MÖBIUS FUNCTIONS AND SEMIGROUP REPRESENTATION THEORY II: CHARACTER FORMULAS AND MULTIPLICITIES

BENJAMIN STEINBERG

Abstract. We generalize the character formulas for multiplicities of irreducible constituents from group theory to semigroup theory using Rota's theory of Möbius inversion. The technique works for inverse semigroups, semigroups with commuting idempotents, idempotent semigroups, semigroups with basic algebras and even more generally. These results include the representation theoretic tools used by Brown to analyze certain random walks on chambers of hyperplane arrangements and allow us to complete the analysis of eigenvalues with multiplicity for random walks on triangularizable semigroups. Applications are also given to decomposing tensor powers and exterior products of rook matrix representations of inverse semigroups, generalizing and simplifying earlier results of Solomon for the rook monoid.

Contents

1. Introduction 2
2. Inverse Semigroups 4
2.1. Definition and basic properties 4
2.2. Idempotents and order 6
3. Incidence Algebras and Möbius Functions 7
4. Inverse Semigroup Algebras 9
4.1. The groupoid algebra 9
4.2. Decomposition into matrix algebras over group rings 11
5. Character Formulas for Multiplicities 13
6. Applications to Decomposing Representations 15
6.1. Tensor powers 15
6.2. Exterior powers 19
6.3. Direct products 20
6.4. Decomposing partial permutation representations 21
7. The Character Table and Solomon's Approach 25
8. Semigroups with commuting idempotents and generalizations 29

Date: March 19, 2006.

1991 *Mathematics Subject Classification.* 06A07,20M25,20M18,60B15.

Key words and phrases. Inverse semigroups, representation theory, characters, semigroup algebras, Möbius functions.

The author gratefully acknowledges the support of NSERC.

8.1. Semigroups with commuting idempotents 29
8.2. Multiplicities for some more general classes of semigroups 33
8.3. Random walks on triangularizable finite semigroups 35
Acknowledgments 37
References 37

1. Introduction

Group representation theory has been crucial in so many areas of mathematics that there is essentially no need to speak further of its successes. The same is not the case for semigroup representation theory at the present time. This is beginning to change, in a large part due to work of Brown [8, 9] and Bidgaire *et al.* [7] who found applications to random walks and connections with Solomon's descent algebra. Solomon, himself, has also found interest in representations of semigroups, in particular inverse semigroups, for the purpose of algebraic combinatorics and representation theory of the symmetric group [41, 42]. Putcha has applied semigroup representation theory to finding weights for finite groups of Lie type [30] and has explored other connections with modern representation theory [27, 29]. Recently Aguiar and Rosas [1] have used the inverse monoid of uniform block permutations to study Malvenuto and Reutenauer's Hopf algebra of permutations and the Hopf algebra of non-commutative symmetric functions. Recently representations of infinite inverse semigroups on Hilbert spaces have received a lot of attention in the C^*-algebra community, see the book of Paterson [25] for more details. This author has been trying over the past couple of years to apply semigroup representations to finite semigroup theory and to automata theory [3, 4].

One of the great successes of group representation theory is character theory. Thanks to Maschke's theorem and the orthogonality relations and their consequences, much of group representation theory boils down to the combinatorics of characters and character sums [39]. Despite intensive work in the fifties and sixties on representations of finite semigroups [10, 23, 22, 24, 20, 21, 34], culminating in the reduction of calculating irreducible representations to group representation theory and combinatorics in matrix algebras over group rings, very few results on characters and how to calculate multiplicities of irreducible constituents have been obtained until now. This is, of course, complicated by the fact that semigroup algebras are almost never semisimple. But even in cases where they are known to be so, like for inverse semigroups, no nice combinatorial formulas were known in general until this paper.

There are three notable exceptions to this. First there are Munn's results on characters of the symmetric inverse monoid (also known as the rook monoid) [24]. These results were extended by Solomon, who obtained multiplicity formulas for irreducible constituents using combinatorics associated to partitions, Ferrer's diagrams, symmetric functions and symmetric groups [42]. Then there is Putcha's work on the characters of the full transformation semigroup [27] and his work on monoid quivers [29], in which he develops multiplicity formulas for certain representations. In particular he obtained a formula in terms of the Möbius function on the $\mathcal{J}$-order for multiplicities of irreducible constituents for representations of idempotent semigroups acting on their left ideals. This same formula, in the special case of a minimal left ideal, was obtained independently by Brown [8, 9] and made much more famous because the formulas were applied to random walks on chambers of hyperplane arrangements and to other Markov chains to obtain absolutely amazing results! Also Brown's work, based on the work of Bidigaire *et al.* [7] for hyperplane face semigroups, developed the theory from scratch, making it accessible to the general public. Putcha's work, however goes much deeper from the representation theoretic point-of-view: he shows that regular semigroups have quasi-hereditary algebras and he calculates the blocks in terms of character formulas [29].

In our previous paper [44], we showed how Solomon's [40] approach to the semigroup algebra of a semilattice, that is an idempotent inverse semigroup, via the Möbius algebra can be extended to inverse semigroups via the groupoid algebra. This allowed us to obtain an explicit decomposition of the algebra of an inverse semigroup into a direct sum of matrix algebras over algebras of maximal subgroups. Also we were able to explicitly determine the central primitive idempotents of the semigroup algebra in terms of character sums and the Möbius function of the inverse semigroup.

In this paper we use the above decomposition to give a character formula for multiplicities of irreducible constituents in representations of inverse semigroups. In particular, we recover and greatly generalize Solomon's results [42] for the symmetric inverse monoid concerning decompositions of tensor and exterior powers of rook matrix representations. Moreover, we obtain the results in a more elementary fashion.

We also give character theoretic proofs of the description of the decomposition of partial permutation representations into irreducible constituents (this last result can also be obtained by the classical semigroup techniques [10, 34], with greater effort).

Just as the irreducible representations of idempotent semigroups correspond to irreducible representations of semilattices (this has been known to semigroup theorists since [10, 22, 23, 34] and has recently been popularized by Brown [9]), there is a large class of semigroups whose irreducible representations essentially factor through inverse semigroups; this includes all finite semigroups with basic algebras. Our results therefore extend to this domain, and in particular we recover the case of idempotent semigroups [8, 9, 29] and semigroups with basic algebras [44]. In the process we calculate an explicit basis for the radical of the semigroup algebra of a finite semigroup with commuting idempotents as well as identifying the semisimple quotient as a certain retract.

Our aim is to make this paper accessible to people interested in algebraic combinatorics, semigroups and representation theory and so we shall try to keep specialized semigroup notions to a minimum. In particular, we shall try to prove most results from [10] that we need, or refer to [3], where many results that we need are proved in a less semigroup theoretic language than [10]. Putcha [27] gives a nice survey of semigroup representation theory, but for the inverse semigroup case our methods handle things from scratch.

The paper is organized as follows. We begin with a brief review of inverse semigroups. This is followed by a review of Rota's theory of incidence algebras and Möbius inversion. The results of our first paper [44] are then summarized. The main argument of [44] is proved in a simpler (and at the same time more complete) manner. The following section gives the general formula for inverse semigroup intertwining numbers. To demonstrate the versatility of our formula, we compute several examples involving tensor and exterior products of rook matrix representations. We then compare our method for computing multiplicities to Solomon's method (properly generalized) via character tables. Finally we explain how to use the inverse semigroup results to handle more general semigroups. This last section will be more demanding of the reader in terms of semigroup theoretic background, but most of the necessary background can be found in [10, 16, 3]. In this last section, we also finish the work begun in [44] on analyzing random walks on triangularizable semigroups. In that paper, we calculated the eigenvalues, but were unable to determine multiplicities under the most general assumptions. In this paper we can handle the general case.

