} \xrightarrow{\cong} (\mathbb{K}\mathrm{L})^W$. Since the algebra structure and the bilinear form on $\mathbb{K}\mathrm{L}$ commute with the action of W, we obtain an induced algebra structure and bilinear form on $\mathbb{K}\overline{\mathrm{L}}$. The specific choice $\mathbb{K}\overline{\mathrm{L}} \hookrightarrow \mathbb{K}\mathrm{L}$ for the map is given in Definition 5.7.4. In analogy with Section 5.6.9, we now make explicit the significance of the q basis of $\mathbb{K}\overline{\mathrm{L}}$.

For $\lambda \in \overline{\mathrm{L}}$, let

$$z_\lambda = \frac{|W|}{|\mathcal{O}_\lambda| n_X},$$

with X any element of L in the orbit $\mathcal{O}_\lambda$ and n_X as in Definition 2.5.1.

Example For the example of type A_{n-1}, for a partition $\lambda = (1^{r_1}2^{r_2}\ldots m^{r_m})$, we obtain the standard formula

$$z_\lambda = \prod_{i=1}^m i^{r_i} r_i!$$

as in Macdonald [59] or Stanley [94, Equation 7.17]. This follows from Equations (5.20) and (2.15).

Going back to the general case, we have the following analogue of Lemma 5.6.3.

Lemma 5.7.1 *Elements of the q basis of $\mathbb{K}\overline{\mathrm{L}}$ are the primitive idempotents for the split semisimple algebra $\mathbb{K}\overline{\mathrm{L}}$. Further, the q basis is an orthogonal basis with respect to the bilinear form on $\mathbb{K}\overline{\mathrm{L}}$. More precisely,*

$$\langle q_\lambda, q_\mu\rangle = z_\lambda^{-1}\delta_{\lambda,\mu}.$$

In particular, the form on $(\mathbb{K}\mathrm{L})^W$ is nondegenerate $\iff$ $n_X \neq 0$ for each $X \in \mathrm{L}$. This was also written in Lemma 2.6.5.

Proof The first part follows from Lemma 5.6.3 and Definition 5.7.4 on the q basis. For the second part, we have

$$\begin{aligned}\langle q_\lambda, q_\mu \rangle &= \frac{1}{|W|}\langle \sum_{X\in\mathcal{O}_\lambda} q_X, \sum_{X\in\mathcal{O}_\mu} q_X\rangle_{\mathrm{L}} \\ &= \frac{|\mathcal{O}_\lambda|}{|W|}\langle q_X, q_X\rangle_{\mathrm{L}}\delta_{\lambda,\mu} \\ &= \frac{|\mathcal{O}_\lambda|}{|W|} n_X \delta_{\lambda,\mu} \\ &= z_\lambda^{-1}\delta_{\lambda,\mu}\end{aligned}$$

The second and third equalities follow from Lemma 5.6.3.

□

Corollary 5.7.1 *We have $\phi(q_\lambda) = z_\lambda^{-1} p_\lambda$.*

5.7.6 A different viewpoint relating diagrams (5.8) and (5.17)

So far, we regarded diagram (5.17) as a partly invariant and partly coinvariant picture of the diagram (5.8), see Propositions 5.7.1 and 5.7.2. One may also consider it as just coinvariants of diagram (5.8). This viewpoint, though unnecessary for the internal structure, is well suited for studying external structure, which is initiated in the next two chapters.

Proposition 5.7.5 *With Definitions 5.7.5-5.7.8 and Definitions 5.7.16-5.7.18 as below, the diagram* (5.8) *projects onto the diagram* (5.17).

The new maps we need to define are:

Definition 5.7.16 On the H basis, the map $\mathbb{K}\mathrm{Q} \to \mathbb{K}\overline{\mathrm{Q}}$ is given by

$$H_{(P,C)} \mapsto H_{\mathrm{type}\,P}.$$

Observe that the map has the same expression on the K basis as well.

Definition 5.7.17 On the K basis, the map $\mathbb{K}(\mathcal{C}\times\mathcal{C}) \to \mathbb{K}W$ is given by

$$K_{(C,D)} \mapsto K_{d(C,D)}.$$

The map has the same expression on the H basis also. On the R basis,

$$R_{(C,D)} \mapsto \kappa\, H_{d(C,D)},$$

where κ is the number of elements less than (C,D) in the partial order $\leq'$.

Definition 5.7.18 The map $\mathbb{K}\mathrm{Z} \to \mathbb{K}\overline{\mathrm{L}}$ is given by

$$h_L \mapsto h_\lambda,$$

where λ represents the orbit of the element L in Z.

Proof of Proposition 5.7.5 We recall some notation and facts from Sections 2.2 and 2.5. For $P \in \Sigma$, let $c_P = |\mathcal{C}_P|$ be the number of chambers $C \in \mathcal{C}$ such that $P \leq C$. We know from Lemma 2.2.1 that c_P depends only on $\operatorname{supp} P$. Hence for each $X \in \mathrm{L}$, let c_X be the number of chambers $C \in \mathcal{C}$ such that $C \geq P$, where P is any fixed element of Σ having support X. As explained in the proof of Lemma 2.6.2, if $\operatorname{type} P = T$ then $c_P\ |\Sigma_T| = |W|$.

Now using the definitions of the maps in this section, we make some simple computations. The map $\mathbb{K}W \hookrightarrow \mathbb{K}(\mathcal{C} \times \mathcal{C}) \twoheadrightarrow \mathbb{K}W$ sends

$$K_w \longmapsto \sum_{d(D,C)=w} K_{(D,C)} \longmapsto |W| K_w.$$

Next the map $\mathbb{K}\overline{\mathrm{Q}} \hookrightarrow \mathbb{K}\mathrm{Q} \twoheadrightarrow \mathbb{K}\overline{\mathrm{Q}}$ sends

$$H_T \longmapsto \sum_{P \leq C,\ P \in \Sigma_T} H_{(P,C)} \longmapsto c_P\ |\Sigma_T| h_T = |W| h_T.$$

Next the map $\mathbb{K}\overline{\mathrm{L}} \hookrightarrow \mathbb{K}\mathrm{L} \twoheadrightarrow \mathbb{K}\overline{\mathrm{L}}$ sends

$$h_\lambda \longmapsto |\lambda| \sum_{X \in \mathcal{O}_\lambda} h_X \longmapsto |\lambda||\mathcal{O}_\lambda| c_X h_\lambda = |\mathcal{O}_T| c_X h_\lambda = |W| h_\lambda,$$

where $X \in \mathcal{O}_\lambda$ and $T = \operatorname{type} P$ for any P such that $\operatorname{supp} P = X$.

Thus in each case, the composite map is simply multiplication by $|W|$. The result is now implied by Propositions 5.7.1-5.7.4.

$\square$

Chapter 6

The construction of Hopf algebras

6.1 Introduction

In the introduction to Chapter 5, we outlined a program for organizing together some graded Hopf algebras of recent interest. These included the Malvenuto-Reutenauer Hopf algebra [61] and the Hopf algebras of symmetric functions [34, 102], quasi-symmetric functions [36, 61, 46] and noncommutative symmetric functions [35]. These Hopf algebras, which we denote by $\mathrm{S}\Lambda$, Λ, $\mathrm{Q}\Lambda$ and $\mathrm{N}\Lambda$ respectively, fit in a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{N}\Lambda \arrow[r, hook] \arrow[d, two heads] & \mathrm{S}\Lambda \arrow[d, two heads] \\ \Lambda \arrow[r, hook] & \mathrm{Q}\Lambda \end{tikzcd} \tag{6.1}$$

In this chapter, we construct some new graded Hopf algebras and show that they fit in a commutative diagram (Theorem 6.1.3). The diagram (6.4) projects onto diagram (6.1) and hence is more fundamental. In Coxeter or Lie theory, the symmetric group S_n is known as the group of type A_{n-1}. All the above Hopf algebras are related to the family of symmetric groups S_n, $n \geq 0$ in some way; hence one may say that they are objects of type A.

6.1.1 A diagram of vector spaces for a LRB

We saw in Chapter 5 that the graded pieces of the above Hopf algebras display a rich internal structure. This internal theory is not specific to type A; it works for any Coxeter group, and more generally, for any left regular band (LRB). We recall some notation from Chapter 2.

Let Σ be a LRB, and $\mathcal{C}$ be the set of chambers in Σ,

$$\mathrm{Q} = \{(F, D) \mid F \leq D\} \subseteq \Sigma \times \mathcal{C}$$

be the set of pointed faces, and L and Z be the support and lune posets associated to Σ and Q respectively. For P a poset, let $\mathbb{K}P$ be the vector space over the field $\mathbb{K}$ with basis the elements of P, and let $\mathbb{K}P^*$ be its dual space. Now recall Theorem 5.6.1.

Theorem *For Σ a LRB, the following diagram of vector spaces commutes.*

$$
\begin{tikzcd}
\mathbb{K}\Sigma \arrow[r, "b^*"] \arrow[dd, twoheadrightarrow, "\mathrm{supp}"'] \arrow[dddrr, "\Upsilon^*"] & \mathbb{K}\mathrm{Q} \arrow[rr, "\Theta"] \arrow[dd, twoheadrightarrow, "\mathrm{lune}"] \arrow[dddrrr, "\Psi"] & & \mathbb{K}(\mathcal{C}\times\mathcal{C}) \arrow[dr, "s", "\cong"'] & \\
& & & & \mathbb{K}(\mathcal{C}\times\mathcal{C})^* \arrow[dd, "\mathrm{Road}"] \\
\mathbb{K}\mathrm{L} \arrow[r, "b^*"] \arrow[ddrr, "\Phi"'] & \mathbb{K}\mathrm{Z} \arrow[ddrrr, "\Upsilon"] & & & \\
& & \mathbb{K}\mathrm{Z}^* \arrow[rr, hook, "\mathrm{lune}^*"] \arrow[d, "b"] & & \mathbb{K}\mathrm{Q}^* \arrow[d, "b"] \\
& & \mathbb{K}\mathrm{L}^* \arrow[rr, hook, "\mathrm{supp}^*"'] & & \mathbb{K}\Sigma^*
\end{tikzcd}
\tag{6.2}
$$

The maps in the above diagram were defined in Sections 5.6.2-5.6.3 and Sections 5.6.5-5.6.7. The part of the diagram excluding $\mathbb{K}\mathrm{L}$, $\mathbb{K}\mathrm{Z}$ and their duals generalized further to projection posets, a concept introduced in Section 2.7.

6.1.2 A diagram of coalgebras and algebras for a family of LRBs

The goal of this chapter is to study external structure related to diagram (6.2). For that, let $\{\Sigma^n\}_{n\geq 0}$ be a family of LRBs. As mentioned in Section 6.1.1, for each LRB Σ^n, one can define

$$\mathrm{Q}^n = \{(F, D) \mid F \leq C\} \subseteq \Sigma^n \times \mathcal{C}^n,$$

the set of pairs of chambers $\mathcal{C}^n \times \mathcal{C}^n$, and the support and lune posets L^n and Z^n. Using them, one can construct graded vector spaces as shown in Table 6.1.

Table 6.1: Graded vector spaces for a family of LRBs.

Name	Vector space	Basis	Name	Dual space	Dual basis
$\mathcal{M}$	$\oplus\mathbb{K}\Sigma^n$	H	$\mathcal{P}$	$\oplus\mathbb{K}(\Sigma^n)^*$	M
$\mathcal{N}$	$\oplus\mathbb{K}\mathrm{Q}^n$	H	$\mathcal{Q}$	$\oplus\mathbb{K}(\mathrm{Q}^n)^*$	M
$\mathcal{R}$	$\oplus\mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)$	K	$\mathcal{S}$	$\oplus\mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$	F
$A_{\mathcal{L}}$	$\oplus\mathbb{K}\mathrm{L}^n$	h	$A_{\mathcal{L}^*}$	$\oplus\mathbb{K}(\mathrm{L}^n)^*$	m
$A_{\mathcal{Z}}$	$\oplus\mathbb{K}\mathrm{Z}^n$	h	$A_{\mathcal{Z}^*}$	$\oplus\mathbb{K}(\mathrm{Z}^n)^*$	m

For example,

$$\mathcal{M} = \underset{n\geq 0}{\oplus} \mathbb{K}\Sigma^n, \qquad \mathcal{P} = \underset{n\geq 0}{\oplus} \mathbb{K}(\Sigma^n)^*,$$

and so on. The letters H and M, and K and F for the bases are used for book-keeping, see the discussion in Section 5.6.1 for more information on this notation. In Section 5.6.1, we consider more than one basis for a given vector space. That is not the case in this chapter; only the basis mentioned in Table 6.1 will be used. Now by taking direct sums of the diagrams (6.2) for varying n, and using the notation of Table 6.1, we obtain the following commutative diagram.

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "b^*"] \arrow[dd, "\mathrm{supp}"'] \arrow[dddrr, "\Upsilon"] & \mathcal{N} \arrow[rr, "\Theta"] \arrow[dd, "\mathrm{lune}"] \arrow[dddrrr, "\Psi"] & & \mathcal{R} \arrow[dr, "s", "\cong"'] & \\
& & & & \mathcal{S} \arrow[dd, "\mathrm{Road}"] \\
A_{\mathcal{L}} \arrow[r, "b^*"] \arrow[ddrr, "\Phi"] & A_{\mathcal{Z}} \arrow[ddrrr, "\Upsilon^*"] & & & \\
& & A_{\mathcal{Z}^*} \arrow[rr, hook, "\mathrm{lune}^*"] \arrow[d, "b"] & & \mathcal{Q} \arrow[d, "b"] \\
& & A_{\mathcal{L}^*} \arrow[rr, hook, "\mathrm{supp}^*"'] & & \mathcal{P}
\end{tikzcd}
\tag{6.3}
$$

For the moment, the above is just a diagram of graded vector spaces. We show that under suitable assumptions on the family $\{\Sigma^n\}_{n\geq 0}$, one can say more:

Theorem 6.1.1 *Let $\{\Sigma^n\}_{n\geq 0}$ be a family of LRBs, that satisfies all coalgebra axioms $(C1)-(CP)$. Then diagram* (6.3) *is a diagram of coalgebras.*

Theorem 6.1.2 *Let $\{\Sigma^n\}_{n\geq 0}$ be a family of LRBs, that satisfies all algebra axioms $(A1)-(AP)$. Then diagram* (6.3) *is a diagram of algebras.*

The coalgebra and algebra axioms, along with examples, are defined in Sections 6.3 and 6.6 respectively. The above theorems are proved in Sections 6.5 and 6.8 respectively.

6.1.3 The example of type A

A beautiful illustration of the above theorems is given by the type A example, that is, when Σ^n is the Coxeter complex of type A_{n-1}. They can then be put together in one theorem as below.

Theorem 6.1.3 *The following is a commutative diagram of graded Hopf algebras.*

$$
\begin{tikzcd}
\mathrm{M\Pi} \arrow[r, "b^*"] \arrow[dd, "\mathrm{supp}"'] \arrow[dddrr, "\Upsilon"] & \mathrm{N\Pi} \arrow[rr, hook, "\Theta"] \arrow[dd, "\mathrm{lune}"] \arrow[dddrrr, "\Psi"] & & \mathrm{R\Pi} \arrow[dr, "s", "\cong"'] & \\
& & & & \mathrm{S\Pi} \arrow[dd, "\mathrm{Road}"] \\
\Pi_{\mathrm{L}} \arrow[r, "b^*"] \arrow[ddrr, "\Phi", "\cong"'] & \Pi_{\mathrm{Z}} \arrow[ddrrr, "\Upsilon^*"] & & & \\
& & \Pi_{\mathrm{Z}^*} \arrow[rr, hook, "\mathrm{lune}^*"] \arrow[d, "b"] & & \mathrm{Q\Pi} \arrow[d, "b"] \\
& & \Pi_{\mathrm{L}^*} \arrow[rr, hook, "\mathrm{supp}^*"'] & & \mathrm{P\Pi}
\end{tikzcd}
\tag{6.4}
$$

The graded vector spaces MΠ, NΠ, *etc. and the algebra and coalgebra structures on them are defined in Section 6.2.*

Proof There are three steps in the proof.

- For $n \geq 0$, let Σ^n be the Coxeter complex of type A_{n-1}. For this example, the vector spaces $\mathcal{M}$, $\mathcal{N}$, etc. in diagram (6.3) specialize to MΠ, NΠ, etc. The material in Section 5.4 is useful to make this translation. Hence, as vector spaces, diagram (6.4) is a special case of diagram (6.3) and so it commutes.

- For type A, we show that the family $\{\Sigma^n\}_{n\geq 0}$ satisfies the coalgebra and algebra axioms (Lemmas 6.3.2 and 6.6.2). Specializing Theorems 6.1.1 and 6.1.2, one concludes that diagram (6.4) is a diagram of both coalgebras and algebras, see the corollaries to Propositions 6.5.1 and 6.8.1 for more details.
- The algebra and coalgebra structures on $\mathrm{S\Pi}$, or equivalently $\mathrm{R\Pi}$, are compatible; hence it is a Hopf algebra. This can be checked directly from the definition. We omit this computation. From the surjectivity of the maps Road, base, supp and lune and the injectivity of their duals, it follows that the remaining spaces are also Hopf algebras.

□

Remark The Hopf algebra Π_{L^*}, or equivalently Π_{L}, is the algebra of symmetric functions in noncommuting variables introduced by Wolf [100] and further studied by Gebhard, Rosas and Sagan [83, 33]. In addition to Π_{L}, some of the other Hopf algebras, particularly $\mathrm{M\Pi}$ and $\mathrm{P\Pi}$, have been considered recently in the literature from various points of view, independent of our work. Chapoton [21] defines a Hopf algebra structure on the faces of the permutahedron (dual of the Coxeter complex of type A) that is the same as $\mathrm{P\Pi}$. Hivert considered a polynomial realization of this algebra in his thesis [45]. Palacios and Ronco [72] and Novelli and Thibon [68] deal with $\mathrm{P\Pi}$ as a dendriform trialgebra. Patras and Reutenauer [74] and Patras and Schocker [75] deal with the internal structure of $\mathrm{M\Pi}$. Bergeron, Hohlweg, Rosas and Zabrocki [8] and Bergeron and Zabrocki [9] deal with Π_{L} and $\mathrm{P\Pi}$.

The remaining Hopf algebras in diagram (6.4) are probably considered here for the first time. It is our unified geometric approach to these objects that we consider most valuable.

Theorem 6.1.4 *The diagram* (6.4) *projects onto diagram* (6.1) *as Hopf algebras, or more precisely onto the diagram below.*

$$\begin{tikzcd}
\mathrm{N\Lambda} \arrow[rr, hook, "\theta"] \arrow[dd, "\mathrm{supp}"'] \arrow[dddrrr, "\psi"] & & \mathrm{R\Lambda} \arrow[dr, "s", "\cong"'] & \\
& & & \mathrm{S\Lambda} \arrow[dd, "\mathrm{des}"] \\
\Lambda_{\mathrm{L}} \arrow[dr, "\cong", "\phi"'] & & & \\
& \Lambda_{\mathrm{L}^*} \arrow[rr, hook, "\mathrm{supp}^*"'] & & \mathrm{Q\Lambda}
\end{tikzcd} \tag{6.5}$$

The maps involved in the projection, namely $\mathrm{N\Pi}, \mathrm{M\Pi} \to \mathrm{N\Lambda}$, *and* $\mathrm{S\Pi} \to \mathrm{S\Lambda}$, *and* $\mathrm{R\Pi} \to \mathrm{R\Lambda}$, *and* $\mathrm{Q\Pi}, \mathrm{P\Pi} \to \mathrm{Q\Lambda}$, *and* $\Pi_{\mathrm{Z}}, \Pi_{\mathrm{L}} \to \Lambda_{\mathrm{L}}$ *and* $\Pi_{\mathrm{Z}^*}, \Pi_{\mathrm{L}^*} \to \Lambda_{\mathrm{L}^*}$ *are given in Definitions 5.7.5-5.7.8, 5.7.16-5.7.18.*

Proof We showed in Proposition 5.7.5 that diagram (6.4) projects onto diagram (6.5) as vector spaces. In this regard, we recall that S_n acts on the nth graded piece of each Hopf algebra in diagram (6.4), and taking the coinvariants of this action, we obtain the corresponding Hopf algebra in diagram (6.5). Now to finish the proof, one needs to check that the quotient maps $\mathrm{N\Pi} \to \mathrm{N\Lambda}$, etc. are maps of Hopf algebras. These are simple checks that we leave to the reader.

□

Remark Let $H = \underset{n\geq 0}{\oplus} \mathbb{K}H^n$ be a graded Hopf algebra, with the symmetric group S_n acting on the set H^n. Then under what conditions on H do we get an induced Hopf algebra on the quotient

$$\overline{H} = \underset{n\geq 0}{\oplus} \mathbb{K}(H^n)_{\mathrm{S}_n}?$$

The Hopf algebras in the above theorem, for example $H = \mathrm{Q}\Pi$ and $\overline{H} = \mathrm{Q}\Lambda$, all fit in this framework. Hence a good answer to this question can simplify matters a lot. This issue will be taken up in a future work.

In light of the theorems presented so far, one can ask the following question.

Open Question For a family $\{\Sigma^n\}_{n\geq 0}$ of LRBs, can one define compatibility axioms such that if a family satisfies the coalgebra, algebra and compatibility axioms then diagram (6.3) is a diagram of Hopf algebras?

We make two comments about this question. Firstly, there must be enough examples to justify an axiomatic approach; at the moment, there is only one example (Theorem 6.1.3). Secondly, we believe that the solution may require modification of the present coalgebra and algebra axioms themselves.

6.2 The Hopf algebras of type A

In this section, we define (without proof) the Hopf algebras in diagram (6.4). That is, we do not check the (co)associativity of the (co)products or the compatibility of the products and coproducts. These facts follow from the general considerations of later sections, see the proof of Theorem 6.1.3 for a summary.

6.2.1 Summary

Table 6.2: Hopf algebras and their indexing sets.

Hopf algebras		Indexing set
$\mathrm{P}\Pi$	$\mathrm{M}\Pi$	Σ^n = compositions of $[n]$
$\mathrm{Q}\Pi$	$\mathrm{N}\Pi$	Q^n = fully nested compositions of $[n]$
Π_{L^*}	Π_{L}	L^n = partitions of $[n]$
Π_{Z^*}	Π_{Z}	Z^n = fully nested partitions of $[n]$
$\mathrm{S}\Pi$	$\mathrm{R}\Pi$	$\mathcal{C}^n \times \mathcal{C}^n$ = pairs of permutations of $[n]$

Table 6.3: Unified description of the Hopf algebras.

Hopf algebras	Coproduct	Product
$\mathrm{P}\Pi$, $\mathrm{Q}\Pi$, Π_{L^*}, Π_{Z^*}	Local vertex	Quasi-shuffle
$\mathrm{M}\Pi$, $\mathrm{N}\Pi$, Π_{L}, Π_{Z}	Global vertex	Join

We showed in Section 5.4 that for the example of type A, the objects in diagram (6.3) can be described combinatorially, using the notions of (fully nested) set compositions and partitions. Each one of them indexes two Hopf algebras, as shown in Table 6.2. We recall these four definitions in Sections 6.2.3, 6.2.6, 6.2.9 and 6.2.12, and for each one define the notion of a local vertex, a global vertex, a quasi-shuffle and a join. The geometric meaning of the first two notions can be found in Tables 6.5 and 6.6. With these notions, the first eight Hopf algebras in Table 6.2 can be described in a unified way as summarized in Table 6.3. The formal definitions are given in Sections 6.2.4-6.2.5, 6.2.7-6.2.8, 6.2.10-6.2.11, and 6.2.13-6.2.14. For example, the coproduct of $\mathrm{P}\Pi$, $\mathrm{Q}\Pi$, Π_{L^*}

and Π_{Z^*} can be written using the notion of a local vertex of a composition, a fully nested composition, a partition and a fully nested partition of $[n]$ respectively, and so on. The Hopf algebras $\mathrm{S}\Pi$ and $\mathrm{R}\Pi$ are a little different and are defined towards the end of this section (Sections 6.2.15-6.2.16).

6.2.2 The structure of the Hopf algebras of type A

Table 6.4: Hopf algebras and their structure.

Hopf algebra	Comm.	Cocomm.	Structure
$\mathrm{R}\Pi$, $\mathrm{S}\Pi$	No	No	Free and cofree
$\mathrm{M}\Pi$	No	Yes	Free
$\mathrm{P}\Pi$	No	No	Free and cofree
$\mathrm{N}\Pi$	No	No	Free
$\mathrm{Q}\Pi$	No	No	Cofree
Π_{L}, Π_{L^*}	No	Yes	Free
Π_{Z}	No	No	Free
Π_{Z^*}	No	Yes	

We now make a few comments about the structure of the Hopf algebras that we construct. Table 6.4 summarizes what is known so far. We note that the Hopf algebra $\mathrm{M}\Pi$ is not dual to $\mathrm{P}\Pi$, the Hopf algebra Π_{Z} is not dual to Π_{Z^*}; similarly the Hopf algebra Π (short for Π_{L^*} and Π_{L}) is not self-dual. This is clear by looking at the commutativity and cocommutativity columns. This nonduality mystery will be explained in a future work.

The fact that $\mathrm{M}\Pi$ and $\mathrm{N}\Pi$, and Π and Π_{Z} are free follows directly from the definitions on the H and h basis respectively. The freeness and cofreeness of $\mathrm{S}\Pi$, or equivalently, $\mathrm{R}\Pi$, as well as the cofreeness of $\mathrm{P}\Pi$ and $\mathrm{Q}\Pi$ will be proved in Chapter 7. The freeness of $\mathrm{P}\Pi$ will be a consequence of further theory that we will develop in a follow-up to this work.

Remark The freeness and cofreeness of $\mathrm{P}\Pi$ also appears in Bergeron and Zabrocki [9]. Many other interesting Hopf algebras of a similar combinatorial nature are defined by Hivert, Novelli and Thibon [45, 67, 68, 69, 44]. They are constructed as subalgebras of the free algebra on countably many generators. This important point of view is not pursued in this monograph. For other related work, see the references given in Section 6.1.3.

6.2.3 Set compositions

Definition 6.2.1 A vertex of a set N is an ordered splitting of N into two subsets. If one of the subsets is empty then we have a virtual vertex. For example,

$$134\overset{\downarrow}{\ }25 \quad \rightsquigarrow \quad 134 \quad \text{and} \quad 25$$

is a vertex of the set [5]. And $\overset{\downarrow}{\ }12345$ and $12345\overset{\downarrow}{\ }$ are the two virtual vertices of [5].

Definition 6.2.2 A set composition is an ordered set partition. For example, $6|34|125$ is a composition of [6].

Definition 6.2.3 There is a unique order preserving map st from any n-set N of the integers to the standard n-set $[n]$. Using this map, one can standardize a composition of N to a composition of $[n]$. For example,

$$\mathrm{st}(9|36|58) = 5|13|24.$$

In the same way, we may standardize a composition of N to a composition of any n-set, say A, of the integers. We denote this map by st_A.

Definition 6.2.4 A local vertex of a set composition F is an ordered splitting of F into two set compositions. For example,

$$F^1|F^2|F^3\overset{\downarrow}{|}F^4 \quad \rightsquigarrow \quad F^1|F^2|F^3 \quad \text{and} \quad F^4.$$

If one of the two set compositions is empty then we have a virtual local vertex. Every set composition F has exactly two virtual local vertices.

$$F^1|F^2|F^3|F^4\,{}^{\downarrow} \quad \rightsquigarrow \quad F^1|F^2|F^3|F^4 \quad \text{and} \quad \emptyset.$$

$${}^{\downarrow}\,F^1|F^2|F^3|F^4 \quad \rightsquigarrow \quad \emptyset \quad \text{and} \quad F^1|F^2|F^3|F^4.$$

Definition 6.2.5 A "shuffle" of set compositions F_1 and F_2 is a shuffle of the components of F_1 and F_2. For example,

$$5|mar|ce|21|34|lo|6 \;\text{ is a "shuffle" of }\; 5|21|34|6 \;\text{ and }\; mar|ce|lo.$$

This definition is not fully precise since we need to ensure that the shuffled sets are disjoint. For that, we work with the following definition.

Let $K = K_1\,{}^{\downarrow}\,K_2$ be a vertex of the set N. And let F_1 and F_2 be compositions of the sets N_1 and N_2 such that $|N_1| = |K_1|$ and $|N_2| = |K_2|$. Then a K-shuffle of F_1 and F_2 is a shuffle of the components of $\mathrm{st}_{K_1}(F_1)$ and $\mathrm{st}_{K_2}(F_2)$. For example, for $K = 245679\,{}^{\downarrow}\,138$,

$$26|18|479|3|5 \text{ is a } K\text{-shuffle of } 14|256|3 \text{ and } 13|2.$$

We mostly deal with the special case when F_1 and F_2 are compositions of $[g_1]$ and $[g_2]$, and we shuffle them by shifting the indices of F_2 by g_1 and then shuffling their components.

Definition 6.2.6 A "quasi-shuffle" of set compositions F_1 and F_2 is a shuffle of the components of F_1 and F_2, where in addition we are allowed to substitute a disjoint set of pairs of components (F_1^i, F_2^j) for $F_1^i \cup F_2^j$, if they are adjacent in the shuffle. For example,

$$5sw|ap|21|34nil|6 \;\text{ is a "quasi-shuffle" of }\; 5|21|34|6 \;\text{ and }\; sw|ap|nil.$$

We leave it to the reader to make this precise by defining the notion of a K-quasi-shuffle, as in the previous definition.

Definition 6.2.7 A global vertex of a set composition $P = P^1|P^2|\cdots|P^l$ is a choice of a vertex for each set P^i for $1 \leq i \leq l$. This allows us to split P into two ordered set compositions. For example,

$$12\,{}^{\downarrow}\,5|347\,{}^{\downarrow}|8\,{}^{\downarrow}\,69 \quad \rightsquigarrow \quad 12|347|8 \quad \text{and} \quad 5|69.$$

Note that there are two virtual global vertices that create an empty set composition after splitting.

Definition 6.2.8 The join of a composition P_1 of $[g_1]$ and a composition P_2 of $[g_2]$ is the composition $j(P_1 \times P_2)$ of $[g_1 + g_2]$ obtained by shifting up the indices of P_2 by g_1 and then placing it after P_1. For example,

$$j(31|2 \times 23|14|5) = 31|2|56|47|8.$$

Remark The notion of a standardization map (Definition 6.2.3) and a virtual vertex (both local and global) exists and plays the same role for nested set compositions and (nested) set partitions also. We omit their definitions to avoid unnecessary repetition. Similarly, we do not repeat the issue of index shifting in the subsequent definitions of quasi-shuffles.

6.2.4 The Hopf algebra $\mathrm{P\Pi}$

Let $\mathrm{P\Pi} = \bigoplus_{n\geq 0} \mathbb{K}(\Sigma^n)^*$, where Σ^n is the set of compositions of $[n]$. Write M_F for the basis element corresponding to $F \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.9 The coproduct on $\mathrm{P\Pi}$ is given by

$$\Delta(M_F) = \sum_{\text{local vertices of } F} M_{F_1} \otimes M_{F_2}. \tag{6.6}$$

A local vertex, by definition, splits F into two ordered parts. The F_1 and F_2 in the above formula are obtained by standardizing these two parts respectively. The two virtual local vertices of F contribute to the terms $1 \otimes M_F$ and $M_F \otimes 1$. More explicitly,

$$\Delta(M_{F^1|\cdots|F^l}) = \sum_{i=0}^{l} M_{\mathrm{st}(F^1|\cdots|F^i)} \otimes M_{\mathrm{st}(F^{i+1}|\cdots|F^l)}.$$

For example,

$$\Delta(M_{136|25|47}) = 1 \otimes M_{136|25|47} + M_{123} \otimes M_{13|24} + M_{135|24} \otimes M_{12} + M_{136|25|47} \otimes 1.$$

Definition 6.2.10 The product on $\mathrm{P\Pi}$ is given by

$$M_{F_1} * M_{F_2} = \sum_{F\colon\ F \text{ a quasi-shuffle of } F_1 \text{ and } F_2} M_F. \tag{6.7}$$

If F_1 and F_2 are compositions of $[g_1]$ and $[g_2]$ respectively then the understanding is that we shift up the indices of F_2 by g_1 and then quasi-shuffle. For example,

$$M_{13|2} * M_{12} = M_{13|2|45} + M_{13|45|2} + M_{45|13|2} + M_{13|245} + M_{1345|2}.$$

These formulas for the product and coproduct agree with Formulas (1) and (2) of Chapoton [21].

6.2.5 The Hopf algebra $\mathrm{M\Pi}$

Let $\mathrm{M\Pi} = \bigoplus_{n\geq 0} \mathbb{K}\Sigma^n$, where Σ^n is the set of compositions of $[n]$. Write H_P for the basis element corresponding to $P \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.11 The coproduct on $\mathrm{M\Pi}$ is given by

$$\Delta(H_P) = \sum_{\text{global vertices of } P} H_{P_1} \otimes H_{P_2}. \tag{6.8}$$

A global vertex splits P into two ordered parts. The P_1 and P_2 in the above formula are obtained by standardizing these two parts respectively. The two virtual global vertices of P contribute to the terms $1 \otimes H_P$ and $H_P \otimes 1$. For example,

$$\Delta(H_{1|2|3}) = 1 \otimes H_{1|2|3} + 3(H_1 \otimes H_{1|2} + H_{1|2} \otimes H_1) + H_{1|2|3} \otimes 1.$$

The global vertices $1\overset{\downarrow}{\,}|\overset{\downarrow}{\,}2|\overset{\downarrow}{\,}3$, $\overset{\downarrow}{\,}1|2\overset{\downarrow}{\,}|\overset{\downarrow}{\,}3$, $\overset{\downarrow}{\,}1|\overset{\downarrow}{\,}2|3\overset{\downarrow}{\,}$ each give the term $H_1 \otimes H_{1|2}$, the global vertices $1\overset{\downarrow}{\,}|2\overset{\downarrow}{\,}|\overset{\downarrow}{\,}3$, $\overset{\downarrow}{\,}1|2\overset{\downarrow}{\,}|3\overset{\downarrow}{\,}$, $1\overset{\downarrow}{\,}|\overset{\downarrow}{\,}2|3\overset{\downarrow}{\,}$ each give the term $H_{1|2} \otimes H_1$, and the two virtual global vertices give the end terms.

In contrast to the local case, different global vertices of P can give rise to the same P_1 and P_2.

Remark The Hopf algebra MΠ is cocommutative.

$$5\overset{\downarrow}{\,}12|3\overset{\downarrow}{\,}74|689\overset{\downarrow}{\,} \quad \rightsquigarrow \quad 5|3|689 \quad \text{and} \quad 12|74.$$

$$12\overset{\downarrow}{\,}5|74\overset{\downarrow}{\,}3|\overset{\downarrow}{\,}689 \quad \rightsquigarrow \quad 12|74 \quad \text{and} \quad 5|3|689.$$

As illustrated above, there is an involution without fixed points on the set of global vertices of P obtained by switching the order of the two parts in each component of P. If a global vertex K splits P into P_1 and P_2 then $\overline{K}$, the image of K under the involution, splits P into P_2 and P_1. This shows that MΠ is cocommutative.

