ON THE IRREDUCIBLE REPRESENTATIONS OF A FINITE SEMIGROUP

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Abstract. Work of Clifford, Munn and Ponizovskiĭ parameterized the irreducible representations of a finite semigroup in terms of the irreducible representations of its maximal subgroups. Explicit constructions of the irreducible representations were later obtained independently by Rhodes and Zalcstein and by Lallement and Petrich. All of these approaches make use of Rees's theorem characterizing 0-simple semigroups up to isomorphism. Here we provide a short modern proof of the Clifford-Munn-Ponizovskiĭ result based on a lemma of J. A. Green, which allows us to circumvent the theory of 0-simple semigroups. A novelty of this approach is that it works over any base ring.

1. Introduction and preliminaries

Work of Clifford [3,4], Munn [11,12] and Ponizovskiĭ [13] parameterized the irreducible representations of a finite semigroup in terms of the irreducible representations of its maximal subgroups. (See [5, Chapter 5] for a full account of this work.) Explicit constructions of the irreducible representations were later obtained independently by Rhodes and Zalcstein [18] and by Lallement and Petrich [10] in terms of the Schützenberger representation by monomial matrices [19]. All of these approaches make use of Rees's theorem [16] characterizing 0-simple semigroups up to isomorphism, thereby rendering the results somewhat inaccessible to the non-specialist in semigroup theory. As a consequence, it seems that when researchers from other areas need to use semigroup representation theory, they are forced to reinvent parts of the theory for themselves, e.g. [1,2]. This paper is a continuation of the author's program of revitalizing semigroup representation theory [21,22].

The aim of this note is to give a self-contained accounting of the theory of simple modules over the semigroup algebra of a finite semigroup using only the tools of associative algebras. This should make the results accessible to the general mathematician for the first time. Our key tool is a lemma of J. A. Green [8]. An advantage of this approach is that it avoids Wedderburn theory and hence works over an arbitrary commutative ring with unit.

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We collect here some basic definitions and facts concerning finite semigroups that can be found in any of [5, 9, 17]. Let S be a (fixed) finite semigroup. If e is an idempotent, then eSe is a monoid with identity e; its group of units G_e is called the *maximal subgroup* of S at e. Two idempotents e, f are said to be *isomorphic* if there exist $x \in eSf$ and $x' \in fSe$ such that $xx' = e, x'x = f$. In this case one can show that eSe is isomorphic to fSf as monoids and hence $G_e \cong G_f$.

If $s \in S$, then $J(s) = S^1 s S^1$ is the principal (two-sided) ideal generated by s (here S^1 means S with an adjoined identity). Following Green [7], two elements s, t of a semigroup S are said to be *$\mathscr{J}$-equivalent* if $J(s) = J(t)$. In this case one writes $s \mathrel{\mathscr{J}} t$. In fact there is a preorder on S given by $s \leq_{\mathscr{J}} t$ if $J(s) \subseteq J(t)$. This preorder induces an ordering on $\mathscr{J}$-classes in the usual way.

Fact 1. *In a finite semigroup, idempotents e, f are isomorphic if and only if $e \mathrel{\mathscr{J}} f$, that is, $SeS = SfS$.*

An element s of a semigroup S is said to be (von Neumann) *regular* if $s = sts$ for some $t \in S$. Each idempotent is of course regular.

Fact 2. *Let S be a finite semigroup and J a $\mathscr{J}$-class of S. Then the following are equivalent:*

(1) *J contains an idempotent;*
(2) *J contains a regular element;*
(3) *all elements of J are regular;*
(4) *$J^2 \cap J \neq \emptyset$.*

A $\mathscr{J}$-class satisfying the equivalent conditions in Fact 2 is called a *regular* $\mathscr{J}$-class. The poset of regular $\mathscr{J}$-classes is denoted $\mathscr{U}(S)$. It was shown by Putcha [15] that the module category of a regular semigroup (one in which all elements are regular) is a highest weight category [6] with weight poset $\mathscr{U}(S)$. We need one last fact about finite semigroups in order to state and prove the Clifford-Munn-Ponizovskiĭ theorem.

Fact 3. *Let S be a finite semigroup and J a regular $\mathscr{J}$-class. Let $e \in J$ be an idempotent. Then $eSe \cap J = G_e$.*

Let J be a $\mathscr{J}$-class of S. Set $I_J = \{s \in S \mid J \nsubseteq J(s)\}$; it is the ideal of all elements of S that are not $\mathscr{J}$-above some (equals any) element of J.