In this paper we take the convention that all transformation groups and semigroups act on the right of sets. We also consider only right modules.

2. Inverse Semigroups

Inverse semigroups capture partial symmetry in much the same way that groups capture symmetry; see Lawson's book [18] for more on this viewpoint and the abstract theory of inverse semigroups.

2.1. **Definition and basic properties.** Let X be a set. We shall denote by $\mathfrak{S}_X$ the symmetric group on X. If n is a natural number, we shall set $[n] = \{1, \ldots, n\}$. The symmetric group on $[n]$ will be denoted $\mathfrak{S}_n$, as usual. What is a partial permutation? An example of a partial permutation of the set $\{1, 2, 3, 4\}$ is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 3 & - & 1 \end{pmatrix}.$$

The domain of σ is $\{2, 4\}$ and the range of σ is $\{1, 3\}$. More formally a *partial permutation* of a set X is a bijection $\sigma : Y \to Z$ with $Y, Z \subseteq X$. We admit the possibility that Y and Z are empty. Partial permutations can be composed via the usual rule for composition of partial functions and the monoid of all partial permutations on a set X is called the *symmetric inverse monoid* on X, denoted $\mathfrak{I}_X$. We shall write $\mathfrak{I}_n$ for the symmetric inverse monoid on $[n]$. The empty partial permutation is the zero element of $\mathfrak{I}_X$, and so will be denoted 0. Clearly $\mathfrak{S}_n$ is the group of units of $\mathfrak{I}_n$.

For reasons that will be come apparent later, we shall use the term *subgroup* to mean any subsemigroup of a semigroup that happens to be a group. For instance, if $Y \subseteq X$, then the collection of all partial permutations of X with domain and range Y is a subgroup of $\mathfrak{I}_X$, which is isomorphic to $\mathfrak{S}_Y$. This is a nice feature of $\mathfrak{I}_n$: it contains in a natural way all symmetric groups of degree at most n and its representations relate to the representations of each of these symmetric groups.

The monoid $\mathfrak{I}_X$ comes equipped with a natural involution that takes a bijection $\sigma : Y \to Z$ to its inverse $\sigma^{-1} : Z \to Y$. The key properties of the involution are, for $\sigma, \tau \in \mathfrak{I}_X$:

- $\sigma\sigma^{-1}\sigma = \sigma$;
- $\sigma^{-1}\sigma\sigma^{-1} = \sigma^{-1}$;
- $(\sigma^{-1})^{-1} = \sigma$;
- $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$;
- $\sigma\sigma^{-1}\tau\tau^{-1} = \tau\tau^{-1}\sigma\sigma^{-1}$.

Let use define a (concrete) *inverse semigroup* to be an involution-closed subsemigroup of some $\mathfrak{I}_X$. There is an abstract characterization due to Preston and Vagner [10, 18] that says that S is an inverse semigroup if and only if, for all $s \in S$, there is a unique $t \in T$ such that $sts = s$ and $tst = t$; t is called the *inverse* of s and is denoted by s^{-1}. Moreover, S in this case has a faithful representation (called the Preston-Vagner representation) $\rho_S : S \to \mathfrak{I}_S$ by partial permutations via partial right multiplication [10, 18]. Thus we can view finite inverse semigroups as involution closed subsemigroups of $\mathfrak{I}_n$ and we shall draw our intuition from there. Later we shall give a new interpretation of the Preston-Vagner representation for finite inverse semigroups in terms of their semigroup algebras.

Another way to think of inverse semigroups is via so-called rook matrices. An $n \times n$ *rook matrix* is a matrix of zeroes and ones with the constraint that if we view the ones as rooks on an $n \times n$ chessboard, then the rooks must all be non-attacking. In other words a rook matrix is what you obtain from a permutation matrix by replacing some of the ones by zeros. One might equally well call these partial permutation matrices and it is clear that the monoid of $n \times n$ rook matrices R_n, called by Solomon the *rook monoid* [42], is just the symmetric inverse monoid. If $\sigma \in \mathfrak{I}_n$, the corresponding element of R_n has a one in position i, j if $i\sigma = j$ and zero otherwise. In R_n the involution is given by the transpose. It follows that any representation of an inverse semigroup S by rook matrices is completely reducible over any subfield of $\mathbb{C}$ since if one uses the usual inner product, then the orthogonal complement of an S-invariant subspace will be S-invariant (as is the case for permutation representations of groups). If $\sigma \in \mathfrak{I}_n$, then by the *rank* of

σ, denoted $\mathrm{rk}(\sigma)$, we mean the cardinality of the range of σ; this is precisely the rank of the associated rook matrix.

Let $M_n(K)$ be the monoid of $n \times n$ matrices over a field K. Let B be the Borel subgroup of invertible $n \times n$ upper triangular matrices over K. Then one easily verifies that the Bruhat decomposition of $Gl_n(K)$ extends to $M_n(K)$ via R_n: that is $M_n(K) = \biguplus_{r \in R_n} BrB$. Renner showed more generally that, for any reductive algebraic monoid, there is a Bruhat decomposition involving the Borel subgroup of the reductive group of units and a finite inverse monoid R, now called the Renner monoid [26, 31, 32]. Another example of a Renner monoid is the signed symmetric inverse monoid $\mathbb{Z}/2\mathbb{Z} \wr I_n$. Solomon defined a Hecke algebra in this context [41] and Putcha [28] studied the relationship of the Hecke algebra with the algebra of the Renner monoid.

2.2. **Idempotents and order.** If S is a semigroup, then we denote by $E(S)$ the set of idempotents of S. Observe that the idempotents of $\mathfrak{I}_n$ are the partial identities 1_X, $X \subseteq [n]$. Also the equality

$$1_X 1_Y = 1_{X \cap Y} = 1_Y 1_X \tag{2.1}$$

holds. Thus $E(\mathfrak{I}_n)$ is a commutative semigroup isomorphic to the Boolean lattice $\mathfrak{B}_n$ of subsets of $[n]$ with the meet operation. In general, if $S \leq \mathfrak{I}_n$ is an inverse semigroup, then $E(S)$ is a meet subsemilattice of $\mathfrak{B}_n$ (it will be a lattice if S is a submonoid). The order can be defined intrinsically by observing that

$$e \leq f \iff ef = e, \text{ for } e, f \in E(S). \tag{2.2}$$

The ordering on idempotents extends naturally to the whole semigroup: let's again take $\mathfrak{I}_n$ as our model. We can define $\sigma \leq \tau$, for $\sigma, \tau \in \mathfrak{I}_n$, if σ is a restriction of τ. This order is clearly compatible with multiplication and $\sigma \leq \tau$ implies $\sigma^{-1} \leq \tau^{-1}$. Thus if $S \leq \mathfrak{I}_n$ is an inverse semigroup, it too has an ordering by restriction. Again the order can be defined intrinsically: it is easy to see that if $s, t \in S$, then

$$s \leq t \iff s = et, \text{ some } e \in E(S) \iff s = tf, \text{ some } e \in E(S). \tag{2.3}$$

If one likes, one can take (2.3) as the definition of the *natural partial order* on S [10, 18].