Definition 6.2.12 The product on MΠ is given by

$$H_{P_1} * H_{P_2} = H_{j(P_1 \times P_2)}. \tag{6.9}$$

For example,

$$H_{31|2} * H_{12} = H_{13|2|45}.$$

6.2.6 Nested set compositions

Definition 6.2.13 A nested set composition is a sequence $F = F^1 \Big| F^2 \Big| \dots \Big| F^l$, in which each F^i is a set composition of A^i, and $A^1|\dots|A^l$ is a set composition of $[n]$. For example,

$$(3|15\Big|7|48|29\Big|6)$$

is a nested composition of $[9]$. A fully nested set composition is a nested set composition with singleton blocks.

Equivalently, a nested set composition is a pair (F, H) of set compositions such that $F \leq H$. In the above example, the pair is $(135|24789|6, 3|15|7|48|29|6)$. And a fully nested set composition is a pair (F, D) with $F \leq D$ and D a chamber.

We use both the notations depending on our convenience.

Definition 6.2.14 A local vertex of a nested set composition (F, H) is simply a local vertex of F, as given in Definition 6.2.4. For example,

$$3|15\overset{\downarrow}{\Big|}7|48|29\Big|6 \quad \rightsquigarrow \quad 3|15 \quad \text{and} \quad 7|48|29\Big|6.$$

In other words, a local vertex is either a choice of a big bar, or one of the two extreme choices on either side corresponding to the virtual local vertices.

Definition 6.2.15 A quasi-shuffle of two fully nested set compositions (F_1, D_1) and (F_2, D_2) is a fully nested set composition (F, D), where F is a quasi-shuffle of F_1 and F_2 and D is the underlying shuffle of D_1 and D_2. For example,

$$3|1|b\Big|a|c\Big|5|4|2|d\Big|6 \text{ is a quasi-shuffle of } 3|1\Big|5|4|2\Big|6 \text{ and } b\Big|a|c\Big|d.$$

In other words, we quasi-shuffle with respect to the big bars, and if two blocks get merged in the quasi-shuffle then we write them one after the other with a small bar in between.

Definition 6.2.16 A global vertex of a fully nested set composition $P^1\Big|P^2\Big|\dots\Big|P^l$ is a choice of a local vertex for each set composition P^i. This allows us to split P into two ordered fully nested set compositions. For example,

$$3\overset{\downarrow}{|}1\Big|5|4\overset{\downarrow}{|}2|7\Big|\overset{\downarrow}{}6|8 \quad \rightsquigarrow \quad 3\Big|5|4 \quad \text{and} \quad 1\Big|2|7\Big|6|8.$$

Definition 6.2.17 The join of a nested set composition (P_1, Q_1) of $[g_1]$ and a nested set composition (P_2, Q_2) of $[g_2]$ is the nested set composition of $[g_1 + g_2]$ given by $(j(P_1 \times P_2), j(Q_1 \times Q_2))$, with the notation as in Definition 6.2.8. For example,

$$j(3|1\Big|2 \times 23\Big|1|4\Big|5) = 3|1\Big|2\Big|56\Big|4|7\Big|8.$$

Remark Note that we defined a quasi-shuffle and global vertex only for fully nested set compositions. These notions for nested set compositions are more complicated. Since they are not necessary for this chapter, we omit them.

6.2.7 The Hopf algebra $\mathrm{Q\Pi}$

Let $\mathrm{Q\Pi} = \bigoplus_{n\geq 0} \mathbb{K}(\mathrm{Q}^n)^*$, where Q^n is the set of fully nested compositions of $[n]$. Write $M_{(F,D)}$ for the basis element corresponding to $(F, D) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.18 The coproduct and product on $\mathrm{Q\Pi}$ are given by Equations (6.6) and (6.7), but where local vertex and quasi-shuffle are now given by Definitions 6.2.14 and 6.2.15. As an example, for the coproduct,

$$\Delta(M_{3|1|5\big|4|2\big|6}) = 1 \otimes M_{3|1|5\big|4|2\big|6} + M_{3|1|5} \otimes M_{2|1\big|3} + M_{3|1|5\big|4|2} \otimes M_1 + M_{3|1|5\big|4|2\big|6} \otimes 1.$$

The number of terms is two more than the number of big bars in the nested set composition.

As an example, for the product,

$$M_{2|3\big|1|4} * M_{2|1} = M_{2|3\big|1|4\big|6|5} + M_{2|3\big|6|5\big|1|4} + M_{6|5\big|2|3\big|1|4} + M_{2|3\big|1|4|6|5} + M_{2|3|6|5\big|1|4}.$$

6.2.8 The Hopf algebra $\mathrm{N\Pi}$

Let $\mathrm{N\Pi} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Q}^n$, where Q^n is the set of fully nested compositions of $[n]$. Write $H_{(P,C)}$ for the basis element corresponding to $(P, C) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.19 The coproduct and product on $\mathrm{N\Pi}$ are given by Equations (6.8) and (6.9), but where global vertex and join are now given by Definitions 6.2.16 and 6.2.17. For example, for the coproduct,

$$\Delta(H_{3|1\big|2}) = 1 \otimes H_{3|1\big|2} + H_1 \otimes H_{2|1} + H_1 \otimes H_{1\big|2} + H_{2\big|1} \otimes H_1 + H_{2|1} \otimes H_1 + H_{3|1\big|2} \otimes 1.$$

The fully nested set composition $3|1\Big|2$ has $3 * 2 = 6$ global vertices, namely,

$$\overset{\downarrow}{}3|1\Big|\overset{\downarrow}{}2\ ,\ \overset{\downarrow}{}3|1\Big|2\overset{\downarrow}{}\ ,\ 3\overset{\downarrow}{|}1\Big|\overset{\downarrow}{}2\ ,\ 3\overset{\downarrow}{|}1\Big|2\overset{\downarrow}{}\ ,\ 3|1\overset{\downarrow}{}\Big|\overset{\downarrow}{}2\ ,\ 3|1\overset{\downarrow}{}\Big|2\overset{\downarrow}{}.$$

Each one of these contributes to a term in the coproduct.

The product is quite simple and involves only one term. For example,

$$H_{2|3\big|1|4} * H_{2|1} = H_{2|3\big|1|4\big|6|5}.$$

6.2.9 Set partitions

We denote a set partition by $X = \{X^1, X^2, \ldots, X^l\}$. For example, $\{134, 56, 2\}$ is a partition of the set $[6]$.

Definition 6.2.20 A local vertex of a set partition X is an ordered splitting of X into two set partitions. For example,

$$\{X^1, X^2, X^3, X^4\} \quad \rightsquigarrow \quad \{X^1, X^2, X^4\} \quad \text{and} \quad \{X^3\}.$$

Definition 6.2.21 A quasi-shuffle of set partitions X_1 and X_2 is the union of X_1 and X_2, where in addition we are allowed to substitute a disjoint set of pairs of components (X_1^i, X_2^j) for $X_1^i \cup X_2^j$. For example,

$$\{34a, 15bd, 6, c\} \ \text{ is a quasi-shuffle of } \ \{15, 34, 6\} \ \text{ and } \ \{a, bd, c\}.$$

Definition 6.2.22 A global vertex of a set partition $Y = \{Y^1, Y^2, \ldots, Y^l\}$ is a choice of a vertex for each set Y^i. For example,

$$\{36\overset{\downarrow}{}4, 7\overset{\downarrow}{}1, 2\overset{\downarrow}{}5\} \quad \rightsquigarrow \quad \{36, 7, 2\} \quad \text{and} \quad \{4, 1, 5\}.$$

Definition 6.2.23 The join of a partition Y_1 of $[g_1]$ and a partition Y_2 of $[g_2]$ is a partition $j(Y_1 \times Y_2)$ of $[g_1 + g_2]$ obtained by shifting up the indices of Y_2 by g_1 and then taking union with Y_1. For example,

$$j(\{31, 2\} \times \{23, 14, 5\}) = \{31, 2, 56, 47, 8\}.$$

6.2.10 The Hopf algebra Π_{L^*}

Let $\Pi_{\mathrm{L}^*} = \bigoplus_{n \geq 0} \mathbb{K}(\mathrm{L}^n)^*$, where L^n is the set of partitions of $[n]$. Write m_X for the basis element corresponding to $X \in \mathrm{L}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.24 The coproduct and product on Π_{L^*} are given by Equations (6.6) and (6.7), but where local vertex and quasi-shuffle are now given by Definitions 6.2.20 and 6.2.21. As an example, for the coproduct,

$$\Delta(m_{\{13,2\}}) = 1 \otimes m_{\{13,2\}} + m_{\{12\}} \otimes m_{\{1\}} + m_{\{1\}} \otimes m_{\{12\}} + m_{\{13,2\}} \otimes 1.$$

As an example, for the product,

$$m_{\{13,2\}} * m_{\{12\}} = m_{\{13,2,45\}} + m_{\{13,245\}} + m_{\{1345,2\}}.$$

It is clear that the coproduct is cocommutative.

6.2.11 The Hopf algebra Π_{L}

Let $\Pi_{\mathrm{L}} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{L}^n$, where L^n is the set of partitions of $[n]$. Write h_Y for the basis element corresponding to $Y \in \mathrm{L}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.25 The coproduct and product on Π_{L} are given by Equations (6.8) and (6.9), but where global vertex and join are now given by Definitions 6.2.22 and 6.2.23. For example, for the coproduct,

$$\begin{aligned}\Delta(h_{\{12,3\}}) \quad &= \quad 1 \otimes h_{\{12,3\}} + 2(h_{\{1\}} \otimes h_{\{1,2\}} + h_{\{1,2\}} \otimes h_{\{1\}}) + \\ &\qquad (h_{\{1\}} \otimes h_{\{12\}} + h_{\{12\}} \otimes h_{\{1\}}) + h_{\{12,3\}} \otimes 1.\end{aligned}$$

The product is quite simple and involves only one term. For example,

$$h_{\{134,2\}} * h_{\{1,23\}} = h_{\{134,2,5,67\}}.$$

Note that using the reasoning given for $\mathrm{M}\Pi$, it follows that the coalgebra Π_{L} is also cocommutative. We may also use the fact that the quotient of a cocommutative coalgebra is again cocommutative. As mentioned before, this Hopf algebra is not self-dual.

Remark The Hopf algebras Π_{L} and Π_{L^*} are isomorphic. This is a part of the claim made in Theorem 6.1.3.

6.2.12 Nested set partitions

Definition 6.2.26 A nested set partition is a set $L = \{L^1, \ldots, L^l\}$, in which each L^i is a set composition of A^i, and $\{A^1, \ldots, A^l\}$ is a set partition of $[n]$. For example,

$$\{3|56, 2|17, 4\}$$

is a nested partition of [7]. A fully nested set partition is a nested set partition with singleton blocks.

Definition 6.2.27 A local vertex of a nested set partition L is an ordered splitting of L into two nested set partitions. For example,

$$\{L^1, L^2, L^3, L^4\} \quad \rightsquigarrow \quad \{L^1, L^2, L^4\} \quad \text{and} \quad \{L^3\}.$$

Definition 6.2.28 A quasi-shuffle of two fully nested set partitions L_1 and L_2 is a union of L_1 and L_2, where in addition we are allowed to substitute a disjoint set of pairs of components (L_1^i, L_2^j) for the composition $L_1^i|L_2^j$. For example,

$$\{3|4|a, 1|6|c|d, b, 2|7\} \text{ is a quasi-shuffle of } \{1|6, 3|4, 2|7\} \text{ and } \{a, c|d, b\}.$$

Definition 6.2.29 A global vertex of a fully nested set partition $M = \{M^1, \ldots, M^l\}$ is a choice of a local vertex for each set composition M^i. For example,

$$\{\overset{\downarrow}{\,}3|6|4, 7|8|1\overset{\downarrow}{|}9, 2\overset{\downarrow}{|}5\} \quad \rightsquigarrow \quad \{7|8|1, 2\} \quad \text{and} \quad \{3|6|4, 9, 5\}.$$

Definition 6.2.30 The join of nested partitions M_1 of $[g_1]$ and M_2 of $[g_2]$ is a nested partition $j(M_1 \times M_2)$ of $[g_1 + g_2]$ obtained by shifting up the indices of M_2 by g_1 and then taking union with M_1. For example,

$$j(\{3|1, 2\} \times \{23, 1|4, 5\}) = \{3|1, 2, 56, 4|7, 8\}.$$

Remark As for nested set compositions, to avoid complications, we have defined a quasi-shuffle and global vertex only for fully nested set partitions.

6.2.13 The Hopf algebra Π_{Z^*}

Let $\Pi_{\mathrm{Z}^*} = \bigoplus_{n\geq 0} \mathbb{K}(\mathrm{Z}^n)^*$, where Z^n is the set of fully nested partitions of $[n]$. Write m_L for the basis element corresponding to $L \in \mathrm{Z}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.31 The coproduct and product on Π_{Z^*} are given by Equations (6.6) and (6.7), but where local vertex and quasi-shuffle are now given by Definitions 6.2.27 and 6.2.28. As an example, for the coproduct,

$$\Delta(m_{\{1|3,4|2\}}) = 1 \otimes m_{\{1|3,4|2\}} + m_{\{1|2\}} \otimes m_{\{2|1\}} + m_{\{2|1\}} \otimes m_{\{1|2\}} + m_{\{1|3,4|2\}} \otimes 1.$$

As an example, for the product,

$$m_{\{1|4|3,2\}} * m_{\{1|2\}} = m_{\{1|4|3,2,5|6\}} + m_{\{1|4|3,2|5|6\}} + m_{\{1|4|3|5|6,2\}}.$$

It is clear that the coproduct is cocommutative.

6.2.14 The Hopf algebra Π_{Z}

Let $\Pi_{\mathrm{Z}} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Z}^n$, where Z^n is the set of fully nested partitions of $[n]$. Write h_M for the basis element corresponding to $M \in \mathrm{Z}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.2.32 The coproduct and product on Π_{Z} are given by Equations (6.8) and (6.9), but where global vertex and join are now given by Definitions 6.2.29 and 6.2.30. For example,

$$\begin{aligned} \Delta(h_{\{3|1,2\}}) \quad = \quad & 1 \otimes h_{\{3|1,2\}} + h_{\{1\}} \otimes h_{\{2|1\}} + h_{\{2|1\}} \otimes h_{\{1\}} \\ & + h_{\{1\}} \otimes h_{\{1,2\}} + h_{\{1,2\}} \otimes h_{\{1\}} + h_{\{3|1,2\}} \otimes 1. \end{aligned}$$

The product is quite simple and involves only one term. For example,

$$h_{\{1|4|3,2\}} * h_{\{1,2|3\}} = h_{\{1|4|3,2,5,6|7\}}.$$

Remark The Hopf algebras Π_{Z} and Π_{Z^*} are neither isomorphic nor dual to each other.

6.2.15 The Hopf algebra $\mathrm{S}\Pi$

Let $\mathrm{S}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ are the chambers in Σ^n and can be identified with permutations. Write $F_{(C,D)}$ for the basis element $(C, D) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0. Let $C = C^1|C^2|\ldots|C^n$ and $D = D^1|D^2|\ldots|D^n$.

Definition 6.2.33 The coproduct on $\mathrm{S}\Pi$ is given by

$$\Delta(F_{(C,D)}) = \sum_{i=0}^{n} F_{\mathrm{st}(\tilde{C}^1|\cdots|\tilde{C}^i, D^1|\cdots|D^i)} \otimes F_{\mathrm{st}(\tilde{C}^{i+1}|\cdots|\tilde{C}^n, D^{i+1}|\cdots|D^n)},$$

where $\tilde{C}^1, \ldots, \tilde{C}^i$ are the letters in the set $\{D^1, \ldots, D^i\}$ and $\tilde{C}^{i+1}, \ldots, \tilde{C}^n$ are the letters in the set $\{D^{i+1}, \ldots, D^n\}$ written in the order in which they appear in $C^1|\cdots|C^n$. For example,

$$\begin{aligned} \Delta(F_{2|3|1|4,4|2|1|3}) \quad = \quad & 1 \otimes F_{2|3|1|4,4|2|1|3} + F_{1,1} \otimes F_{2|3|1,2|1|3} + F_{1|2,2|1} \otimes F_{2|1,1|2} + \\ & F_{2|1|3,3|2|1} \otimes F_{1,1} + F_{2|3|1|4,4|2|1|3} \otimes 1. \end{aligned}$$

Definition 6.2.34 The product on SΠ is given by

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} = \sum_{D:\ D \text{ a shuffle of } D_1 \text{ and } D_2} F_{(j(C_1 \times C_2),D)}.$$

The term $j(C_1 \times C_2)$ refers to the join of C_1 and C_2 given in Definition 6.2.8. If D_1 and D_2 are compositions of $[g_1]$ and $[g_2]$ respectively then the understanding is that we shift up the indices of D_2 by g_1 and then shuffle. For example,

$$\begin{aligned} F_{(2|1,2|1)} * F_{(2|1,1|2)} &= F_{(2|1|4|3,2|1|3|4)} + F_{(2|1|4|3,2|3|1|4)} + F_{(2|1|4|3,3|2|1|4)} + \\ & \quad F_{(2|1|4|3,2|3|4|1)} + F_{(2|1|4|3,3|2|4|1)} + F_{(2|1|4|3,3|4|2|1)}. \end{aligned}$$

6.2.16 The Hopf algebra RΠ

Let $\mathrm{R}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)$, where $\mathcal{C}^n$ are the chambers in Σ^n and can be identified with permutations. Write $K_{(D,C)}$ for the basis element $(D,C) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0. We define a product and coproduct on RΠ (see below) such that:

Proposition 6.2.1 *The switch map* $s : \mathrm{R}\Pi \to \mathrm{S}\Pi$ *which sends* $K_{(D,C)} \to F_{(C,D)}$ *is an isomorphism of Hopf algebras.*

Definition 6.2.35 The coproduct on RΠ is given by

$$\Delta(K_{(D,C)}) = \sum_{i=0}^{n} K_{\mathrm{st}(D^1|\cdots|D^i,\tilde{C}^1|\cdots|\tilde{C}^i)} \otimes K_{\mathrm{st}(D^{i+1}|\cdots|D^n,\tilde{C}^{i+1}|\cdots|\tilde{C}^n)}.$$

where $\tilde{C}^1, \ldots, \tilde{C}^n$ are as in Definition 6.2.33.

Definition 6.2.36 The product on RΠ is given by

$$K_{(D_1,C_1)} * K_{(D_2,C_2)} = \sum_{D:\ D \text{ a shuffle of } D_1 \text{ and } D_2} K_{(D,j(C_1 \times C_2))}.$$

Remark As mentioned above, the Hopf algebras RΠ and SΠ are clearly isomorphic. However for book-keeping purposes, we prefer to keep them separate.

6.3 The coalgebra axioms and examples

In this section, we give the coalgebra axioms for a family of posets (Section 6.3.1). They appear very abstract at first glance; however they are natural, that is, there are many examples (Sections 6.3.2-6.3.4). The motivation is that one can construct the coalgebras in diagram (6.3) starting with a family of posets that satisfies these axioms.

Our first example (Section 6.3.2) is based on compositions. It is simple yet nontrivial enough to help the reader gain some understanding of the axioms. The next example (Section 6.3.3) arises from the Coxeter group of type A. It is the motivating example which led us to the axioms. We give two viewpoints for this example, one geometric and the other combinatorial, stressing the former. In Section 6.3.4, we give another example from Coxeter theory.

6.3.1 The coalgebra axioms

Consider the family $\{\Sigma^n\}_{n\geq 0}$, where Σ^n is a finite graded poset of rank $n-1$ with a unique minimum element that we denote $\emptyset$. Further let Σ^0 and Σ^1 be singleton sets with the unique element $\emptyset$. For K a face of Σ, let

$$\Sigma^n_K = \{F \in \Sigma^n \mid K \leq F\}.$$

Let $\mathcal{C}^n$ be the set of chambers (maximal elements) in Σ^n and $\mathcal{C}^n_F = \{D \in \mathcal{C}^n \mid F \leq D\}$. Let $\deg K = \operatorname{rank} K + 1$, where rank denotes the rank of an element.

We give three coalgebra axioms for such a family of posets.

Axiom $(C1)$**.** For every face F of Σ^n, there exists a composition $(f_1, f_2, \ldots, f_{\deg F})$ of n and a poset isomorphism

$$b_F : \Sigma^n_F \to \Sigma^{f_1} \times \Sigma^{f_2} \times \ldots \times \Sigma^{f_{\deg F}}. \tag{6.10}$$

For $F = \emptyset$, the composition is (n) and the poset isomorphism $b_\emptyset$ is the identity map $\mathrm{id} : \Sigma^n \to \Sigma^n$.

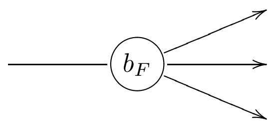


Figure 6.1: The break map b_F.

Figure 6.1 shows a schematic picture for the break map b_F for $\deg F = 3$. The circle at the center is an apparatus which takes one input larger than F and produces three ordered outputs.

Axiom $(C2)$**.** The maps b_F in axiom $(C1)$ are "associative" in the following sense.

Let K be a face of F. First apply the axiom $(C1)$ to the face K of Σ^n to get a composition $(k_1, k_2, \ldots, k_{\deg K})$ of n and a poset isomorphism

$$b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2} \times \ldots \times \Sigma^{k_{\deg K}}.$$

Under this map, let $F \mapsto F_1 \times F_2 \times \ldots \times F_{\deg K}$. This induces an isomorphism

$$b_K : \Sigma^n_F \to \Sigma^{k_1}_{F_1} \times \Sigma^{k_2}_{F_2} \times \ldots \times \Sigma^{k_{\deg K}}_{F_{\deg K}}.$$

Apply the axiom $(C1)$ to the face F_i of Σ^{k_i} to get a composition $(f_{i1}, f_{i2}, \ldots, f_{i\deg F_i})$ of k_i and a poset isomorphism

$$b_{F_i} : \Sigma^{k_i}_{F_i} \to \Sigma^{f_{i1}} \times \Sigma^{f_{i2}} \times \ldots \times \Sigma^{f_{i\deg F_i}} \quad \text{for} \quad 1 \leq i \leq \deg K.$$

Then, we have

$$b_F = (b_{F_1} \times b_{F_2} \times \ldots \times b_{F_{\deg K}}) \circ b_K, \tag{6.11}$$

where b_F is the poset map given by the axiom $(C1)$ for the face F. In particular, we require that

$$(f_1, f_2, \ldots, f_{\deg F}) = (f_{11}, f_{12}, \ldots, f_{1\deg F_1}, f_{21}, \ldots, f_{m\deg F_m}), \tag{6.12}$$

where $m = \deg K$.

Figure 6.2 shows a schematic picture for the associativity of the break map. It summarizes what is going on without using the above menagerie of symbols.

Axiom $(C3)$**.** Every rank 2 face H of Σ^n has exactly two rank 1 faces, say K and G. Further the poset isomorphisms $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$ and $b_G : \Sigma^n_G \to \Sigma^{g_1} \times \Sigma^{g_2}$ map H to $\emptyset \times K'$ and $G' \times \emptyset$ respectively for some rank 1 faces $K' \in \Sigma^{k_2}$ and $G' \in \Sigma^{g_1}$.

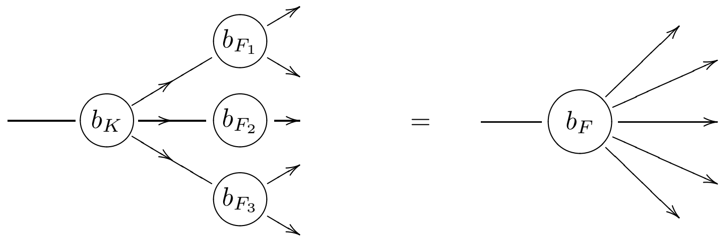


Figure 6.2: The break map is associative.

Remark The content of this axiom is not the existence of K' and G' but the fact that they belong to the second and first coordinate respectively of the range.

In addition to the above three coalgebra axioms, we define a projection axiom. For that assume, in addition, that each Σ^n is a projection poset, as defined in Section 2.7. This includes the case when Σ^n is a LRB.

Axiom (CP)**.** The poset maps respect the product structure of Σ^n; that is, for any $F \leq H, N \in \Sigma^n$, we have

$$b_F(HN) = b_F(H)b_F(N),$$

where b_F is the poset isomorphism associated to F by the axiom $(C1)$.

Remark The elements of rank 1, or equivalently, degree 2 in Σ^n have a special role to play when we consider coproducts. We also refer to them as vertices, since Σ^n is a simplicial complex in our examples.

Proposition 6.3.1 *Let $\{\Sigma^n\}_{n\geq 0}$ satisfy the coalgebra axioms $(C1)$ and $(C2)$. Then for each $n > 0$, there are poset maps*

$$\text{type} : \Sigma^n \to \overline{\mathrm{Q}}^n,$$

where $\overline{\mathrm{Q}}^n$ is the poset of compositions of n.

Proof The map type sends a face $F \in \Sigma^n$ to the composition $(f_1, f_2, \ldots, f_{\deg F})$ of n as given by the axiom $(C1)$. The axiom $(C2)$, in particular Equation (6.12), implies that this is a poset map.

□

Remark In our examples, the map type : $\Sigma^n \to \overline{\mathrm{Q}}^n$ is a map from a simplicial complex of rank $n-1$ to the simplex of rank $n-1$. It will always be surjective and have a "nice" section $\overline{\mathrm{Q}}^n \hookrightarrow \Sigma^n$. This section is important for the algebra axioms that we consider in Section 6.6.

6.3.2 The warm-up example of compositions

Let $\Sigma^n = \overline{\mathrm{Q}}^n$ be the simplex of rank $n-1$, $n \geq 0$. As a poset, it consists of subsets of $[n-1]$, or equivalently, compositions of n. Recall that a composition of n of degree l is a sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of positive integers such that $\sum_{i=1}^{l} \alpha_i = n$. The partial order of $\overline{\mathrm{Q}}^n$ is given by subset inclusion, or equivalently, refinement of compositions. The poset $\overline{\mathrm{Q}}^n$ has only one chamber D, namely the n-tuple

$$(\underbrace{1, 1, \ldots, 1}_{n})$$

and a unique minimum element $\emptyset = (n)$. It is a commutative LRB with the product being union of subsets. Let

$$\overline{\mathrm{Q}}^n_\alpha = \{\beta \in \overline{\mathrm{Q}}^n \mid \alpha \leq \beta\}.$$

Our goal is to show the following.

Lemma 6.3.1 *The family of LRBs $\{\overline{\mathrm{Q}}^n\}_{n\geq 0}$ satisfies all coalgebra axioms $(C1) - (CP)$.*

Proof The map in Proposition 6.3.1, in this case, is the identity isomorphism $\overline{\mathrm{Q}}^n \to \overline{\mathrm{Q}}^n$. So the situation is as simple as can be. The break map is defined as below.

Definition 6.3.1 For $\alpha = (\alpha_1, \ldots, \alpha_k)$, a composition of n and degree k, we define a poset isomorphism

$$b_\alpha : \overline{\mathrm{Q}}^n_\alpha \to \overline{\mathrm{Q}}^{\alpha_1} \times \overline{\mathrm{Q}}^{\alpha_2} \times \ldots \times \overline{\mathrm{Q}}^{\alpha_k}$$

as follows.

Let $\beta \in \overline{\mathrm{Q}}^n_\alpha$ be a refinement of α. Then the image of β on the ith factor is obtained by lumping together the parts of β that refine α_i. For example, for $n = 9$ and $\alpha = (2, 4, 3)$, the map $\overline{\mathrm{Q}}^9_\alpha \to \overline{\mathrm{Q}}^2 \times \overline{\mathrm{Q}}^4 \times \overline{\mathrm{Q}}^3$ sends $\beta = (2, 1, 3, 2, 1)$ to $(2) \times (1, 3) \times (2, 1)$.

It is easy but instructive to check the axioms $(C2)$ and (CP) from this definition. This is left to the reader. We check the axiom $(C3)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a rank 2 face. Then it has two rank 1 faces, namely, $\beta = (\alpha_1, \alpha_2 + \alpha_3)$ and $\gamma = (\alpha_1 + \alpha_2, \alpha_3)$. Further, the maps $\overline{\mathrm{Q}}^n_\beta \to \overline{\mathrm{Q}}^{\alpha_1} \times \overline{\mathrm{Q}}^{\alpha_2+\alpha_3}$ and $\overline{\mathrm{Q}}^n_\gamma \to \overline{\mathrm{Q}}^{\alpha_1+\alpha_2} \times \overline{\mathrm{Q}}^{\alpha_3}$ send α to $(\alpha_1) \times (\alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2) \times (\alpha_3)$ respectively, in accordance with the axiom $(C3)$. Remember that (n) stands for the empty face $\emptyset$ in $\overline{\mathrm{Q}}^n$.

$\square$

Remark The idea behind Definition 6.3.1 is better seen in the language of subsets. It will also be useful for the next example. Let V be a subset of $[n-1]$ of cardinality $k-1$. Then

$$[n-1] \setminus V \ = \bigsqcup_{j=1}^{k} I_j$$

is a disjoint union of k intervals. To every T such that $V \subseteq T \subseteq [n-1]$, one can associate $T_j \subseteq I_j$ by the identity

$$T = V \bigsqcup \left(\bigsqcup_{j=1}^{k} T_j \right). \tag{6.13}$$

This defines a poset isomorphism

$$\overline{\mathrm{Q}}^n_V \ \longrightarrow \ \mathop{\times}_{j=1}^{k} \{\text{subsets of } I_j\},$$

equivalent to the one in Definition 6.3.1, that sends T to $T_1 \times T_2 \times \ldots \times T_k$.

6.3.3 The motivating example of type A_{n-1}

The motivation for the coalgebra axioms comes from the theory of Coxeter groups. Let (W, S) be a Coxeter system, Σ its Coxeter complex and $\mathcal{C}$ the set of chambers in Σ. We recall some key facts from Chapters 1 and 2.

Fact 6.3.1 A Coxeter complex Σ is a left regular band (LRB). For $F \in \Sigma$ and $C \in \mathcal{C}$, the chamber FC is that chamber in $\mathcal{C}_F = \{D \mid F \leq D\}$ which is closest in the gallery metric to C. This is shown in Figure 1.2. The product in Σ is then given by

$$FP = \bigcap_{P \leq C} FC.$$

We call FP the projection of P on F.

Fact 6.3.2 Let $F \in \Sigma$ be a face of type $T \leq S$. Then the link of F in Σ, denoted link(F), is again a Coxeter complex. In our notation, link$(F) \cong \Sigma_F$ as posets. Further the Coxeter diagram of link(F) is obtained from the Coxeter diagram of Σ by deleting all the vertices whose type is contained in T.

Fact 6.3.3 The join $\Sigma^1 * \Sigma^2$ of two Coxeter complexes is again a Coxeter complex, whose diagram is the disjoint union of the diagrams of Σ^1 and Σ^2. Further, the join operation is compatible with the projection maps, that is,

$$(H_1 * N_1)(H_2 * N_2) = (H_1 H_2 * N_1 N_2), \text{ where } H_i, N_i \in \Sigma^i.$$

Geometry

Recall from Section 1.4 that for type A_{n-1}, the Coxeter group W is S_n, the symmetric group on n letters and $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the transposition that interchanges i and $i+1$. Let Σ^n be the Coxeter complex of S_n. Identify $\overline{\mathrm{Q}}^n$, the poset of compositions of n under refinement, with the poset of subsets of S under inclusion in the usual way.

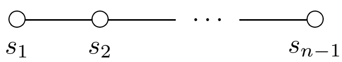


Figure 6.3: The Coxeter diagram of type A_{n-1}.

Lemma 6.3.2 *The family of LRBs $\{\Sigma^n\}_{n\geq 0}$ satisfies all coalgebra axioms $(C1)-(CP)$.*

Proof Our goal here is two-fold, to prove the above lemma; at the same time, to use this example, to motivate the axioms themselves.

Axiom $(C1)$**.** Let $F \in \Sigma$ be a face of type $T \leq S$. Observe that deleting $|T| = \operatorname{rank} F$ vertices breaks the Coxeter diagram of type A, shown in Figure 6.3, into

$$\deg F = \operatorname{rank} F + 1,$$

possibly empty, disjoint parts, each part again being of type A. Further, these parts can be ordered in a natural way. In other words, using Facts 6.3.2 and 6.3.3,

$$b_F : \Sigma^n_F \xrightarrow{\cong} \Sigma^{f_1} \times \Sigma^{f_2} \times \ldots \times \Sigma^{f_{\deg F}}, \tag{6.14}$$

where $f_i = 1+$ (number of vertices in the ith part) and where $\times$ refers to the join of simplicial complexes. It is the cartesian product at the level of posets. This is the origin of the coalgebra axiom $(C1)$. There is still one difference though. The above isomorphism b_F is specified only upto the symmetric groups involved. To pick out a specific b_F, one needs to do the following.

Fix a fundamental chamber C_0^n in Σ^n for all $n \geq 1$. Now define b_F to be the unique isomorphism that maps

$$FC_0^n \mapsto C_0^{f_1} \times C_0^{f_2} \times \ldots \times C_0^{f_{\deg F}}. \tag{6.15}$$

In this case, Proposition 6.3.1 specializes to the following.

Proposition 6.3.2 *The poset map* type $: \Sigma^n \to \overline{\mathrm{Q}}^n$ *sends* $F \in \Sigma^n$ *to* $\mathrm{type}(F) \leq S$.

This explains the origin of the name "type". The above map is clearly surjective and has a "nice" section that sends $T \leq S$ to the face of the fundamental chamber C_0^n of type T.

Axiom (CP)**.** With b_F as defined in Equation (6.14), it follows from Fact 6.3.3 that $\{\Sigma^n\}_{n\geq 0}$ satisfies the axiom (CP).

Axiom $(C2)$**.** We now show that $\{\Sigma^n\}_{n\geq 0}$ satisfies the axiom $(C2)$. In other words, we show that Equation (6.11) holds. We use the notation of the axioms $(C1)$ and $(C2)$, and further set $m = \deg K$ for notational simplicity. The first requirement is that Equation (6.12) holds.

To get the composition of n in the left hand side of Equation (6.12) , we delete those vertices from the Coxeter diagram of Σ^n whose type is contained in type(F). And to get the composition of n in the right hand side, we first delete those vertices whose type is contained in type(K) and then from the ith part delete those vertices whose type is contained in type(F_i), for $1 \leq i \leq m$. The point is that in both cases we delete the same vertices because

$$\mathrm{type}(F) = \mathrm{type}(K) \bigsqcup \left(\bigsqcup_{j=1}^{m} \mathrm{type}(F_j) \right). \tag{6.16}$$

since $b_K(F) = F_1 \times F_2 \times \ldots \times F_m$. This proves Equation (6.12). It is also instructive to compare Equations (6.16) and (6.13).

Hence both sides of Equation (6.11) specify an isomorphism

$$\Sigma_F^n \to \Sigma^{f_{11}} \times \Sigma^{f_{12}} \times \ldots \Sigma^{f_{1\deg F_1}} \times \Sigma^{f_{21}} \times \ldots \times \Sigma^{f_{m\deg F_m}}.$$

To see that they specify the same isomorphism, one needs to check that

$$b_F(FC_0^n) = (b_{F_1} \times b_{F_2} \times \ldots \times b_{F_m}) \circ b_K(FC_0^n). \tag{6.17}$$

For the left hand side, by (6.15), we have

$$b_F(FC_0^n) = C_0^{f_{11}} \times C_0^{f_{12}} \times \ldots C_0^{f_{1\deg F_1}} \times C_0^{f_{21}} \times \ldots \times C_0^{f_{m\deg F_m}}.$$

The right hand side requires more work. Since $K \leq F$, we have $FC_0^n = FKC_0^n$. Hence, first by the axiom (CP), which we proved, and then applying (6.15) to b_K, we obtain

$$b_K(FC_0^n) = b_K(F) b_K(KC_0^n) = F_1 C_0^{k_1} \times F_2 C_0^{k_2} \times \ldots \times F_m C_0^{k_m}.$$

Applying $(b_{F_1} \times b_{F_2} \times \ldots \times b_{F_m})$ to both sides and applying (6.15) to each b_{F_i}, we obtain

$$(b_{F_1} \times b_{F_2} \times \ldots \times b_{F_m}) \circ b_K(FC_0^n) = C_0^{f_{11}} \times C_0^{f_{12}} \times \ldots C_0^{f_{1\deg F_1}} \times C_0^{f_{21}} \times \ldots \times C_0^{f_{m\deg F_m}}.$$

This proves Equation (6.17) and consequently the axiom $(C2)$.