2. Characterization and Construction of Simple Modules

Fix a finite semigroup S and a commutative ring with unit K. The semigroup algebra KS need not be unital. If A is a K-algebra, not necessarily unital, then by a *simple module* M, we mean a (right) A-module M such that $MA \neq 0$ and M contains no proper non-zero submodules, or equivalently for all $0 \neq m \in M$, the cyclic module $mA = M$. Of course if K is a field and A is finite-dimensional, then every simple A-module is finite dimensional, being cyclic and hence a quotient of the regular module A. The category of (right) A-modules will be denoted mod-A. We adopt the convention that if A is unital, then by mod-A we mean the category of unital A-modules. The reader should verify that all functors considered in this paper respect this convention.

If M is a KS-module, then $\mathrm{Ann}_S(M) = \{s \in S \mid Ms = 0\}$. Clearly $\mathrm{Ann}_S(M)$ is an ideal of S. The following definition, due to Munn [12], is crucial to semigroup representation theory.

Definition 4 (Apex)**.** *A regular $\mathscr{J}$-class J is said to be the apex of a KS-module M if* $\mathrm{Ann}_S(M) = I_J$.

It is easy to see that M has apex J if and only if J is the unique $\leq_{\mathscr{J}}$-minimal $\mathscr{J}$-class that does not annihilate M.

Fix an idempotent transversal $E = \{e_J \mid J \in \mathscr{U}(S)\}$ to the set $\mathscr{U}(S)$ of regular $\mathscr{J}$-classes and set $G_J = G_{e_J}$. Let $A_J = KS/KI_J$. Notice that the category of KS-modules with apex J can be identified with the full subcategory of mod-A_J whose objects are modules M such that $Me_J \neq 0$.

Our first goal is to show that every simple module has an apex. This result is due independently to Munn and Ponizovskiĭ [11–13].

Theorem 5. *Let M be a simple KS-module. Then M has an apex.*

Proof. Since $MKS \neq 0$, there is a $\leq_{\mathscr{J}}$-minimal $\mathscr{J}$-class J so that $J \nsubseteq \mathrm{Ann}_S(M)$. Let $I = S^1JS^1$; of course, I is an ideal of S. Since $I \setminus J$ annihilates M by minimality of J, it follows $0 \neq MKJ = MKI$. From the fact that I is an ideal of S, we may deduce that MKI is a KS-submodule and so by simplicity

$$M = MKI = MKJ. \tag{2.1}$$

Therefore, since $JI_J \subseteq I \setminus J \subseteq \mathrm{Ann}_S(M)$, it follows from (2.1) that $I_J = \mathrm{Ann}_S(M)$. Now if J is not regular, then Fact 2 implies $J^2 \subseteq I \setminus J$ and hence J annihilates M by (2.1), a contradiction. Thus J is regular and is an apex for M. □

Now we wish to establish a bijection between simple KS-modules with apex J and simple KG_J-modules. This relies on a well-known result of Green [8]. Let A be an algebra and e an idempotent of A. Then eA is an eAe-A-bimodule and Ae is an A-eAe-bimodule. Hence we have a restriction functor $\mathrm{Res} : \text{mod-}A \to \text{mod-}eAe$ and induction/coinduction functors $\mathrm{Ind}, \mathrm{Coind} : \text{mod-}eAe \to \text{mod-}A$ given by

$$\mathrm{Ind}(M) = M \otimes_{eAe} eA,\ \mathrm{Res}(M) = Me \text{ and } \mathrm{Coind}(M) = \mathrm{Hom}_{eAe}(Ae, M).$$

Moreover, Ind is right exact, Res is exact, Coind is left exact and Ind and Coind are the left and right adjoints of Res, respectively. This follows from observing that $\mathrm{Hom}_A(eA, M) = Me = M \otimes_A Ae$ and the usual adjunction between hom and the tensor product. Moreover, it is well known that unit of the first adjunction gives a natural isomorphism $M \cong \mathrm{Ind}(M)e$ while the counit of the second adjunction gives a natural isomorphism $\mathrm{Coind}(M)e \cong M$.

There also two important functors $N, L : \text{mod-}A \to \text{mod-}A$ given by $N(M) = \{m \in M \mid mAe = 0\}$ and $L(M) = MeA$. It is easily verified that $N(M)$ is the largest A-submodule of M that is annihilated by e, while $L(M)$ is the smallest A-submodule of M with $L(M)e = Me$. Our next result can be found in [8, 6.2], but we reproduce the proof here for convenience of the reader.