Let us use the notation $\mathrm{dom}(\sigma)$ for the domain of $\sigma \in \mathfrak{I}_n$ and $\mathrm{ran}(\sigma)$ for the range. If $\sigma \in \mathfrak{I}_n$, then (recalling that $\mathfrak{I}_n$ acts on the right of $[n]$)

$$\sigma\sigma^{-1} = 1_{\mathrm{dom}(\sigma)} \tag{2.4}$$

$$\sigma^{-1}\sigma = 1_{\mathrm{ran}(\sigma)}. \tag{2.5}$$

Thus if s is an element of an inverse semigroup, then it is natural to think of ss^{-1} as the "domain" of s and $s^{-1}s$ as the "range" of s and so we shall write

$$\begin{aligned} ss^{-1} &= \mathrm{dom}(s) \\ s^{-1}s &= \mathrm{ran}(s) \end{aligned} \tag{2.6}$$

This means that we are going to abuse the distinction between a partial identity and the corresponding subset. So if $S \leq \mathfrak{I}_n$, we shall write $x \in \mathrm{dom}(s)$ to mean x belongs to the domain of s. With this viewpoint it is natural to think of s as an isomorphism from $\mathrm{dom}(s)$ to $\mathrm{ran}(s)$. So let us define $e, f \in E(S)$ to be *isomorphic* if there exists $s \in S$ with $\mathrm{dom}(s) = e$ and $\mathrm{ran}(s) = f$: that is $e = ss^{-1}$, $f = s^{-1}s$. Following long standing semigroup tradition, going back to Green [13, 10, 18], we shall write $e \mathrel{\mathcal{D}} f$. One can extend this relation to all of S by defining $s \mathrel{\mathcal{D}} t$ if $\mathrm{dom}(s)$ is isomorphic to $\mathrm{dom}(t)$ (or equivalently $\mathrm{ran}(s) \mathrel{\mathcal{D}} \mathrm{ran}(t)$). One can easily verify that $\mathcal{D}$ is an equivalence relation on S. With a bit more work, one can verify that if S is a finite inverse semigroup, then $s \mathrel{\mathcal{D}} t$ if and only if s and t generate the same two-sided ideal [10, 18]. The equivalence classes with respect to the $\mathcal{D}$-relation are called *$\mathcal{D}$-classes*, although connected components in this context would be a better word.

Let $S \leq \mathfrak{I}_n$ and let $e \in E(S)$. Then e is the identity of a subset $X \subseteq [n]$. It is then clear that the set

$$G_e = \{s \in S \mid \mathrm{dom}(s) = e = \mathrm{ran}(s)\}$$

is a permutation group of degree $\mathrm{rk}(e)$. Actually G_e makes perfectly good sense via (2.6), and is a group, without any reference to an embedding of S into $\mathfrak{I}_n$. It is called the *maximal subgroup* of S at e. It is straightforward to see that if $e, f \in E(S)$ are isomorphic idempotents, then $G_e \cong G_f$. In fact if $s \in S$ with $\mathrm{dom}(s) = e$, $\mathrm{ran}(s) = f$, then conjugation by s implements the isomorphism. It was first observed by Munn [22, 23, 10] that the representation theory of S is in fact controlled by the representations of its maximal subgroups. We shall see this more explicitly below.

Recall that an *order ideal* in a partially ordered set P is a subset I such that $x \leq y \in I$ implies $x \in I$. If $p \in P$, then $p^{\downarrow}$ denotes the *principal order ideal* generated by p. So

$$p^{\downarrow} = \{x \in P \mid x \leq p\}.$$

As usual, for $p_1, p_2 \in P$, the closed interval from p_1 to p_2 will be denoted $[p_1, p_2]$.

It is easy to see that if S is an inverse semigroup then the idempotent set $E(S)$ is an order ideal of S: any restriction of a partial identity is a partial identity. So if $e, f \in E(S)$, then the interval $[e, f]$ in S and in $E(S)$ coincide. Also one can verify that if $s, t \in S$, then the following intervals are isomorphic posets:

$$[s, t] \cong [\mathrm{dom}(s), \mathrm{dom}(t)] \cong [\mathrm{ran}(s), \mathrm{ran}(t)]. \tag{2.7}$$

3. Incidence Algebras and Möbius Functions

Let $(P, \leq)$ be a finite partially ordered set and A a commutative ring with unit. The *incidence algebra* of P over A, which we denote $A[\![P]\!]$, is the algebra of all functions $f : P \times P \to A$ such that

$$f(x,y) \neq 0 \implies x \leq y$$

equipped with the convolution product

$$(f * g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

In other words you can think of $A[\![P]\!]$ as the algebra of all $P \times P$ upper triangular matrices over A, where upper triangular is defined relative to the partial order on P.

With this product and pointwise addition $A[\![P]\!]$ is an A-algebra with unit the Kronecker delta function δ [14, 43]. An element $f \in A[\![P]\!]$ is invertible if and only if $f(x,x)$ is a unit of A for all $x \in P$ [14, 43]. One can define the inverse inductively by

$$\begin{aligned} f^{-1}(x,x) &= f(x,x)^{-1} \\ f^{-1}(x,y) &= -f(x,x)^{-1} \sum_{x < z \leq y} f(x,z)f^{-1}(z,y), \text{ for } x \leq y. \end{aligned} \tag{3.1}$$

The *zeta function* ζ of P is the element of $A[\![P]\!]$ that takes on the value of 1 whenever $x \leq y$ and 0 elsewhere. It is invertible over any ring A and its inverse is called the *Möbius function*. It only depends on the characteristic; the characteristic zero version is called the *Möbius function* of P and is denoted μ, or μ_P if we wish to emphasize the partially ordered set. From (3.1), it follows that $\mu(x,y)$ depends only on the isomorphism class of the interval $[x,y]$. In particular, for an inverse semigroup S, the Möbius function for S is determined by the Möbius function for $E(S)$ via (2.7). The following is Rota's Möbius Inversion Theorem [14, 43].

Theorem 3.1 (Möbius Inversion Theorem)**.** *Let $(P, \leq)$ be a finite partially ordered set and G be an Abelian group. Suppose that $f : P \to G$ is a function and define $g : P \to G$ by*

$$g(x) = \sum_{y \leq x} f(y).$$

Then

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x).$$

Returning to our motivating example $\mathfrak{I}_n$, the Möbius function for $\mathfrak{B}_n$ is well known [14, 43]: if $Y \subseteq Z$, then

$$\mu_{\mathfrak{B}_n}(Y,Z) = (-1)^{|Z|-|Y|}.$$

Hence, for $\mathfrak{I}_n$, the Möbius function is determined by

$$\mu_{\mathfrak{I}_n}(s,t) = (-1)^{\mathrm{rk}(t)-\mathrm{rk}(s)} \tag{3.2}$$

for $s \leq t$.

The semilattice of idempotents of any Renner monoid R is the face lattice of a rational polytope [26, 32]. Such a partially ordered set is Eulerian [43] and so has a well-defined rank function. This rank function extends to R and in this situation the Möbius function is given by (3.2).

4. Inverse Semigroup Algebras

Let A be a commutative ring with unit. Solomon [40] assigned to each partially ordered set P an A-algebra called the Möbius algebra of P. When P is a semilattice, he used Möbius inversion to show that the Möbius algebra is isomorphic to the semigroup algebra of P. In [44] we extended this to arbitrary finite inverse semigroups. We review the construction here from a different viewpoint; further details can be found in [44].

4.1. **The groupoid algebra.** Let S be a finite inverse semigroup. We define an A-algebra called the groupoid algebra of S over A for reasons to become clear. Let us denote by $G(S)$ the set $\{\lfloor s\rfloor \mid s \in S\}$, a formal disjoint copy of S. To motivate the definition of the groupoid algebra, let us point out that there is another way to model partial bijections: allowing composition if and only if the domains and ranges line up. The groupoid algebra encodes this. So let $AG(S)$ be a free A-module with basis $G(S)$ and define a multiplication on $AG(S)$ by setting, for $s, t \in S$,

$$\lfloor s\rfloor\lfloor t\rfloor = \begin{cases} \lfloor st\rfloor & \text{if } \mathrm{ran}(s) = \mathrm{dom}(t) \\ 0 & \text{else.} \end{cases} \tag{4.1}$$

It is easy to check that (4.1) extends to an associative multiplication on $AG(S)$. In fact, the set $G(S)$ with the above multiplication (where 0 is interpreted as undefined) is a groupoid in the sense of a small category [19] in which all arrows are invertible; see [18] for details. We shall use this fact and its consequences without further comment. In particular, we use that if $\lfloor s\rfloor\lfloor t\rfloor$ is defined, then $\mathrm{dom}(st) = \mathrm{dom}(s)$ and $\mathrm{ran}(st) = \mathrm{ran}(t)$ c.f. [18]. The algebra $AG(S)$ is termed the *groupoid algebras* of S [44].