Axiom $(C3)$**.** Note that Σ^n is a simplicial complex. Hence, a rank 2 face H in an edge in Σ^n and has 2 rank 1 faces, or vertices, say K and G. If type$(H) = \{s_i, s_j\}$ with $i < j$ then assume for concreteness that K has type s_i and G has type s_j. Note that if we delete the vertex of type s_i (resp. s_j) then the Coxeter diagram of Σ^n, shown in Figure 6.3, breaks into two ordered parts with s_j (resp. s_i) in the second (resp. first) part. This is the content of the axiom $(C3)$.

□

Combinatorics

We now explain the combinatorial content of the geometry. The Coxeter complex Σ^n can be identified with $\mathcal{B}^n$, the poset of set compositions. Namely, the elements of $\mathcal{B}^n$ are compositions $F = F^1|\dots|F^l$ of the set $[n]$. For example, $347|16|258$ is an element of $\mathcal{B}^8$. The equivalence between the geometry and the combinatorics is explained in detail in Section 1.4, with further details in Section 5.4.

Definition 6.3.2 Multiply two set compositions by taking intersections and ordering them lexicographically; more precisely, if $F = F^1|\dots|F^l$ and $H = H^1|\dots|H^m$, then

$$FH = (F^1 \cap H^1|\dots|F^1 \cap H^m|\dots|F^l \cap H^1|\dots|F^l \cap H^m)\widehat{\ },$$

where the hat means "delete empty intersections". For example,

$$(347|16|258)(6|157|28|34) = (7|34|6|1|5|28).$$

For the partial order of $\mathcal{B}^n$, we say $F \le H$ if H is a refinement of F. The minimum element in $\mathcal{B}^n$ is the one block partition $12\dots n$. An element of rank 1 in $\mathcal{B}^n$ is a two block ordered partition $F^1|F^2$ of $[n]$. The set of chambers $\mathcal{C}^n$ in Σ^n consists of set compositions with singleton blocks, so they correspond to permutations of $[n]$.

Lemma 6.3.3 *The family of LRBs $\{\mathcal{B}^n\}_{n\ge 0}$ satisfies all coalgebra axioms $(C1)-(CP)$.*

In principle, we already proved this in Lemma 6.3.2. So what is happening is language translation. For the fundamental chamber C_0^n in Σ^n, we take the set composition $1|2|\dots|n$. The geometric procedure then yields the following.

Definition 6.3.3 For $F = F^1|\dots|F^l$, a composition of the set $[n]$, we have $l = \deg F$. Let f_i be the cardinality of F^i for $1 \le i \le l$. The poset isomorphism

$$b_F : \mathcal{B}^n_F \to \mathcal{B}^{f_1} \times \mathcal{B}^{f_2} \times \dots \times \mathcal{B}^{f_l}$$

is defined as follows.

Let $H \in \mathcal{B}^n_F$ be a refinement of F. Then the image of H on the ith factor is obtained by lumping together the blocks of H that refine F^i and standardizing (see Definition 6.2.3) so that the result is a composition of the set $[f_i]$.

For example, for $n = 9$ and $F = 18|2357|469$, the map $b_F : \mathcal{B}^9_F \to \mathcal{B}^2 \times \mathcal{B}^4 \times \mathcal{B}^3$ sends $H = 18|5|3|27|9|46$ to the triplet $12 \times 3|2|14 \times 3|12$.

The combinatorial content of Proposition 6.3.2 is the following.

Proposition 6.3.3 *The poset map* $\mathrm{type} : \mathcal{B}^n \twoheadrightarrow \overline{\mathrm{Q}}^n$, *sends a composition of the set $[n]$ to its underlying composition of n.*

The "nice" section $\overline{\mathrm{Q}}^n \hookrightarrow \mathcal{B}^n$ sends a composition $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of n to the set composition

$$12\dots\alpha_1|\alpha_1+1\dots\alpha_1+\alpha_2|\dots|\dots n.$$

We know from Lemma 6.3.2 that the axioms $(C2)-(CP)$ hold. But it is fun and instructive to check them directly using Definitions 6.3.2 and 6.3.3. For example, to check the axiom $(C3)$, let $H = H^1|H^2|H^3$ be a rank 2 face. As a concrete example, take $H = 136|25|47$. Then its two rank 1 faces are $K = 136|2457$ and $G = 12356|47$. The map b_K maps H to $123 \times 13|24$ and b_G maps H to $135|24 \times 12$. Remember that 123 and 12 are the empty faces in Σ^3 and Σ^2 respectively.

6.3.4 The example of type $A_1^{\times(n-1)}$

Geometry

Recall from Section 5.5 that for type $A_1^{\times(n-1)}$, the Coxeter group is $\mathbb{Z}_2^{n-1}$ and $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the generator in the ith coordinate. Let Σ^n be the Coxeter complex of $\mathbb{Z}_2^{n-1}$. Its Coxeter diagram is shown in Figure 6.4. Identify $\overline{\mathrm{Q}}^n$, the poset of compositions of n under refinement, with the poset of subsets of S under inclusion in the usual way.

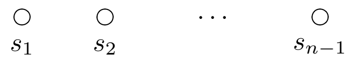


Figure 6.4: The Coxeter diagram of type $A_1^{\times(n-1)}$.

Lemma 6.3.4 *The family of LRBs $\{\Sigma^n\}_{n\geq 0}$ satisfies all coalgebra axioms $(C1)-(CP)$.*

Proof The situation is quite similar to the type A example. Recall that for type A, the poset map b_F in (6.14) was defined using the following key property.

> For $F \in \Sigma$, a face of type $T \leq S$, deleting $|T| = \operatorname{rank} F$ vertices breaks the Coxeter diagram of type A, shown in Figure 6.3, into $\deg F = \operatorname{rank} F + 1$, possibly empty, disjoint parts, each part again being of type A. Further, these parts can be ordered in a natural way.

Now note the similarity between the Coxeter diagrams of type $A_1^{\times(n-1)}$ and A_{n-1}. In both cases, there are $n-1$ vertices labelled, $s_1, \ldots, s_{n-1}$. For type $A_1^{\times(n-1)}$, there are no edges in the diagram; however, the above property holds for the same reason as before. The proof is now identical to the proof of Lemma 6.3.2.

$\square$

In this case, Proposition 6.3.1 specializes to Proposition 6.3.2, with the understanding that Σ^n now refers to the Coxeter complex of type $A_1^{\times(n-1)}$.

Combinatorics

We make things combinatorially explicit (Section 5.5). The Coxeter complex Σ^n of type $A_1^{\times(n-1)}$ can be identified with $\mathcal{W}^n$, the poset of words of length $n-1$ in the alphabet $\{+,-,0\}$. Namely, elements of $\mathcal{W}^n$ are sequences $F = F^1F^2\ldots F^{n-1}$ of length $n-1$ in the alphabet $\{+,0,-\}$. The poset $\mathcal{W}^1$ consists of the empty word, which we write as ϵ.

Definition 6.3.4 The product FP is the face with sign sequence

$$(FP)^i = \begin{cases} F^i & \text{if } F^i \neq 0, \\ P^i & \text{if } F^i = 0. \end{cases}$$

This is the usual product rule in an oriented matroid.

For the partial order, we say that $F \leq H$ in $\mathcal{W}^n$ if H is obtained by replacing some of the zeroes in F by either a $+$ or a $-$. The minimum element in $\mathcal{W}^n$ is the word

$$\underbrace{00\ldots0}_{n-1}.$$

The chambers in $\mathcal{W}^n$ are sequences of length $n-1$ in the alphabet $\{+,-\}$. For the fixed chamber C_0^n in Σ^n, we take

$$\underbrace{++\ldots+}_{n-1}.$$

Definition 6.3.5 The rank of a word F in $\mathcal{W}^n$ is the number of nonzero letters in it. For $1 \leq i \leq \operatorname{rank} F$, let p_i be the position of the ith nonzero letter in F. Also let $p_0 = 0$ and $p_{\deg F} = n$. Let $f_i = p_i - p_{i-1}$ for $1 \leq i \leq \deg F$. We define

$$b_F : \mathcal{W}^n_F \to \mathcal{W}^{f_1} \times \mathcal{W}^{f_2} \times \ldots \times \mathcal{W}^{f_{\deg F}}$$

as follows.

Let $H \geq F$. Then the image of H on the ith factor is the subword of H between positions p_{i-1} and p_i. For example, for $F = +00-000+-$ and $H = ++0-0+0+-$, the map $b_F : \mathcal{W}^{10}_F \to \mathcal{W}^1 \times \mathcal{W}^3 \times \mathcal{W}^4 \times \mathcal{W}^1 \times \mathcal{W}^1$ sends H to $\epsilon \times (+0) \times (0+0) \times \epsilon \times \epsilon$.

We know from Lemma 6.3.4 that the axioms $(C2)-(CP)$ hold. However it is also easy to check them directly using Definitions 6.3.4 and 6.3.5.

6.4 From coalgebra axioms to coalgebras

Starting with a family $\{\Sigma^n\}_{n\geq 0}$ of projection posets that satisfies the coalgebra axioms in Section 6.3, one can construct many graded coalgebras. The formal coalgebra constructions are given in the next section. However in this section, we do explain the basic idea behind the coproducts and then establish some results that are central to the coassociativity issue. The reader may just skim this section and refer back to these results, as necessary.

6.4.1 The coproducts

Let $\{\Sigma^n\}_{n\geq 0}$ be a family of projection posets that satisfies the coalgebra axioms. Let $K \in \Sigma^n$ and $\operatorname{rank} K = 1$. Then by the axiom $(C1)$, there exists a composition (k_1, k_2) of n and a poset isomorphism

$$b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}. \tag{6.18}$$

Using this map, one can define coproducts on the graded vector spaces $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{M}$ that occur in Table 6.1. In other words, the rank 1 faces, or vertices, in Σ^n hold the key. The remaining axioms are necessary to prove coassociativity and that the maps relating these objects are morphisms of coalgebras. Details are given in the next section, see Theorem 6.5.1.

To get the full picture in diagram (6.3), one needs to assume that $\{\Sigma^n\}_{n\geq 0}$ is a family of LRBs. This is because the supp and lune maps are defined only for LRBs. The coproducts on $A_{\mathcal{L}}$ and $A_{\mathcal{Z}}$ are forced by the surjectivity of the supp and lune maps; similarly, the coproducts on $A_{\mathcal{Z}^*}$ and $A_{\mathcal{L}^*}$ are forced by the injectivity of the supp* and lune* maps.

6.4.2 Coassociativity of the coproducts

No matter which coalgebra we are considering, the coproduct Δ of an element of degree n is always related to the rank 1 faces of Σ^n. Similarly, the k-fold coproduct $\Delta^{(k)}$ is related to the rank k faces of Σ^n. But before one can talk of $\Delta^{(k)}$, one needs to show that $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ coincide and can be written unambiguously as $\Delta^{(2)}$. For this purpose, it is useful to formulate a special case of the axiom $(C2)$, which we state as a proposition below.

Proposition 6.4.1 *Let $\{\Sigma^n\}$ satisfy the coalgebra axioms $(C1)-(C3)$. Let H be a rank 2 face of Σ^n with type (h_1, h_2, h_3) and poset map b_H given by the axiom $(C1)$. Let K, G, K', G' be as defined by the axiom $(C3)$. They have rank 1 in their respective posets. Namely, K and G are the two rank 1 faces of H and*

$$b_K(H) = \emptyset \times K' \quad \textit{and} \quad b_G(H) = G' \times \emptyset, \tag{6.19}$$

where b_K and b_G are the poset maps of K and G respectively. Following the notation of the axiom $(C1)$, let K have type (k_1, k_2) and poset map b_K and so on. Then

$$b_H = (\mathrm{id} \times b_{K'}) \circ b_K \quad \textit{and} \quad h = (b_{G'} \times \mathrm{id}) \circ b_G. \tag{6.20}$$

Further, we have $(h_1, h_2, h_3) = (k_1, k_1', k_2') = (g_1', g_2', g_2)$, where (k_1', k_2') and (g_1', g_2') are compositions of k_2 and g_1 respectively.

Definition 6.4.1 Define three sets, namely Left, Middle and Right as follows.

$$\begin{aligned}
&\mathrm{Left} = \{(G', G) \mid G \in \Sigma^n,\ \mathrm{rank}\, G = 1,\ G' \in \Sigma^{g_1},\ \mathrm{rank}\, G' = 1\} \\
&\mathrm{Middle} = \{H \mid H \in \Sigma^n,\ \mathrm{rank}\, H = 2\} \ \text{ and} \\
&\mathrm{Right} = \{(K, K') \mid K \in \Sigma^n,\ \mathrm{rank}\, K = 1,\ K' \in \Sigma^{k_2},\ \mathrm{rank}\, K' = 1\},
\end{aligned}$$

where G has type (g_1, g_2) and K has type (k_1, k_2).

We define a map Middle $\to$ Right that sends H to (K, K') and a map Middle $\to$ Left that sends H to (G', G), where K, G, K', G' are as defined by the axiom $(C3)$. Since the poset maps are isomorphisms, it follows that:

Proposition 6.4.2 *The maps* Middle $\to$ Left *and* Middle $\to$ Right *are bijections.*

The sets Left and Right will show up as index sets when we consider $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ respectively. This should be somewhat clear from the discussion so far. As a requirement for coassociativity, one needs a bijection between Left and Right. We showed this in the above proposition by using the third set Middle as a go-between the two. The set Middle will then give an unbiased index set for the definition of $\Delta^{(2)}$.

6.4.3 Useful results for coassociativity

In Section 6.4.2, we explained the basic principle behind coassociativity. However, depending on the coalgebra at hand, one requires more specialized results, which we give here. To avoid repetition, we omit the set Left from the discussion.

Proposition 6.4.3 *Let P be a fixed face of Σ^n. Then the bijection* Middle $\to$ Right *restricts to a bijection between the subsets*

$$\{H \in \mathrm{Middle} \mid H \le P\} \quad \textit{and} \quad \{(K, K') \in \mathrm{Right} \mid K \le P,\ K' \le P'\},$$

where P', which depends on K, is defined by

$$b_K(P) = P_1 \times P'.$$

Proof The proposition follows from the fact that b_K is a poset isomorphism. □

Proposition 6.4.4 *Let H be a rank 2 face of Σ^n and $H \mapsto (K, K')$ under the bijection* Middle $\to$ Right. *Then, we have*

$$b_H(HP) = (\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KP)),$$

for any face P of Σ^n.

Proof The proposition follows from the following sequence of equalities.

$$\begin{aligned}
b_H(HP) &= (\mathrm{id} \times b_{K'}) \circ b_K(HP) && \text{(Equation (6.20))}\\
&= (\mathrm{id} \times b_{K'}) \circ b_K(HKP) && (K \leq H)\\
&= (\mathrm{id} \times b_{K'})(b_K(H) b_K(KP)) && \text{(Coalgebra axiom } (CP))\\
&= (\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KP)) && \text{(Equation (6.19))}
\end{aligned}$$

For the second equality, we used that $HP = HKP$ for $K \leq H$, which holds for projection posets by Lemma 2.7.2.

□

Proposition 6.4.5 *Let H be a rank 2 face of Σ^n and $H \mapsto (K, K')$ under the bijection* Middle $\to$ Right. *Let $P \leq C$ and*

$$b_K(KP) = P_1 \times P' \quad \textit{and} \quad b_K(KC) = C_1 \times C'.$$

Then

$$P'K' \leq C' \iff KPH \leq KC.$$

Proof The proposition follows from the following sequence of equalities.

$$\begin{aligned}
P'K' \leq C' &\iff (P_1 \times P')(\emptyset \times K') \leq C_1 \times C' && (P_1 \leq C_1)\\
&\iff b_K(KP) b_K(H) \leq b_K(KC) && \text{(Equation (6.19))}\\
&\iff b_K(KPH) \leq b_K(KC) && \text{(Coalgebra axiom } (CP))\\
&\iff KPH \leq KC. && (b_K \text{ a poset isomorphism})
\end{aligned}$$

□

Corollary 6.4.1 *With the notation as above,*

$$PH \leq C \iff PK \leq C \quad \textit{and} \quad P'K' \leq C'.$$

Proof It was shown in Lemma 2.7.8 that for P, K, H, C such that $K \leq H$,

$$PH \leq C \iff PK \leq C \quad \text{and} \quad KPH \leq KC.$$

The result now follows from the previous proposition.

□

6.5 Construction of coalgebras

The goal of this section is to prove Theorem 6.1.1. In particular, starting with a family $\{\Sigma^n\}_{n \geq 0}$ of LRBs that satisfies the coalgebra axioms defined in Section 6.3, we construct many graded coalgebras. Throughout this section, we use the notation and definitions in Section 6.1.2. We first show the following.

Theorem 6.5.1 *Let $\{\Sigma^n\}_{n \geq 0}$ be a family of projection posets, that satisfies all coalgebra axioms $(C1) - (CP)$. Then*

$$\mathcal{M} \xrightarrow{\text{base}^*} \mathcal{N} \xrightarrow{\Theta} \mathcal{R} \xrightarrow{s} \mathcal{S} \xrightarrow{\text{Road}} \mathcal{Q} \xrightarrow{\text{base}} \mathcal{P}$$

is a diagram of coalgebras.

Proof There are two parts to this theorem. Firstly, we define coproducts on the objects (see Definitions 6.5.1-6.5.6) and show that they are coassociative (see Lemmas 6.5.1-6.5.6). Secondly, we show that the maps are morphisms of coalgebras (see Propositions 6.5.2-6.5.6).

□

We restate Theorem 6.1.1 here for convenience.

Theorem *Let* $\{\Sigma^n\}_{n\geq 0}$ *be a family of LRBs, that satisfies all coalgebra axioms* $(C1)-(CP)$*. Then diagram* (6.3) *is a diagram of coalgebras.*

Proof A LRB is a special case of a projection poset. Hence, in view of Theorem 6.5.1, one only needs to show that the surjective supp and lune maps induce coproducts on $A_{\mathcal{L}}$ and $A_{\mathcal{Z}}$ and similarly the injective supp* and lune* maps induce coproducts on $A_{\mathcal{L}^*}$ and $A_{\mathcal{Z}^*}$. This is the content of Definitions 6.5.7, 6.5.8 and Proposition 6.5.7.

□

6.5.1 Examples

In Sections 6.3.2-6.3.4, we gave three examples that satisfy the coalgebra axioms. So the coalgebra constructions in this section can be applied to them. We mainly concentrate on the example of type A.

Proposition 6.5.1 *For the example of type* A*, the coalgebras* $\mathcal{P}$*,* $\mathcal{Q}$*,* $\mathcal{S}$*,* $\mathcal{R}$*,* $\mathcal{N}$ *and* $\mathcal{M}$ *and* $A_{\mathcal{Z}}$*,* $A_{\mathcal{L}}$*,* $A_{\mathcal{Z}^*}$ *and* $A_{\mathcal{L}^*}$ *as defined in this section respectively give the coalgebras* $\mathrm{P\Pi}$, $\mathrm{Q\Pi}$, $\mathrm{S\Pi}$, $\mathrm{R\Pi}$, $\mathrm{N\Pi}$ *and* $\mathrm{M\Pi}$ *and* Π_{Z}*,* Π_{L}*,* Π_{Z^*} *and* Π_{L^*} *as defined in Section 6.2.*

Proof The definitions of the coalgebras $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{M}$ involve two ingredients: the product of the projection poset Σ^n and the poset isomorphisms in the axiom $(C1)$. For the example of type A, explicit descriptions of these two were given in Definitions 6.3.2 and 6.3.3. Hence the proof is a matter of unwinding definitions, see Facts 6.5.2, 6.5.4, 6.5.5, 6.5.6 and 6.5.7.

The part of the proposition dealing with the coalgebras $A_{\_}$ and $\Pi_{\_}$ is left as an exercise to the reader, also see the remark before Proposition 6.5.7.

□

Corollary 6.5.1 *Diagram* (6.4) *is a diagram of coalgebras.*

6.5.2 The coproducts and local and global vertices

The coproducts we define have the form

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \Delta_+(x),$$

where 1 is the basis element of degree 0 and Δ_+ only involves terms of degree greater than 0. If x is of degree n then $\Delta_+(x)$ is written as a sum over a subset of the vertices of Σ^n. This was partly motivated in Section 6.4.1. The specific subset of vertices to use will depend on the kind of object that x is.

Now recall that the coproducts in Section 6.2 were defined by summing over either the local or global vertices. These notions are not specific to type A, and can be defined in more generality. Such an approach also clarifies the geometric meaning of local or global vertices, see Tables 6.5 and 6.6, which are valid for projection posets and LRBs respectively. Needless to say, these ideas will play a key role in the coalgebras that we define in this section. We recall that

$$\mathrm{reg}(F, D) = \{N \mid FN \leq D\} \tag{6.21}$$

Table 6.5: Local and global vertex of a face and pointed face.

Object		Local vertex	Global vertex
Face	$F \in \Sigma$	A vertex of F	Any vertex of Σ
Pointed face	$(F, D) \in \mathrm{Q}$	A vertex of F	A vertex in $\mathrm{reg}(F, D)$

Table 6.6: Local and global vertex of a flat and lune.

Object		Local vertex	Global vertex
Flat	$X \in \mathrm{L}$	A vertex with support in X	Any vertex of Σ
Lune	$L \in \mathrm{Z}$	A vertex with support in $\mathrm{base}(L)$	A vertex in $\mathrm{zone}(L)$

is the lunar region of F and D. The definition of the maps base and zone and related information can be found in Sections 2.3 and 2.4.

6.5.3 The coalgebra $\mathcal{P}$

Let $\mathcal{P} = \bigoplus_{n\geq 0} \mathbb{K}(\Sigma^n)^*$. Write M_F for the basis element corresponding to $F \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.5.1 The coproduct on $\mathcal{P}$ is given by

$$\Delta(M_F) = 1 \otimes M_F + M_F \otimes 1 + \Delta_+(M_F), \quad \text{where}$$

$$\Delta_+(M_F) = \sum_{K:\ \mathrm{rank}\, K=1,\ K \leq F} M_{F_1} \otimes M_{F_2},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(F) = F_1 \times F_2$. In other words, to compute $\Delta_+(F)$, we sum over all the vertices of F and for each vertex K, the map b_K specifies a way to break F into two ordered parts F_1 and F_2.

We now explain how this definition works in the examples (Sections 6.3.2-6.3.4).

Example of compositions

For the example in Section 6.3.2, the LRB Σ^n is the poset of compositions $\overline{\mathrm{Q}}^n$ of n. Let $P\Delta = \bigoplus_{n\geq 0} \mathbb{K}(\overline{\mathrm{Q}}^n)^*$.

Fact 6.5.1 The coproduct on $P\Delta$ given by Definition 6.5.1 is as follows.

$$\Delta_+(M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}) = \sum_{i=1}^{k-1} M_{(\alpha_1,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\ldots,\alpha_k)}.$$

Note that this is precisely how the coproduct is defined on the M basis of the Hopf algebra $\mathrm{Q}\Lambda$ of quasi-symmetric functions (Section 3.2.2).

Example of type A_{n-1}

Recall from Section 6.3.3 or from Table 6.2 in Section 6.2 that for type A, the LRB Σ^n is the poset of compositions $\mathcal{B}^n$ of the set $[n]$. Let $\mathrm{P\Pi} = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{B}^n)^*$.

Fact 6.5.2 The coproduct on $\mathrm{P\Pi}$ given by Definition 6.5.1 coincides with the one given by Definition 6.2.9.

Proof To be concrete, take $F = 347|18|29|56 \in \Sigma^9$. The three vertices of F are $347|125689, 13478|2569$ and $1234789|56$. Each one of them contributes one term to $\Delta_+(M_F)$ by Definition 6.5.1. Namely,

$$\Delta_+(M_{347|18|29|56}) = M_{123} \otimes M_{15|26|34} + M_{234|15} \otimes M_{14|23} + M_{345|16|27} \otimes M_{12}.$$

These terms were computed using the definition of the poset maps for type A given by Definition 6.3.3. Namely, for each vertex, we split F at the bar indicated by that vertex and standardize the two parts. This is precisely how the coproduct on $\mathrm{P\Pi}$ was given in Definition 6.2.9. This proves the first of the 10 parts of Proposition 6.5.1.

□

Example of type $A_1^{\times(n-1)}$

Now we consider the example in Section 6.3.4. Let $P\Gamma = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{W}^n)^*$, where $\mathcal{W}^n$ is the poset of words of length $n-1$ in the alphabet $\{+,-,0\}$. The letter Γ, which is the greek equivalent of the letter C, stands for "cube" to remind us that the faces of the Coxeter complex in this case correspond to the faces of the cube. Now observe that:

Fact 6.5.3 The coproduct on $P\Gamma$ given by Definitions 6.5.1 and 6.3.5 is as follows.

$$\Delta_+(M_{F^1F^2\ldots F^n}) = \sum_{F^i \text{ is nonzero}} M_{F^1\ldots F^{i-1}} \otimes M_{F^{i+1}\ldots F^n}.$$

For example,

$$\Delta_+(M_{+-0-0}) = M_\epsilon \otimes M_{-0-0} + M_+ \otimes M_{0-0} + M_{+-0} \otimes M_0.$$

We now return to the general definition.

Lemma 6.5.1 *The coproduct on $\mathcal{P}$ is coassociative. We have*

$$\Delta_+^{(2)}(M_F) = \sum_{H:\ \mathrm{rank}\, H=2,\ H\leq F} M_{F_1} \otimes M_{F_2} \otimes M_{F_3},$$

where for $b_H : \Sigma^n_H \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(F) = F_1 \times F_2 \times F_3$.

Proof The skeleton of the computation is as follows. The index sets Middle and Right that occur in the computation are as in Definition 6.4.1. The term $(\mathrm{id}\otimes\Delta_+)\circ\Delta_+(M_F)$ is equal to

$$\begin{aligned}
&= \sum_{K:\ \mathrm{rank}\, K=1,\ K\leq F} M_{F_1} \otimes \Delta_+(M_{F'}) && (b_K(F) = F_1 \times F') \\
&= \sum_{\substack{(K,K')\in \mathrm{Right}:\\ K\leq F,\ K'\leq F'}} M_{F_1} \otimes M_{F_2} \otimes M_{F_3} && ((\mathrm{id}\times b_{K'})b_K(F) = F_1 \times F_2 \times F_3) \\
&= \sum_{H\in \mathrm{Middle}:\ H\leq F} M_{F_1} \otimes M_{F_2} \otimes M_{F_3}. && (b_H(F) = F_1 \times F_2 \times F_3)
\end{aligned}$$

The first two equalities are clear; but the third requires some justification. By Proposition 6.4.3, the indexing sets in the second and third sum are in bijection. This is the restriction of the bijection Middle → Right given by Definition 6.4.1. Further from Equation (6.20), we see that

$$b_H(F) = (\mathrm{id} \times b_{K'}) \circ b_K(F). \tag{6.22}$$

This shows that corresponding indices give rise to the same term. This justifies the third equality and shows that

$$(\mathrm{id} \otimes \Delta_+) \circ \Delta_+(M_F) = \Delta_+^{(2)}(M_F),$$

with $\Delta_+^{(2)}(M_F)$ as defined in the lemma. The term $(\Delta_+ \otimes \mathrm{id}) \circ \Delta_+(M_F)$ can be handled similarly and shown equal to $\Delta_+^{(2)}(M_F)$.

□

6.5.4 The coalgebra $\mathcal{M}$

Let $\mathcal{M} = \bigoplus\limits_{n \geq 0} \mathbb{K}\Sigma^n$. Write H_P for the basis element corresponding to $P \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.5.2 The coproduct on $\mathcal{M}$ is given by

$$\Delta(H_P) = 1 \otimes H_P + H_P \otimes 1 + \Delta_+(H_P), \quad \text{where}$$

$$\Delta_+(H_P) = \sum_{K:\ \mathrm{rank}\, K=1} H_{P_1} \otimes H_{P_2},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(KP) = P_1 \times P_2$.

Example of type A_{n-1}

Recall from Section 6.3.3 that for type A, the LRB Σ^n is the poset of compositions $\mathcal{B}^n$ of the set $[n]$. Let $\mathrm{M\Pi} = \bigoplus\limits_{n \geq 0} \mathbb{K}\mathcal{B}^n$.

Fact 6.5.4 The coproduct on $\mathrm{M\Pi}$ given by Definition 6.5.2 coincides with the one given by Definition 6.2.11.

Proof In Definition 6.2.11, to compute $\Delta_+(H_P)$, we sum over the nonvirtual global vertices of $P \in \Sigma^n$; whereas in Definition 6.5.2, we sum over all the vertices of Σ^n. There is a bijection between these two indexing sets given as follows.

A global vertex v of P specifies a way to split P into two ordered set compositions, say P_1' and P_2', see Definition 6.2.7. We define a bijection that sends v to the vertex $K = K^1|K^2$ of Σ^n, where K^1 (resp. K^2) contains the letters that occur in P_1' (resp. P_2').

For example, for $P = 125|347|689$,

$$v = 12\overset{\downarrow}{\ }5|347\overset{\downarrow}{\ }|8\overset{\downarrow}{\ }69 \quad \mapsto \quad K = 123478|569.$$

Carrying this example further,

$$KP = 12|347|8|5|69,$$

which is simply $P_1' = 12|347|8$ and $P_2' = 5|69$ placed next to each other. In Definitions 6.5.2 and 6.2.11, the next step is to standardize P_1' and P_2' (Definition 6.2.3). Hence both v and K give rise to the same term, namely $H_{12|345|6} \otimes H_{1|23}$. The general proof should be clear from this illustration.

□

Remark It is clear from the proof that

$$\{P \mid b_K(KP) = P_1 \times P_2\} = \{P \mid P \text{ is a } K\text{-quasi-shuffle of } P_1 \text{ and } P_2\}. \tag{6.23}$$

One can say that the left hand set is the geometric meaning of a K-quasi-shuffle (Definition 6.2.6). Similarly, it is easy to show that

$$b_K(KP) = P_1 \times P_2,\ PK = P \iff P \text{ is a } K\text{-shuffle of } P_1 \text{ and } P_2,$$

which gives the geometric meaning of a K-shuffle.

Let us compare the coproducts $\Delta_+(M_F)$ in $\mathcal{P}$ and $\Delta_+(H_P)$ in $\mathcal{M}$. There are two differences.

- In $\Delta_+(M_F)$, we sum only over the vertices of F, whereas in $\Delta_+(H_P)$, we sum over all the vertices. This is precisely the difference between the local and global vertices of an element of Σ given in Table 6.5.
- The poset map b_K can only be applied to faces $\geq K$. This does not create any problem for $\Delta_+(M_F)$; however for $\Delta_+(H_P)$, we first modify P to KP and then apply b_K. Note that $K \leq KP$ for any P.

Lemma 6.5.2 *The coproduct on $\mathcal{M}$ is coassociative. We have*

$$\Delta_+^{(2)}(H_P) = \sum_{H:\ \operatorname{rank} H = 2} H_{P_1} \otimes H_{P_2} \otimes H_{P_3},$$

where for $b_H : \Sigma_H^n \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(HP) = P_1 \times P_2 \times P_3$.

Proof Let the sets Middle and Right be as in Definition 6.4.1. Following the lines of the computation in Lemma 6.5.1,

$$(\mathrm{id} \otimes \Delta_+) \circ \Delta_+(H_P) = \sum_{(K,K') \in \mathrm{Right}} H_{P_1} \otimes H_{P_2} \otimes H_{P_3}, \quad \text{where}$$

$$(\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KP)) = P_1 \times P_2 \times P_3.$$

To see that this is the same as $\Delta_+^{(2)}(H_P)$, one needs to show that

$$b_H(HP) = (\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KP)), \tag{6.24}$$

where $H \mapsto (K, K')$ under the bijection Middle $\to$ Right of Definition 6.4.1. This is true by Proposition 6.4.4.

□

6.5.5 The coalgebra $\mathcal{Q}$

Let $\mathcal{Q} = \underset{n \geq 0}{\oplus} \mathbb{K}(\mathrm{Q}^n)^*$. Write $M_{(F,D)}$ for the basis element corresponding to $(F, D) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.5.3 The coproduct on $\mathcal{Q}$ is given by

$$\Delta(M_{(F,D)}) = 1 \otimes M_{(F,D)} + M_{(F,D)} \otimes 1 + \Delta_+(M_{(F,D)}), \quad \text{where}$$

$$\Delta_+(M_{(F,D)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ K \leq F} M_{(F_1,D_1)} \otimes M_{(F_2,D_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(F) = F_1 \times F_2$ and $b_K(D) = D_1 \times D_2$.

Remark Note that the above coproduct is similar in spirit to the one that we gave for $\mathcal{P}$. Since $K \leq F \leq D$, the poset map b_K can be applied to both F and D. Also, b_K being a poset isomorphism sends a chamber D to a pair of chambers D_1 and D_2.

Example of type A_{n-1}

Recall from Table 6.2 in Section 6.2 that for type A, the set Q^n is the poset of fully nested compositions of the set $[n]$. Let $\mathrm{Q}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathrm{Q}^n)^*$.

Fact 6.5.5 The coproduct on $\mathrm{Q}\Pi$ given by Definition 6.5.3 coincides with the one given by Definition 6.2.18.

Proof This is an easy extension of the proof of Fact 6.5.2.

$\square$

Lemma 6.5.3 *The coproduct on $\mathcal{Q}$ is coassociative. We have*

$$\Delta^{(2)}_+(M_{(F,D)}) = \sum_{H:\ \mathrm{rank}\, H=2,\ H \leq F} M_{(F_1,D_1)} \otimes M_{(F_2,D_2)} \otimes M_{(F_3,D_3)},$$

where for $b_H : \Sigma^n_H \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(F) = F_1 \times F_2 \times F_3$ and $b_H(D) = D_1 \times D_2 \times D_3$.

Proof We want to show that

$$(\mathrm{id} \otimes \Delta_+) \circ \Delta_+(M_{(F,D)}) = \Delta^{(2)}_+(M_{(F,D)}).$$

For that, repeat the steps in the proof of Lemma 6.5.1. The difference is that now we need both

$$b_H(F) = (\mathrm{id} \times b_{K'}) \circ b_K(F) \quad \text{and} \quad b_H(D) = (\mathrm{id} \times b_{K'}) \circ b_K(D)$$

to hold. As before, this follows from the identity $b_H = (\mathrm{id} \times b_{K'}) \circ b_K$ given by Equation (6.20).

$\square$

Proposition 6.5.2 *The map* base : $\mathcal{Q} \to \mathcal{P}$ *given by $M_{(F,D)} \mapsto M_F$ is a morphism of coalgebras.*

Proof Clear from Definitions 6.5.1 and 6.5.3.