Lemma 6 (Green)**.** *Let A be an algebra and e an idempotent of A.*

(1) *If M is a simple A-module, then $Me = 0$ or Me is a simple eAe-module.*
(2) *If V is a simple eAe-module, then $\mathrm{Ind}(V)$ has unique maximal submodule $N(\mathrm{Ind}(V))$ and $\mathrm{Ind}(V)/N(\mathrm{Ind}(V))$ is the unique simple A-module M with $Me \cong V$.*
(3) *If V is a simple eAe-module, then $\mathrm{Coind}(V)$ has a unique minimal A-submodule $L(\mathrm{Coind}(V))$ and $L(\mathrm{Coind}(V))$ is the unique simple A-module M with $Me \cong V$.*

Consequently, restriction yields a bijection between simple A-modules that are not annihilated by e and simple eAe-modules.

Proof. To prove (1), assume $Me \neq 0$. Let $m \in Me$ be non-zero. Then $meA = mA = M$, so $meAe = Me$. Thus Me is simple.

Now we turn to (2). Let V be a simple eAe-module and suppose $w \in \mathrm{Ind}(V)$ with $w \notin N(\mathrm{Ind}(V))$. Then $0 \neq wAe \subseteq \mathrm{Ind}(V)e$. But $\mathrm{Ind}(V)e \cong V$ is a simple eAe-module, so

$$(wAeAe)A = \mathrm{Ind}(V)eA = (V \otimes_{eAe} eA)eA = (V \otimes_{eAe} e)A = \mathrm{Ind}(V)$$

and thus $N(\mathrm{Ind}(V))$ is the unique maximal A-submodule of $\mathrm{Ind}(V)$. In particular, $\mathrm{Ind}(V)/N(\mathrm{Ind}(V))$ is a simple A-module. Since restriction is exact and $N(\mathrm{Ind}(V))e = 0$ by construction, it follows

$$[\mathrm{Ind}(V)/N(\mathrm{Ind}(V))]e \cong \mathrm{Ind}(V)e/N(\mathrm{Ind}(V))e \cong \mathrm{Ind}(V)e \cong V.$$

It remains to prove uniqueness. Suppose that M is a simple A-module such that $Me \cong V$. Then, using the adjunction between induction and restriction, we have $\mathrm{Hom}_{eAe}(V, Me) \cong \mathrm{Hom}_A(V \otimes_{eAe} eA, M)$. Hence the isomorphism $V \to Me$ corresponds to a non-zero homomorphism $\varphi : \mathrm{Ind}(V) \to M$, which is necessarily onto as M is simple. But $N(\mathrm{Ind}(V))$ is the unique maximal submodule of $\mathrm{Ind}(V)$, so $\ker \varphi = N(\mathrm{Ind}(V))$ and hence $M \cong \mathrm{Ind}(V)/N(\mathrm{Ind}(V))$, as required.

Finally, to prove (3) first observe that if M is any non-zero A-submodule of $\mathrm{Coind}(V)$, then $Me \neq 0$. Indeed, suppose $Me = 0$ and let $\varphi \in M$. Then, for any x in Ae, we have $\varphi(x) = (\varphi xe)(e) = 0$ since $\varphi xe \in Me = 0$. It follows that $M = 0$. Since $\mathrm{Coind}(V)e \cong V$ is a simple eAe-module, it now follows that if M is a non-zero A-submodule of $\mathrm{Coind}(V)$, then $Me = \mathrm{Coind}(V)e$ and hence

$$L(\mathrm{Coind}(V)) = \mathrm{Coind}(V)eA \subseteq MeA \subseteq M.$$

This establishes that $L(\mathrm{Coind}(V))$ is the unique minimal A-submodule. Since $L(\mathrm{Coind}(V))e = \mathrm{Coind}(V)eAe = \mathrm{Coind}(V)e \cong V$, it just remains to prove uniqueness. Suppose M is a simple A-module with $Me \cong V$. Then the existence of a non-zero element of $\mathrm{Hom}_{eAe}(Me, V) \cong \mathrm{Hom}_A(M, \mathrm{Coind}(V))$ implies that M admits a non-zero homomorphism to $\mathrm{Coind}(V)$. Hence M is isomorphic to a simple A-submodule of $\mathrm{Coind}(V)$. But $L(\mathrm{Coind}(V))$ is the unique simple submodule of $\mathrm{Coind}(V)$ and so $M \cong L(\mathrm{Coind}(V))$, as required. □

We may now complete the proof of the Clifford-Munn-Ponizovskiĭ theorem, with an explicit construction of the simple modules equivalent to the one found in [10, 18].