Define temporarily, for $s \in S$, an element $v_s \in AG(S)$ by

$$v_s = \sum_{t \leq s} \lfloor t\rfloor.$$

Then by Möbius inversion:

$$\lfloor s\rfloor = \sum_{t \leq s} v_t \mu(t, s).$$

Hence $\{v_s \mid s \in S\}$ is a basis for $AG(S)$. The following result from [44] can be viewed as a coordinate-free form of the Schützenberger representation [37] or Preston-Vagner representation [18]. We include the proof for completeness. The proof here includes a small detail that was unfortunately omitted from [44]. Nonetheless, the proof is more compact, as we don't obtain it from considering explicitly the direct sum of Schützenberger representations, as was done in [44].

Lemma 4.1 ([44]). *Let $s, t \in S$. Then $v_s v_t = v_{st}$.*

Proof. First we compute

$$\begin{aligned} v_s v_t &= \left(\sum_{s'\leq s}\lfloor s'\rfloor\right)\left(\sum_{t'\leq t}\lfloor t'\rfloor\right) \\ &= \sum_{s'\leq s,\ t'\leq t,\ \mathrm{ran}(s')=\mathrm{dom}(t')}\lfloor s't'\rfloor. \end{aligned}$$

Since the natural partial order is compatible with multiplication if $s'\leq s$ and $t'\leq t$, it follows that $s't'\leq st$. Thus to obtain the desired result it suffices to show that each $u\leq st$ can be written uniquely as a product $s't'$ with $s'\leq s$, $t'\leq t$ and $\mathrm{ran}(s')=\mathrm{dom}(t')$. Note that $u\leq st$ implies that $uu^{-1}st=u=stu^{-1}u$.

To obtain such a factorization $u=s't'$ we must have $\mathrm{dom}(s')=\mathrm{dom}(u)$ and $\mathrm{ran}(t')=\mathrm{ran}(u)$. That is we must have $s'=uu^{-1}s$ and $t'=tu^{-1}u$. Clearly $s't'=uu^{-1}stu^{-1}u=uu^{-1}u=u$. Let's check that $\mathrm{ran}(s')=\mathrm{dom}(t')$. First observe that $(s')^{-1}=tu^{-1}$. Indeed,

$$\begin{aligned} s'(tu^{-1})s' &= (uu^{-1}stu^{-1}u)u^{-1}s = uu^{-1}s = s' \\ (tu^{-1})s'(tu^{-1}) &= tu^{-1}(uu^{-1}st)u^{-1} = tu^{-1}uu^{-1} = tu^{-1}. \end{aligned}$$

Similarly, one can verify that $(t')^{-1}=u^{-1}s$. Thus we see

$$(s')^{-1}s' = tu^{-1}(uu^{-1}s) = (tu^{-1}u)u^{-1}s = t'(t')^{-1},$$

as desired. □

As a consequence, it follows that $AG(S)\cong AS$ and from now on we identify AS with $AG(S)$ by identifying s with v_s. In particular we shall drop the notation $AG(S)$. So the viewpoint is that we have two natural bases for the semigroup algebra AS: the usual basis and the basis $\{\lfloor s\rfloor \mid s\in S\}$. Let us make this precise.

Theorem 4.2 ([44])**.** *Let A be a unital algebra and S a finite inverse semigroup. Define, for $s\in S$,*

$$\lfloor s\rfloor = \sum_{t\leq s} t\mu(t,s). \tag{4.2}$$

Then $\{\lfloor s\rfloor \mid s\in S\}$ is a basis for AS and the multiplication with respect to this basis is given by (4.1)*.*

It is worth remarking that Theorem 4.2 remains valid for infinite inverse semigroups S so long as one assumes descending chain condition on the set of idempotents of S. We now give a new proof of the semisimplicity of KS for S a finite semigroup and K a subfield of $\mathbb{C}$ by comparing the linear representations associated to the two bases of KS. If we use the usual basis $\{s \mid s\in S\}$, then the associated linear representation is just the regular representation of S. What we now wish to show is that if we use the basis $\{\lfloor s\rfloor \mid s\in S\}$, then we obtain the rook matrix representation associated to the Preston-Vagner representation $\rho_S : S \to \mathfrak{I}_S$. In particular, these two representations will be equivalent and so, since the latter (as observed earlier) is completely reducible, we shall obtain that the regular representation of KS is completely reducible, whence KS is a semisimple algebra.

Let us recall the definition of $\rho_S : S \to \mathfrak{I}_S$. For $s, t \in S$,

$$s\rho_S(t) = \begin{cases} st & s^{-1}s \leq tt^{-1} \\ \text{undefined} & \text{else.} \end{cases}$$

Our claim in the above paragraph is then an immediate consequence of the following proposition.

Proposition 4.3. *Let S be a finite inverse semigroup and A a commutative ring with unit. Then for $s, t \in S$,*

$$\lfloor s \rfloor t = \begin{cases} \lfloor st \rfloor & s^{-1}s \leq tt^{-1} \\ 0 & \textit{else.} \end{cases}$$

Proof. By Möbius inversion, $t = \sum_{u \leq t} \lfloor u \rfloor$. So

$$\begin{aligned} \lfloor s \rfloor t &= \lfloor s \rfloor \sum_{u \leq t} \lfloor u \rfloor \\ &= \sum_{u \leq t} \lfloor s \rfloor \lfloor u \rfloor. \end{aligned}$$

Now $\lfloor s \rfloor \lfloor u \rfloor = \lfloor su \rfloor$ if $s^{-1}s = uu^{-1}$ and is zero else. There can be at most one element $u \leq t$ with $s^{-1}s = uu^{-1}$ [18]. Now if $u \leq t$ and $s^{-1}s = uu^{-1}$, then $s^{-1}s \leq tt^{-1}$. Conversely, if $s^{-1}s \leq tt^{-1}$, define $u = s^{-1}st \leq t$. Then $u \leq t$ and

$$uu^{-1} = s^{-1}stt^{-1}s^{-1}s = s^{-1}s.$$

Moreover, $su = s(s^{-1}st) = st$. Thus

$$\lfloor s \rfloor t = \sum_{u \leq t} \lfloor s \rfloor \lfloor u \rfloor = \begin{cases} \lfloor st \rfloor & s^{-1}s \leq tt^{-1} \\ 0 & \text{else,} \end{cases}$$

completing the proof of the proposition. □

Corollary 4.4. *Let S be a finite inverse semigroup and K any field. Then the regular representation of S and the Preston-Vagner representation of S are equivalent as linear representations. In particular, if K is a subfield of $\mathbb{C}$, then KS is semisimple.*

4.2. **Decomposition into matrix algebras over group rings.** We now turn to decomposing KS into matrix algebras over group algebras. Let S be a finite inverse semigroup with $\mathcal{D}$-classes $D_1, \ldots, D_r$. Recall these are the equivalence classes corresponding to isomorphic idempotents. Let AD_i be the A-span of $\{\lfloor s \rfloor \mid s \in D_i\}$. The following result is immediate from (4.1) and Theorem 4.2.