$\square$

6.5.6 The coalgebra $\mathcal{N}$

Let $\mathcal{N} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Q}^n$. Write $H_{(P,C)}$ for the basis element corresponding to $(P,C) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0. We would like to define a coproduct on $\mathcal{N}$ using the one on $\mathcal{M}$ just as we defined a coproduct on $\mathcal{Q}$ using the one on $\mathcal{P}$. However, the definition is not the one that would immediately come to mind.

Definition 6.5.4 The coproduct on $\mathcal{N}$ is given by

$$\Delta(H_{(P,C)}) = 1 \otimes H_{(P,C)} + H_{(P,C)} \otimes 1 + \Delta_+(H_{(P,C)}), \quad \text{where}$$

$$\Delta_+(H_{(P,C)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ PK\leq C} H_{(P_1,C_1)} \otimes H_{(P_2,C_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(KP) = P_1 \times P_2$ and $b_K(KC) = C_1 \times C_2$.

Remark Let us compare the coproducts $\Delta_+(M_{(F,D)})$ in $\mathcal{Q}$ and $\Delta_+(H_{(P,C)})$ in $\mathcal{N}$. The comparison is similar to the one we made for $\mathcal{P}$ and $\mathcal{M}$.

- In $\Delta_+(M_{(F,D)})$, we sum over the vertices of F, whereas in $\Delta_+(H_{(P,C)})$, we sum over the vertices which lie in the lunar region $\mathrm{reg}(P,C) = \{N \mid PN \leq C\}$, see Equation (6.21). This is precisely the difference between the local and global vertices of an element of Q given in Table 6.5.

- The poset map b_K does not create any problem for $\Delta_+(M_{(F,D)})$, since $K \leq F \leq D$; however for $\Delta_+(H_{(P,C)})$, we first need to modify P and C to KP and KC respectively and then apply b_K.

Example of type A_{n-1}

Recall from Table 6.2 in Section 6.2 that for type A, the set Q^n is the poset of fully nested compositions of the set $[n]$. Let $\mathrm{N\Pi} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Q}^n$.

Fact 6.5.6 The coproduct on NΠ given by Definition 6.5.4 coincides with the one given by Definition 6.2.19.

Proof This is an extension of the proof of Fact 6.5.4. The key fact is that there is a bijection between the nonvirtual global vertices of a nested set composition (P,C) (see Definition 6.2.16) and the vertices which lie in the lunar region $\mathrm{reg}(P,C) = \{N \mid PN \leq C\}$. It is defined the same way as in the proof of Fact 6.5.4. The rest is simple checking and left to the reader.

□

Lemma 6.5.4 *The coproduct on $\mathcal{N}$ is coassociative. We have*

$$\Delta^{(2)}_+(H_{(P,C)}) = \sum_{H:\ \mathrm{rank}\, H=2,\ PH\leq C} H_{(P_1,C_1)} \otimes H_{(P_2,C_2)} \otimes H_{(P_3,C_3)},$$

where for $b_H : \Sigma^n_H \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(HP) = P_1 \times P_2 \times P_3$ and $b_H(HC) = C_1 \times C_2 \times C_3$.

Proof Let the sets Middle and Right be as in Definition 6.4.1 and $P', C' \in \Sigma^{k_2}$ be as in Proposition 6.4.5. Following the lines of the computation for the coalgebra $\mathcal{M}$ in Lemma 6.5.2, one obtains

$$(\mathrm{id} \otimes \Delta_+) \circ \Delta_+(H_{(P,C)}) = \sum_{\substack{(K,K') \in \text{Right:} \\ PK \leq C,\ P'K' \leq C'}} H_{(P_1,C_1)} \otimes H_{(P_2,C_2)} \otimes H_{(P_3,C_3)},$$

where

$$(\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KP)) = P_1 \times P_2 \times P_3 \quad \text{and}$$
$$(\mathrm{id} \times b_{K'})((\emptyset \times K') b_K(KC)) = C_1 \times C_2 \times C_3.$$

Comparing with the proof of Lemma 6.5.2, the additional fact one needs to show is

$$PH \leq C \iff PK \leq C \quad \text{and} \quad P'K' \leq C'.$$

This is precisely the content of the Corollary to Proposition 6.4.5.

$\square$

Proposition 6.5.3 *The map* $\mathrm{base}^* : \mathcal{M} \to \mathcal{N}$ *given by* $H_P \mapsto \sum_{C:\ P \leq C} H_{(P,C)}$ *is a morphism of coalgebras.*

Proof This is a little more complicated than the proof of Proposition 6.5.2. We outline all the steps without getting into details. For similar ideas, see the proofs of Propositions 6.5.5 and 6.5.6. We need to show that

$$\Delta \circ \mathrm{base}^*(H_P) = (\mathrm{base}^* \otimes \mathrm{base}^*) \circ \Delta(H_P).$$

The LHS is a sum over the set

$$I_C = \{(K, C) \mid \operatorname{rank} K = 1,\ PK \leq C\}$$

and the pair (K, C) yields the term $H_{(P_1,C_1)} \otimes H_{(P_2,C_2)}$, where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(KP) = P_1 \times P_2$ and $b_K(KC) = C_1 \times C_2$.

The RHS is a sum over the set

$$J_C = \{(K, C_1, C_2) \mid \operatorname{rank} K = 1,\ P_1 \leq C_1,\ P_2 \leq C_2\},$$

where P_1 and P_2 are given by $b_K(KP) = P_1 \times P_2$ and the triple (K, C_1, C_2) yields the term $H_{(P_1,C_1)} \otimes H_{(P_2,C_2)}$.

There is an obvious map $I_C \to J_C$ between the index sets, which sends (K, C) to (K, C_1, C_2), where C_1 and C_2 are given by $b_K(KC) = C_1 \times C_2$. One needs to check that this map is well-defined and a bijection. This is a consequence of the bijection between the set of chambers $\mathcal{C}_{PK}$ and $\mathcal{C}_{KP}$ that maps C to KPC with inverse sending D to PKD, see the corollary to Lemma 2.7.7. And clearly, corresponding elements contribute the same term to the coproduct.

$\square$

6.5.7 The coalgebra $\mathcal{S}$

Let $\mathcal{S} = \bigoplus_{n \geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in Σ^n. Write $F_{(C,D)}$ for the basis element $(C, D) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.5.5 Define a coproduct on $\mathcal{S}$ by

$$\Delta_+(F_{(C,D)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ K\leq D} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

Note that the above coproduct combines the ideas used in defining the coproducts for $\mathcal{M}$ and $\mathcal{P}$. In the pair (C, D), the first (resp. second) coordinate is treated like the coordinate in $\mathcal{M}$ (resp. $\mathcal{P}$).

- We again emphasize that the poset map b_K can be applied only to faces containing K. Hence to compute C_1 and C_2 from C in the above formula, we first modify C to KC and then apply b_K. For D, there is no problem since $K \leq D$.

- The set of chambers $\mathcal{C}$ is a left ideal in Σ. Hence if $C \in \mathcal{C}$ then $KC \in \mathcal{C}$.

- The map b_K being a poset isomorphism sends a chamber D to a pair of chambers D_1 and D_2.

Remark Observe that Definition 6.5.5 can be used to define a coproduct on

$$\bigoplus_{n\geq 0} \mathbb{K}(\Sigma^n \times \Sigma^n)^*.$$

This will be taken up in a future work.

Example of type A_{n-1}

Recall from Section 6.3.3 that for type A, the LRB Σ^n is the poset of compositions of the set $[n]$. Let $\mathrm{S}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in Σ^n and can be identified with the set of permutations S_n.

Fact 6.5.7 The coproduct on $\mathrm{S}\Pi$ given by Definition 6.5.5 coincides with the one given by Definition 6.2.33.

Proof Let $D = D^1|\cdots|D^n$ be an element of $\mathcal{C}^n$, that is, a permutation. We begin with Definition 6.5.5. The vertices of D are

$$K = D^1 \cdots D^i | D^{i+1} \cdots D^n, \quad \text{for} \quad 1 \leq i \leq n-1.$$

From Definition 6.3.3, if $b_K(D) = D_1 \times D_2$ then $D_1 = \mathrm{st}(D^1|\cdots|D^i)$ and $D_2 = \mathrm{st}(D^{i+1}|\cdots|D^n)$. This is precisely how the second coordinate works in Definition 6.2.33. The proof so far is identical to the proof of Fact 6.5.2. The first coordinate involves an extra step, which is to compute KC. Note that by Definition 6.3.2,

$$KC = \tilde{C}^1|\cdots|\tilde{C}^i|\tilde{C}^{i+1}|\cdots|\tilde{C}^n,$$

with notation as in Definition 6.2.33. Now repeating the above argument finishes the proof.

□

Example of type $A_1^{\times(n-1)}$

Now we consider the example in Section 6.3.4. Let $S\Gamma = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in $\mathcal{W}^n$, the poset of words of length $n-1$ in the alphabet $\{+,-,0\}$. Then observe that:

Fact 6.5.8 The coproduct on $S\Gamma$ given by Definitions 6.5.5 and 6.3.5 is as follows.

$$\Delta_+(F_{(C^1C^2\ldots C^n,D^1D^2\ldots D^n)}) = \sum_{i=1}^{n} F_{(C^1\ldots C^{i-1},D^1\ldots D^{i-1})} \otimes F_{(C^{i+1}\ldots C^n,D^{i+1}\ldots D^n)}.$$

For example,

$$\Delta_+(F_{+-+,--+}) = F_{\epsilon,\epsilon} \otimes F_{-+,-+} + F_{+,-} \otimes F_{+,+} + F_{+-,--} \otimes F_{\epsilon,\epsilon}.$$

Lemma 6.5.5 *The coproduct on $\mathcal{S}$ is coassociative. We have*

$$\Delta_+^{(2)}(F_{(C,D)}) = \sum_{H:\ \mathrm{rank}\, H=2,\ H\leq D} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)} \otimes F_{(C_3,D_3)},$$

where for $b_H : \Sigma_H^n \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(D) = D_1 \times D_2 \times D_3$ and $b_H(HC) = C_1 \times C_2 \times C_3$.

Proof This is simply putting together the proofs of coassociativity for the coalgebras $\mathcal{P}$ and $\mathcal{M}$. To compute $(\mathrm{id} \otimes \Delta_+) \circ \Delta_+(F_{(C,D)})$, repeat the three steps in Lemma 6.5.1. The indexing sets in the sums remain the same, with D instead of F. The summands are a little different with (C_1, D_1) for F_1 and so on. To now justify the third equality, use Equation (6.22) with D for F and Equation (6.24) with C for F.

□

6.5.8 The coalgebra $\mathcal{R}$

Let $\mathcal{R} = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)$, where $\mathcal{C}^n$ is the set of chambers in Σ^n. Write $K_{(D,C)}$ for the basis element $(D,C) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0. We define a coproduct on $\mathcal{R}$ such that:

Proposition 6.5.4 *The switch map $s : \mathcal{R} \to \mathcal{S}$ which sends $K_{(D,C)} \to F_{(C,D)}$ is an isomorphism of coalgebras.*

Definition 6.5.6 Define a coproduct on $\mathcal{R}$ by

$$\Delta_+(K_{(D,C)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ K\leq D} K_{(D_1,C_1)} \otimes K_{(D_2,C_2)},$$

where for $b_K : \Sigma_K^n \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

Lemma 6.5.6 *The coproduct on $\mathcal{R}$ is coassociative. We have*

$$\Delta_+^{(2)}(K_{(D,C)}) = \sum_{H:\ \mathrm{rank}\, H=2,\ H\leq D} K_{(D_1,C_1)} \otimes K_{(D_2,C_2)} \otimes K_{(D_3,C_3)},$$

where for $b_H : \Sigma_H^n \to \Sigma^{h_1} \times \Sigma^{h_2} \times \Sigma^{h_3}$, we have $b_H(D) = D_1 \times D_2 \times D_3$ and $b_H(HC) = C_1 \times C_2 \times C_3$.

6.5.9 The maps $\mathrm{Road} : \mathcal{S} \to \mathcal{Q}$ and $\Theta : \mathcal{N} \to \mathcal{R}$

So far, we have defined the coalgebras that occur in Theorem 6.5.1, and showed that they are coassociative. We also showed that the maps base*, s and base are morphisms of coalgebras. To complete the proof, we now show that the maps Road and Θ are morphisms of coalgebras.

Proposition 6.5.5 *The map* $\mathrm{Road} : \mathcal{S} \to \mathcal{Q}$ *that sends* $F_{(C,D)}$ *to* $\sum_{F:\ FC=D} M_{(F,D)}$ *is a morphism of coalgebras.*

Proof We need to show that

$$\Delta \circ \mathrm{Road}(F_{(C,D)}) = (\mathrm{Road} \otimes \mathrm{Road}) \circ \Delta(F_{(C,D)}).$$

From the definitions,

$$\Delta \circ \mathrm{Road}(F_{(C,D)}) = \sum_{(K,F)\in I_C} M_{(F_1,D_1)} \otimes M_{(F_2,D_2)},$$

where

$$I_C = \{(K,F) \mid K \le F \le D,\ \mathrm{rank}\, K = 1,\ FC = D\},$$

and where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(F) = F_1 \times F_2$ and $b_K(D) = D_1 \times D_2$.

Similarly,

$$(\mathrm{Road} \otimes \mathrm{Road}) \circ \Delta(F_{(C,D)}) = \sum_{(K,F_1,F_2)\in J_C} M_{(F_1,D_1)} \otimes M_{(F_2,D_2)},$$

where

$$J_C = \{(K,F_1,F_2) \mid K \le D,\ \mathrm{rank}\, K = 1,\ F_1C_1 = D_1,\ F_2C_2 = D_2\},$$

and where the chambers C_i and D_i vary with K and are given by $b_K(D) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

There is an obvious map $I_C \to J_C$ between the index sets, which sends (K,F) to (K,F_1,F_2), with F_1 and F_2 given by $b_K(F) = F_1 \times F_2$. This map is well-defined and a bijection because for $K \le F$, we have

$$FC = D \iff FKC = D \iff b_K(F)b_K(KC) = b_K(D) \iff F_iC_i = D_i.$$

For the first equivalence, we used $F = FK$ for $K \le F$. For the second equivalence, we used the coalgebra axiom (CP) and the fact that b_K is a poset isomorphism.

□

Proposition 6.5.6 *The map* $\Theta : \mathcal{N} \to \mathcal{R}$ *that sends* $H_{(P,C)}$ *to* $\sum_{D:\ PD=C} K_{(D,C)}$ *is a morphism of coalgebras.*

Proof The argument is very similar to the previous proof. Hence we will be brief. We need to show that

$$\Delta \circ \Theta(H_{(P,C)}) = (\Theta \otimes \Theta) \circ \Delta(H_{(P,C)}).$$

The LHS is a sum over the set

$$I_C = \{(K,D) \mid K \le D,\ \mathrm{rank}\, K = 1,\ PD = C\},$$

and the pair (K,D) yields the term $K_{(D_1,C_1)} \otimes K_{(D_2,C_2)}$, where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

The RHS is a sum over the set

$$J_C = \{(K, D_1, D_2) \mid PK \leq C,\ \operatorname{rank} K = 1,\ P_1D_1 = C_1,\ P_2D_2 = C_2\},$$

where P_i and C_i vary with K and are given by $b_K(KP) = P_1 \times P_2$ and $b_K(KC) = C_1 \times C_2$. The triple (K, D_1, D_2) yields the term $K_{(D_1,C_1)} \otimes K_{(D_2,C_2)}$.

As before, there is an obvious map $I_C \to J_C$ between the index sets, which sends (K, D) to (K, D_1, D_2) given by $b_K(D) = D_1 \times D_2$. This map is well-defined and a bijection because for $K \leq D$, we have

$$\begin{array}{rcll} PD = C & \iff & PK \leq C \text{ and } KPD = KC & \text{(Lemma (2.7.8))} \\ & \iff & PK \leq C \text{ and } b_K(KP)b_K(D) = b_K(KC) & \text{(Axiom } (CP)\text{)} \\ & \iff & PK \leq C \text{ and } P_iD_i = C_i. & \end{array}$$

□

6.5.10 The coalgebras $A_{\mathcal{Z}}$, $A_{\mathcal{L}}$, $A_{\mathcal{Z}^*}$ and $A_{\mathcal{L}^*}$

Thus far, we have proved Theorem 6.5.1. In the rest of this section, we prove the remaining part of Theorem 6.1.1. It involves the maps supp and lune and their duals. For that we recall the content of Lemmas 2.4.1 and 2.4.2.

For a vertex $K \in \Sigma^n$ with type (k_1, k_2), one has the following two commutative diagrams.

$$\begin{tikzcd} \Sigma^n_K \arrow[r, hook] \arrow[d, "\mathrm{supp}^n_K"] & \Sigma^n \arrow[r, two heads, "K\cdot"] \arrow[d, "\mathrm{supp}^n"] & \Sigma^n_K \arrow[d, "\mathrm{supp}^n_K"] \\ \mathrm{L}^n_K \arrow[r, hook] & \mathrm{L}^n \arrow[r, two heads, "K\cdot"'] & \mathrm{L}^n_K \end{tikzcd} \qquad \begin{tikzcd} \Sigma^n_K \arrow[r, "b_K", "\cong"'] \arrow[d, "\mathrm{supp}^n_K"'] & \Sigma^{k_1} \times \Sigma^{k_2} \arrow[d, "\mathrm{supp}^{k_1} \times \mathrm{supp}^{k_2}"] \\ \mathrm{L}^n_K \arrow[r, "\cong", "b_K"'] & \mathrm{L}^{k_1} \times \mathrm{L}^{k_2} \end{tikzcd} \tag{6.25}$$

where

$$\mathrm{L}^n_K = \{X \in \mathrm{L}^n \mid \mathrm{supp}^n(K) \leq X\}$$

is the poset of flats of Σ^n_K, and supp^n_K, supp^{k_1} and supp^{k_2} are the support maps of Σ^n_K, Σ^{k_1} and Σ^{k_2} respectively. The map $K\cdot : \Sigma^n \to \Sigma^n_K$ sends F to KF. One also has the same two diagrams above with Σ and L replaced by Q and Z respectively, and the map supp replaced by the map lune.

Definition 6.5.7 Define a coproduct on $A_{\mathcal{L}^*}$ and $A_{\mathcal{Z}^*}$ by

$$\begin{aligned} \Delta_+(m_X) &= \sum_{K:\ \operatorname{rank} K=1,\ X \in \mathrm{L}^n_K} m_{X_1} \otimes m_{X_2}, \\ \Delta_+(m_L) &= \sum_{K:\ \operatorname{rank} K=1,\ L \in \mathrm{Z}^n_K} m_{L_1} \otimes m_{L_2}, \end{aligned}$$

where for $b_K : \mathrm{L}^n_K \to \mathrm{L}^{k_1} \times \mathrm{L}^{k_2}$, we have $b_K(X) = X_1 \times X_2$ and for $b_K : \mathrm{Z}^n_K \to \mathrm{Z}^{k_1} \times \mathrm{Z}^{k_2}$, we have $b_K(L) = L_1 \times L_2$.

Definition 6.5.8 Define a coproduct on $A_{\mathcal{L}}$ and $A_{\mathcal{Z}}$ by

$$\begin{aligned} \Delta_+(h_X) &= \sum_{K:\ \operatorname{rank} K=1} h_{X_1} \otimes h_{X_2}, \\ \Delta_+(h_L) &= \sum_{K:\ \operatorname{rank} K=1,\ K \in \mathrm{zone}(L)} h_{L_1} \otimes h_{L_2}, \end{aligned}$$

where for $b_K : \mathrm{L}^n_K \to \mathrm{L}^{k_1} \times \mathrm{L}^{k_2}$, we have $b_K(K \cdot X) = X_1 \times X_2$ and for $b_K : \mathrm{Z}^n_K \to \mathrm{Z}^{k_1} \times \mathrm{Z}^{k_2}$, we have $b_K(K \cdot L) = L_1 \times L_2$.

The set $\mathrm{zone}(L)$ is discussed in Section 2.3.4. If $\mathrm{lune}(P, C) = L$ then

$$K \in \mathrm{zone}(L) \iff K \in \mathrm{reg}(P, C) \iff PK \leq C.$$

A comparison with Definition 6.5.4 should make the presence of the condition $K \in \mathrm{zone}(L)$ clear.

Remark For the coproducts in $A_{\mathcal{L}^*}$ and $A_{\mathcal{L}}$, we sum over the local and global vertices respectively of an element of L, as defined in Table 6.6. The same statement holds for the coproducts in $A_{\mathcal{Z}^*}$ and $A_{\mathcal{Z}}$.

Remark For the example of type A_{n-1}, one can make explicit the maps $K\cdot$ in the diagram (6.25) above, as also the break maps $b_K : \mathrm{L}^n_K \to \mathrm{L}^{k_1} \times \mathrm{L}^{k_2}$ and $b_K : \mathrm{Z}^n_K \to \mathrm{Z}^{k_1} \times \mathrm{Z}^{k_2}$. It is then easy to check that the above formulas reduce to the definitions of the corresponding coalgebras in Section 6.2.

Proposition 6.5.7 *Let $\{\Sigma^n\}$ be a family of LRBs, and let $\mathcal{P}$, $\mathcal{M}$, $\mathcal{Q}$ and $\mathcal{N}$ be the coalgebras as given in Definitions 6.5.1-6.5.4.*

(1) The injective maps $\mathrm{supp}^* : A_{\mathcal{L}^*} \to \mathcal{P}$ *that sends* m_X *to* $\sum_{F:\ \mathrm{supp}\, F = X} M_F$ *and* $\mathrm{lune}^* : A_{\mathcal{Z}^*} \to \mathcal{Q}$ *that sends* m_L *to* $\sum_{(F,D):\ \mathrm{lune}(F,D) = L} M_{(F,D)}$ *are maps of coalgebras.*

(2) The surjective maps $\mathrm{supp} : \mathcal{M} \twoheadrightarrow A_{\mathcal{L}}$ *that sends* H_P *to* $h_{\mathrm{supp}\, P}$ *and* $\mathrm{lune} : \mathcal{N} \twoheadrightarrow A_{\mathcal{Z}}$ *that sends* $H_{(P,C)}$ *to* $h_{\mathrm{lune}(P,C)}$ *are maps of coalgebras.*

Proof We only give the proof for the supp and supp* maps.

(1) The following computation shows that the map $\mathrm{supp}^* : A_{\mathcal{L}^*} \twoheadrightarrow \mathcal{P}$ is a map of coalgebras. The term $\Delta_+(\mathrm{supp}^*(m_X))$ equals

$$\begin{aligned}
&= \Delta_+\left(\sum_{F:\ \mathrm{supp}\, F = X} M_F\right) \\
&= \sum_{F:\ \mathrm{supp}\, F = X} \ \sum_{K: \mathrm{rank}\, K = 1,\ K \leq F} M_{F_1} \otimes M_{F_2} && (b_K(F) = F_1 \times F_2) \\
&= \sum_{K:\ \mathrm{rank}\, K = 1,\ \mathrm{supp}\, K \leq X} \ \sum_{F:\ K \leq F,\ \mathrm{supp}\, F = X} M_{F_1} \otimes M_{F_2} && \text{(Switching sums)} \\
&= \sum_{K:\ \mathrm{rank}\, K = 1,\ X \in \mathrm{L}_K} \ \sum_{F_i:\ \mathrm{supp}\, F_i = X_i} M_{F_1} \otimes M_{F_2}, && \text{(Diagram (6.25))}
\end{aligned}$$

which is equal to $(\mathrm{supp}^* \otimes \mathrm{supp}^*)(\Delta_+(m_X))$.

(2) We now deal with the map $\mathrm{supp} : \mathcal{M} \twoheadrightarrow A_{\mathcal{L}}$. Let $P \in \Sigma^n$ with $\mathrm{supp}^n(P) = X$. Using Definition 6.5.2,

$$(\mathrm{supp} \otimes \mathrm{supp})(\Delta_+(H_P)) = \sum_{K:\ \mathrm{rank}\, K = 1} h_{X_1} \otimes h_{X_2},$$

where $X_i = \mathrm{supp}(P_i)$ for $b_K(KP) = P_1 \times P_2$. Now from diagram (6.25), we have

$$b_K(K \cdot X) = b_K(K \cdot \mathrm{supp}\, P) = b_K(\mathrm{supp}(KP)) = (\mathrm{supp} \times \mathrm{supp})(b_K(KP)).$$

This shows that the right hand side above equals $\Delta_+(h_X)$, thus completing the proof. $\square$

6.6 The algebra axioms and examples

In the rest of this chapter, we deal with the algebra analogues of Sections 6.3, 6.4 and 6.5. In this section, we give the algebra axioms for a family of posets (Section 6.6.1). The motivation is that one can construct the algebras in diagram (6.3) starting with a family of posets that satisfies these axioms. In Sections 6.6.2-6.6.4, we revisit our earlier examples (Sections 6.3.2-6.3.4), and show that they satisfy the algebra axioms.

6.6.1 The algebra axioms

Consider the family $\{\Sigma^n\}_{n\geq 0}$, where Σ^n is a finite graded poset of rank $n-1$ with a unique minimum element that we denote $\emptyset$. Further let Σ^0 and Σ^1 be singleton sets with the unique element $\emptyset$. For K a face of Σ, let

$$\Sigma^n_K = \{F \in \Sigma^n \mid K \leq F\}.$$

Let $\mathcal{C}^n$ be the set of chambers (maximal elements) in Σ^n and $\mathcal{C}^n_F = \{D \in \mathcal{C}^n \mid F \leq D\}$. Let $\deg K = \operatorname{rank} K + 1$, where rank denotes the rank of an element.

We give two algebra axioms for such a family of posets.

Axiom $(A1)$**.** For every composition $(f_1, f_2, \ldots, f_k)$ of n, there exists a face F of Σ^n of degree k, and a poset isomorphism

$$j_F : \Sigma^{f_1} \times \Sigma^{f_2} \times \ldots \times \Sigma^{f_k} \to \Sigma^n_F. \qquad (6.26)$$

Further, distinct compositions give distinct faces; so it is unambiguous to use the notation j_F for the poset map. If the composition is (n) then $F = \emptyset$ and the poset isomorphism $j_\emptyset$ is the identity map $\mathrm{id} : \Sigma^n \to \Sigma^n$.

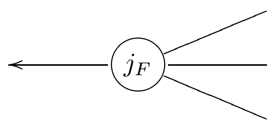


Figure 6.5: The join map j_F.

Axiom $(A2)$**.** The maps j_F in axiom $(A1)$ are "associative" in the following sense.

Let $(g_1, g_2, \ldots, g_m)$ be a composition of n and for $1 \leq i \leq m$, let $(f_{i1}, f_{i2}, \ldots, f_{ik_i})$ be a composition of g_i. First apply the axiom $(A1)$ to the composition of g_i to get a face F_i of Σ^n of degree k_i, and a poset isomorphism

$$j_{F_i} : \Sigma^{f_{i1}} \times \Sigma^{f_{i2}} \times \ldots \times \Sigma^{f_{ik_i}} \to \Sigma^{g_i}_{F_i} \quad \text{for} \quad 1 \leq i \leq m.$$

Next apply the axiom $(A1)$ to the composition $(g_1, g_2, \ldots, g_m)$ of n to get a face G of degree m and a poset isomorphism

$$j_G : \Sigma^{g_1} \times \Sigma^{g_2} \times \ldots \times \Sigma^{g_m} \to \Sigma^n_G.$$

Under this map, let $F_1 \times F_2 \times \ldots \times F_m \mapsto F'$ (say). This induces an isomorphism

$$j_G : \Sigma^{g_1}_{F_1} \times \Sigma^{g_2}_{F_2} \times \ldots \times \Sigma^{g_m}_{F_m} \to \Sigma^n_{F'} \quad \text{with } G \leq F'.$$

Now consider the composition of n

$$(f_{11}, f_{12}, \ldots, f_{1k_1}, f_{21}, \ldots, f_{2k_2}, \ldots, f_{m1}, \ldots, f_{mk_m}),$$

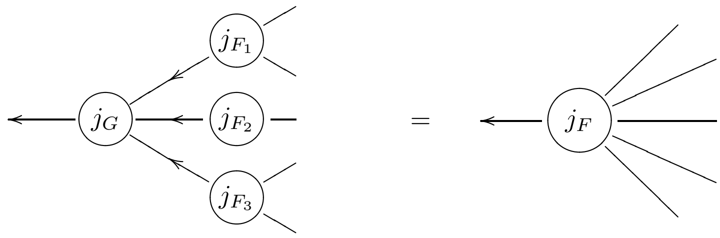


Figure 6.6: The join map is associative.

which is the concatenation of the m compositions that we started with. Apply the axiom $(A1)$ to it to get a face $F \in \Sigma^n$. Then,

$$j_F = j_G \circ (j_{F_1} \times j_{F_2} \times \ldots \times j_{F_m}). \tag{6.27}$$

In particular, we require that $F = F'$.

Remark Figures 6.5 and 6.6 give schematic pictures for the axioms $(A1)$ and $(A2)$. They are obtained by reversing the arrows in Figures 6.1 and 6.2. Just as we used the letter b for "break", the letter j stands for "join".

In addition to the above two algebra axioms, we define a projection axiom. For that assume, in addition, that each Σ^n is a projection poset, as defined in Section 2.7. This includes the case when Σ^n is a LRB.

Axiom (AP)**.** The poset maps in the axiom $(A1)$ respect the product structure of Σ^n, that is,

$$j_F^{-1}(HN) = j_F^{-1}(H) j_F^{-1}(N),$$

for $F \leq H, N \in \Sigma^n$.

Proposition 6.6.1 *Let $\{\Sigma^n\}_{n \geq 0}$ satisfy the algebra axioms $(A1)$ and $(A2)$. Then for each $n \geq 0$, there are injective poset maps*

$$\overline{\mathrm{Q}}^n \to \Sigma^n,$$

where $\overline{\mathrm{Q}}^n$ is the poset of compositions of n, defined in Section 6.3.2.

Proof For a given n, the above map sends the composition $(f_1, f_2, \ldots, f_k)$ of n to the face $F \in \Sigma^n$ as given by the axiom $(A1)$. The axiom $(A2)$ implies that this is a poset map.

□

Remark In our examples, the map in Proposition 6.6.1 is a section to the map in Proposition 6.3.1. We refer to the elements in its image as the *fundamental faces* of Σ^n. The terminology is motivated by the theory of Coxeter groups (Proposition 6.6.2).

6.6.2 The warm-up example of compositions

Let $\Sigma^n = \overline{\mathrm{Q}}^n$ be the poset of compositions of n, see the beginning of Section 6.3.2 for more details.

Lemma 6.6.1 *The family of LRBs $\{\overline{\mathrm{Q}}^n\}_{n \geq 0}$ satisfies all algebra axioms $(A1) - (AP)$.*

Proof The map in Proposition 6.6.1, in this case, is an isomorphism and the inverse to the map in Proposition 6.3.1. The definition of the join map is as below.

Definition 6.6.1 For $(\alpha_1, \ldots, \alpha_k) = \alpha$, a composition of n and of degree k, we define a poset isomorphism

$$j_\alpha : \overline{\mathrm{Q}}^{\alpha_1} \times \overline{\mathrm{Q}}^{\alpha_2} \times \ldots \times \overline{\mathrm{Q}}^{\alpha_k} \to \overline{\mathrm{Q}}^n_\alpha$$

as follows.

Let β_i be a composition of α_i for $1 \leq i \leq k$. Then the above map sends $\beta_1 \times \beta_2 \times \ldots \times \beta_k$ to $(\beta_1, \beta_2, \ldots, \beta_k)$. For example, for $n = 9$ and $(\alpha_1, \alpha_2, \alpha_3) = (2, 4, 3)$, the map $\overline{\mathrm{Q}}^2 \times \overline{\mathrm{Q}}^4 \times \overline{\mathrm{Q}}^3 \to \overline{\mathrm{Q}}^9_\alpha$ sends $(2) \times (1, 3) \times (2, 1)$ to $\beta = (2, 1, 3, 2, 1)$.

With this definition, the reader may check the axioms $(A2)$ and (AP) directly.

$\square$

6.6.3 The motivating example of type A_{n-1}

The motivation for the algebra axioms comes from the theory of Coxeter groups. In Section 6.3.3, we recalled some of their key properties, see Facts 6.3.1-6.3.3.

Geometry

Recall from Section 6.3.3 that for type A_{n-1}, the Coxeter group W is S_n, the symmetric group on n letters and $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the transposition that interchanges i and $i+1$. Let Σ^n be the Coxeter complex of S_n and fix a fundamental chamber $C_0^n \in \Sigma^n$. Identify $\overline{\mathrm{Q}}^n$, the poset of compositions of n under refinement, with the poset of subsets of S under inclusion in the usual way.

Lemma 6.6.2 *The family of LRBs $\{\Sigma^n\}_{n\geq 0}$ satisfies all algebra axioms $(A1) - (AP)$.*

Proof This lemma is proved in the same way as Lemma 6.3.2, shedding light on the geometry that underlies the axioms.

Axiom $(A1)$**.** Let $(f_1, f_2, \ldots, f_k)$ be a composition of n, or equivalently, a subset $T \leq S$. Associate to this subset, the face F of C_0^n of type T. Then F has degree k and there is a unique poset isomorphism

$$j_F : \Sigma^{f_1} \times \Sigma^{f_2} \times \ldots \times \Sigma^{f_{\deg F}} \xrightarrow{\cong} \Sigma^n_F, \tag{6.28}$$

such that

$$C_0^{f_1} \times C_0^{f_2} \times \ldots \times C_0^{f_{\deg F}} \mapsto C_0^n. \tag{6.29}$$

For more details, see the explanation for the poset isomorphism in (6.14). In this case, Proposition 6.6.1 specializes to the following.

Proposition 6.6.2 *The poset map $\overline{\mathrm{Q}}^n \to \Sigma^n$ sends $T \leq S$ to the face of the fundamental chamber C_0^n of type T.*

Note that the above map is injective and a section to the type map $\Sigma^n \twoheadrightarrow \overline{\mathrm{Q}}^n$ given by Proposition 6.3.2.

Axiom (AP)**.** With j_F as defined in (6.28), it follows from Fact 6.3.3 that $\{\Sigma^n\}_{n\geq 0}$ satisfies the projection axiom (AP).

Axiom $(A2)$**.** We now show that $\{\Sigma^n\}_{n\geq 0}$ satisfies the axiom $(A2)$. In other words, we show that Equation (6.27) holds. Let G, F, F' and F_i for $1 \leq i \leq m$ with $m = \deg G$ be as defined in the axiom $(A2)$. Then the type of both F and F' is given by

$$\operatorname{type}(G) \bigsqcup \left(\bigsqcup_{j=1}^{m} \operatorname{type}(F_j) \right),$$

and they are both faces of C_0^n; hence $F = F'$. It is instructive to look at the derivation of Equation (6.16) in this context. Hence both sides of Equation (6.27) specify an isomorphism

$$\Sigma^{f_{11}} \times \Sigma^{f_{12}} \times \dots \Sigma^{f_{1\deg F_1}} \times \Sigma^{f_{21}} \times \dots \times \Sigma^{f_{m\deg F_m}} \to \Sigma^n_F.$$

And further they specify the same isomorphism because

$$C_0^{f_{11}} \times C_0^{f_{12}} \times \dots C_0^{f_{1\deg F_1}} \times C_0^{f_{21}} \times \dots \times C_0^{f_{m\deg F_m}} \mapsto C_0^n$$

in both cases. This proves the axiom $(A2)$.