Theorem 7 (Clifford, Munn, Ponizovskiĭ)**.** *Let S be a finite semigroup, K a commutative ring with unit and $E = \{e_J \mid J \in \mathscr{U}(S)\}$ an idempotent transversal to the set $\mathscr{U}(S)$ of regular $\mathscr{J}$-classes of S. Let G_J be the maximal subgroup G_{e_J}. Define functors $\mathrm{Ind}_{G_J}^S, \mathrm{Coind}_{G_J}^S : \text{mod-}KG_J \to \text{mod-}KS$ by*

$$\mathrm{Ind}_{G_J}^S(V) = V \otimes_{KG_J} e_J(KS/KI_J)$$
$$\mathrm{Coind}_{G_J}^S(V) = \mathrm{Hom}_{KG_J}((KS/KI_J)e_J, V).$$

Then:

1. *If M is a simple KS-module with apex J, then Me_J is a simple KG_J-module;*
2. *If V is a simple KG_J-module and $N = \{w \in \mathrm{Ind}_{G_J}^S(V) \mid wKSe_J = 0\}$, then N is the unique maximal KS-submodule of $\mathrm{Ind}_{G_J}^S(V)$ and $\mathrm{Ind}_{G_J}^S(V)/N$ is the unique simple KS-module M with apex J such that $Me_J = V$;*
3. *If V is a simple KG_J-module, then $\mathrm{Coind}_{G_J}^S(V)eA$ is the unique minimal A-submodule of $\mathrm{Coind}_{G_J}^S(V)$ and moreover is the unique simple KS-module M with apex J such that $Me_J = V$.*

Consequently, if K is a field there is a bijection between irreducible representations of S and irreducible representations of the maximal subgroups G_J of S, $J \in \mathscr{U}(S)$.

Proof. Theorem 5 implies that every simple KS-module M has an apex. Again setting $A_J = KS/I_J$ for a regular $\mathscr{J}$-class J, we know that simple KS-modules with apex J are in bijection with simple A_J-modules M such that $Me_J \neq 0$. It follows directly from Fact 3 that

$$e_JA_Je_J = Ke_JSe_J/Ke_JI_Je_J = KG_J.$$

Lemma 6 then yields that simple A_J-modules not annihilated by e_J, that is simple KS-modules with apex J, are in bijection with simple KG_J-modules in the prescribed manner. □

Let us make a remark to relate the above construction of the simple modules to the ones found in [10, 18]. All the facts about finite semigroups used in this discussion can be found in the appendix of [17] or in [9]. According to Green [7], two elements s, t of a semigroup are said to be *$\mathscr{R}$-equivalent* if they generate the same principal right ideal. Dually s, t are said to be *$\mathscr{L}$-equivalent* if they generate the same principal left ideal. If S is a finite semigroup, then it is well known (retaining our previous notation) that $e_JS \cap J$ is the $\mathscr{R}$-class R_{e_J} of e_J and $Se_J \cap J$ is the $\mathscr{L}$-class L_{e_J} of e_J. Moreover, left multiplication yields a free action of G_J on the left of R_{e_J} by automorphisms of the action of S on the right of R_{e_J} by partial transformations (induced by right multiplication). Moreover, the G_J-orbits on R_{e_J} are in bijection with the set of $\mathscr{L}$-classes of J. Let T be a transversal to the G_J-orbits. Now e_JKS/I_J can be identified as a vector space with KR_{e_J} and the right KS-module structure is the linearization of the right action of S on R_{e_J} described above. Moreover, under this identification, the left KG_J-module structure on KR_{e_J} is induced by the free left action of G_J on R_{e_J} and so KR_{e_J} is a free left KG-module with basis T. Hence $\mathrm{End}_{KG_J}(KR_{e_J}) \cong M_n(KG_J)$ where n is the number of $\mathscr{L}$-classes in J and so there results a representation $\rho_J : S \to M_n(KG_J)$, which is easily checked to be the classical right Schützenberger representation by row monomial matrices [5, 19] since if $s \in S$ and $t \in T$, then either $ts = 0$ or $ts = gt'$ for unique elements $t' \in T$ and $g \in G_J$.