Theorem 4.5. *Let A be a unital algebra and let S be a finite inverse semigroup with $\mathcal{D}$-classes $D_1, \ldots, D_r$. Then $AS = \bigoplus_{i=1}^{r} AD_i$.*

For each i, fix an idempotent e_i of D_i and let G_{e_i} be the corresponding maximal subgroup. Since the idempotents of D_i are isomorphic, this group does not depend on the choice of e_i up to isomorphism. Let $n_i = |E(D_i)|$, that is n_i denotes the number of idempotents in D_i. We recall the argument from [44] (essentially due to Munn [22, 23, 10]) that $AD_i \cong M_{n_i}(AG_{e_i})$. We view $n_i \times n_i$ matrices as indexed by pairs of idempotents of D_i. Fix, for each $e \in D_i$, an element $p_e \in S$ with $\mathrm{dom}(p_e) = e_i$ and $\mathrm{ran}(p_e) = e$. We take $p_{e_i} = e_i$. Define a map $\varphi : AD_i \to M_{n_i}(AG_{e_i})$ on a basis element $\lfloor s \rfloor \in AD_i$ with $\mathrm{dom}(s) = e$ and $\mathrm{ran}(s) = f$ by

$$\varphi(\lfloor s \rfloor) = p_e s p_f^{-1} E_{e,f} \tag{4.3}$$

where $E_{e,f}$ is the standard matrix unit with 1 in position e, f and zero in all other positions. Observe that $p_e s p_f^{-1} \in G_{e_i}$ by construction. It is straightforward [44] to show that φ is an isomorphism and to write down its inverse. The reader should compare (4.3) to the calculation of the fundamental group of a graph.

As a consequence of the above isomorphism, we obtain the following result, which is implicit in the work of Munn [23, 22, 10] and was made explicit in [44].

Theorem 4.6. *Let S be a finite inverse semigroup with representatives $e_1, \ldots, e_r$ of the isomorphism classes of idempotents. Let n_i be the number of idempotents isomorphic to e_i. Let A be a commutative ring with unit. Then $AS \cong \bigoplus_{i=1}^{r} M_{n_i}(AG_{e_i})$.*

This decomposition implies the well known fact that the size of S is $\sum_{i=1}^{r} n_i^2 |G_{e_i}|$. One may also deduce the following theorem of Munn [23, 10].

Corollary 4.7 (Munn)**.** *Let K be a field and S a finite inverse semigroup. If the characteristic of K is 0, then KS is semisimple. If the characteristic of K is a prime p, then KS is semisimple if and only if p does not divide the order of any maximal subgroup of S.*

We mention that Solomon [42] gives the exact decomposition obtained above for the special case of $\mathfrak{I}_n$ (this decomposition was obtained independently by V. Dlab in unpublished work). In [44] we went on to describe explicitly central idempotents and central primitive idempotents. However, the author, somewhat embarrassingly, missed that the above decomposition allows one to obtain a formula for character multiplicities (Solomon also seems to have missed this [42] since he uses a different approach to obtain multiplicity formulas for $\mathfrak{I}_n$; we shall compare the two approaches in Section 7). The goal of the next few sections is to rectify this.

5. Character Formulas for Multiplicities

In this section we assume that K is a field of characteristic zero. The most interesting case is when $K = \mathbb{C}$, the complex field. Fix a finite inverse semigroup S and, for each $\mathcal{D}$-class $D_1, \ldots, D_r$ of S, fix an idempotent e_i and set $G_i = G_{e_i}$. Let again n_i be the number of idempotents in D_i.

From Theorem 4.6 it is clear that the irreducible representations of S over K correspond to elements of the set $\biguplus_{i=1}^{r} \mathrm{Irr}(G_i)$, where $\mathrm{Irr}(G_i)$ is the set of irreducible representations of G_i (up to equivalence). Namely, if φ is an irreducible representation of G_i, then we can tensor it up to $KD_i \cong M_{n_i}(KG_i)$ and then extend to KS by making it zero on the other summands. Let us call this representation φ^*. If χ is the character of G associated to φ, denote by χ^* the character of S associated to φ^*. Note that $\deg(\varphi^*) = n_i \deg(\varphi)$. Thus over an algebraically closed field we see, as in the group case, that the size of S is the sum over all inequivalent irreducible representations of S of the squares of the degrees of the irreducible representations.

Let G be a finite group and $\psi, \alpha : G \to K$ be a functions. We define on the space of K-valued functions on G the usual bilinear form given by

$$(\psi, \alpha)_G = \frac{1}{|G|} \sum_{g \in G} \psi(g^{-1})\alpha(g).$$

If χ is an irreducible character of G and α is any character of G, then standard group representation theory says that $(\chi, \alpha)_G = md$ where m is the multiplicity of χ as a constituent of α and d is the degree over K of the division algebra of KG-endomorphisms of the simple KG-module corresponding to χ.

Let us consider now an irreducible representation of $M_n(KG)$. We identify G with $G \cdot E_{1,1}$. If we have a character θ of $M_n(KG)$ and an irreducible character χ of G, then clearly

$$(\chi, \theta|_G)_G = md \tag{5.1}$$

where m is the multiplicity of χ^* in θ and d is the dimension of the associated division algebra to χ.

We now wish to generalize the above formula to our finite inverse semigroup S. Given a character θ of S and an idempotent $f \in S$, define θ_f by $\theta_f(s) = \theta(fs)$. If idempotent $e \in E(S)$, then θ_f restricts to a K-valued function on G_e (for which we use the same notation); it need not in general be a class function. Now if χ is a K-valued function on G_e, define

$$(\chi, \theta)_S = \sum_{f \leq e} (\chi, \theta_f)_{G_e} \mu(f, e). \tag{5.2}$$

Of course, if S is a group this reduces to the usual formula. Here is the main new result of this paper.

Theorem 5.1. *Let S be a finite inverse semigroup and $e \in E(S)$. Let θ be a character of S and let χ be an irreducible character of G_e. Denote by d the* *dimension of the division algebra associated to χ and by m the multiplicity of the induced irreducible character χ^* of S in θ. Then*

$$(\chi,\theta)_S = \sum_{f\leq e}(\chi,\theta_f)_{G_e}\mu(f,e) = md.$$

Proof. Our discussion above about matrix algebras over group algebras and the explicit isomorphism we have constructed between KS and the direct sum of matrix algebras over maximal subgroups tells us how to calculate multiplicities using the basis $\{\lfloor s\rfloor \mid s\in S\}$ for KS. Namely (5.1) is transformed via our isomorphism to:

$$md = \frac{1}{|G_e|}\sum_{g\in G_e}\chi(g^{-1})\theta(\lfloor g\rfloor) \tag{5.3}$$

$$= \frac{1}{|G_e|}\sum_{g\in G_e}\chi(g^{-1})\theta\left(\sum_{t\leq g} t\mu(t,g)\right) \tag{5.4}$$

$$= \frac{1}{|G_e|}\sum_{g\in G_e}\chi(g^{-1})\sum_{t\leq g}\theta(t)\mu(t,g) \tag{5.5}$$

But recall that the order ideal $g^{\downarrow}$ can be parameterized as

$$\{fg \mid f\leq \mathrm{dom}(g)=e\}$$

and that, for $f\leq \mathrm{dom}(g)$,

$$\mu(fg,g) = \mu(\mathrm{dom}(fg),\mathrm{dom}(g)) = \mu(f,e).$$

Thus the right hand side of (5.5) is equal to

$$\begin{aligned}
&= \frac{1}{|G_e|}\sum_{g\in G_e}\chi(g^{-1})\sum_{f\leq e}\theta(fg)\mu(f,e)\\
&= \sum_{f\leq e}\left(\frac{1}{|G_e|}\sum_{g\in G_e}\chi(g^{-1})\theta_f(g)\right)\mu(f,e)\\
&= \sum_{f\leq e}(\chi,\theta_f)_{G_e}\mu(f,e)\\
&= (\chi,\theta)_S.
\end{aligned}$$

Thus we obtain $(\chi,\theta)_S = md$, as desired. □

Of course if K is algebraically closed, or more generally if χ is an absolutely irreducible representation, then $(\chi,\theta)_S$ is the multiplicity of χ^* in θ. This lets us generalize the following well-known result from groups to inverse semigroups.