□

Combinatorics

Recall from Section 6.3.3 that the Coxeter complex Σ^n can be identified with $\mathcal{B}^n$, the poset of compositions of $[n]$.

Lemma 6.6.3 *The family of LRBs $\{\mathcal{B}^n\}_{n\geq 0}$ satisfies all algebra axioms $(A1) - (AP)$.*

This is a restatement of Lemma 6.6.2. For the fixed chamber C_0^n in Σ^n, we take the set composition $1|2|\dots|n$. The translation from geometry to combinatorics yields the following.

Definition 6.6.2 For a composition $(f_1, \dots, f_k)$ of n, let $F = F^1|\dots|F^k$ be the composition of the set $[n]$, where $F^1 = 12\dots f_1$, $F^2 = f_1 + 1 \dots f_1 + f_2$, and so on. Then define a poset isomorphism

$$j_F : \mathcal{B}^{f_1} \times \mathcal{B}^{f_2} \times \dots \times \mathcal{B}^{f_k} \to \mathcal{B}^n_F$$

as follows.

Let $F_i \in \mathcal{B}^{f_i}$ be a composition of the set $[f_i]$. Then the image of $F_1 \times F_2 \times \dots \times F_k$ is obtained by shifting up the indices of F_2 by f_1, F_3 by $f_1 + f_2$, and so on and then placing them next to one another. The case $k = 2$ was also explained in Definition 6.2.8. For example, for the composition $(2, 4, 3)$ of 9, we have $F = 12|3456|789$, and the map $j_F : \mathcal{B}^2 \times \mathcal{B}^4 \times \mathcal{B}^3 \to \mathcal{B}^9_F$ sends $12 \times 3|2|14 \times 3|12$ to the set composition $12|5|4|36|9|78$.

The combinatorial content of Proposition 6.6.2 is the following. It is already present in Definition 6.6.2.

Proposition 6.6.3 *The poset map $\overline{\mathrm{Q}}^n \to B^n$ sends a composition $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of n to the set composition*

$$12\dots\alpha_1|\alpha_1 + 1 \dots \alpha_1 + \alpha_2|\dots|\dots n.$$

The reader may also check the axioms $(A2)$ and (AP) directly using Definitions 6.3.2 and 6.6.2.

6.6.4 The example of type $A_1^{\times(n-1)}$

Recall from Section 6.3.4 that for type $A_1^{\times(n-1)}$, the Coxeter group is $\mathbb{Z}_2^{n-1}$ and the generating set $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the generator in the ith coordinate. Let Σ^n be the Coxeter complex of $\mathbb{Z}_2^{n-1}$.

Geometry

Lemma 6.6.4 *The family of LRBs $\{\Sigma^n\}_{n\geq 0}$ satisfies all algebra axioms $(A1) - (AP)$.*

Proof Follow the proof of Lemma 6.6.2. The point of similarity between the examples of type A_{n-1} and $A_1^{\times(n-1)}$ was explained in the proof of Lemma 6.3.4.

□

Combinatorics

The Coxeter complex Σ^n of type $A_1^{\times(n-1)}$ can be identified with $\mathcal{W}^n$, the poset of words of length $n-1$ in the alphabet $\{+, -, 0\}$. For more details, see the beginning of Section 6.3.4.

Definition 6.6.3 For a composition $(f_1, \ldots, f_k)$ of n, let

$$F = \underbrace{0\ldots0}_{f_1-1}+\underbrace{0\ldots0}_{f_2-1}+\ldots+\underbrace{0\ldots0}_{f_k-1} \quad \in \quad \Sigma^n$$

be the word of length $n-1$. Define a poset isomorphism

$$j_F : \Sigma^{f_1} \times \Sigma^{f_2} \times \ldots \times \Sigma^{f_k} \to \Sigma^n_F,$$

which sends $F_1 \times F_2 \times \ldots \times F_k$ to the word $F_1 + F_2 + \ldots + F_k$. For example, for the composition $(1, 3, 4, 1, 1)$ of 10, we have $F = +00 + 000 + +$ and the map

$$j_F : \Sigma^1 \times \Sigma^3 \times \Sigma^4 \times \Sigma^1 \times \Sigma^1 \to \Sigma^{10}_F$$

sends $\epsilon \times (+0) \times (0 + 0) \times \epsilon \times \epsilon$ to the word $+ + 0 + 0 + 0 + +$.

The reader may also check the axioms $(A2)$ and (AP) directly using Definitions 6.3.4 and 6.6.3.

6.7 From algebra axioms to algebras

In this section, we explain how the algebra axioms allow us to define associative products on the vector spaces in diagram (6.3). The formal algebra constructions are given in the next section.

6.7.1 The products

Let $\{\Sigma^n\}_{n\geq 0}$ be a family of projection posets that satisfies the algebra axioms. Then by the axiom $(A1)$, for every composition (g_1, g_2) of n, there is a poset inclusion

$$j_G : \Sigma^{g_1} \times \Sigma^{g_2} \hookrightarrow \Sigma^n. \tag{6.30}$$

Further, the image of this map is Σ^n_G for some rank 1 face $G \in \Sigma^n$. Analogous to Section 6.4.1, using this map, one can define products on the graded vector spaces in diagram (6.3).

6.7.2 Associativity of the products

We first formulate a special case of the axiom $(A2)$, which we state as a proposition.

Proposition 6.7.1 *Let $\{\Sigma^n\}$ satisfy the algebra axioms $(A1)$ and $(A2)$. For the composition (h_1, h_2, h_3) of n, let H be the rank 2 fundamental face of Σ^n and j_H, the poset map given by the axiom $(A1)$. Similarly, let G, K, G', K' be the fundamental faces corresponding to the compositions $(h_1, h_2 + h_3)$, $(h_1 + h_2, h_3)$, (h_2, h_3) and (h_1, h_2) respectively. Note that they have rank 1 in their respective posets and G and K are faces of H. Let j_G, j_K, $j_{G'}$, $j_{K'}$ denote the respective poset maps. Then*

$$j_H = j_G \circ (\mathrm{id} \times j_{G'}) \quad \textit{and} \quad j_H = j_K \circ (j_{K'} \times \mathrm{id}). \tag{6.31}$$

In particular, we have

$$H = j_G(\emptyset \times G') \quad \textit{and} \quad H = j_K(K' \times \emptyset). \tag{6.32}$$

No matter which algebra we are considering, the following simple principle controls associativity.

Proposition 6.7.2 *Let $F_i \in \Sigma^{h_i}$ for $1 \leq i \leq 3$. With the notation as above,*

$$j_G(F_1 \times j_{G'}(F_2 \times F_3)) = j_K(j_{K'}(F_1 \times F_2) \times F_3) \quad \in \quad \Sigma^{h_1+h_2+h_3}.$$

Proof From Equation (6.31), both the sides are equal to $j_H(F_1 \times F_2 \times F_3)$.

□

6.7.3 Useful results for associativity

In Section 6.7.2, we explained the basic principle for associativity. However, depending on the algebra at hand, one requires more specialized results, which we now give. To avoid repetition, we omit K and K' from the discussion.

Proposition 6.7.3 *Let H, G and G' be the fundamental faces for the compositions (h_1, h_2, h_3), $(h_1, h_2 + h_3)$ and (h_2, h_3) respectively. Then we have*

$$j_H^{-1}(HF) = (\mathrm{id} \times j_{G'}^{-1})((\emptyset \times G')j_G^{-1}(GF)),$$

for any face F.

Proof By definition, we have $G \leq H$. The proposition now follows from the following sequence of equalities.

$$\begin{aligned}
j_H^{-1}(HF) &= (\mathrm{id} \times j_{G'}^{-1}) \circ j_G^{-1}(HF) && \text{(Equation (6.31))} \\
&= (\mathrm{id} \times j_{G'}^{-1}) \circ j_G^{-1}(HGF) && (HG = H) \\
&= (\mathrm{id} \times j_{G'}^{-1})(j_G^{-1}(H) j_G^{-1}(GF)) && \text{(Algebra axiom } (AP)) \\
&= (\mathrm{id} \times j_{G'}^{-1})((\emptyset \times G') j_G^{-1}(GF)). && \text{(Equation (6.32))}
\end{aligned}$$

□

Proposition 6.7.4 *Let H, G and G' be the fundamental faces for the compositions (h_1, h_2, h_3), $(h_1, h_2 + h_3)$ and (h_2, h_3) respectively. Let $GF = F_1 \times F'$ and $D_i \in \Sigma^{h_i}$ with $F_1 \leq D_1$. Then, we have*

$$F j_G(D_1 \times F' j_{G'}(D_2 \times D_3)) = F j_H(D_1 \times D_2 \times D_3).$$

Proof This follows from the sequence of equalities below. The left hand side is equal to

$$\begin{aligned}
&= Fj_G(F_1 \times F')j_G(D_1 \times j_{G'}(D_2 \times D_3)) && (F_1 \leq D_1 \text{ and axiom}(AP))\\
&= FGFj_H(D_1 \times D_2 \times D_3) && (GF = j_G(F_1 \times F'))\\
&= FGj_H(D_1 \times D_2 \times D_3) && (FGF = FG)\\
&= Fj_H(D_1 \times D_2 \times D_3), && (G \leq j_H(D_1 \times D_2 \times D_3))
\end{aligned}$$

which equals the right hand side.

□

6.8 Construction of algebras

This section is the algebra analogue of Section 6.5 and the goal is to prove Theorem 6.1.2. In particular, starting with a family $\{\Sigma^n\}_{n\geq 0}$ of LRBs that satisfies the algebra axioms defined in Section 6.6, we construct many graded algebras. Throughout this section, we use the notation and definitions in Section 6.1.2. We first show the following.

Theorem 6.8.1 *Let $\{\Sigma^n\}_{n\geq 0}$ be a family of projection posets, that satisfies all algebra axioms $(A1)-(AP)$. Then*

$$\mathcal{M} \xrightarrow{\text{base}^*} \mathcal{N} \xrightarrow{\Theta} \mathcal{R} \xrightarrow{s} \mathcal{S} \xrightarrow{\text{Road}} \mathcal{Q} \xrightarrow{\text{base}} \mathcal{P}$$

is a diagram of algebras.

Proof There are two parts to this theorem. Firstly, we define products on the objects (see Definitions 6.8.1-6.8.6) and show that they are associative (see Lemmas 6.8.1-6.8.6). Secondly, we show that the maps are morphisms of algebras (see Propositions 6.8.2-6.8.6).

□

We restate Theorem 6.1.2 here for convenience.

Theorem *Let $\{\Sigma^n\}_{n\geq 0}$ be a family of LRBs, that satisfies all algebra axioms $(A1)-(AP)$. Then diagram (6.3) is a diagram of algebras.*

Proof A LRB is a special case of a projection poset. Hence, in view of the above theorem, one only needs to show that the surjective supp and lune maps induce products on $A_{\mathcal{L}}$ and $A_{\mathcal{Z}}$ and similarly the injective supp* and lune* maps induce products on $A_{\mathcal{L}^*}$ and $A_{\mathcal{Z}^*}$. This is the content of Definitions 6.8.7, 6.8.8 and Proposition 6.8.7.

□

6.8.1 Examples

In Sections 6.6.2-6.6.4, we gave three examples that satisfy the algebra axioms. So the algebra constructions in this section can be applied to them. We mainly concentrate on the example of type A.

Proposition 6.8.1 *For the example of type A, the algebras $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{M}$ and $A_{\mathcal{Z}}$, $A_{\mathcal{L}}$, $A_{\mathcal{Z}^*}$ and $A_{\mathcal{L}^*}$ as defined in this section respectively give the algebras* $\mathrm{P\Pi}$, $\mathrm{Q\Pi}$, $\mathrm{S\Pi}$, $\mathrm{R\Pi}$, $\mathrm{N\Pi}$ *and* $\mathrm{M\Pi}$ *and* Π_{Z}, Π_{L}, Π_{Z^*} *and* Π_{L^*} *as defined in Section 6.2.*

Proof The definitions of the algebras $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{M}$ involve two ingredients: the product of the projection poset Σ^n and the poset isomorphism in the axiom $(A1)$. For the example of type A, explicit descriptions of these two were given in Definitions 6.3.2 and 6.6.2. Hence the proof is a matter of unwinding definitions, see Facts 6.8.2, 6.8.4, 6.8.5, 6.8.6 and 6.8.7.

The part of the proposition dealing with the algebras $A_{\_}$ and $\Pi_{\_}$ is left as an exercise to the reader, also see the remark before Proposition 6.8.7.

□

Corollary 6.8.1 *Diagram* (6.4) *is a diagram of algebras.*

6.8.2 The algebra $\mathcal{P}$

Let $\mathcal{P} = \bigoplus_{n\geq 0} \mathbb{K}(\Sigma^n)^*$. Write M_F for the basis element corresponding to $F \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.8.1 The product on $\mathcal{P}$ is given by

$$M_{F_1} * M_{F_2} = \sum_{F:\ GF = j_G(F_1 \times F_2)} M_F.$$

The vertex $G \in \Sigma^n$ is fixed in the above sum. It is the fundamental vertex given by the axiom $(A1)$ for the composition (g_1, g_2) of n, where $F_i \in \Sigma^{g_i}$, also see (6.30). We say that a F as above is a quasi-shuffle of F_1 and F_2, see (6.33) below.

We now explain how this definition works in the examples (Sections 6.6.2-6.6.4).

Example of compositions

Recall from Section 6.6.2 that $\overline{\mathrm{Q}}^n$ is the poset of compositions of n. Let $P\Delta = \bigoplus_{n\geq 0} \mathbb{K}(\overline{\mathrm{Q}}^n)^*$.

Fact 6.8.1 The product on $P\Delta$ given by Definition 6.8.1 is as follows.

$$M_{(\alpha_1,\alpha_2,\ldots,\alpha_i)} * M_{(\beta_1,\beta_2,\ldots,\beta_j)} = M_{(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_j)} + M_{(\alpha_1,\ldots,\alpha_k+\beta_1,\ldots,\beta_j)}.$$

We point out that, in contrast to the coproduct on $P\Delta$, this is not how the product is defined on the M basis of the Hopf algebra $\mathrm{Q}\Lambda$ of quasi-symmetric functions (Section 3.2.2). Rather this is how the product is defined on the K basis of the Hopf algebra $\mathrm{N}\Lambda$ (Section 3.2.4). We also remark that the product and coproduct we have obtained on $P\Delta$ from the warm-up example of compositions are not compatible.

Example of type A_{n-1}

Recall from Section 6.6.3 that for type A, the LRB Σ^n is the poset of compositions $\mathcal{B}^n$ of the set $[n]$. Let $\mathrm{P}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{B}^n)^*$.

Fact 6.8.2 The product on $\mathrm{P}\Pi$ given by Definition 6.8.1 coincides with the one given by Definition 6.2.10.

Proof With the notations as in Definitions 6.8.1 and 6.2.10, we want to show that

$$\{F \mid GF = j_G(F_1 \times F_2)\} = \{F \mid F \text{ is a quasi-shuffle of } F_1 \text{ and } F_2\}. \tag{6.33}$$

One can say that the left hand set is the geometric meaning of a quasi-shuffle. We illustrate this with an example.

Let $F_1 = 23|14|5$ and $F_2 = 3|124$. Then in the notation of Definition 6.8.1, we have $(g_1, g_2) = (5, 4)$, the vertex $G = 12345|6789 \in \mathcal{B}^9$ and $j_G(F_1 \times F_2) = 23|14|5|8|679$ by Definition 6.6.2. Now let $F \in \mathcal{B}^9$ be such that $GF = j_G(F_1 \times F_2)$. Then by the definition of the product (see Definition 6.3.2), this occurs precisely if 23, 14 and 5 (resp. 8 and 679) occur in different blocks of F and the blocks are in the order $23 < 14 < 5$ (resp. $8 < 679$). An example of this is $F = 238|14|5679$. This is precisely a quasi-shuffle of F_1 and F_2 as given in Definition 6.2.6.

$\square$

Remark It is an easy exercise to see that

$$GF = j_G(F_1 \times F_2),\ FG = F \iff F \text{ is a shuffle of } F_1 \text{ and } F_2.$$

This gives the geometric meaning of a shuffle.

Example of type $A_1^{\times(n-1)}$

Now we consider the example in Section 6.6.4. Let $P\Gamma = \underset{n\geq 0}{\oplus} \mathbb{K}(\mathcal{W}^n)^*$, where $\mathcal{W}^n$ is the poset of words of length $n-1$ in the alphabet $\{+,-,0\}$. Then observe that:

Fact 6.8.3 The product on $P\Gamma$ given by Definitions 6.8.1 and 6.6.3 is as follows.

$$M_{F^1\ldots F^i} * M_{H^1\ldots H^j} = M_{F^1\ldots F^i + H^1\ldots H^j} + M_{F^1\ldots F^i - H^1\ldots H^j} + M_{F^1\ldots F^i 0 H^1\ldots H^j}.$$

For example,

$$M_{+-0} * M_0 = M_{+-0+0} + M_{+-0-0} + M_{+-000}.$$

We remark that the product and coproduct we have obtained on $P\Gamma$ from the example of type $A_1^{\times(n-1)}$ are not compatible.

We now return to the general definition.

Lemma 6.8.1 *The product on $\mathcal{P}$ is associative. We have*

$$M_{F_1} * M_{F_2} * M_{F_3} = \sum_{F:\ HF = j_H(F_1\times F_2\times F_3)} M_F.$$

The face $H \in \Sigma^n$ is fixed in the above sum. It is the rank 2 fundamental face given by the axiom $(A1)$ for the composition (h_1, h_2, h_3) of n, where $F_i \in \Sigma^{h_i}$.

Proof Let G and G' be the fundamental faces of Σ^n and $\Sigma^{h_2+h_3}$ given by the axiom $(A1)$ for the compositions $(h_1, h_2 + h_3)$ and (h_2, h_3) respectively. They are fixed throughout this computation.

$$\begin{aligned} M_{F_1} * (M_{F_2} * M_{F_3}) &= \sum_{F':\ G'F' = j_{G'}(F_2\times F_3)} M_{F_1} * M_{F'} \\ &= \sum_{F':\ G'F' = j_{G'}(F_2\times F_3)} \ \sum_{F:\ GF = j_G(F_1\times F')} M_F \\ &= \sum_{F:\ GF = j_G(F_1\times F'),\ G'F' = j_{G'}(F_2\times F_3)} M_F. \end{aligned}$$

An alternate way of writing the condition on F in the last sum is

$$(\mathrm{id} \times j_{G'}^{-1})((\emptyset \times G') j_G^{-1}(GF)) = F_1 \times F_2 \times F_3.$$

In order to see that $M_{F_1} * (M_{F_2} * M_{F_3}) = M_{F_1} * M_{F_2} * M_{F_3}$, with the right hand side as defined in the lemma, one needs to show that

$$j_H^{-1}(HF) = (\mathrm{id} \times j_{G'}^{-1})((\emptyset \times G')j_G^{-1}(GF)). \tag{6.34}$$

This is true by Proposition 6.7.3. The term $(M_{F_1} * M_{F_2}) * M_{F_3}$ can be handled similarly.

$\square$

6.8.3 The algebra $\mathcal{M}$

Let $\mathcal{M} = \bigoplus_{n \geq 0} \mathbb{K}\Sigma^n$. Write H_P for the basis element corresponding to $P \in \Sigma^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.8.2 The product on $\mathcal{M}$ is given by

$$H_{P_1} * H_{P_2} = H_{j_G(P_1 \times P_2)}.$$

The vertex $G \in \Sigma^n$ is the fundamental vertex given by the axiom $(A1)$ for the composition (g_1, g_2) of n, where $F_i \in \Sigma^{g_i}$. We say that $j_G(P_1 \times P_2)$ is the join of the faces P_1 and P_2.

Example of type A_{n-1}

Recall from Section 6.6.3 that for type A, the LRB Σ^n is the poset of compositions $\mathcal{B}^n$ of the set $[n]$. Let $\mathrm{M\Pi} = \bigoplus_{n \geq 0} \mathbb{K}\mathcal{B}^n$.

Fact 6.8.4 The product on MΠ given by Definition 6.8.2 coincides with the one given by Definition 6.2.12.

Proof Follows directly from Definitions 6.6.2 and 6.2.8.

$\square$

Lemma 6.8.2 *The product on $\mathcal{M}$ is associative. We have*

$$H_{P_1} * H_{P_2} * H_{P_3} = H_{j_H(P_1 \times P_2 \times P_3)}.$$

The face $H \in \Sigma^n$ is the rank 2 fundamental face given by the axiom $(A1)$ for the composition (h_1, h_2, h_3) of n, where $P_i \in \Sigma^{h_i}$.

Proof Clear from Proposition 6.7.2.

$\square$

6.8.4 The algebra $\mathcal{Q}$

Let $\mathcal{Q} = \bigoplus_{n \geq 0} \mathbb{K}(\mathrm{Q}^n)^*$. Write $M_{(F,D)}$ for the basis element corresponding to $(F, D) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.8.3 The product on $\mathcal{Q}$ is given by

$$M_{(F_1,D_1)} * M_{(F_2,D_2)} = \sum_{F:\ GF=j_G(F_1\times F_2)} M_{(F,Fj_G(D_1\times D_2))}.$$

The vertex $G \in \Sigma^n$ is fixed in the above sum. It is the fundamental vertex given by the axiom $(A1)$ for the composition (g_1, g_2) of n, where $F_i \in \Sigma^{g_i}$.

Remark Note that the above product is quite similar to the one that we gave for $\mathcal{P}$. The index set in the summation is identical in both cases. The additional ingredient is the presence of the second coordinate.

Observe that $D = Fj_G(D_1 \times D_2)$ satisfies $GD = j_G(D_1 \times D_2)$ and $G \in \operatorname{reg}(F, D)$. One may readily check that

$$M_{(F_1,D_1)} * M_{(F_2,D_2)} = \sum_{\substack{(F,D):\\ GF=j_G(F_1\times F_2)\\ GD=j_G(D_1\times D_2)\\ G\in\operatorname{reg}(F,D)}} M_{(F,D)}.$$

This way of writing will be useful later. We say that a (F, D) as above is a quasi-shuffle of the pointed faces (F_1, D_1) and (F_2, D_2).

Example of type A_{n-1}

Recall from Section 6.3.3 that for type A, the set Q^n is the poset of fully nested compositions of the set $[n]$. Let $\mathrm{Q\Pi} = \bigoplus_{n\geq 0} \mathbb{K}(\mathrm{Q}^n)^*$.

Fact 6.8.5 The product on $\mathrm{Q\Pi}$ given by Definition 6.8.3 coincides with the one given by Definition 6.2.18.

Proof This is an easy extension of the proof of Fact 6.8.2.

□

Lemma 6.8.3 *The product on $\mathcal{Q}$ is associative. We have*

$$M_{(F_1,D_1)} * M_{(F_2,D_2)} * M_{(F_3,D_3)} = \sum_{F:\ HF=j_H(F_1\times F_2\times F_3)} M_{(F,Fj_H(D_1\times D_2\times D_3))}.$$

The face $H \in \Sigma^n$ is fixed in the above sum. It is the rank 2 fundamental face given by the axiom $(A1)$ for the composition (h_1, h_2, h_3) of n, where $F_i \in \Sigma^{h_i}$.

Proof In view of the above remark, the work on the first coordinate is done in the proof of Lemma 6.8.1. The extra ingredient is to check that the second coordinate works out correctly. This follows from Proposition 6.7.4.

□

Proposition 6.8.2 *The map* base : $\mathcal{Q} \to \mathcal{P}$ *given by $M_{(F,D)} \mapsto M_F$ is a morphism of algebras.*

Proof Clear from Definitions 6.8.1 and 6.8.3.

□

6.8.5 The algebra $\mathcal{N}$

Let $\mathcal{N} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Q}^n$. Write $H_{(P,C)}$ for the basis element corresponding to $(P,C) \in \mathrm{Q}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.8.4 The product on $\mathcal{N}$ is given by

$$H_{(P_1,C_1)} * H_{(P_2,C_2)} = H_{(j_G(P_1\times P_2), j_G(C_1\times C_2))}.$$

The vertex $G \in \Sigma^n$ is the fundamental vertex given by the axiom $(A1)$ for the composition (g_1, g_2) of n, where $P_i \in \Sigma^{g_i}$. We say that $(j_G(P_1 \times P_2), j_G(C_1 \times C_2))$ is the join of the pointed faces (P_1, C_1) and (P_2, C_2).

Example of type A_{n-1}

Recall from Section 6.3.3 that for type A, the set Q^n is the poset of fully nested compositions of the set $[n]$. Let $\mathrm{N\Pi} = \bigoplus_{n\geq 0} \mathbb{K}\mathrm{Q}^n$.

Fact 6.8.6 The product on $\mathrm{N\Pi}$ given by Definition 6.8.4 coincides with the one given by Definition 6.2.19.

Proof Follows directly from Definitions 6.6.2 and 6.2.17.

□

Lemma 6.8.4 *The product on $\mathcal{N}$ is associative. We have*

$$H_{(P_1,C_1)} * H_{(P_2,C_2)} * H_{(P_3,C_3)} = H_{(j_H(P_1\times P_2\times P_3), j_H(C_1\times C_2\times C_3))}.$$

The face $H \in \Sigma^n$ is the rank 2 fundamental face given by the axiom $(A1)$ for the composition (h_1, h_2, h_3) of n, where $P_i \in \Sigma^{h_i}$.

Proof Clear from Proposition 6.7.2.

□

Proposition 6.8.3 *The map* $\mathrm{base}^* : \mathcal{M} \to \mathcal{N}$ *given by* $H_P \mapsto \sum_{C:\ P\leq C} H_{(P,C)}$ *is a morphism of algebras.*

Proof This is a simple consequence of the axiom $(A1)$.

□

6.8.6 The algebra $\mathcal{S}$

Let $\mathcal{S} = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in Σ^n. Write $F_{(C,D)}$ for the basis element $(C,D) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0.

Definition 6.8.5 Define a product on $\mathcal{S}$ by

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} = \sum_{D:\ GD=j_G(D_1\times D_2)} F_{(j_G(C_1\times C_2),D)}.$$

The vertex $G \in \Sigma^n$ is fixed in the above sum. It is the fundamental vertex given by the axiom $(A1)$ for the composition (g_1, g_2) of n, where $C_i \in \Sigma^{g_i}$.

Note that the above product combines the ideas used in defining the products for $\mathcal{M}$ and $\mathcal{P}$. In the pair (C, D), the first (resp. second) coordinate is treated like the coordinate in $\mathcal{M}$ (resp. $\mathcal{P}$).

Remark Observe that Definition 6.8.5 can be used to define a product on

$$\bigoplus_{n\geq 0} \mathbb{K}(\Sigma^n \times \Sigma^n)^*.$$

This will be taken up in a future work.

Example of type A_{n-1}

Recall from Section 6.6.3 that for type A, the LRB Σ^n is the poset of compositions of the set $[n]$. Let $\mathrm{S\Pi} = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in Σ^n and can be identified with permutations.

Fact 6.8.7 The product on $\mathrm{S\Pi}$ given by Definition 6.8.5 coincides with the one given by Definition 6.2.34.

Proof Comparing Definitions 6.8.5 and 6.2.34, we see that the first coordinate presents no difficulty. To complete the proof, we want to show that

$$\{D : D \text{ a permutation}, GD = j_G(D_1 \times D_2)\} = \{D : D \text{ a shuffle of } D_1 \text{ and } D_2\}.$$

This follows from Equation (6.33) and the following simple observation.

A quasi-shuffle D of two permutations D_1 and D_2 is a permutation if and only if D is a shuffle of D_1 and D_2.

$\square$

Example of type $A_1^{\times(n-1)}$

Now we consider the example in Section 6.6.4. Let $S\Gamma = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$, where $\mathcal{C}^n$ is the set of chambers in $\mathcal{W}^n$, the poset of words of length $n-1$ in the alphabet $\{+, -, 0\}$. Then observe that:

Fact 6.8.8 The product on $S\Gamma$ given by Definitions 6.8.5 and 6.6.3 is as follows.

$$\begin{aligned} F_{(C^1\cdots C^i, D^1\cdots D^i)} * F_{(\tilde{C}^1\cdots\tilde{C}^j, \tilde{D}^1\cdots\tilde{D}^j)} &= F_{(C^1\cdots C^i + \tilde{C}^1\cdots\tilde{C}^j, D^1\cdots D^i + \tilde{D}^1\cdots\tilde{D}^j)} \\ &\quad + F_{(C^1\cdots C^i + \tilde{C}^1\cdots\tilde{C}^j, D^1\cdots D^i - \tilde{D}^1\cdots\tilde{D}^j)}. \end{aligned}$$

For example,

$$F_{(+,-)} * F_{(+-,-+)} = F_{(+++-,-+-+)} + F_{(+++-,---+)}.$$

We remark that the product and coproduct we have obtained on $S\Gamma$ from the example of type $A_1^{\times(n-1)}$ are not compatible.

Lemma 6.8.5 *The product on $\mathcal{S}$ is associative. We have*

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} * F_{(C_3,D_3)} = \sum_{D:\ HD=j_H(D_1\times D_2\times D_3)} F_{(j_H(C_1\times C_2\times C_3),D)}.$$

The face $H \in \Sigma^n$ is fixed in the above sum. It is the fundamental face given by the axiom (A1) for the composition (h_1, h_2, h_3) of n, where $F_i \in \Sigma^{h_i}$.

Proof One can look at the two coordinates separately. The associativity in the first coordinate is clear (Proposition 6.7.2). For the second coordinate, we repeat the proof of Lemma 6.8.1 with F_i and F' replaced by D_i and D' respectively.

□

6.8.7 The algebra $\mathcal{R}$

Let $\mathcal{R} = \underset{n\geq 0}{\oplus} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)$, where $\mathcal{C}^n$ is the set of chambers in Σ^n. Write $K_{(D,C)}$ for the basis element $(D, C) \in \mathcal{C}^n \times \mathcal{C}^n, n > 0$ and 1 for the basis element of degree 0. We define a product on $\mathcal{R}$ such that:

Proposition 6.8.4 *The switch map $s : \mathcal{R} \to \mathcal{S}$ which sends $K_{(D,C)} \to F_{(C,D)}$ is an isomorphism of algebras.*

Definition 6.8.6 Define a product on $\mathcal{R}$ by

$$K_{(D_1,C_1)} * K_{(D_2,C_2)} = \sum_{D:\ GD=j_G(D_1\times D_2)} K_{(D,j_G(C_1\times C_2))}.$$

Lemma 6.8.6 *The product on $\mathcal{R}$ is associative. We have*

$$K_{(D_1,C_1)} * K_{(D_2,C_2)} * K_{(D_3,C_3)} = \sum_{D:\ HD=j_H(D_1\times D_2\times D_3)} K_{(D,j_H(C_1\times C_2\times C_3))}.$$

6.8.8 The maps $\mathrm{Road} : \mathcal{S} \to \mathcal{Q}$ and $\Theta : \mathcal{N} \to \mathcal{R}$

So far, we have defined the algebras that occur in Theorem 6.8.1, and showed that they are associative. We also showed that the maps base*, s and base are morphisms of algebras. To complete the proof, we now show that the maps Road and Θ are morphisms of algebras.

Proposition 6.8.5 *The map* $\mathrm{Road} : \mathcal{S} \to \mathcal{Q}$ *that sends* $F_{(C,D)}$ *to* $\sum_{F:\ FC=D} M_{(F,D)}$ *is a morphism of algebras.*

Proof We need to show that

$$\mathrm{Road}(F_{(C_1,D_1)} * F_{(C_2,D_2)}) = \mathrm{Road}(F_{(C_1,D_1)}) * \mathrm{Road}(F_{(C_2,D_2)}).$$

From the definitions,

$$\begin{aligned} LHS &= \sum_{D:\ GD=j_G(D_1\times D_2)} \mathrm{Road}(F_{(j_G(C_1\times C_2),D)}) \\ &= \sum_{D:\ GD=j_G(D_1\times D_2)} \ \sum_{F:\ Fj_G(C_1\times C_2)=D} M_{(F,D)} \\ &= \sum_{F:\ GFj_G(C_1\times C_2)=j_G(D_1\times D_2)} M_{(F,Fj_G(C_1\times C_2))}. \end{aligned}$$

Similarly, from the definitions,

$$\begin{aligned}
RHS &= \sum_{F_1,F_2:\ F_iC_i=D_i} M_{(F_1,D_1)} * M_{(F_2,D_2)} \\
&= \sum_{F_1,F_2:\ F_iC_i=D_i} \ \sum_{F:\ GF=j_G(F_1\times F_2)} M_{(F,Fj_G(D_1\times D_2))} \\
&= \sum_{F:\ GFj_G(C_1\times C_2)=j_G(D_1\times D_2)} M_{(F,Fj_G(D_1\times D_2))}.
\end{aligned}$$

Using $FGF = FG$ and $G \leq j_G(C_1 \times C_2)$, observe that

$$Fj_G(D_1 \times D_2) = FGFj_G(C_1 \times C_2) = FGj_G(C_1 \times C_2) = Fj_G(C_1 \times C_2).$$

This shows that the left and right hand sides are equal.

□

Proposition 6.8.6 *The map* $\Theta : \mathcal{N} \to \mathcal{R}$ *that sends* $H_{(P,C)}$ *to* $\sum_{D:\ PD=C} K_{(D,C)}$ *is a morphism of algebras.*

Proof We need to show that

$$\Theta(H_{(P_1,C_1)} * H_{(P_2,C_2)}) = \Theta(H_{(P_1,C_1)}) * \Theta(H_{(P_2,C_2)}).$$

From the definitions,

$$LHS = \Theta(H_{(j_G(P_1\times P_2),j_G(C_1\times C_2))}) = \sum_{D:\ j_G(P_1\times P_2)D=j_G(C_1\times C_2)} K_{(D,j_G(C_1\times C_2))}.$$

Similarly, from the definitions,

$$\begin{aligned}
RHS &= \sum_{D_1,D_2:\ P_iD_i=C_i} K_{(D_1,C_1)} * K_{(D_2,C_2)} \\
&= \sum_{D_1,D_2:\ P_iD_i=C_i} \ \sum_{D:\ GD=j_G(D_1\times D_2)} K_{(D,j_G(C_1\times C_2))} \\
&= \sum_{D:\ j_G(P_1\times P_2)GD=j_G(C_1\times C_2)} K_{(D,j_G(C_1\times C_2))}.
\end{aligned}$$

Since $G \leq j_G(P_1 \times P_2)$, we have $j_G(P_1 \times P_2)G = j_G(P_1 \times P_2)$. Hence the left and right hand sides are equal.

□

6.8.9 The algebras $A_{\mathcal{Z}}$, $A_{\mathcal{L}}$, $A_{\mathcal{Z}^*}$ and $A_{\mathcal{L}^*}$

Thus far, we have proved Theorem 6.8.1. In the rest of this section, we prove the remaining part of Theorem 6.1.2. It involves the maps supp and lune and their duals. For that we recall the content of Lemmas 2.4.1 and 2.4.2.