Now let V be a simple KG_J-module affording the irreducible representation $\varphi : G_J \to GL_r(K)$. Then the matrix representation afforded by the module $V \otimes_{KG_J} KR_{e_J}$ is the tensor product of φ with ρ_J. Now an element of Se_J which does not belong to J automatically annihilates $V \otimes_{KG_J} KR_{e_J}$, so the unique maximal submodule consists of those vectors annihilated by $Se_J \cap J = L_{e_J}$, the $\mathscr{L}$-class of e_J. If one chooses Rees matrix coordinates for J [5, 17], then it is not hard to show that the vectors annihilated by the $\mathscr{L}$-class of e_J are those belonging to the null space of the image of the sandwich matrix under φ. Hence the construction of the simple modules we have provided corresponds exactly to the construction found in [10, 18], but our proof avoids Rees matrix semigroups and Munn algebras.

The coinduced module also has a natural semigroup theoretic interpretation. Indeed, $\mathrm{Hom}_{KG_J}((KS/KI_J)e_J, V) \cong \mathrm{Hom}_{G_J}(L_{e_J}, V)$ where we view L_{e_J} and V as right G_J-sets. The semigroup S acts on the left of L_{e_J} by the left Schützenberger representation and this induces the KS-module structure on $\mathrm{Hom}_{G_J}(L_{e_J}, V)$. Since L_{e_J} is a free right G_{e_J}-set and the orbits are in bijection with the set of $\mathscr{R}$-classes in J, elements of $\mathrm{Hom}_{G_J}(L_{e_J}, V)$ are in bijection with elements of V^m where m is the number of $\mathscr{R}$-classes of J. The space V^m (viewed as row vectors) is naturally a right KS-module via the left Schützenberger representation $\lambda_J : S \to M_m(KG_J)$ and the module structure agrees with the original one. If one chooses Rees matrix coordinates for J [5, 17], then the structure matrix C takes V^n to V^m where n is the number of $\mathscr{L}$-classes of S. One can verify that the image of C is the unique minimal KS-submodule of V^m. (The fact that it is a submodule is a consequence of the so-called linked equations [9, 17].) This yields the other construction of the irreducible representations found in [18].

Putcha has used both the induced and coinduced modules, which he calls the left and right induced modules, in his work on representation theory [14, 15].

As an application, we provide the description of the irreducible representations of an idempotent semigroup that was rediscovered by Brown [1, 2] and put to good effect in the study of random walks. First we establish a well-known lemma.

Lemma 8. *Let S be a semigroup of idempotents and let J be a $\mathscr{J}$-class of S. Then the complement of I_J is a subsemigroup of S.*

Proof. First we show that J is a subsemigroup. Let $e, f \in J$. Then we have $e = ufv$ some $u, v \in S$ and so $efv = ufvfv = ufv = e$, establishing $ef \in J$. Next suppose $J \subseteq SsS \cap Ss'S$. We need $J \subseteq Sss'S$. Let $e \in J$. Then $e = usv$ and $e = u's'v'$ with $u, v, u', v' \in S$. Since $us(vus)v = e$ and $u'(s'v'u')s'v' = e$, it follows $vus \mathrel{\mathscr{J}} e \mathrel{\mathscr{J}} s'v'u'$. Since J is a subsemigroup, $vuss'v'u' \in J$ and hence $J \subseteq Sss'S$, as required. □

Corollary 9. *Let S be a finite semigroup all of whose elements are idempotents. Then all the irreducible representations of S over a field have degree one and the unique irreducible representation φ_J with apex J is given by*

$$\varphi_J(s) = \begin{cases} 0 & s \in I_J \\ 1 & \textit{otherwise.} \end{cases} \tag{2.2}$$

Proof. Let J be a regular $\mathscr{J}$-class. Lemma 8 implies that (2.2) is an irreducible representation with apex J. It is afforded by K with S-action

$$ks = \begin{cases} 0 & s \in I_J \\ k & \text{otherwise.} \end{cases}$$

Since G_J is trivial, there is exactly one simple KS-module with apex J, namely the quotient of $M = e_J KS/KI_J$ by its unique maximal submodule N. Now $R_{e_J} = e_J S \setminus I_J = e_J S \cap J$ is a basis for M. As a consequence of Lemma 8, $R_{e_J} s \subseteq R_{e_J}$ for $s \in S \setminus I_J$ and $R_{e_J} s \subseteq I_J$, otherwise. Thus the augmentation map $\varepsilon : M \to K$ sending each element of R_{e_J} to 1 is a surjective morphism of KS-modules with kernel the unique maximal submodule N of M, as K is of course simple. This completes the proof. □

The above argument applies *mutatis mutandis* to semigroups all of whose subgroups are trivial and whose regular $\mathscr{J}$-classes are subsemigroups. This class of semigroups, known as **DA**, was introduced by Schützenberger in his study of unambiguous products of regular languages [20].

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