Corollary 5.2. *Let K be a field of characteristic 0 and let S be a finite inverse semigroup. Then two representations of S are equivalent if and only if they have the same character.*

Proof. Since KS is semisimple, two representations are equivalent if and only if they have the same multiplicity for each irreducible constituent. But Theorem 5.1 shows that the multiplicities of the irreducible constituents depend only on the character. □

6. Applications to Decomposing Representations

In this section we use the formula from Theorem 5.1 to calculate multiplicities in different settings. For the case of $\mathfrak{I}_n$, our formula looks different than that of [42], but of course gives the same answer. In fact, we shall see that our formula is easier to use than Solomon's and at the same time applies more generally. Throughout this section we assume that K is an algebraically closed field of characteristic zero.

6.1. **Tensor powers.** Let us begin with $S = \mathfrak{I}_n$. We choose as our set of isomorphism class representatives the idempotents $1_{[r]}$ with $r \leq n$. We identify the maximal subgroup at $1_{[r]}$ with $\mathfrak{S}_r$ (where $\mathfrak{S}_0$ is the trivial group and the empty set is viewed as having a unique partition). If μ is a partition of r, let χ^μ denote the irreducible character of $\mathfrak{S}_r$ associated to μ [36]. Then, for θ a character of $\mathfrak{I}_n$, we obtain the following multiplicity formula (which should be contrasted with [42, Lemma 3.17]):

$$(\chi^\mu, \theta)_{\mathfrak{I}_n} = \sum_{X \subseteq [r]} (-1)^{r-|X|} (\chi^\mu, \theta_{1_X})_{\mathfrak{S}_r} \tag{6.1}$$

Solomon decomposes the tensor powers of the rook matrix representation of $\mathfrak{I}_n$ using his formula. We do the same using ours, but more generally. In particular, we can handle wreath products of the form $G \wr S$ with G a finite group and $S \leq \mathfrak{I}_n$ a finite inverse semigroup containing all the idempotents of $\mathfrak{I}_n$. This includes, in addition to the symmetric inverse monoid, the signed symmetric inverse monoid.

First we need a standard lemma about Boolean algebras whose proof we leave to the reader. We use x^c for the complement of x in a Boolean algebra. We continue to use multiplicative notation for the meet in a semilattice.

Lemma 6.1. *Let B be a Boolean algebra and $x \in B$. Then $B \cong x^\downarrow \times (x^c)^\downarrow$ via the maps*

$$y \longmapsto (yx, yx^c)$$
$$y \vee z \longleftarrow\!\shortmid (y, z)$$

To apply Lemma 6.1 we use the well known fact [43] that if P_1, P_2 are finite partially ordered sets, then

$$\mu_{P_1 \times P_2}((x, y), (u, v)) = \mu_{P_1}(x, u)\mu_{P_2}(y, v). \tag{6.2}$$

Combining Lemma 6.1 and (6.2) we obtain:

Corollary 6.2. *Let B be a Boolean algebra and fix $x \in B$. Then for $a, b \in B$, $\mu(a, b) = \mu(ax, bx)\mu(ax^c, bx^c)$.*

In the case of $\mathfrak{B}_n$, Corollary 6.2 just asserts that if $X \subseteq [n]$ is fixed, then for $Y \subseteq Z$, $(-1)^{|Z|-|Y|} = (-1)^{|Z\cap X|-|Y\cap X|} \cdot (-1)^{|Z\cap X^c|-|Y\cap X^c|}$.

We shall also need the following reformulation of the fact that μ is the inverse of ζ:

$$\sum_{x\leq y\leq z} \mu(y,z) = \begin{cases} 1 & x = z \\ 0 & x < z \end{cases} \tag{6.3}$$

If $s \in S \leq \mathfrak{I}_n$, then $\mathrm{Fix}(s)$ denotes the set of fixed points of s on $[n]$. Of course, $\mathrm{Fix}(s) \leq \mathrm{dom}(s)$.

Proposition 6.3. *Let $S \leq \mathfrak{I}_n$ and θ^p be the character of the p^{th}-tensor power of the rook matrix representation of S. Let $e \in E(S)$ and $\chi \in \mathrm{Irr}(G_e)$. Suppose that $\{X \subseteq \mathrm{dom}(e) \mid 1_X \in E(S)\}$ is closed under the operations of union and relative complement, i.e. $X \mapsto \mathrm{dom}(e) \setminus X$. Suppose further that, for all $g \in G_e$, $1_{\mathrm{Fix}(g)} \in E(S)$. Then*

$$(\chi,\theta^p)_S = \frac{1}{|G_e|}\deg(\chi)\sum_{f\leq e}\mathrm{rk}(f)^p\mu(f,e) \tag{6.4}$$

Proof. We begin by computing:

$$\begin{aligned}(\chi,\theta^p)_S &= \sum_{f\leq e}(\chi,\theta^p_f)_{G_e}\mu(f,e) \\ &= \sum_{f\leq e}\frac{1}{|G_e|}\sum_{g\in G}\chi(g^{-1})\theta^p(fg)\mu(f,e) \\ &= \frac{1}{|G_e|}\sum_{g\in G}\chi(g^{-1})\sum_{f\leq e}\theta^p(fg)\mu(f,e)\end{aligned} \tag{6.5}$$

So let us analyze the term $\sum_{f\leq e}\theta^p(fg)\mu(f,e)$. Setting $h = 1_{\mathrm{Fix}(g)} \in E(S)$ and $h^c = 1_{\mathrm{dom}(e)\setminus\mathrm{Fix}(g)} \in E(S)$, we can rewrite $\theta^p(fg)$ as follows:

$$\theta^p(fg) = |\mathrm{Fix}(fg)|^p = |\mathrm{Fix}(g|_{\mathrm{dom}(f)})|^p = |\mathrm{Fix}(g)\cap\mathrm{dom}(f)|^p = \mathrm{rk}(hf)^p.$$

The above equation, together with Corollary 6.2 and the fact that $e^{\downarrow}$ is a Boolean algebra (via our hypotheses), allows us to rewrite our sum:

$$\begin{aligned}\sum_{f\leq e}\theta^p(fg)\mu(f,e) &= \sum_{x\leq h,\ y\leq h^c}\mathrm{rk}(x)^p\mu(x,h)\mu(y,h^c) \\ &= \sum_{x\leq h}\mathrm{rk}(x)^p\mu(x,h)\sum_{0\leq y\leq h^c}\mu(y,h^c)\end{aligned}$$

But by (6.3),

$$\sum_{0\leq y\leq h^c}\mu(y,h^c) = 0$$

unless $h^c = 0$, or equivalently, unless $e = h$. In this latter case, we then have that $\mathrm{dom}(e) = \mathrm{Fix}(g)$ and hence $g = e$. Thus the right hand side of (6.5) becomes:

$$\begin{aligned} &= \frac{1}{|G_e|}\chi(e)\sum_{f\leq e}\mathrm{rk}(f)^p\mu(f,e) \\ &= \frac{1}{|G_e|}\deg(\chi)\sum_{f\leq e}\mathrm{rk}(f)^p\mu(f,e). \end{aligned}$$

This proves (6.4) □

We wish to obtain Solomon's result [42, Example 3.18] in a more general form, in particular for wreath products. If G is a finite group and $S \subseteq \mathfrak{I}_n$, we can define their *wreath product* $G \wr S$ as an inverse semigroup of partial permutations of $G \times [n]$. It consists of all partial permutations that can be expressed in the form (f, σ) where $\sigma \in S$, $f : [n] \to G$ and

$$(g, i)(f, \sigma) = \begin{cases} (gf(i), i\sigma) & \text{if } i\sigma \text{ is defined} \\ \text{undefined} & \text{else.} \end{cases}$$

It is easy to verify that $G \wr S$ is an inverse subsemigroup of $\mathfrak{I}_{G\times[n]} \cong \mathfrak{I}_{|G|n}$. The reader should consult the text of Eilenberg [12] for more on wreath products of partial transformation semigroups. We remark that the representation in the form (f, σ) is not unique if σ is not totally defined. Alternatively, $G \wr S$ can be described as all matrices with entries in $G \cup \{0\}$ that can be obtained by replacing ones in the rook matrices from S by arbitrary elements of G. So, for example, the signed symmetric inverse monoid $\mathbb{Z}_2 \wr \mathfrak{I}_n$ consists of all signed rook matrices.