Let $G \in \Sigma^n$ be the fundamental vertex with type (g_1, g_2). Then one has the following two commutative diagrams.

$$\begin{tikzcd}
\Sigma^n_G \arrow[r, hook] \arrow[d, "\mathrm{supp}^n_G"] & \Sigma^n \arrow[r, two heads, "G\cdot"] \arrow[d, "\mathrm{supp}^n"] & \Sigma^n_G \arrow[d, "\mathrm{supp}^n_G"] \\
\mathrm{L}^n_G \arrow[r, hook] & \mathrm{L}^n \arrow[r, two heads, "G\cdot"'] & \mathrm{L}^n_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{g_1} \times \Sigma^{g_2} \arrow[r, "j_G", "\cong"'] \arrow[d, "\mathrm{supp}^{g_1} \times \mathrm{supp}^{g_2}"'] & \Sigma^n_G \arrow[d, "\mathrm{supp}^n_G"] \\
\mathrm{L}^{g_1} \times \mathrm{L}^{g_2} \arrow[r, "\cong", "j_G"'] & \mathrm{L}^n_G
\end{tikzcd}
\qquad (6.35)$$

where

$$\mathrm{L}^n_G = \{X \in \mathrm{L}^n \mid \operatorname{supp}^n(G) \leq X\}$$

is the poset of flats of Σ^n_G, and supp^n_G, $\operatorname{supp}^{g_1}$ and $\operatorname{supp}^{g_2}$ are the support maps of Σ^n_G, Σ^{g_1} and Σ^{g_2} respectively. The map $G\cdot : \Sigma^n \to \Sigma^n_G$ sends F to GF. One also has the same two diagrams above with Σ and L replaced by Q and Z respectively, and the map supp replaced by the map lune.

Definition 6.8.7 Define a product on $A_{\mathcal{L}^*}$ and $A_{\mathcal{Z}^*}$ by

$$\begin{aligned} m_{X_1} * m_{X_2} &= \sum_{X:\ G\cdot X = j_G(X_1\times X_2)} m_X, \\ m_{L_1} * m_{L_2} &= \sum_{L:\ G\cdot L = j_G(L_1\times L_2),\ G\in \operatorname{zone}(L)} m_L. \end{aligned}$$

The set zone(L) is discussed in Section 2.3.4. It also made an appearance in the coproduct formula for $A_{\mathcal{Z}}$, see Definition 6.5.8. One way to justify its presence is by the condition $G \in \operatorname{reg}(F, D)$ which occurs in the second product formula for $\mathcal{Q}$, see the remark after Definition 6.8.3.

We say that a X (resp. L) as above is a quasi-shuffle of the flats X_1 and X_2 (resp. lunes L_1 and L_2).

Definition 6.8.8 Define a product on $A_{\mathcal{L}}$ and $A_{\mathcal{Z}}$ by

$$\begin{aligned} h_{X_1} * h_{X_2} &= h_{j_G(X_1\times X_2)}, \\ h_{L_1} * h_{L_2} &= h_{j_G(L_1\times L_2)}. \end{aligned}$$

We say that $j_G(X_1 \times X_2)$ (resp. $j_G(L_1 \times L_2)$) is the join of the flats X_1 and X_2 (resp. lunes L_1 and L_2).

Remark For the example of type A_{n-1}, one can make explicit the maps $G\cdot$ in the diagram (6.35) above, as also the join maps $j_G : \mathrm{L}^{g_1} \times \mathrm{L}^{g_2} \to \mathrm{L}^n_G$ and $j_G : \mathrm{Z}^{g_1} \times \mathrm{Z}^{g_2} \to \mathrm{Z}^n_G$. It is then easy to check that the above formulas reduce to the definitions of the corresponding algebras in Section 6.2.

Proposition 6.8.7 *Let $\{\Sigma^n\}$ be a family of LRBs, and let $\mathcal{P}$, $\mathcal{M}$, $\mathcal{Q}$ and $\mathcal{N}$ be the algebras as given in Definitions 6.8.1-6.8.4.*

(1) The injective maps $\operatorname{supp}^* : A_{\mathcal{L}^*} \to \mathcal{P}$ *that sends* m_X *to* $\sum_{F:\ \operatorname{supp} F = X} M_F$ *and* $\operatorname{lune}^* : A_{\mathcal{Z}^*} \to \mathcal{Q}$ *that sends* m_L *to* $\sum_{(F,D):\ \operatorname{lune}(F,D)=L} M_{(F,D)}$ *are maps of algebras.*

(2) The surjective maps $\operatorname{supp} : \mathcal{M} \twoheadrightarrow A_{\mathcal{L}}$ *that sends* H_P *to* $h_{\operatorname{supp} P}$ *and* $\operatorname{lune} : \mathcal{N} \twoheadrightarrow A_{\mathcal{Z}}$ *that sends* $H_{(P,C)}$ *to* $h_{\operatorname{lune}(P,C)}$ *are maps of algebras.*

Proof We only give the proof for the supp and supp* maps.

(1) The following computation shows that the map $\operatorname{supp}^* : A_{\mathcal{L}^*} \hookrightarrow \mathcal{P}$ is a morphism of algebras. The term $\operatorname{supp}^*(m_{X_1}) * \operatorname{supp}^*(m_{X_2})$ equals

$$\begin{aligned} &= \sum_{F_i:\ \operatorname{supp} F_i = X_i} M_{F_1} * M_{F_2} \\ &= \sum_{F_i:\ \operatorname{supp} F_i = X_i} \ \sum_{F:\ GF = j_G(F_1\times F_2)} M_F \\ &= \sum_{F:\ \operatorname{supp}(GF) = j_G(X_1\times X_2)} M_F \qquad \text{(Diagram (6.35))} \\ &= \sum_{F:\ \operatorname{supp}(G)\vee \operatorname{supp}(F) = j_G(X_1\times X_2)} M_F \\ &= \sum_{X:\ G\cdot X = j_G(X_1\times X_2)} \ \sum_{F:\ \operatorname{supp} F = X} m_F, \end{aligned}$$

which equals $\operatorname{supp}^*(m_{X_1} * m_{X_2})$.

(2) It follows from diagram (6.35) that

$$\operatorname{supp}(j_G(P_1 \times P_2)) = j_G(\operatorname{supp}(P_1) \times \operatorname{supp}(P_2)).$$

This proves that the map $\operatorname{supp} : \mathcal{M} \twoheadrightarrow A_{\mathcal{L}}$ is a morphism of algebras.

$\square$

Chapter 7

The Hopf algebra of pairs of permutations

7.1 Introduction

In this chapter, we study in detail the Hopf algebra $\mathrm{S}\Pi$ indexed by pairs of permutations that was introduced in Chapter 6.

7.1.1 The basic setup

We recall some notation from Chapter 1, see Section 1.4 in particular. More background material for this chapter can be found in Section 5.4.

Let S_n be the symmetric group on n letters and Σ^n be its Coxeter complex. The group S_n is generated by $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the transposition that interchanges i and $i+1$. Identify $\overline{\mathrm{Q}}^n$, the poset of compositions of n under refinement, with the poset of subsets of S under inclusion in the usual way. Let $\mathcal{C}^n$ be the set of chambers in Σ^n. Since S_n acts simply transitively on $\mathcal{C}^n$, one can identify

$$\mathcal{C}^n \longleftrightarrow \mathrm{S}_n \tag{7.1}$$

after fixing a fundamental chamber $C_0^n \in \mathcal{C}^n$. We use the standard notation that wC_0^n is the chamber that corresponds to $w \in \mathrm{S}_n$.

Let $d : \mathcal{C}^n \times \mathcal{C}^n \to \mathrm{S}_n$ be the distance map, and type : $\Sigma^n \to \overline{\mathrm{Q}}^n$ map a face to its type. Note that $d(C_0^n, wC_0^n) = w$. We sometimes drop the superscript n to avoid overloading the notation. We write $\overline{F}$ for the opposite of the face $F \in \Sigma^n$. The letters C, D and E always stand for chambers.

7.1.2 The main result

Standard material

In [61], Malvenuto introduced the Hopf algebra

$$\mathrm{S}\Lambda = \sum_{n \geq 0} \mathbb{K}(\mathrm{S}_n)^*,$$

indexed by permutations, and related it to the Hopf algebra of quasi-symmetric functions $\mathrm{Q}\Lambda$ via the descent map. Recall that:

Theorem 7.1.1 $\mathrm{S}\Lambda$ *is a self-dual free and cofree graded Hopf algebra.*

Table 7.1: Hopf algebras and their indexing sets and bases.

Name	Vector space	Bases	Name	Dual space	Dual bases
$\mathrm{R}\Pi$	$\mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)$	R, H, K	$\mathrm{S}\Pi$	$\mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*$	S, M, F
$\mathrm{R}\Lambda$	$\mathbb{K}\mathrm{S}_n$	H, K	$\mathrm{S}\Lambda$	$\mathbb{K}(\mathrm{S}_n)^*$	M, F

The self-duality appears in Malvenuto's thesis [61, Section 5.2] and Malvenuto and Reutenauer [62, Theorem 3.3] and the freeness was established by Poirier and Reutenauer [78]. For ideas related to the Hopf algebra $\mathrm{S}\Lambda$, see Reutenauer [82], Patras and Reutenauer [73], Loday and Ronco [56, 57], Duchamp, Hivert and Thibon [26, 27], Foissy [30] and Aguiar and Sottile [4, 3, 5].

New material

In Chapter 5, we initiated a systematic study of the descent theory of Coxeter groups. This pointed us to the fact that one should be able to realize the Hopf algebra $\mathrm{S}\Lambda$ as a quotient of a bigger Hopf algebra indexed by pairs of permutations. This goal was realized in Chapter 6, where we constructed many new Hopf algebras following this philosophy. Among them was the Hopf algebra

$$\mathrm{S}\Pi = \bigoplus_{n\geq 0} \mathbb{K}(\mathcal{C}^n \times \mathcal{C}^n)^*, \tag{7.2}$$

indexed by pairs of permutations. The definitions of the Hopf algebras $\mathrm{S}\Pi$ and $\mathrm{S}\Lambda$ are recalled in Section 7.2.

The goal of this chapter is to study in detail the Hopf algebra $\mathrm{S}\Pi$. Using the quotient map $\mathrm{S}\Pi \twoheadrightarrow \mathrm{S}\Lambda$, one can then quickly derive some of the results obtained in [4] for $\mathrm{S}\Lambda$. The main result of this chapter is:

Theorem 7.1.2 $\mathrm{S}\Pi$ *is a free and cofree graded Hopf algebra.*

Note that we do not claim $\mathrm{S}\Pi$ to be self-dual. The proof of Theorem 7.1.2 is a consequence of Theorems 7.4.3 and 7.5.4 that we establish in this chapter. In Sections 7.1.3-7.1.5 below, we recall relevant ideas from Chapters 5 and 6. In Section 7.1.6, we provide an outline of the proof of Theorem 7.1.2, and explain the organization of the rest of the chapter.

7.1.3 The Hopf algebras $\mathrm{R}\Pi$ and $\mathrm{R}\Lambda$

For book-keeping purposes, we introduce two more Hopf algebras $\mathrm{R}\Pi$ and $\mathrm{R}\Lambda$, which are isomorphic copies of $\mathrm{S}\Pi$ and $\mathrm{S}\Lambda$ respectively, see Table 7.1. The nth graded piece of each Hopf algebra is indicated in the vector space column, see Equation (7.2) in this regard. For each Hopf algebra, we consider a number of useful bases. They are related to one another via certain partial orders on $\mathcal{C}^n \times \mathcal{C}^n$ and S_n. This theory was developed in Chapter 5 for any Coxeter group. We recall the main definitions in Sections 7.1.4 and 7.1.5 below.

The precise relation among the above four Hopf algebras is given by the following special case of Theorem 6.1.4.

Proposition 7.1.1 *The following is a commutative diagram of graded Hopf algebras.*

$$\begin{tikzcd} \mathrm{R}\Pi \arrow[r, "s", "\cong"'] \arrow[d, "d"'] & \mathrm{S}\Pi \arrow[d, "d"] \\ \mathrm{R}\Lambda \arrow[r, "\cong", "s"'] & \mathrm{S}\Lambda \end{tikzcd}$$

where the switch map $s : \mathrm{R}\Pi \to \mathrm{S}\Pi$ *sends* $K_{(D,C)}$ *to* $F_{(C,D)}$*, and* $s : \mathrm{R}\Lambda \to \mathrm{S}\Lambda$ *sends* K_w *to* $F_{w^{-1}}$*. The distance map* d *is given by*

$$A_{(C,D)} \mapsto A_{d(C,D)}$$

for $A = F, M, H$ *and* K*. On the* S *basis, one has*

$$S_{(C,D)} \mapsto \begin{cases} M_{d(C,D)} & \text{if } D = \overline{C}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof The commutativity of the diagram follows from the definitions of the maps. The Hopf algebras $\mathrm{R}\Pi$ and $\mathrm{R}\Lambda$ are defined from the Hopf algebras $\mathrm{S}\Pi$ and $\mathrm{S}\Lambda$ respectively, using the isomorphism s. So the content of the proposition is that the distance map $d : \mathrm{S}\Pi \to \mathrm{S}\Lambda$ is a map of Hopf algebras. This is an easy check.

□

Remark It follows directly from the definitions in the F and K basis that $\mathrm{R}\Lambda \cong \mathrm{S}\Lambda^*$ as Hopf algebras with $K_w = F_w^*$.

7.1.4 Three partial orders on $\mathcal{C}^n \times \mathcal{C}^n$

We recall some definitions and facts from Section 5.2.4 and Section 1.3.5.

Definition 7.1.1 Let $\leq$ be the weak left Bruhat order on S_n. Using the identification in (7.1), define a partial order $\leq_b$ on the set $\mathcal{C}^n$ as follows.

$$uC_0^n \leq_b vC_0^n \quad \text{in} \quad \mathcal{C}^n \qquad \Longleftrightarrow \qquad u \leq v \quad \text{in} \quad \mathrm{S}_n.$$

This occurs precisely if there is a minimum gallery

$$C_0^n - vu^{-1}C_0^n - vC_0^n.$$

The subscript "b" stands for Bruhat. Note that this partial order depends on the choice of the fundamental chamber C_0^n in $\mathcal{C}^n$.

Similarly, we write $\leq_{rb}$ for the partial order on $\mathcal{C}^n$ defined using the weak right Bruhat order on S_n. The only purpose of the partial order $\leq_{rb}$ is to allow us to make a comment about the product of $\mathrm{S}\Lambda$, see Definition 7.2.13.

Definition 7.1.2 We define three partial orders on $\mathcal{C}^n \times \mathcal{C}^n$.

$$\begin{aligned} (C_1, D_1) \leq (C_2, D_2) \quad &\Longleftrightarrow \quad D_1 = D_2 = D \text{ and } C_2 - C_1 - D. \\ &\Longleftrightarrow \quad D_1 = D_2 = D \text{ and } d(C_1, D_1) \leq d(C_2, D_2). \end{aligned}$$

$$(C_1, D_1) \leq' (C_2, D_2) \quad \Longleftrightarrow \quad D_1 \leq_b D_2 \text{ and } d(C_1, D_1) = d(C_2, D_2).$$

$$(C_1, D_1) \preceq (C_2, D_2) \quad \Longleftrightarrow \quad \exists E \ni (C_1, D_1) \leq (E, D_1) \text{ and } (E, D_1) \leq' (C_2, D_2).$$

Note that in the definition of $\preceq$, only one E can satisfy the required condition; namely the one that satisfies $d(E, D_1) = d(C_2, D_2)$. Unlike $\leq$, the partial orders $\leq'$ and $\preceq$ depend on the choice of the fundamental chamber C_0^n.

7.1.5 The different bases of $\mathrm{S}\Pi$ and $\mathrm{S}\Lambda$

We recall some definitions from Section 5.6.1 and Section 5.7.1. The M basis of $\mathrm{S}\Lambda$ was introduced in [4]. It is related to the F basis by the weak left Bruhat order $\leq$ on S_n as follows.

$$F_v = \sum_{u:\ v\leq u} M_u.$$

Let $\mathrm{R}\Lambda$ be the graded dual of $\mathrm{S}\Lambda$ as a Hopf algebra, and H and K be the bases dual to M and F respectively. Then

$$H_u = \sum_{v:\ v\leq u} K_v.$$

In the literature, it is $\mathrm{R}\Lambda$ rather than $\mathrm{S}\Lambda$ that is sometimes referred to as the Malvenuto-Reutenauer Hopf algebra.

Similarly, using the partial orders in Definition 7.1.2, we define the S, M and F bases of $\mathrm{S}\Pi$. They are related by

$$F_{(E,D)} = \sum_{C:\ (E,D)\leq(C,D)} M_{(C,D)}, \qquad M_{(C',D')} = \sum_{(C',D')\leq'(C,D)} S_{(C,D)}. \tag{7.3}$$

Observe from the above formulas that the F basis is related to the S basis via the partial order $\preceq$ as follows.

$$F_{(C',D')} = \sum_{(C',D')\preceq(C,D)} S_{(C,D)}.$$

Let $\mathrm{R}\Pi$ be the graded dual of $\mathrm{S}\Pi$ as a vector space. Then the R, H and K dual bases are related by

$$H_{(E,C)} = \sum_{D:\ (D,C)\leq(E,C)} K_{(D,C)}, \qquad R_{(D',C')} = \sum_{(D,C)\leq'(D',C')} H_{(D,C)}. \tag{7.4}$$

We draw the reader's attention to the fact that $\mathrm{R}\Pi$ is not defined as the graded Hopf algebra dual of $\mathrm{S}\Pi$.

7.1.6 The proof method and the organization of the chapter

Standard material

We first review the method for proving Theorem 7.1.1. It follows directly from the definitions in the F and K basis that $\mathrm{R}\Lambda \cong \mathrm{S}\Lambda^*$ as Hopf algebras with $K_w = F_w^*$. This implies that $\mathrm{S}\Lambda$ is a self-dual Hopf algebra with $\mathrm{S}\Lambda^* \xrightarrow{\cong} \mathrm{S}\Lambda$ given by $F_w^* \mapsto F_{w^{-1}}$. As mentioned before, freeness and cofreeness of $\mathrm{S}\Lambda$ follows from results of Malvenuto, Poirier, and Reutenauer [61, 62, 78]. Other proofs are given in [27, 4]. Explicit expressions for the product and coproduct in the M basis are given in [4].

New material

In order to prove Theorem 7.1.2, we apply similar ideas. However it is not true that $\mathrm{R}\Pi \cong \mathrm{S}\Pi^*$ as Hopf algebras with $K_{(D,C)} = F^*_{(D,C)}$. So one cannot conclude that $\mathrm{S}\Pi$ is self-dual. This doubles our work and one has to prove the freeness and cofreeness of $\mathrm{S}\Pi$ separately.

In Section 7.2, we recall the definition of the Hopf algebra $\mathrm{S}\Pi$ in the F basis as given in Chapter 6. In Section 7.3, we compute the coproduct and product in the M basis of $\mathrm{S}\Pi$. The formulas for $\mathrm{S}\Lambda$ in the M basis obtained in [4] follow directly from these. However, unlike for $\mathrm{S}\Lambda$, one cannot conclude cofreeness of $\mathrm{S}\Pi$ from the coproduct formula in the M basis, since it produces some unwanted terms.

This flaw is rectified by replacing the M basis by the S basis. The coproduct computation in the S basis in Section 7.4 produces exactly the terms that we want. In particular, it shows that SΠ is cofree, thus proving one part of Theorem 7.1.2. For completeness, we also write down the product in the S basis.

In Section 7.5, we compute the coproduct and product in the H basis of RΠ. The freeness of RΠ, and hence SΠ, follows from the product formula in the H basis of RΠ, proving the second part of Theorem 7.1.2.

Open Question For completeness, one would also like to compute the formulas in the R basis of RΠ and formulas for the antipode in the various bases. These are open problems at the moment.

Remark There is a way to recover the "self-duality" of SΠ by viewing it as a Hopf monoid in a certain category of species. One can also derive the antipode formulas for SΛ obtained in [4] by this approach. These ideas will be explained in a future work.

7.2 The Hopf algebra SΠ

In this section, we recall the definition of the Hopf algebra SΠ in the F basis in both combinatorial and geometric terms as given in Chapter 6. Then we recall the combinatorial definition of SΛ in the F basis and give a new geometric definition.

7.2.1 Preliminary definitions

We first give a few preliminary definitions, some of which were written in Section 6.2.3.

Definition 7.2.1 A set composition is an ordered set partition. For example, $6|34|125$ is a composition of $[6]$.

Definition 7.2.2 There is a unique order preserving map from any n-set N of integers to the standard n-set $[n]$. Using this map, one can standardize compositions of N to compositions of $[n]$. For example,

$$\mathrm{st}(8|36|59) = 4|13|25.$$

Definition 7.2.3 The join of a composition F_1 of $[g_1]$ and a composition F_2 of $[g_2]$ is a composition $j(F_1 \times F_2)$ of $[g_1 + g_2]$ obtained by shifting up the indices of F_2 by g_1 and then placing it after F_1. For example,

$$j(31|2 \times 23|14|5) = 31|2|56|47|8.$$

We also define $\overline{j}(F_1 \times F_2)$ to be the composition of $[g_1 + g_2]$ obtained by shifting up the indices of F_2 by g_1 and then placing it before F_1. For example,

$$\overline{j}(31|2 \times 23|14|5) = 56|47|8|31|2.$$

Further, we define $j'(F_1 \times F_2)$ to be the composition of $[g_1 + g_2]$ obtained by shifting up the indices of F_1 by g_2 and then placing it before F_2. For example,

$$j'(31|2 \times 23|14|5) = 86|7|23|14|5.$$

Further, we define $j''(F_1 \times F_2)$ to be the composition of $[g_1 + g_2]$ obtained by shifting up the indices of F_1 by g_2 and then placing it after F_2. For example,

$$j''(31|2 \times 23|14|5) = 23|14|5|86|7.$$

Definition 7.2.4 A shuffle of set compositions F_1 and F_2 is a shuffle of the components of F_1 and F_2. For example,

$$5|bc|af|21|34|deg|6 \text{ is a shuffle of } 5|21|34|6 \text{ and } bc|af|deg.$$

Definition 7.2.5 A quasi-shuffle of set compositions F_1 and F_2 is a shuffle of the components of F_1 and F_2, where in addition we are allowed to substitute a disjoint set of pairs of components (F_1^i, F_2^j) for $F_1^i \cup F_2^j$, if they are adjacent in the shuffle. For example,

$$5bc|af|21|34deg|6 \text{ is a quasi-shuffle of } 5|21|34|6 \text{ and } bc|af|deg.$$

The above definitions of a shuffle and quasi-shuffle are not fully precise since one needs to ensure that the shuffled sets are disjoint, see Section 6.2.3 for more details.

7.2.2 Combinatorial definition

The Coxeter complex Σ^n can be identified with the poset of set compositions under refinement. This allows us to define $\mathrm{S}\Pi$ in combinatorial terms, without any reference to Coxeter complexes, as below. Let $C = C^1|C^2|\ldots|C^n$ and $D = D^1|D^2|\ldots|D^n$ be two permutations.

Definition 7.2.6 The coproduct on $\mathrm{S}\Pi$ is given by

$$\Delta(F_{(C,D)}) = \sum_{i=0}^{n} F_{\mathrm{st}(\tilde{C}^1|\cdots|\tilde{C}^i, D^1|\cdots|D^i)} \otimes F_{\mathrm{st}(\tilde{C}^{i+1}|\cdots|\tilde{C}^n, D^{i+1}|\cdots|D^n)},$$

where $\tilde{C}^1, \ldots, \tilde{C}^i$ are the letters in the set $\{D^1, \ldots, D^i\}$ and $\tilde{C}^{i+1}, \ldots, \tilde{C}^n$ are the letters in the set $\{D^{i+1}, \ldots, D^n\}$ written in the order in which they appear in $C^1|\cdots|C^n$. For example,

$$\begin{aligned}\Delta(F_{2|3|1|4,4|1|2|3}) \quad = \quad & 1 \otimes F_{2|3|1|4,4|1|2|3} + F_{1,1} \otimes F_{2|3|1,1|2|3} + F_{1|2,2|1} \otimes F_{1|2,1|2} + \\ & F_{2|1|3,3|1|2} \otimes F_{1,1} + F_{2|3|1|4,4|1|2|3} \otimes 1.\end{aligned}$$

Definition 7.2.7 The product on $\mathrm{S}\Pi$ is given by

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} = \sum_{D:\ D \text{ a shuffle of } D_1 \text{ and } D_2} F_{(j(C_1 \times C_2),D)}.$$

The term $j(C_1 \times C_2)$ refers to the join of C_1 and C_2 given in Definition 7.2.3. If D_1 and D_2 are compositions of $[g_1]$ and $[g_2]$ respectively then the understanding is that we shift up the indices of D_2 by g_1 and then shuffle. For example,

$$\begin{aligned}F_{(2|1,1|2)} * F_{(1|2,2|1)} \quad = \quad & F_{(2|1|3|4,1|2|4|3)} + F_{(2|1|3|4,1|4|2|3)} + F_{(2|1|3|4,1|4|3|2)} + \\ & F_{(2|1|3|4,4|1|2|3)} + F_{(2|1|3|4,4|1|3|2)} + F_{(2|1|3|4,4|3|1|2)}.\end{aligned}$$

Note that the first coordinate is the same in each term in the right hand side.

Proposition 7.2.1 *With the coproduct and product as above,* $\mathrm{S}\Pi$ *is a Hopf algebra.*

We leave the proof to the reader.

7.2.3 The break and join operations

We recall the construction of the break and join operations for the family $\{\Sigma^n\}$, as n varies (Sections 6.3.3 and 6.6.3). Let

$$\Sigma^n_K = \{F \in \Sigma^n \mid K \leq F\}$$

be the star region of the face K. It can be identified with link(K), which is a Coxeter complex in its own right.

For a vertex $K \in \Sigma^n$ of type (k_1, k_2), we have an isomorphism

$$\Sigma^n_K \xrightarrow{\cong} \Sigma^{k_1} * \Sigma^{k_2},$$

where $\Sigma^{k_1} * \Sigma^{k_2}$ is the join of Σ^{k_1} and Σ^{k_2}. This is because deleting a vertex in the Coxeter diagram of type A results in two smaller diagrams both of type A. However there is a choice involved in this isomorphism because of the action of the symmetric groups S_{k_1} and S_{k_2}. We fix an isomorphism

$$b_K : \Sigma^n_K \xrightarrow{\cong} \Sigma^{k_1} \times \Sigma^{k_2}, \quad \text{by demanding that} \quad KC^n_0 \mapsto C^{k_1}_0 \times C^{k_2}_0,$$

where KC^n_0 is the projection of C^n_0 on K, see Figure 1.2. Recall from Section 1.3.1 that these projection maps define a semigroup structure on Σ^n. For $F, G \in \Sigma^n$, we denote the product by FG, and call it the projection of G on F.

Similarly, one defines an isomorphism

$$j_G : \Sigma^{g_1} \times \Sigma^{g_2} \xrightarrow{\cong} \Sigma^n_G, \quad \text{by demanding that} \quad C^{g_1}_0 \times C^{g_2}_0 \mapsto C^n_0,$$

where $G \in C^n_0$ is the fundamental vertex of type (g_1, g_2).

The letters b and j stand for "break" and "join" respectively. In our arguments, we often refer to the compatibility of galleries with the maps b_K and j_G. This is same as the compatibility of galleries with joins (Section 1.3.7).

7.2.4 Geometric definition

The Hopf algebra SΠ can be described geometrically using the semigroup structure of Σ^n and the maps b_K and j_G as below.

Definition 7.2.8 The coproduct on SΠ is given by

$$\Delta(F_{(C,D)}) = 1 \otimes F_{(C,D)} + F_{(C,D)} \otimes 1 + \Delta_+(F_{(C,D)}), \text{ where}$$

$$\Delta_+(F_{(C,D)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ K \leq D} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

Definition 7.2.9 The product on SΠ is given by

$$F_{(C_1,D_1)} * F_{(C_2,D_2)} = \sum_{D:\ GD = j_G(D_1 \times D_2)} F_{(j_G(C_1 \times C_2), D)}.$$

The vertex $G \in \Sigma^n$ is fixed in the above sum. It is the vertex of C^n_0 of type (g_1, g_2), where $C_i \in \Sigma^{g_i}$.

Equivalently, the sum ranges over the chambers D in reg(G, D'), which is the lunar region of G and $D' = j_G(D_1 \times D_2)$, as shown in Figure 7.1. A more concrete example is given in Figure 7.2.

For a discussion of lunes, see Section 2.3 and the references therein. The maps b_K, j_G and the semigroup structure of Σ^n can be described combinatorially. Using them, one can recover the combinatorial definitions of the product and coproduct for SΠ given earlier. This is explained in some detail in Chapter 6.

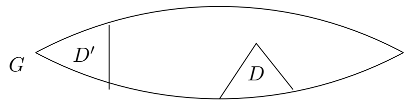


Figure 7.1: A chamber D in $\operatorname{reg}(G, D')$, the lunar region of G and D'.

7.2.5 The Hopf algebra SΛ

In this subsection, we use chambers rather than permutations to index SΛ. We first recall the combinatorial definition of SΛ in the F basis.

Definition 7.2.10 The coproduct on SΛ is given by

$$\Delta(F_D) = \sum_{i=0}^{n} F_{\mathrm{st}(D^1|\cdots|D^i)} \otimes F_{\mathrm{st}(D^{i+1}|\cdots|D^n)}.$$

Definition 7.2.11 The product on SΛ is given by

$$F_{D_1} * F_{D_2} = \sum_{D\colon D \text{ a shuffle of } D_1 \text{ and } D_2} F_D.$$

In geometric terms, the definitions are as follows.

Definition 7.2.12 The coproduct on SΛ is given by

$$\Delta(F_D) = 1 \otimes F_D + F_D \otimes 1 + \sum_{K\colon \operatorname{rank} K = 1,\ K \leq D} F_{D_1} \otimes F_{D_2},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$.

Definition 7.2.13 The product on SΛ is given by

$$F_{D_1} * F_{D_2} = \sum_{D\colon GD = j_G(D_1 \times D_2)} F_D.$$

The vertex $G \in \Sigma^n$ is the vertex of C^n_0 of type (g_1, g_2), where $C_i \in \Sigma^{g_i}$.

By definition, the sum ranges over the chambers D in $\operatorname{reg}(G, D')$, which is the lunar region of G and $D' = j_G(D_1 \times D_2)$. Equivalently, the sum ranges over D such that

$$D' \leq_{rb} D \leq_{rb} \overline{G}D'$$

in the weak right Bruhat order $\leq_{rb}$ on $\mathcal{C}^n$ given in Definition 7.1.1. This relation of the product in SΛ to the weak right Bruhat order appears in Loday and Ronco [57]. From the remark after Lemma 5.3.2, it is clear that one can also define the product in SΛ by summing over T-shuffles where $T = \operatorname{type} G$.

As an example,

$$F_{2|1} * F_{1|2} = F_{2|1|3|4} + F_{2|3|1|4} + F_{2|3|4|1} + F_{3|2|1|4} + F_{3|2|4|1} + F_{3|4|2|1}.$$

The chambers that occur in the right hand side are precisely those in $\operatorname{reg}(12|34, 2|1|3|4)$, which is the shaded region in Figure 7.2.

From the discussion in this section, one also observes that:

Proposition 7.2.2 *The map* SΠ $\to$ SΛ *that sends* $F_{(C,D)} \to F_D$ *is a map of Hopf algebras.*

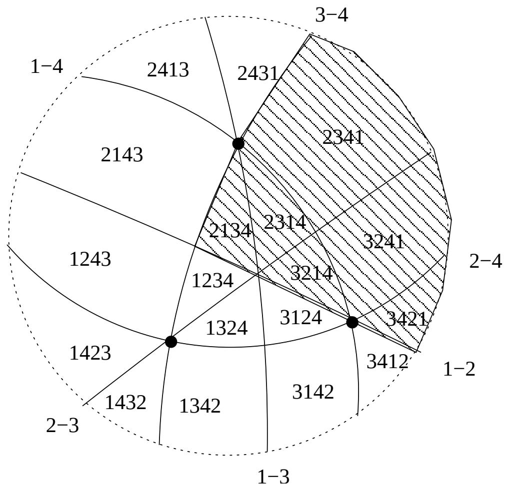


Figure 7.2: A lunar region in the Coxeter complex Σ^4.

7.3 The Hopf algebra SΠ in the M basis

In this section, we write down formulas for the coproduct and product in the M basis of SΠ. We also show how they can be used to deduce the corresponding formulas for SΛ obtained in [4]. The global descent maps $\mathrm{GDes} : \mathcal{C}^n \times \mathcal{C}^n \to \Sigma^n$ and $\mathrm{gdes} : \mathrm{S}_n \to S$ appear in the computations. For more details on these maps, see Sections 5.2-5.4.

7.3.1 A preliminary result

In order to compare the coproducts in the F and M basis, we need to relate the partial order $\leq$ in Σ^n to the same partial order restricted to a star region Σ^n_K. The precise fact which is needed is stated below and it is valid for the poset of faces of any central hyperplane arrangement.

Fact 7.3.1 Let $K \leq \overline{C}, D$. Then

$$(E, D) \leq (C, D) \iff (KE, D) \leq (KC, D).$$

Proof The reader may refer to Figure 7.3. The assumption $K \leq \overline{C}, D$ implies that KC and $\overline{C}$ are opposite chambers in $\mathrm{star}(K)$. The lemma follows from the following equivalences.

$$\begin{aligned}
(E, D) \leq (C, D) &\iff C - E - D && \text{(Definition 7.1.1)}\\
&\iff C - E - KE - D && \text{(Gate property)}\\
&\iff C - KE - D - \overline{C} && \\
&\iff C - KC - KE - D - \overline{C} && \text{(Gate property)}\\
&\iff KC - KE - D && \\
&\iff (KE, D) \leq (KC, D) && \text{(Definition 7.1.1).}
\end{aligned}$$

For the backward implication of the third equivalence, note that we always have the gallery $C - E - \overline{C}$. By the gate property, we then get $C - E - KE - \overline{C}$, which along with $KE - D - \overline{C}$ gives the gallery $C - E - KE - D$. For the second to last equivalence, we use the observation made at the beginning of the proof.

□

7.3.2 Coproduct in the M basis

Theorem 7.3.1 *The coproduct on* $\mathrm{S\Pi}$ *is given by*

$$\Delta(M_{(C,D)}) = 1 \otimes M_{(C,D)} + M_{(C,D)} \otimes 1 + \Delta_+(M_{(C,D)}), \ \textit{where}$$

$$\Delta_+(M_{(C,D)}) = \sum_{K:\ \mathrm{rank}\, K=1,\ K\le D,\ K\le\overline{C}} M_{(C_1,D_1)} \otimes M_{(C_2,D_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$*, we have* $b_K(D) = D_1 \times D_2$ *and* $b_K(KC) = C_1 \times C_2$*.*

Comparing with Definition 7.2.8, the coproduct in the M basis has fewer terms than in the F basis due to the additional condition $K \le \overline{C}$ on K. Since $\mathrm{GDes}(C,D) = D \cap \overline{C}$, one can equivalently write the condition on K as $K \le \mathrm{GDes}(C,D),\ \mathrm{rank}\, K = 1$.

Proof To prove the theorem, we start with the above formula and derive the coproduct in the F basis. The chambers D and E are fixed in the computation and K is a vertex that varies.

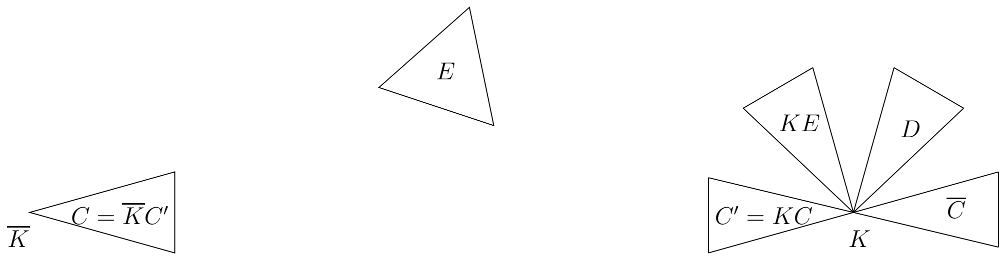


Figure 7.3: The close relation between the star regions of K and $\overline{K}$.