Suppose now that $S \leq \mathfrak{I}_n$ contains all the idempotents of $\mathfrak{I}_n$. Then the idempotents of $G \wr S$ are precisely the identities at the subsets of the form $G \times X$ with $X \subseteq [n]$. Hence, $E(G \wr S) \cong \mathfrak{B}_n$ and so if $Y \subseteq X$, then

$$\mu(1_{G\times Y}, 1_{G\times X}) = (-1)^{|X|-|Y|}.$$

The maximal subgroup of $G\wr S$ at $G\times X$ is isomorphic to the wreath product $G \wr G_X$ where G_X is the maximal subgroup of S at X. In particular, for $G \wr \mathfrak{I}_n$, we see that the maximal subgroup at $G \times [r]$ is $G \wr \mathfrak{S}_r$. So for the signed symmetric inverse monoid, the maximal subgroups are the signed symmetric groups of appropriate ranks. Clearly if $X \subseteq [n]$, then the set of subsets of the form $G \times Y$ with $Y \subseteq X$ is closed under union and relative complement. Thus to verify that Proposition 6.3 applies to $G \wr S$, we must show that if (f, s) represents an element g of $G \wr G_X$, then $\mathrm{Fix}(g)$ is of the form $G \times Y$ with $Y \subseteq X$. But $\mathrm{Fix}(g) = G \times (1f^{-1} \cap \mathrm{Fix}(s))$, which is of the required form.

Let $S(p, r)$ be the Stirling number of the second kind [43]. It is given by

$$S(p, r) = \frac{1}{r!}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}k^p.$$

Then we obtain the following generalization of [42, Example 3.18], where $S = \mathfrak{I}_n$ and G is trivial.

Theorem 6.4. *Let $S \leq \mathfrak{I}_n$ contain all the idempotents and let G be a finite group. Let θ^p be the character of the p-tensor power of the representation of $G \wr \mathfrak{I}_n \leq \mathfrak{I}_{|G|n}$ by rook matrices. Let $X \subseteq [n]$ with $|X| = r$ and let G_X be the associated maximal subgroup of S. Let $\chi \in \mathrm{Irr}(G \wr G_X)$. Then*

$$(\chi, \theta^p)_S = \frac{1}{|G|^{r-p}|G_X|}\deg(\chi) r! S(p,r). \tag{6.6}$$

In particular, for $S = \mathfrak{I}_n$, we obtain

$$(\chi, \theta^p)_S = \frac{1}{|G|^{r-p}}\deg(\chi) S(p,r). \tag{6.7}$$

Proof. Since $\mathrm{rk}(1_{G\times Y}) = |G||Y|$, we calculate, using Proposition 6.3:

$$\begin{aligned}(\chi, \theta^p)_S &= \frac{1}{|G|^r|G_X|}\deg(\chi)\sum_{Y\subseteq X}(-1)^{|X|-|Y|}(|G||Y|)^p \\ &= \frac{1}{|G|^{r-p}|G_X|}\deg(\chi)\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}k^p \\ &= \frac{1}{|G|^{r-p}|G_X|}\deg(\chi) r! S(p,r).\end{aligned}$$

In particular, if $G_X = \mathfrak{S}_r$, (6.6) reduces to (6.7). $\square$

Specializing to the case that G is trivial we obtain:

Corollary 6.5. *Let $S \leq \mathfrak{I}_n$ with $E(S) = E(\mathfrak{I}_n)$. Let $X \subseteq [n]$ with $|X| = r$ and let G_X be the maximal subgroup of S with identity 1_X. Let χ be an irreducible character of G_X. Then the multiplicity of χ^* in the p-tensor power θ^p of the rook matrix representation of S is given by*

$$(\chi, \theta^p)_S = \frac{1}{|G_X|}\deg(\chi) r! S(p,r).$$

In particular, if $S = \mathfrak{I}_r$, $X = [r]$ and χ^λ is the irreducible character corresponding to a partition λ of $[r]$, then the multiplicity of $(\chi^\lambda)^*$ in θ^p is $f_\lambda S(p,r)$ where f_λ is the number of standard Young tableaux of type λ (c.f. [36]*).*

Two other examples for which the formula from Corollary 6.5 is valid are $\mathfrak{B}_n$ and for the inverse semigroup of all order-preserving partial permutations of $[n]$. In both these cases all the maximal subgroups are trivial. So the formula then comes down to saying that if e is an idempotent of rank r in either of these semigroups, then the multiplicity of the unique irreducible representation associated to e in θ^p is simply $r!S(p,r)$.

6.2. **Exterior powers.** We continue to take K to be an algebraically closed field of characteristic zero. Solomon showed [42, Example 3.22] that the exterior powers of the rook matrix representation of $\mathfrak{I}_n$ are irreducible and are induced by the alternating representations of the maximal subgroups. The proof is quite involved: he makes heavy use of the combinatorics of Ferrer's diagrams and the theory of symmetric functions. Here we obtain a more general result with a completely elementary proof.

Let us use $\mathfrak{A}_X$ for the alternating group on a finite set X. If $S \leq \mathfrak{I}_n$ and G_e is a maximal subgroup with identity e, then define, for $g \in G_e$,

$$\mathrm{sgn}_e(g) = \begin{cases} 1 & g \in \mathfrak{A}_{\mathrm{dom}(e)} \\ -1 & \text{else.} \end{cases}$$

That is, $\mathrm{sgn}_e(g)$ is the sign of g as a permutation of $\mathrm{dom}(e)$. Then the map $\mathrm{sgn}_e : G_e \to K$ is an irreducible representation (it could be the trivial representation if $G_e \subseteq \mathfrak{A}_{\mathrm{dom}(e)}$).

Theorem 6.6. *Let $S \leq \mathfrak{I}_n$ and let $\theta^{\wedge p}$ denote the character of the p^{th}-exterior power of the rook matrix representation of S. Suppose that S contains all the rank p idempotents of $\mathfrak{I}_n$. Let G_e be a maximal subgroup and $\chi \in \mathrm{Irr}(G_e)$. Then*

$$(\chi, \theta^{\wedge p})_S = \begin{cases} 1 & \mathrm{rk}(e) = p, \chi = \mathrm{sgn}_e \\ 0 & \textit{else.} \end{cases}$$

In particular, under these hypotheses $\theta^{\wedge p}$ is an irreducible character if and only if all rank p idempotents of S are isomorphic. So for $\mathfrak{I}_n$, all the exterior powers are irreducible.