It is useful to keep Figure 7.3 in mind. The steps in the computation are as below. The justification for the main steps are provided after the computation.

$$\begin{aligned}
\Delta_+(F_{(E,D)}) &= \sum_{C:\ (E,D)\le(C,D)} \Delta_+(M_{(C,D)}) && \text{(Relation (7.3))}\\
&= \sum_{\substack{(K,C):\ \mathrm{rank}\, K=1\\ K\le D,\ K\le\overline{C},\ (E,D)\le(C,D)}} M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} && \left(\begin{smallmatrix} b_K(D)=D_1\times D_2\\ b_K(KC)=C_1\times C_2.\end{smallmatrix}\right)\\
&= \sum_{\substack{(K,C'):\ \mathrm{rank}\, K=1\\ K\le D,\ K\le C',\ (KE,D)\le(C',D)}} M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} && \left(\begin{smallmatrix} b_K(D)=D_1\times D_2\\ b_K(C')=C_1\times C_2.\end{smallmatrix}\right)\\
&= \sum_{\substack{(K,C_1,C_2):\ \mathrm{rank}\, K=1\\ K\le D,\ (E_i,D_i)\le(C_i,D_i)}} M_{(C_1,D_1)} \otimes M_{(C_2,D_2)} && \left(\begin{smallmatrix} b_K(D)=D_1\times D_2\\ b_K(KE)=E_1\times E_2.\end{smallmatrix}\right)\\
&= \sum_{K:\ \mathrm{rank}\, K=1,\ K\le D} F_{(E_1,D_1)} \otimes F_{(E_2,D_2)}. && \text{(Relation (7.3))}
\end{aligned}$$

For the third equality, given K, the chambers C and C' determine each other by the relations $C' = KC$ and $C = \overline{K}C'$, as indicated in Figure 7.3. The equivalence between the conditions $(E,D) \le (C,D)$ and $(KE,D) \le (C',D)$ in this situation is the content of Fact 7.3.1, which was proved above. We now recall that star regions are convex. Since the partial order $\le$ is defined by a gallery condition, we have

$$(KE,D) \le (C',D) \text{ in the complex } \Sigma^n \iff (KE,D) \le (C',D) \text{ in the complex } \Sigma^n_K.$$

Hence all the action is now happening in the smaller Coxeter complex Σ^n_K.

For the fourth equality, given K, the chamber C' and the pair (C_1, C_2) determine each other because the map b_K is an isomorphism. The equivalence of the conditions $(KE, D) \leq (C', D)$ and $(E_i, D_i) \leq (C_i, D_i)$ is due to the compatibility of galleries with the map b_K.

□

Let $C = C^1|C^2|\ldots|C^n$ and $D = D^1|D^2|\ldots|D^n$ be two permutations. Recall that Definition 5.4.4 implies that

$$s_i \in \operatorname{type}(\operatorname{GDes}(C, D)) \iff D^j \text{ appears after } D^k \text{ in } C \text{ for all } j \leq i \text{ and } i+1 \leq k.$$

If this happens then one says that the chamber D has a global descent of type s_i, or at position i, with respect to the chamber C. This fact can be used to derive the combinatorial content of Theorem 7.3.1 given below. We omit the details, since they are similar to those involved in showing that Definitions 7.2.6 and 7.2.8 are equivalent.

Theorem 7.3.2 *The coproduct on* SΠ *is given by* $\Delta_+(M_{(C,D)}) =$

$$\sum_{s_i:\ s_i \in \operatorname{type}(\operatorname{GDes}(C,D))} M_{\operatorname{st}(C^{n-i+1}|\cdots|C^n, D^1|\cdots|D^i)} \otimes M_{\operatorname{st}(C^1|\cdots|C^{n-i}, D^{i+1}|\cdots|D^n)}.$$

As an example,

$$\Delta(M_{(2|1|3,1|3|2)}) = 1 \otimes M_{(2|1|3,1|3|2)} + M_{(2|1|3,1|3|2)} \otimes 1 + M_{(1|2,1|2)} \otimes M_{(1,1)}.$$

Remark From the above theorem, one cannot conclude that SΠ is cofree. This is because, given $M_{(C_1,D_1)}$ and $M_{(C_2,D_2)}$, one cannot uniquely recover $M_{(C,D)}$. However, the M basis is much better than the F basis to study the coradical filtration of SΠ. For example, one sees directly that the elements in the set

$$\{M_{(C,D)} \mid \operatorname{GDes}(C, D) = D \cap \overline{C} = \emptyset\}$$

are primitive. The graded cardinalities of the above set are 1, 2, 18, etc. Direct hand computation from the coproduct formula shows that the graded dimensions of the space of primitive elements are 1, 3, 29, etc. In the first two degrees, $M_{(1,1)}$, $M_{(1|2,1|2)}$, $M_{(2|1,2|1)}$, and $M_{(1|2,2|1)} - M_{(2|1,1|2)}$ provide a basis for the primitive elements.

Theorem 7.3.3 *[4, Theorem 3.1] Let $w = w^1|\ldots|w^n$ be a permutation. The coproduct on* SΛ *is given by*

$$\Delta(M_w) = 1 \otimes M_w + M_w \otimes 1 + \sum_{s_i:\ s_i \in \operatorname{gdes}(w)} M_{\operatorname{st}(w^1|\ldots|w^i)} \otimes M_{\operatorname{st}(w^{i+1}|\ldots|w^n)}.$$

Proof Take $C = C_0^n = 1|2|\ldots|n$ and $D = wC_0^n$ in the previous theorem and then project down the formula to SΛ via the distance map. The result follows by noting that $d(C_0^n, uC_0^n) = u$ and $\operatorname{type}(\operatorname{GDes}(C_0^n, wC_0^n)) = \operatorname{gdes}(w)$.

□

For $s_i \in \operatorname{gdes}(w)$, given $w_1 = \operatorname{st}(w^1|\ldots|w^i)$ and $w_2 = \operatorname{st}(w^{i+1}|\ldots|w^n)$, one can uniquely recover w as $j'(w_1 \times w_2)$, with j' as in Definition 7.2.3. Hence, in contrast to the situation for SΠ, the formula in the M basis shows that SΛ is cofree. This is explained in detail in the proof of [4, Theorem 6.1].

7.3.3 Product in the M basis

Theorem 7.3.4 *The product on* SΠ *is given by*

$$M_{(C_1,D_1)} * M_{(C_2,D_2)} = \sum_{\substack{(C,D): \\ M(i):\ GD=j_G(D_1\times D_2) \\ M(ii):\ GC=j_G(C_1\times C_2) \\ M(iii):\ C-GC-GD-D}} M_{(C,D)}.$$

The vertex $G \in \Sigma^n$ is as in Definition 7.2.9.

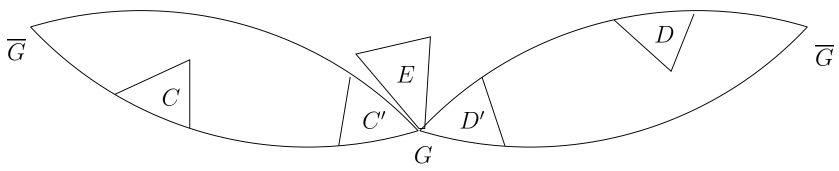


Figure 7.4: The term $M_{(C,D)}$ occurring in the product $M_{(C_1,D_1)} * M_{(C_2,D_2)}$.

In Figure 7.4, we have used the notation $C' = j_G(C_1 \times C_2)$ and $D' = j_G(D_1 \times D_2)$. The two lunar regions shown in the figure lie on a sphere and are supposed to meet at $\overline{G}$, the vertex opposite to G. For simplicity, this is not shown in the figure, where there are two vertices labeled $\overline{G}$.

Proof As in Theorem 7.3.1, to prove this theorem, we start with the above formula and derive the product in the F basis. We use the following simple consequence of the gate property.

$$\text{For} \quad G \le E, \qquad C - GC - E - GD - D \iff C - E - D. \tag{7.5}$$

Let $M(1)$ denote the condition

$$(j_G(E_1 \times E_2), j_G(D_1 \times D_2)) \le (j_G(C_1 \times C_2), j_G(D_1 \times D_2)),$$

or equivalently, $j_G(C_1 \times C_2) - j_G(E_1 \times E_2) - j_G(D_1 \times D_2)$.

The vertex G and the chambers E_1, E_2, D_1 and D_2 are fixed in the computation below. For the simplicity of notation, we denote the conditions $M(1)$, $M(i)$, $M(ii)$ and $M(iii)$ simply as (1),(i),(ii) and (iii) respectively.

$$\begin{aligned}
F_{(E_1,D_1)} * F_{(E_2,D_2)} &= \sum_{\substack{C_1,C_2: \\ (1)}} M_{(C_1,D_1)} * M_{(C_2,D_2)} && \begin{pmatrix}\text{Compatibility of} \\ \text{galleries with the} \\ \text{map } j_G.\end{pmatrix} \\
&= \sum_{\substack{C_1,C_2: \\ (1)}} \sum_{\substack{C,D: \\ (i),(ii),(iii)}} M_{(C,D)} && \text{(Theorem 7.3.4)} \\
&= \sum_{\substack{D: \\ (i)}} \sum_{\substack{C: \\ (1),(ii),(iii)}} M_{(C,D)} && \begin{pmatrix}\text{Switching the order of} \\ \text{the summations.}\end{pmatrix} \\
&= \sum_{\substack{D: \\ (i)}} \sum_{\substack{C: \\ C-j_G(E_1\times E_2)-D}} M_{(C,D)} && (7.5) \\
&= \sum_{\substack{D: \\ (i)}} F_{(j_G(E_1\times E_2),D)}. && \text{(Relation (7.3))}
\end{aligned}$$

For the third equality, also note that C determines C_1 and C_2, since j_G is an isomorphism. Hence we can get rid of them from the summation.

□

The combinatorial content of the above statement is as below. Let $(C_i, D_i) \in \Sigma^{g_i}$ for $i = 1, 2$ and let $[g_1 + 1, g_1 + g_2] = \{g_1 + 1, \dots, g_1 + g_2\}$.

Theorem 7.3.5 *The product on* SΠ *is given by*

$$M_{(C_1,D_1)} * M_{(C_2,D_2)} = \sum_{(C,D):\ —} M_{(C,D)},$$

where — says that C (resp. D) is a shuffle of C_1 and C_2 (resp. D_1 and D_2) such that if $*_1 \in [g_1]$ and $*_2 \in [g_1 + 1, g_1 + g_2]$ then $*_2$ does not appear before $*_1$ in both C and D.

As an example,

$$\begin{aligned} M_{(1|2,2|1)} * M_{(1,1)} &= M_{(1|2|3,2|1|3)} + M_{(1|3|2,2|1|3)} + M_{(3|1|2,2|1|3)} \\ &\quad + M_{(1|2|3,2|3|1)} + M_{(1|3|2,2|3|1)} + M_{(1|2|3,3|2|1)}. \end{aligned}$$

Theorem 7.3.6 *[4, Theorem 4.1] Let $u \in \mathrm{S}_{g_1}$, $v \in \mathrm{S}_{g_2}$ with $g_1 + g_2 = n$. Also let $G = 1 \dots g_1 | g_1 + 1 \dots n$ be the fundamental vertex in Σ^n of type (g_1, g_2). The product on* SΛ *is given by*

$$M_u * M_v = \sum_w |S^0_w(u \times v)|\ M_w,$$

where

$$S^0_w(u \times v) = \left\{ (C, D) \in \mathcal{C}^n \times C^n, d(C, D) = w, GD = C^n_0 \;\middle|\; \begin{array}{l} C - GC - GD - D \\ GC = (u^{-1} \times v^{-1})C^n_0. \end{array} \right\}$$

There are two more sets $S^+_w(u \times v)$ and $S^-_w(u \times v)$, more combinatorial in nature, with the same cardinality as $S^0_w(u \times v)$. This is explained in detail in Lemma 5.3.2. The description in [4] for the above product is given using the set $S^+_w(u \times v)$.

Proof Specialize Theorem 7.3.4 to the chambers $C_1 = u^{-1}C^{g_1}_0$, $C_2 = v^{-1}C^{g_2}_0$, $D_1 = C^{g_1}_0$ and $D_2 = C^{g_2}_0$. Then note that $d(C_1, D_1) = u$ and $d(C_2, D_2) = v$. Now the result follows by projecting to SΛ via the distance map and noting that $j_G(C^{g_1}_0 \times C^{g_2}_0) = C^n_0$.

□

7.3.4 The switch map on the M basis

We note a striking feature of the formulas in the M basis of SΠ. In contrast to the formulas in the F basis, the two coordinates of $\mathcal{C} \times \mathcal{C}$ now play a symmetric role. We formulate this precisely as a corollary to Theorems 7.3.1 and 7.3.4.

Corollary 7.3.1 *The map* SΠ → SΠ *that sends $M_{(C,D)}$ to $M_{(D,C)}$ is a map of algebras and a map of anti-coalgebras.*

Proof The algebra part is clear. The coalgebra part follows from the following fact.

$$b_K(D) = D_1 \times D_2,\ b_{\overline{K}}(\overline{K}D) = D'_1 \times D'_2 \quad\Longrightarrow\quad D'_1 = D_2,\ D'_2 = D_1.$$

In other words, shifting from K to $\overline{K}$ has the effect of interchanging the factors.

□

Applying the map SΠ ↠ SΛ of Hopf algebras that sends $M_{(C,D)}$ to $M_{d(C,D)}$, one obtains:

Corollary 7.3.2 *The map* SΛ → SΛ *that sends M_w to $M_{w^{-1}}$ is a map of algebras and a map of anti-coalgebras.*

7.4 The Hopf algebra SΠ in the S basis

In this section, we write down formulas for the coproduct and product in the S basis of SΠ. From the coproduct formula, we conclude that SΠ is a cofree coalgebra, thus proving one half of Theorem 7.1.2.

7.4.1 Two preliminary results

In order to compare the coproducts in the M and S basis, we need to relate the partial order $\leq'$ in Σ^n to the same partial order restricted to a star region $\Sigma^n_{K'}$. In particular, we need to undertake such a study for the distance map d and the partial order $\leq_b$, which are a part of $\leq'$. The crucial facts about d and $\leq_b$ which are needed are stated below. They are valid for any Coxeter complex Σ. A special case of the second fact was proved in Chapter 5, see Proposition 5.3.7.

Fact 7.4.1 Let K and K' be faces of Σ of the same type. Also let C, D, C' and D' be chambers with $K \leq D, \overline{C}$ and $K' \leq D', \overline{C'}$. Then

$$d(C, D) = d(C', D') \iff d(KC, D) = d(K'C', D').$$

Proof The reader may refer to Figure 7.6.

For the forward implication, let $w \in W$ be such that $wC = C'$ and $wD = D'$. Then since the action of W on Σ is type-preserving, we get $wK = K'$. This implies that $w(KC) = w(K'C')$, which proves the right hand side.

For the backward implication, let $w \in W$ be such that $w(KC) = K'C'$ and $wD = D'$. Then since the action of W on Σ is type-preserving, we get $wK = K'$, which implies $w\overline{K} = \overline{K'}$. Now $\overline{K} \leq C$ and $\overline{K'} \leq C'$ implies that $C = \overline{K}(KC)$ and $C' = \overline{K'}(K'C)$. Hence we obtain $wC = C'$, which proves the left hand side.

Fact 7.4.2 Let C_0 be the chosen fundamental chamber in Σ. Also let K and K' be faces of Σ of the same type, with $K \leq \overline{C}_0$. Then using the type-preserving action of W on Σ, we have an isomorphism

$$\Sigma_K \xrightarrow{\cong} \Sigma_{K'},$$

sending $KC_0 \mapsto K'C_0$, or equivalently, $K\overline{C}_0 \mapsto K'\overline{C}_0$. Further, let

$$D \in \Sigma_K \longleftrightarrow D'' \in \Sigma_{K'}$$

correspond to each other under the above isomorphism. In other words,

$$d(KC_0, D) = d(K'C_0, D'') \quad \text{or equivalently,} \quad d(D, \overline{C}_0) = d(D'', K'\overline{C}_0). \tag{7.6}$$

Let $K' \leq D'$. Then

$$D' \leq_b D \text{ in the complex } (\Sigma, C_0) \iff D' \leq_b D'' \text{ in the complex } (\Sigma_{K'}, K'C_0).$$

We recall that the partial order $\leq_b$ depends on the choice of a fundamental chamber. This should clarify our use of the notation (Σ, C_0) and $(\Sigma_{K'}, K'C_0)$.

As a special case, we obtain

$$D'' \leq_b D, \quad \text{and in particular,} \quad K'C_0 \leq_b KC_0.$$

The last conclusion was the content of Proposition 5.3.7; its alternative proof is being generalized here.

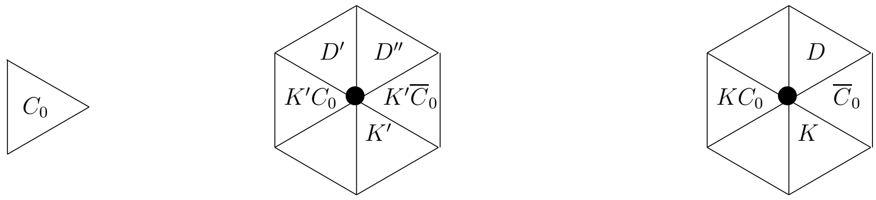


Figure 7.5: A comparison of two star regions.

Proof To get an idea of what is going on, the reader may refer to Figure 7.5. It shows two bold dots, which are the faces K and K' of the same type. The hexagonal regions give a schematic picture of their star regions.

In order to use the definition of $\leq_b$, we first need to name chambers by group elements. Accordingly, let $D = wC_0$, $D' = w'C_0$, and $D'' = w''C_0$. Then observe that $w(w')^{-1}D' = D$ and hence $w(w')^{-1}K' = K$. The fact now follows from the following sequence of equivalences.

$$\begin{aligned}
D' \leq_b D \text{ in } (\Sigma, C_0) &\iff C_0 - w(w')^{-1}C_0 - D && \text{(Definition 7.1.1)}\\
&\iff KC_0 - K(w(w')^{-1}C_0) - D && \text{(Fact 7.3.1)}\\
&\iff KC_0 - w(w')^{-1}(K'C_0) - D && (w(w')^{-1}K' = K)\\
&\iff K'C_0 - w''(w')^{-1}(K'C_0) - D'' && \begin{pmatrix}\text{Multiplying}\\ \text{by } w''w^{-1}\end{pmatrix}\\
&\iff D' \leq_b D'' \text{ in } (\Sigma_{K'}, K'C_0) && \text{(Definition 7.1.1).}
\end{aligned}$$

We made use of the hypothesis $K \leq \overline{C}_0$ in the second equivalence. In the fourth equivalence, we used Equation (7.6) to conclude that $w''(w')^{-1}$ sends KC_0 to $K'C_0$.

□

7.4.2 Coproduct in the S basis

Theorem 7.4.1 *The coproduct on* SΠ *is given by*

$$\Delta(S_{(C,D)}) = 1 \otimes S_{(C,D)} + S_{(C,D)} \otimes 1 + \Delta_+(S_{(C,D)}), \ \textit{where}$$

$$\Delta_+(S_{(C,D)}) = \sum_{K:\ \operatorname{rank} K = 1,\ K \leq D,\ K \leq \overline{C}, \overline{C}_0} S_{(C_1,D_1)} \otimes S_{(C_2,D_2)},$$

where for $b_K : \Sigma_K^n \to \Sigma^{k_1} \times \Sigma^{k_2}$, *we have* $b_K(D) = D_1 \times D_2$ *and* $b_K(KC) = C_1 \times C_2$.

Comparing with Theorem 7.3.1, the coproduct in the S basis has fewer terms than in the M basis due to the additional condition $K \leq \overline{C}_0$ on K. In other words, the term $\mathrm{GDes}(C, D)$ is now replaced by $\overline{C} \cap D \cap \overline{C}_0$.

Proof To prove the theorem, we start with the above formula and derive the coproduct in the M basis given by Theorem 7.3.1. The chambers C' and D' are fixed in the computation and K and K' are vertices that vary.

It is useful to bear Figure 7.6 in mind. The main steps, which are as below, are

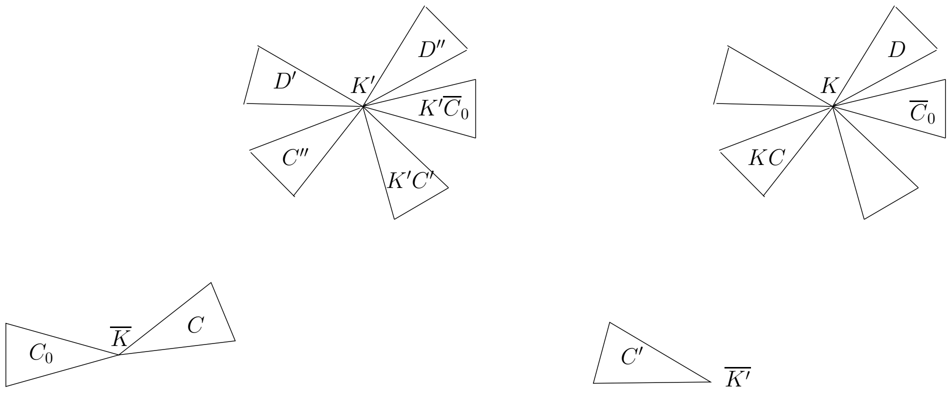


Figure 7.6: The relation between the coproducts in the M and S basis.

justified after the computation.

$$\begin{aligned}
\Delta_+(M_{(C',D')}) &= \sum_{(C,D):\ (C',D')\leq'(C,D)} \Delta_+(S_{(C,D)}) && \text{(Relation (7.3))}\\
&= \sum_{\substack{(K,C,D):\ \mathrm{rank}\,K=1\\ K\leq D,\overline{C},\overline{C}_0,\ (C',D')\leq'(C,D)}} S_{(C_1,D_1)}\otimes S_{(C_2,D_2)} && \left(\begin{smallmatrix} b_K(D)=D_1\times D_2\\ b_K(KC)=C_1\times C_2.\end{smallmatrix}\right)\\
&= \sum_{\substack{(K',C'',D''):\ \mathrm{rank}\,K'=1\\ K'\leq D',\overline{C'},\ (K'C',D')\leq'(C'',D'')}} S_{(C_1,D_1)}\otimes S_{(C_2,D_2)} && \left(\begin{smallmatrix} b_{K'}(D'')=D_1\times D_2\\ b_{K'}(C'')=C_1\times C_2.\end{smallmatrix}\right)\\
&= \sum_{\substack{(K',C_1,C_2,D_1,D_2):\ \mathrm{rank}\,K'=1\\ K'\leq D',\overline{C'},\ (C_i',D_i')\leq'(C_i,D_i)}} S_{(C_1,D_1)}\otimes S_{(C_2,D_2)} && \left(\begin{smallmatrix} b_{K'}(D')=D_1'\times D_2'\\ b_{K'}(K'C')=C_1'\times C_2'.\end{smallmatrix}\right)\\
&= \sum_{\substack{K':\ \mathrm{rank}\,K'=1\\ K'\leq D',\overline{C'}}} M_{(C_1',D_1')}\otimes M_{(C_2',D_2')} && \text{(Relation (7.3))}
\end{aligned}$$

We first point out that in the third step above, for lack of space, we have shortened the expression

$$(K'C',D')\leq'(C'',D'') \quad \text{in the complex} \quad (\Sigma^n_{K'},K'C_0),$$

to simply $(K'C',D')\leq'(C'',D'')$. The heart of the proof lies in this step.

The triplets (K,C,D) and (K',C'',D''), which index the summations on either side, are related to each other by the relations

$$\mathrm{type}\,K=\mathrm{type}\,K',\ \ d(D,\overline{C}_0)=d(D'',K'\overline{C}_0) \text{ and } d(KC,\overline{C}_0)=d(C'',K'\overline{C}_0),$$

as indicated in Figure 7.6. The reader should check that under these relations the triplets determine each other. For example, given (K,C,D), one can define K' as the face of D' of the same type as K, and so forth.

The conditions $(C',D')\leq'(C,D)$ and $(K'C',D')\leq'(C'',D'')$ are equivalent under this correspondence. This follows from the following two facts.

$$\begin{aligned}
d(C',D')=d(C,D) &\iff d(K'C',D')=d(KC,D) && \text{(Fact 7.4.1)}\\
&\iff d(K'C',D')=d(C'',D'') && \text{(Equation (1.5))}\\
D'\leq_b D \text{ in } (\Sigma,C_0) &\iff D'\leq_b D'' \text{ in } (\Sigma_{K'},K'C_0) && \text{(Fact 7.4.2).}
\end{aligned}$$

For the fourth equality, we require that

$$(K'C',D')\leq'(C'',D'') \text{ in } (\Sigma_{K'},K'C_0) \iff (C_i',D_i')\leq'(C_i,D_i) \text{ in } (\Sigma^{g_i},C_0^{g_i}),$$

where we have $b_{K'}(D'') = D_1 \times D_2$, $b_{K'}(C'') = C_1 \times C_2$, $b_{K'}(D') = D'_1 \times D'_2$ and $b_{K'}(K'C') = C'_1 \times C'_2$. This follows from the compatibility of the distance map and galleries with joins. Also observe that $b_{K'}(K'C_0) = C_0^{g_1} \times C_0^{g_2}$. This is important since the partial order $\leq'$ depends on the fundamental chamber.

$\square$

Theorem 7.4.2 *The combinatorial formula for the coproduct in the S basis is given by Theorem 7.3.2 with the term* $\mathrm{GDes}(C, D)$ *replaced by* $\overline{C} \cap D \cap \overline{C}_0$.

Also, from the definitions, one has

$$s_i \in \mathrm{type}(\overline{C} \cap D \cap \overline{C}_0) \iff \begin{array}{l} D \text{ has a global descent at position } i \text{ and} \\ C \text{ has a global ascent at position } n-i, \end{array}$$

with the natural definition for a global ascent. As an example,

$$\Delta(S_{(1|3|2,2|3|1)}) = 1 \otimes S_{(1|3|2,2|3|1)} + S_{(1|3|2,2|3|1)} \otimes 1 + S_{(2|1,1|2)} \otimes S_{(1,1)}.$$

The S basis does precisely what the M basis failed to do. Namely, it allows us to conclude that SΠ is cofree. This is because, given $S_{(C_1,D_1)}$ and $S_{(C_2,D_2)}$, one can uniquely recover $S_{(C,D)}$, by the formula $D = j'(D_1 \times D_2)$ and $C = j''(C_1 \times C_2)$, with j' and j'' as in Definition 7.2.3. More precisely, let $\mathtt{S}^0 = \{1\}$ and for $k \geq 1$, let

$$\mathtt{S}^k = \{(C, D) \mid \mathrm{rank}(\overline{C} \cap D \cap \overline{C}_0) = k - 1\}.$$

Let $\mathrm{S\Pi}_k$ be the vector subspace of SΠ spanned by $\{S_{(C,D)} \mid (C, D) \in \mathtt{S}^k\}$.

Theorem 7.4.3 *With the above grading,* SΠ *is a cofree graded coalgebra. A basis for the kth level of the coradical filtration of* SΠ *is*

$$\coprod_{i=0}^{k} \mathtt{S}^k.$$

In particular, a basis for the space of primitive elements is

$$\{S_{(C,D)} \mid \overline{C} \cap D \cap \overline{C}_0 = \emptyset\}.$$

This follows directly from the definitions (Sections 3.1.1-3.1.2).

Remark We mention that Theorem 7.3.3 can also be deduced from Theorem 7.4.1 in the same way as from Theorem 7.3.1.

7.4.3 Product in the S basis

Theorem 7.4.4 *The product on* SΠ *is given by*

$$S_{(C_1,D_1)} * S_{(C_2,D_2)} = \sum_{\substack{(K,C,D): \\ S(i):\ b_K(KD) = D_1 \times D_2 \\ S(ii):\ b_K(KC) = C_1 \times C_2 \\ S(iii):\ C - KC - KD - D \\ S(iv):\ C_0 - KC_0 - KD - D}} S_{(C,D)}.$$

The proof is left as an exercise to the reader. Unlike the product in the M basis given by Theorem 7.3.4, we note that the above product depends on C_0. The coefficient of $S_{(C,D)}$ continues to be nonnegative but it can be greater than 1.

The combinatorial content of the above statement is as below. We recall that a vertex of type (g_1, g_2) is a two block composition of the set $[g_1 + g_2]$, with underlying composition (g_1, g_2). Now let $(C_i, D_i) \in \Sigma^{g_i}$ for $i = 1, 2$.

Theorem 7.4.5 *The product on* SΠ *is given by*

$$S_{(C_1,D_1)} * S_{(C_2,D_2)} = \sum_{\substack{K=K_1|K_2:\\ \text{a vertex of type } (g_1,g_2)}} \sum_{(C,D):\ —} S_{(C,D)},$$

where — says that C (resp. D) is a K-shuffle of C_1 and C_2 (resp. D_1 and D_2) such that if $*_1 \in K_1$ and $*_2 \in K_2$ and $*_2$ appears before $*_1$ in D then $*_2$ appears after $*_1$ in both C and C_0.

The definition of a K-shuffle can be found in Definition 6.2.5, and its relation with conditions such as $S(i)$ and $S(ii)$ can be found in Equation (6.23). The condition above involving $*_1$ and $*_2$ is a translation of the gallery conditions $S(iii)$ and $S(iv)$. As an example, the product $S_{(1|2,2|1)} * S_{(1,1)}$ equals

$$\begin{aligned}
&S_{(1|2|3,2|1|3)} + S_{(1|3|2,2|1|3)} + S_{(3|1|2,2|1|3)} + S_{(1|2|3,2|3|1)} + S_{(1|3|2,2|3|1)} + S_{(1|2|3,3|2|1)}\\
&+S_{(1|3|2,3|1|2)} + S_{(1|2|3,3|1|2)} + S_{(2|1|3,3|1|2)} + S_{(1|3|2,3|2|1)} + S_{(1|2|3,3|2|1)}\\
&+S_{(2|3|1,3|2|1)} + S_{(2|1|3,3|2|1)} + S_{(1|2|3,3|2|1)}.
\end{aligned}$$

The three rows correspond to the vertices 12|3, 13|2 and 23|1 respectively, which are of type $(2,1)$. Note that the term $S_{(1|2|3,3|2|1)}$ appears with coefficient 3.

Remark We mention that Theorem 7.3.6 can also be deduced from Theorem 7.4.4 in the same way as from Theorem 7.3.4.

7.5 The Hopf algebra RΠ in the H basis

In this section, we write down formulas for the coproduct and product in the H basis of RΠ. From the product formula, we conclude that RΠ, and hence SΠ, is a free graded algebra, thus proving the second half of Theorem 7.1.2. We also show how they can be used to deduce the formulas for the quotient RΛ, and observe that, as expected, they are dual to the formulas for SΛ deduced in Section 7.3.

The formulas in the K basis of RΠ are written in Definitions 6.5.6 and 6.8.6. They are simply obtained from Definitions 7.2.8 and 7.2.9 by changing F to K and interchanging the two coordinates. We use this in the computations.

7.5.1 Coproduct in the H basis

Theorem 7.5.1 *The coproduct on* RΠ *is given by*

$$\Delta(H_{(D,C)}) = 1 \otimes H_{(D,C)} + H_{(D,C)} \otimes 1 + \Delta_+(H_{(D,C)}), \text{ where}$$

$$\Delta_+(H_{(D,C)}) = \sum_{K:\ \operatorname{rank} K=1,\ D-KD-KC-C} H_{(D_1,C_1)} \otimes H_{(D_2,C_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(KD) = D_1 \times D_2$ and $b_K(KC) = C_1 \times C_2$.

Proof This is a straightforward computation.

$$\begin{aligned}
\Delta_+(H_{(D,C)}) &= \sum_{E:\ D-E-C} \Delta_+(K_{(E,C)}) && \text{(Relation (7.4))}\\
&= \sum_{\substack{E:\\ D-E-C}} \ \sum_{\substack{K:\ \operatorname{rank} K=1\\ K\le E}} K_{(E_1,C_1)} \otimes K_{(E_2,C_2)} && \left(\substack{b_K(E)=E_1\times E_2\\ b_K(KC)=C_1\times C_2.}\right)\\
&= \sum_{\substack{(K,E):\ \operatorname{rank} K=1,\ K\le E\\ D-KD-E-KC-C}} K_{(E_1,C_1)} \otimes K_{(E_2,C_2)} && \text{(Gate property)}\\
&= \sum_{\substack{K:\ \operatorname{rank} K=1\\ D-KD-KC-C}} \ \sum_{\substack{E:\ K\le E\\ KD-E-KC}} K_{(E_1,C_1)} \otimes K_{(E_2,C_2)}\\
&= \sum_{\substack{K:\ \operatorname{rank} K=1\\ D-KD-KC-C}} \ \sum_{\substack{E_1,E_2:\\ D_i-E_i-C_i}} K_{(E_1,C_1)} \otimes K_{(E_2,C_2)} && \left(\substack{b_K(KD)=D_1\times D_2\\ b_K(KC)=C_1\times C_2.}\right)\\
&= \sum_{\substack{K:\ \operatorname{rank} K=1\\ D-KD-KC-C}} H_{(D_1,C_1)} \otimes H_{(D_2,C_2)}. && \text{(Relation (7.4))}
\end{aligned}$$

For the fifth equality, we used the compatibility of galleries with the map b_K.

□

Theorem 7.5.2 *The coproduct on* RΛ *is given by*

$$\Delta(H_w) = 1 \otimes H_w + H_w \otimes 1 + \sum_{u,v} |S_w^-(u \times v)|\ H_u \otimes H_v.$$

To avoid repetition, we refer the reader to Section 5.3.6 for the definition of the set $S_w^-(u \times v)$. We note that this result is dual to that in Theorem 7.3.6, consistent with the fact that RΛ ≅ SΛ* as Hopf algebras with $H_w = M_w^*$. A proof is strictly not necessary; but we give it to show how this result follows from the previous theorem.

Proof As in the proof of Theorem 7.3.3, we put $D = C_0^n$ in the previous theorem, which implies that $D_1 = C_0^{g_1}$ and $D_2 = C_0^{g_2}$, where type $K = (g_1, g_2)$. Now let $C = wC_0^n$, $C_1 = uC_0^{g_1}$ and $C_2 = vC_0^{g_2}$, and apply the distance map. The coefficient of $H_u \otimes H_v$ is the number of vertices K of type (g_1, g_2) such that

$$C_0 - K(wC_0) - wC_0 \quad \text{and} \quad b_K(K(wC_0)) = (u \times v)C_0.$$

Let σ be the (g_1, g_2)-shuffle which corresponds to K. Then the two conditions above correspond to the conditions (i) and (ii) respectively in the set $S_w^-(u \times v)$.

□

7.5.2 Product in the H basis

Theorem 7.5.3 *The product on* RΠ *is given by*

$$\begin{aligned}
H_{(D_1,C_1)} * H_{(D_2,C_2)} &= H_{(\overline{G} j_G(D_1\times D_2), j_G(C_1\times C_2))}\\
&= H_{(\overline{j}(D_1\times D_2), j(C_1\times C_2))}. && (\textit{Definition } 7.2.3)
\end{aligned}$$

The vertex $G \in \Sigma^n$ is as in Definition 7.2.9.

Proof The second equality above follows from the first by using Definition 7.2.3. We now prove the first equality.

$$\begin{aligned}
H_{(D_1,C_1)} * H_{(D_2,C_2)} &= \sum_{\substack{E_1,E_2:\\ D_i-E_i-C_i}} K_{(E_1,C_1)} \otimes K_{(E_2,C_2)} \\
&= \sum_{\substack{E_1,E_2:\\ D_i-E_i-C_i}} \sum_{\substack{E:\\ GE=j_G(E_1\times E_2)}} K_{(E,j_G(C_1\times C_2))} \\
&= \sum_{\substack{E:\\ j_G(D_1\times D_2)-GE-j_G(C_1\times C_2)}} K_{(E,j_G(C_1\times C_2))} \\
&= \sum_{\substack{E:\\ \overline{G}j_G(D_1\times D_2)-E-j_G(C_1\times C_2)}} K_{(E,j_G(C_1\times C_2))} \\
&= H_{(\overline{G}j_G(D_1\times D_2),j_G(C_1\times C_2))}.
\end{aligned}$$

For the third equality, note that E determines E_1, E_2 because the map j_G is an isomorphism. We also used the compatibility of galleries with the map j_G. For the fourth equality, we used the fact that

$$\text{For} \quad G \le C, D \quad \text{we have} \quad D - GE - C \iff \overline{G}D - E - C.$$

Note that this is a reformulation of Fact 7.3.1.

□

As an illustration of the theorem,

$$H_{(2|1|3,1|3|2)} * H_{(4|2|1|3,2|4|3|1)} = H_{(7|5|4|6|2|1|3,1|3|2|5|7|6|4)}.$$

Note that in the pair of permutations on the right hand side, the first has a global descent at position 4 while the second has a global ascent at position 3. And these two positions are complementary, that is, $3 + 4 = 7$. It should now be fairly clear that:

Theorem 7.5.4 *The Hopf algebra* RΠ, *and hence* SΠ, *is a free graded algebra on the space spanned by*

$$\left\{ H_{(D,C)} \;\middle|\; \begin{array}{l} D \text{ does not have a global descent} \\ \text{and } C \text{ a global ascent at comple-} \\ \text{mentary positions.} \end{array} \right\} = \{H_{(D,C)} \mid \overline{C} \cap D \cap \overline{C}_0 = \emptyset\}.$$

Theorem 7.5.5 *The product on* RΛ *is given by*

$$H_{u^{-1}} * H_{v^{-1}} = H_{\overline{j}(u\times v)^{-1}} \quad \textit{or equivalently,} \quad H_u * H_v = H_{j'(u\times v)}.$$

Proof Specialize Theorem 7.5.3 to the chambers $C_1 = uC_0^{g_1}$, $C_2 = vC_0^{g_2}$, $D_1 = C_0^{g_1}$ and $D_2 = C_0^{g_2}$. Then note that $d(C_1, D_1) = u^{-1}$ and $d(C_2, D_2) = v^{-1}$. Now the first claim follows by projecting to SΛ via the distance map and noting that $j_G(C_0^{g_1} \times C_0^{g_2}) = C_0^n$. The second claim can be proved from the first. We leave that as an exercise.

□

7.5.3 The switch map on the H basis

As was the case with the M basis of SΠ, we note some symmetry in the two coordinates of $\mathcal{C} \times \mathcal{C}$ in the formulas in the H basis as well. More precisely:

Corollary 7.5.1 *The map* RΠ → RΠ *that sends* $H_{(D,C)}$ *to* $H_{(C,D)}$ *is a map of coalgebras.*

Note that the above is not a map of anti-algebras. However, this problem disappears at the level of RΛ. Namely:

Corollary 7.5.2 *The map* $\mathrm{R}\Lambda \to \mathrm{R}\Lambda$ *that sends* H_w *to* $H_{w^{-1}}$ *is a map of coalgebras and a map of anti-algebras.*

This is consistent with Corollary 7.3.2.

Remark It may be a little disconcerting that the switch map on the H basis is not so nice as on the M basis. A conceptual explanation of why this occurs can be given using the theory of species, which will be explained in a future work.

Chapter 8

The Hopf algebra of pointed faces

8.1 Introduction

In Chapter 7, we studied the Hopf algebra $\mathrm{S\Pi}$ of pairs of permutations and related it to the study of the Hopf algebra $\mathrm{S\Lambda}$ of permutations [61]. In this chapter, we study the Hopf algebra of pointed faces $\mathrm{Q\Pi}$, and the Hopf algebra of faces $\mathrm{P\Pi}$, which were introduced in Chapter 6, and relate them to the Hopf algebra $\mathrm{Q\Lambda}$ of quasi-symmetric functions [36, 61, 46], which was discussed in Section 3.2.2. For missing definitions or details, the reader should refer to Chapter 7.

8.1.1 The basic setup

We quickly recall some basic notation and facts from the previous chapters. Let Σ^n be the Coxeter complex of S_n and $\mathcal{C}^n$ be the set of chambers in Σ^n. Let

$$\mathrm{Q}^n = \{(F, D) \mid F \leq D\} \subseteq \Sigma^n \times \mathcal{C}^n$$

be the set of pointed faces and

$$\mathrm{Q\Pi} = \bigoplus_{n \geq 0} \mathbb{K}(\mathrm{Q}^n)^* \quad \text{and} \quad \mathrm{P\Pi} = \bigoplus_{n \geq 0} \mathbb{K}(\Sigma^n)^*.$$

The definitions of $\mathrm{Q\Pi}$ and $\mathrm{P\Pi}$ as Hopf algebras are recalled in Sections 8.2 and 8.3. Recall the following special case of Theorem 6.1.4.

Proposition 8.1.1 *The following is a commutative diagram of graded Hopf algebras.*

$$\begin{tikzcd}
\mathrm{S\Pi} \arrow[r, "\mathrm{Road}"] \arrow[d, two heads, "d"'] & \mathrm{Q\Pi} \arrow[r, two heads] \arrow[d, two heads, "\mathrm{type}"] & \mathrm{P\Pi} \arrow[dl, two heads, "\mathrm{type}"] \\
\mathrm{S\Lambda} \arrow[r, "\mathrm{des}"] & \mathrm{Q\Lambda} &
\end{tikzcd}$$

The map des relating $\mathrm{S\Lambda}$ and $\mathrm{Q\Lambda}$ is the descent map defined by Malvenuto [61]. The map Road relating $\mathrm{S\Pi}$ and $\mathrm{P\Pi}$ is a lift of the descent map, which was explained in detail in Chapter 5. We recall its form in different bases in Section 8.1.5. The Hopf algebra $\mathrm{P\Pi}$ is the quotient of $\mathrm{Q\Pi}$ via the map which forgets the second coordinate on the M basis. The quotient maps d and type are the usual distance and type map in Coxeter theory.

8.1.2 Cofreeness

Our goal in this chapter is to prove the following result. Part of this appears in Bergeron and Zabrocki [9].

Theorem 8.1.1 *The Hopf algebras* $\mathrm{Q\Pi}$ *and* $\mathrm{P\Pi}$ *are cofree graded coalgebras.*

This is a consequence of Theorem 8.2.2 and the remark after Theorem 8.3.1. It gives an analogue to the following well known result, see the remark after Theorem 8.4.1, or the discussion in Section 3.2.2.

Theorem 8.1.2 *The Hopf algebra* $\mathrm{Q\Lambda}$ *is a cofree graded coalgebra.*

The method of proof is the same as for the Hopf algebras $\mathrm{S\Pi}$ and $\mathrm{S\Lambda}$. Namely, we compute the coproduct in a basis different from the standard basis. In Sections 8.1.3-8.1.5, we recall the relevant notions from Chapter 5 and later in Sections 8.2, 8.3 and 8.4, we give product and coproduct formulas in the various bases of $\mathrm{Q\Pi}$, $\mathrm{P\Pi}$ and $\mathrm{Q\Lambda}$ respectively. By using the quotient map $\mathrm{S\Pi} \to \mathrm{Q\Pi}$, the results for $\mathrm{Q\Pi}$ follow in a straightforward manner from those for $\mathrm{S\Pi}$ obtained in Chapter 7. As an example, consider the following analogue of Corollaries 7.3.1 and 7.3.2, related to the switch map.

Theorem 8.1.3 *The map* $\mathrm{Q\Pi} \to \mathrm{Q\Pi}$ *that sends* $M_{(F,D)}$ *to* $M_{(\overline{F},\overline{F}D)}$*, the map* $\mathrm{P\Pi} \to \mathrm{P\Pi}$ *that sends* M_F *to* $M_{\overline{F}}$ *and the map* $\mathrm{Q\Lambda} \to \mathrm{Q\Lambda}$ *that sends* $M_{(\alpha_1,\dots,\alpha_k)}$ *to* $M_{(\alpha_k,\dots,\alpha_1)}$ *are maps of algebras and anti-coalgebras.*

Proof The result for $\mathrm{Q\Pi}$ follows from Corollary 7.3.1 and the expression for the quotient map $\mathrm{S\Pi} \to \mathrm{Q\Pi}$ on the M basis, as recorded in Lemma 8.1.1. The result for $\mathrm{Q\Pi}$ implies the result for $\mathrm{P\Pi}$ and $\mathrm{Q\Lambda}$ by using the quotient maps recorded in Proposition 8.1.1. $\square$

By virtue of studying $\mathrm{Q\Pi}$ and $\mathrm{P\Pi}$, we also derive an interesting expression for the product in the F basis of $\mathrm{Q\Lambda}$, see Theorem 8.4.2.

8.1.3 Three partial orders on Q^n

Recall the following from Definition 5.2.3.

Definition 8.1.1 Define three partial orders on Q^n as follows.

$$(F_1, D_1) \le (F_2, D_2) \iff D_1 = D_2 \text{ and } F_1 \le F_2.$$

$$(F_1, D_1) \le' (F_2, D_2) \iff D_1 \le_b D_2 \text{ and } \operatorname{type} F_1 = \operatorname{type} F_2.$$

$$(F_1, D_1) \preceq (F_2, D_2) \iff \exists H \ni (F_1, D_1) \le (H, D_1) \text{ and } (H, D_1) \le' (F_2, D_2).$$

The partial order $\le_b$ is the weak left Bruhat order of Definition 7.1.1. Note that in the definition of $\preceq$, only one H can satisfy the required condition; namely the face of D_1 whose type is the same as that of F_2. Unlike $\le$, the partial orders $\le'$ and $\preceq$ depend on the choice of the fundamental chamber C_0, since they involve $\le_b$.

8.1.4 The different bases of $Q\Pi$

Next we recall the following from Section 5.6.1. Using the partial orders in Definition 8.1.1, one can define the S, M and F bases of $Q\Pi$. They are related by

$$F_{(F,D)} = \sum_{F \le H \le D} M_{(H,D)}, \qquad M_{(H',D')} = \sum_{(H',D') \le' (H,D)} S_{(H,D)}. \tag{8.1}$$

Observe from the above formulas that the F basis is related to the S basis via the partial order $\preceq$ as follows.

$$F_{(H',D')} = \sum_{(H',D') \preceq (H,D)} S_{(H,D)}.$$

8.1.5 The connection between $S\Pi$ and $Q\Pi$

Recall from Section 5.2 that for the partial orders $\le$ and $\preceq$, we had defined order preserving maps $\text{Road} : \mathcal{C}^n \times \mathcal{C}^n \to Q^n$ and $\Theta : Q^n \to \mathcal{C}^n \times \mathcal{C}^n$, which were adjoint to each other and shown the following in Lemma 5.6.2.

Lemma 8.1.1 *The following are three equivalent definitions of the map* $\text{Road} : S\Pi \to Q\Pi$.

$$\begin{array}{rcll} \text{Road}(F_{(C,D)}) & = & F_{\text{Road}(C,D)} & (F \ \textit{basis}). \\ \text{Road}(M_{(C,D)}) & = & \begin{cases} M_{\text{Road}\,(C,D)} & \textit{if } (C,D) = \Theta(\text{Road}(C,D)) \\ 0 & \textit{otherwise.} \end{cases} & (M \ \textit{basis}). \end{array}$$

By replacing M by S, one gets the expression on the S basis.

8.2 The Hopf algebra $Q\Pi$

In Chapter 6, we had defined $Q\Pi$ on the M basis and shown that the map $S\Pi \to Q\Pi$ is a map of Hopf algebras, see Propositions 6.5.5 and 6.8.5. In Theorems 8.2.1 and 8.2.3 below, we define $Q\Pi$ on the S and F basis.

8.2.1 Geometric definition

We first describe $Q\Pi$ in geometric terms. The proof of the coproduct and product formulas are indicated together.

Theorem 8.2.1 *The following are three equivalent definitions of the coproduct on* $Q\Pi$.

$$\begin{array}{rcl} \Delta_+(M_{(F,D)}) & = & \sum_{K:\ \text{rank}\,K=1,\ K \le F} M_{(F_1,D_1)} \otimes M_{(F_2,D_2)} \\ \Delta_+(S_{(F,D)}) & = & \sum_{K:\ \text{rank}\,K=1,\ K \le F,\overline{C}_0} S_{(F_1,D_1)} \otimes S_{(F_2,D_2)} \\ \Delta_+(F_{(F,D)}) & = & \sum_{K:\ \text{rank}\,K=1,\ K \le D} F_{(F_1,D_1)} \otimes F_{(F_2,D_2)}, \end{array}$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(KF) = F_1 \times F_2$ and $b_K(D) = D_1 \times D_2$. In the first two formulas, $KF = F$.

Consider the coproduct formula on the S basis. Suppose we are given $S_{(F_1,D_1)}$ and $S_{(F_2,D_2)}$. Then one can recover the indexing vertex K as the vertex of $\overline{C}_0$ of the type determined by F_1 and F_2. Since b_K is an isomorphism, one then recovers $F = b_K^{-1}(F_1 \times F_2)$

and $D = b_K^{-1}(D_1 \times D_2)$, and hence $S_{(F,D)}$. This shows that QΠ is cofree. More precisely, let $\mathtt{Q}^0 = \{1\}$ and for $k \geq 1$, let

$$\mathtt{Q}^k = \{(F, D) \mid \operatorname{rank}(F \cap \overline{C}_0) = k - 1\}.$$

Let $\mathrm{Q}\Pi_k$ be the vector subspace of QΠ spanned by $\{S_{(F,D)} \mid (F, D) \in \mathtt{Q}^k\}$.

Theorem 8.2.2 *With the above grading,* QΠ *is a cofree graded coalgebra. A basis for the kth level of the coradical filtration of* QΠ *is*

$$\coprod_{i=0}^{k} \mathtt{Q}^k.$$

In particular, a basis for the space of primitive elements is

$$\{S_{(F,D)} \mid F \cap \overline{C}_0 = \emptyset\}.$$

This follows directly from the definitions (Sections 3.1.1-3.1.2).

Theorem 8.2.3 *The following are three equivalent definitions of the product on* QΠ.

$$\begin{aligned}
M_{(F_1,D_1)} * M_{(F_2,D_2)} &= \sum_{F:\ GF=j_G(F_1\times F_2)} M_{(F,Fj_G(D_1\times D_2))}. \\
S_{(F_1,D_1)} * S_{(F_2,D_2)} &= \sum_{\substack{(K,F):\ \operatorname{rank} K=1 \\ b_K(KF)=F_1\times F_2 \\ C_0-KC_0-Fb_K^{-1}(D_1\times D_2).}} S_{(F,Fb_K^{-1}(D_1\times D_2))}. \\
F_{(F_1,D_1)} * F_{(F_2,D_2)} &= \sum_{D:\ GD=j_G(D_1\times D_2)} F_{\mathrm{Road}(j_G(\overline{F}_1\times\overline{F}_2)D,D)}.
\end{aligned}$$

Proof of Theorems 8.2.1 and 8.2.3 The Hopf algebra QΠ was defined in Chapter 6 using the M basis. So we can take those formulas as a known quantity.

F **basis.** There are two ways to derive the formulas on the F basis. One can either use the formulas on the M basis along with Relation (8.1), or one can project the formulas on the F basis of SΠ using the map SΠ ↠ QΠ in Lemma 8.1.1. We use the second approach.

We know from Propositions 5.2.4 and 5.2.6 that Θ is a section of Road and that $\Theta(F, D) = (\overline{F}D, D)$. Hence $\mathrm{Road}(\overline{F}D, D) = (F, D)$.

We first deal with the coproduct. From Definition 7.2.8, we have

$$\Delta_+(F_{(\overline{F}D,D)}) = \sum_{K:\ \operatorname{rank} K=1,\ K\leq D} F_{(C_1,D_1)} \otimes F_{(C_2,D_2)},$$

where for $b_K : \Sigma_K^n \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(K\overline{F}D) = C_1 \times C_2$. Now let $b_K(KF) = F_1 \times F_2$. Then $b_K(K\overline{F}) = \overline{F}_1 \times \overline{F}_2$. Now using the formula $b_K(K\overline{F}D) = b_K(K\overline{F})b_K(D)$, we obtain $C_1 = \overline{F}_1 D_1$ and $C_2 = \overline{F}_2 D_2$. Hence the above formula can be rewritten as

$$\Delta_+(F_{(\overline{F}D,D)}) = \sum_{K:\ \operatorname{rank} K=1,\ K\leq D} F_{(\overline{F}_1 D_1,D_1)} \otimes F_{(\overline{F}_2 D_2,D_2)},$$

where for $b_K : \Sigma_K^n \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(K\overline{F}) = F_1 \times F_2$. Now applying the map Road, we get the formula for the coproduct in the F basis in Theorem 8.2.1.

We apply the same method to compute the product in the F basis. From Definition 7.2.9, we obtain

$$\begin{aligned} F_{(\overline{F}_1 D_1, D_1)} * F_{(\overline{F}_2 D_2, D_2)} &= \sum_{D:\ GD = j_G(D_1 \times D_2)} F_{(j_G(\overline{F}_1 D_1 \times \overline{F}_2 D_2), D)}. \\ &= \sum_{D:\ GD = j_G(D_1 \times D_2)} F_{(j_G(\overline{F}_1 \times \overline{F}_2) j_G(D_1 \times D_2), D)}. \\ &= \sum_{D:\ GD = j_G(D_1 \times D_2)} F_{(j_G(\overline{F}_1 \times \overline{F}_2) D, D)}. \end{aligned}$$

For the third equality, we used that $j_G(\overline{F}_1 \times \overline{F}_2) \geq G$. Now applying the map Road, we get the formula for the product in the F basis in Theorem 8.2.3.

S **basis.** We derive the formulas on the M basis of QΠ by projecting the formulas on the M basis of SΠ. We leave it to the reader to modify the steps in this proof by replacing M by S, and derive the formulas on the S basis.

For the coproduct formula, from Theorem 7.3.1, we have

$$\Delta_+(M_{(\overline{F}D, D)}) = \sum_{K:\ \mathrm{rank}\, K = 1,\ K \leq F} M_{(C_1, D_1)} \otimes M_{(C_2, D_2)},$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(D) = D_1 \times D_2$ and $b_K(K\overline{F}D) = C_1 \times C_2$. Since

$$D \cap \overline{\overline{F}D} = D \cap F\overline{D} = F,$$

the condition $K \leq F$ in the summation index is justified. The rest is similar to the F basis proof.

For the product formula, from Theorem 7.3.4, we have

$$M_{(\overline{F}_1 D_1, D_1)} * M_{(\overline{F}_2 D_2, D_2)} = \sum_{\substack{(C,D): \\ M(i), M(ii), M(iii)}} M_{(C,D)}.$$

Now applying the map Road, only those $M_{(C,D)}$ survive for which $C = \overline{F}D$ for some $F \leq D$. Further in this situation, we have $GF = j_G(F_1 \times F_2)$; hence dropping condition $M(ii)$, we obtain

$$M_{(\overline{F}_1 D_1, D_1)} * M_{(\overline{F}_2 D_2, D_2)} = \sum_{\substack{(F,D): \\ M(i), M(iii) \\ GF = j_G(F_1 \times F_2)}} M_{(\overline{F}D, D)} + \text{ irrelevant terms.}$$

The condition $M(iii)$ can be replaced by $G \in \mathrm{reg}(F, D)$, that is, $FG \leq D$. And D is determined by F because $Fj_G(D_1 \times D_2) = FGD = D$. Hence

$$M_{(\overline{F}_1 D_1, D_1)} * M_{(\overline{F}_2 D_2, D_2)} = \sum_{\substack{F: \\ GF = j_G(F_1 \times F_2)}} M_{(\overline{F} j_G(D_1 \times D_2), j_G(D_1 \times D_2))},$$

since $M(i)$ and $M(iii)$ are both implied after we substitute for D. This gives the formula on the M basis.

□

8.2.2 Combinatorial definition

We now describe QΠ in combinatorial terms in the M, S and F basis. The translation from geometry to combinatorics is straightforward and is left to the reader.

We begin with the coproduct. Let $F = F^1|F^2|\cdots|F^k$ be a face of the chamber $D = D^1|D^2|\cdots|D^n$.

Theorem 8.2.4 *The following are three equivalent definitions of the coproduct on* QΠ.

$$\begin{aligned}
\Delta_+(M_{(F,D)}) &= \sum_{i=1}^{k-1} M_{\mathrm{st}(F^1|\cdots|F^i,D')} \otimes M_{\mathrm{st}(F^{i+1}|\cdots|F^k,D'')} \\
\Delta_+(S_{(F,D)}) &= \sum_{i\in \mathrm{gdes}(F)} S_{\mathrm{st}(F^1|\cdots|F^i,D')} \otimes S_{\mathrm{st}(F^{i+1}|\cdots|F^k,D'')} \\
\Delta_+(F_{(F,D)}) &= \sum_{i=1}^{n-1} F_{\mathrm{st}(F',D^1|\cdots|D^i)} \otimes F_{\mathrm{st}(F'',D^{i+1}|\cdots|D^n)},
\end{aligned}$$

We explain the above notation.

- D' and D'' are the restrictions of D, to the letters in the sets $\{F^1, \ldots, F^i\}$ and $\{F^{i+1}, \ldots, F^k\}$ respectively,
- F' and F'' are the restrictions of F to the letters in the sets $\{D^1, \ldots, D^i\}$ and $\{D^{i+1}, \ldots, D^n\}$ respectively,
- The map st is the standardization map in Definition 7.2.2.
- gdes(F) consists of those $1 \leq i \leq k-1$ such that the letters in $\{F^1, \ldots, F^i\}$ are greater than those in $\{F^{i+1}, \ldots, F^k\}$. Observe that this generalizes gdes(w), which is the set of global descents of a permutation. In a future work, we will generalize the descent theory in Chapter 5, where the set gdes(F) will appear naturally.

The following examples will make things clearer.

$$\begin{aligned}
\Delta_+(M_{43|25|1,4|3|2|5|1}) &= M_{21,2|1} \otimes M_{23|1,2|3|1} + M_{32|14,3|2|1|4} \otimes M_{1,1}. \\
\Delta_+(F_{43|25|1,4|3|2|5|1}) &= F_{1,1} \otimes F_{3|24|1,3|2|4|1} + F_{21,2|1} \otimes F_{23|1,2|3|1} + \\
&\quad F_{23|1,3|2|1} \otimes F_{2|1,2|1} + F_{32|14,3|2|1|4} \otimes F_{1,1}. \\
\Delta_+(S_{43|25|1,4|3|2|5|1}) &= S_{32|14,3|2|1|4} \otimes S_{1,1}.
\end{aligned}$$

Before describing the product, we review some notation and facts. A vertex K of type (g_1, g_2) is a two block composition $K_1|K_2$ of the set $[g_1 + g_2]$ such that $|K_i| = g_i$. The definition of a K-quasi-shuffle of set compositions can be found in Definition 6.2.5. This concept shows up in the product of QΠ on the S basis. We have already seen this happen for SΠ in Section 7.4.3. A quasi-shuffle, as explained in Definition 7.2.5, is a special case of a K-quasi-shuffle. Now let $(F_i, D_i) \in \Sigma^{g_i}$ for $i = 1, 2$.

Theorem 8.2.5 *The following are two equivalent definitions of the product on* QΠ.

$$\begin{aligned}
M_{(F_1,D_1)} * M_{(F_2,D_2)} &= \sum_{F:\ F \text{ a quasi-shuffle of } F_1 \text{ and } F_2} M_{(F,D')}. \\
S_{(F_1,D_1)} * S_{(F_2,D_2)} &= \sum_{\substack{K=K_1|K_2: \\ \text{a vertex of type } (g_1,g_2)}} \sum_{F:\ —} S_{(F,E)}. \\
F_{(F_1,D_1)} * F_{(F_2,D_2)} &= \sum_{D:\ D \text{ a shuffle of } D_1 \text{ and } D_2} F_{(F',D)}.
\end{aligned}$$

where — says that F is a K-quasi-shuffle of F_1 and F_2 such that if $*_1 \in K_1$ and $*_2 \in K_2$ and $*_2$ appears before $*_1$ in F then $*_1 < *_2$.

We now explain the rest of the notation.

- D' is the unique shuffle of D_1 and D_2, that refines F and keeps the blocks of D_1 before D_2, whenever there is such a choice.

- E is the unique K-shuffle of D_1 and D_2, that refines F and keeps the blocks of D_1 before D_2, whenever there is such a choice.
- F' is the face of $D = D^1|D^2|\cdots|D^n$ obtained as below. D^i and D^{i+1} occur in different blocks of F' iff
  - D^i belongs to F_2' and D^{i+1} belongs to F_1, or
  - D^i and D^{i+1} both belong to F_1 and D^i appears before D^{i+1} in F_1, or
  - D^i and D^{i+1} both belong to F_2' and D^i appears before D^{i+1} in F_2',

  where F_2' is F_2 shifted up by g_1.

The following examples will make things clearer.

$$\begin{aligned}
M_{(21,2|1)} * M_{(1|2,1|2)} &= M_{(21|3|4,2|1|3|4)} + M_{(3|21|4,3|2|1|4)} + M_{(3|4|21,3|4|2|1)} + \\
&\quad M_{(213|4,2|1|3|4)} + M_{(3|214,3|2|1|4)}. \\
S_{(1,1)} * S_{(1|2,1|2)} &= S_{(12|3,1|2|3)} + S_{(12|3,2|1|3)} + S_{(13|2,3|1|2)} + S_{(2|13,2|1|3)} + \\
&\quad S_{(1|2|3,1|2|3)} + S_{(2|3|1,2|3|1)} + 2\ S_{(2|1|3,2|1|3)} + S_{(3|1|2,3|1|2)}. \\
F_{(21,2|1)} * F_{(1|2,1|2)} &= F_{(213|4,2|1|3|4)} + F_{(23|14,2|3|1|4)} + F_{(3|124,3|2|1|4)} + \\
&\quad F_{(23|4|1,2|3|4|1)} + F_{(3|24|1,3|2|4|1)} + F_{(3|4|12,3|4|2|1)}.
\end{aligned}$$

8.3 The Hopf algebra PΠ

We now want a cofreeness result for the Hopf algebra PΠ. Ideally we would like to deduce the result right away by making use of Theorem 8.2.1. However there seem to be some problems with this approach. So we proceed more directly as below.

Definition 8.3.1 Define the partial order $\leq'$ on Σ^n as follows.

$$\begin{aligned}
F_1 \leq' F_2 \quad \text{in} \quad \Sigma^n \quad &\iff \quad (F_1, F_1C_0) \leq' (F_2, F_2C_0) \quad \text{in} \quad \mathrm{Q}^n \\
&\iff \quad F_1C_0 \leq_b F_2C_0 \text{ and } \operatorname{type} F_1 = \operatorname{type} F_2.
\end{aligned}$$

The partial order $\leq_b$ is as given in Definition 7.1.1. Next define the S basis of the Hopf algebra PΠ by the equation
$$M_{F'} = \sum_{F' \leq' F} S_F.$$

We now state without proof the formulas on the M and S basis of PΠ. The interested reader may take this as an exercise. The combinatorial descriptions, which we also omit, are along the lines of QΠ.

Theorem 8.3.1 *The following are two equivalent definitions of the coproduct on* PΠ.

$$\begin{aligned}
\Delta_+(M_F) &= \sum_{K:\ \operatorname{rank} K = 1,\ K \leq F} M_{F_1} \otimes M_{F_2} \\
\Delta_+(S_F) &= \sum_{K:\ \operatorname{rank} K = 1,\ K \leq F, \overline{C}_0} S_{F_1} \otimes S_{F_2}
\end{aligned}$$

where for $b_K : \Sigma^n_K \to \Sigma^{k_1} \times \Sigma^{k_2}$, we have $b_K(F) = F_1 \times F_2$.

Remark The coproduct formula on the S basis shows that PΠ is cofree.

Theorem 8.3.2 *The following are two equivalent definitions of the product on* PΠ.

$$\begin{aligned} M_{F_1} * M_{F_2} &= \sum_{F:\ GF=j_G(F_1\times F_2)} M_F. \\ S_{F_1} * S_{F_2} &= \sum_{\substack{(K,F):\ \operatorname{rank} K=1 \\ b_K(KF)=F_1\times F_2 \\ C_0 - KC_0 - FC_0.}} S_F \end{aligned}$$

The product formula in the S basis is only of academic interest. In any case, it is interesting that the coefficients involved are nonnegative. As an example,

$$S_1 * S_{1|2} = S_{12|3} + S_{2|13} + S_{1|2|3} + 2\ S_{2|1|3} + S_{2|3|1} + S_{3|1|2}.$$

Remark The partial order $\leq'$ on Σ is different from the one defined by Krob, Latapy, Novelli, Phan and Schwer [52] and Palacios and Ronco [78]. In a future work, we will give a more conceptual motivation for the partial order $\leq'$, and prove the above results. Namely, we will define another Hopf algebra isomorphic to PΠ such that its F and M basis correspond to the M and S basis respectively of PΠ.

8.4 The Hopf algebra QΛ of quasi-symmetric functions

In this section, we describe QΛ in the F and M basis. Recall that

$$\mathrm{Q}\Lambda = \underset{n\geq 0}{\oplus}\, \mathbb{K}(\overline{\mathrm{Q}}^n)^*,$$

where $\overline{\mathrm{Q}}^n$ is the poset of compositions of n. Equivalently, it is the set of subsets of $[n-1]$, or the set of words of length $n-1$ in the alphabet $\{+,-\}$. As an illustration,

$$(2,1,2,3) \quad \longleftrightarrow \quad \{2,3,5\} \quad \longleftrightarrow \quad -++-+--$$

The quotient map $\mathrm{Q}\Pi \twoheadrightarrow \mathrm{Q}\Lambda$ is given by $M_{(F,D)} \mapsto M_{\operatorname{type} F}$, or equivalently, $F_{(F,D)} \mapsto F_{\operatorname{type} F}$. It is independent of the second coordinate D. As an example,

$$M_{(25|16|473,2|5|1|6|4|7|3)} \quad \mapsto \quad M_{(2,2,3)} = M_{-+-+--}.$$

Using this map, one can use Theorems 8.2.4 and 8.2.5 to describe the Hopf algebra QΛ in the M and F basis, as below. It is convenient to use the composition notation for the M basis and the alphabet notation for the F basis.

Theorem 8.4.1 *The following are two equivalent definitions of the coproduct on* QΛ.

$$\begin{aligned} \Delta_+(M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}) &= \sum_{i=1}^{k-1} M_{(\alpha_1,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\ldots,\alpha_k)} \\ \Delta_+(F_{(\xi_1\xi_2\ldots\xi_{n-1})}) &= \sum_{i=1}^{n-1} F_{(\xi_1\ldots\xi_{i-1})} \otimes F_{(\xi_{i+1}\ldots\xi_{n-1})} \end{aligned}$$

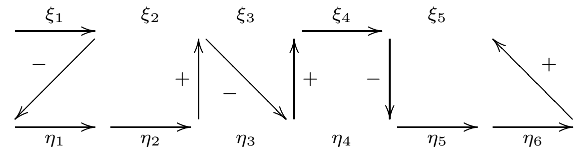
Remark These formulas are well known. The coproduct formula on the M basis shows that QΛ is cofree.

Before we can describe the products, we need a definition. Let S be a shuffle of the compositions $1|2|\cdots|g_1$ and $1|2|\cdots|g_2$. This can be shown by a diagram as below, where we have taken $g_1 = 6$ and $g_2 = 7$ for illustration.

$$S = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[dl] & 3 \arrow[dr] & 4 \arrow[r] & 5 \arrow[d] & 6 & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] & 4 \arrow[u] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[ul] \end{tikzcd}$$

Now suppose we have two sign sequences $\xi = \xi_1\xi_2\ldots\xi_{g_1-1}$ and $\eta = \eta_1\eta_2\ldots\eta_{g_2-1}$ of lengths $g_1 - 1$ and $g_2 - 1$ respectively. Then, using the shuffle S, we can define a sign sequence $S(\xi, \eta)$ of length $g_1 + g_2 - 1$ as illustrated below.

$S(\xi, \eta) =$



Namely, first draw the diagram for the shuffle S. Then put a $-$ sign on the arrows going down and a $+$ sign on the arrows going up. The horizontal arrows get labeled ξ_i or η_i. In the example above, $S(\xi, \eta) = \xi_1 - \eta_1\eta_2 + - + \xi_4 - \eta_5\eta_6 +$.

Theorem 8.4.2 *The following are two equivalent definitions of the product on* QΛ*.*

$$\begin{aligned} M_\alpha * M_\beta &= \sum_{\gamma:\ \gamma \text{ a quasi-shuffle of } \alpha \text{ and } \beta} M_\gamma. \\ F_\xi * F_\eta &= \sum_{S:\ S \text{ a shuffle of } 1|\cdots|g_1 \text{ and } 1|\cdots|g_2} F_{S(\xi,\eta)}, \end{aligned}$$

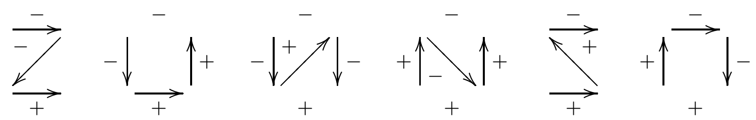
where ξ and η are sign sequences of length $g_1 - 1$ and $g_2 - 1$ respectively.

For example, in the M basis,

$$M_{(2)} * M_{(1,1)} = M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)} + M_{(3,1)} + M_{(1,3)}.$$

In the F basis,

$$F_- * F_+ = F_{--+} + F_{-++} + F_{-+-} + F_{+-+} + F_{++-} + F_{+--}$$



Below each term, we have drawn the diagram from which it arises.

In the literature, the product in the F basis is usually described in terms of permutations, by appealing to the fact that QΛ is a quotient of SΛ. The description given in Theorem 8.4.2 in terms of sign sequences may be new. For an expression in terms of compositions, see Bertet, Krob, et al [10, Proposition 3.7].

Remark The Loday-Ronco Hopf algebra $Y\Lambda$ of planar binary trees [56, 57] sits between SΛ and QΛ. We claim that there is a Hopf algebra $Y\Pi$ that lifts $Y\Lambda$ and sits between SΠ and QΠ; see Chapoton [21], Palacios and Ronco [72], Novelli and Thibon [68] and Loday and Ronco [58] for related ideas. There is a similar story for the Hopf algebra of tableau, see Poirier and Reutenauer [78] and also Taskin [97] for related ideas. The way trees and tableau fit into this theory will be explained in a future work.

Remark The Hopf algebra QΛ satisfies a certain universal property [2]. What can we say about QΠ and PΠ; do they satisfy any universal properties? This will be explained in a future work.

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