Proof. Let V be the module affording the rook matrix representation and $\{e_1, \ldots, e_n\}$ be the standard basis; so a basis for $\bigwedge^p V$ is the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}.$$

If $s \in S$, then

$$(e_{i_1} \wedge \cdots \wedge e_{i_p})s = e_{i_1 s} \wedge \cdots \wedge e_{i_n s}.$$

This will be a non-zero multiple of $e_{i_1} \wedge \cdots \wedge e_{i_p}$ if and only if $Xs = X$, where $X = \{e_{i_1}, \ldots, e_{i_p}\}$. In this case,

$$(e_{i_1} \wedge \cdots \wedge e_{i_p})s = \mathrm{sgn}_X(s|_X)(e_{i_1} \wedge \cdots \wedge e_{i_p}).$$

Thus

$$\theta^{\wedge p}(s) = \sum_{Y \subseteq \mathrm{dom}(s), |Y|=p, Ys=Y} \mathrm{sgn}_Y(s|_Y). \tag{6.8}$$

We now compute for $\chi \in \mathrm{Irr}(G_e)$:

$$\begin{aligned}(\chi, \theta^{\wedge p})_S &= \sum_{f \leq e} \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) \theta^{\wedge p}(fg)\mu(f, e) \\ &= \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) \sum_{f \leq e} \theta^{\wedge p}(fg)\mu(f, e).\end{aligned} \tag{6.9}$$

Using (6.8), we obtain

$$\begin{aligned}\sum_{f \leq e} \theta^{\wedge p}(fg)\mu(f, e) &= \sum_{f \leq e} \left(\sum_{Y \subseteq \mathrm{dom}(f), |Y|=p, Yfg=Y} \mathrm{sgn}_Y((fg)|_Y) \right) \mu(f, e) \\ &= \sum_{f \leq e} \left(\sum_{Y \subseteq \mathrm{dom}(f), |Y|=p, Yg=Y} \mathrm{sgn}_Y(g|_Y)) \right) \mu(f, e) \\ &= \sum_{Y \subseteq \mathrm{dom}(e), |Y|=p, Yg=Y} \mathrm{sgn}_Y(g|_Y) \left(\sum_{1_Y \leq f \leq e} \mu(f, e) \right).\end{aligned}$$

An application of (6.3) then shows that $\sum_{1_Y \leq f \leq e} \mu(f, e)$ is zero unless $1_Y = e$, in which case it is one. So if $\mathrm{rk}(e) \neq p$, $(\chi, \theta^{\wedge p})_S = 0$. Otherwise, (6.9) becomes

$$(\chi, \theta^{\wedge p})_S = \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) \mathrm{sgn}_e(g) = (\chi, \mathrm{sgn}_e)_{G_e}.$$

This establishes the first part of the theorem. The second part is immediate from the first. □

6.3. **Direct products.** Let us consider a direct product $S \times T$ of two inverse semigroups S, T. Then $E(S \times T) = E(S) \times E(T)$. Moreover, the natural order on $S \times T$ is the product ordering. Hence, by (6.2),

$$\mu_{S \times T}((s, t), (s', t')) = \mu_S(s, s')\mu_T(t, t').$$

The $\mathcal{D}$-relation on $S \times T$ is the product of the $\mathcal{D}$-relations on S and T and the maximal subgroup at an idempotent (e, f) is $G_e \times G_f$. So if: K is a commutative ring with unit; $e_1, \ldots, e_s$, respectively $f_1, \ldots, f_t$, represent the $\mathcal{D}$-classes of idempotents of S, respectively T; and n_i, respectively m_j, is the number of idempotents in the $\mathcal{D}$-class of e_i, respectively f_j, then

$$\begin{aligned}
K(S\times T) &\cong \bigoplus_{i,j} M_{n_im_j}(K(G_{e_i}\times G_{f_j}))\\
&\cong \bigoplus_{i,j} M_{n_im_j}(KG_{e_i}\otimes KG_{f_j})\\
&\cong \bigoplus_{i,j} \left(M_{n_i}(KG_{e_i})\otimes M_{m_j}(KG_{f_j})\right)\\
&\cong KS\otimes KT.
\end{aligned}$$

If K is a field, then an irreducible character of $G_{e_i}\times G_{f_j}$ is of the form $\chi\otimes\eta$ where $\chi\in\mathrm{Irr}(G_{e_i})$, $\eta\in\mathrm{Irr}(G_{f_j})$ and $(\chi\otimes\eta)(g,h)=\chi(g)\eta(h)$ [39]. So if θ is a character of $S\times T$, then (assuming characteristic zero) the intertwining number is given by

$$(\chi\otimes\eta,\theta)_{S\times T}=\sum_{e\leq e_i, f\leq f_i}\ \sum_{g\in G_{e_i},h\in G_{f_i}}\chi(g^{-1})\eta(h^{-1})\theta(eg,fh)\mu_S(e,e_i)\mu_T(f,f_j).$$

6.4. **Decomposing partial permutation representations.** Let $S\leq\mathfrak{I}_n$ be a partial permutation inverse semigroup of degree n. We wish to decompose the associated rook matrix representation into irreducible constituents. If S is not transitive, then we can clearly obtain a direct sum decomposition in terms of the transitive components, so we may as well assume that S is a transitive partial permutation semigroup. One could obtain the decomposition below from semigroup theory folklore results: one argues that in this case the rook matrix representation is induced from the permutation representation of the maximal subgroup G of the 0-minimal ideal of S via the Schützenberger representation and hence by [34] and [17] it decomposes via the representations induced from those needed to decompose the associated permutation representation of G. We shall prove this in a more combinatorial way using our multiplicity formulas. Let us first state the result precisely.

Theorem 6.7. *Let $S\leq\mathfrak{I}_n$ be a transitive inverse semigroup. Let $e\in E(S)$ be an idempotent with associated maximal subgroup G_e. Let $\chi\in\mathrm{Irr}(G_e)$ and let θ be the character of the rook matrix representation of S (that is the fixed-point character). Then*

$$(\chi,\theta)_S=\begin{cases}(\chi,\theta|_{G_e})_{G_e} & \text{if } e \text{ has minimal non-zero rank}\\ 0 & \text{else.}\end{cases}\tag{6.10}$$

Moreover, all idempotents of minimal non-zero rank are isomorphic.

This theorem says that decomposing θ corresponds to decomposing $\theta|_{G_e}$ where e is an idempotent of minimal non-zero rank. Before proving the theorem, we consider some of its consequences. The following corollary (well known in the semigroup representation theory community) is quite useful. See the work of Zalcstein [46] for an analogue.

Corollary 6.8. *Suppose that $S \leq \mathfrak{I}_n$ is a transitive inverse semigroup containing a rank* 1 *transformation. Then the rook matrix representation of S is irreducible and is induced from the trivial representation of the maximal subgroup corresponding to a rank one idempotent.*

Proof. The maximal subgroup at a rank 1 idempotent e is trivial and the restriction of the fixed-point character θ to e is the character of the trivial representation of $\{e\}$ since $\theta(e) = \mathrm{rk}(e) = 1$. Theorem 6.7 immediately gives the result. □

The above result then leads to the following corollary identifying the representation of an inverse semigroup S induced by the trivial representation of a maximal subgroup.

Corollary 6.9. *Let S be a finite inverse semigroup and let D be a $\mathcal{D}$-class of S. Then S acts by partial permutations on the idempotents $E(D)$ of D via conjugation as follows:*

$$e^s = \begin{cases} s^{-1}es & e \leq \mathrm{dom}(s) \\ \textit{undefined} & \textit{else} \end{cases}$$

for $e \in E(D)$ and $s \in S$. The associated rook matrix representation of S is the irreducible representation corresponding to the trivial representation of the maximal subgroup of D.

Proof. We first show that the action is well defined. Suppose $e \in D$ and e^s is defined. Then

$$s^{-1}ess^{-1}es = s^{-1}ees = s^{-1}es$$

as $e \leq ss^{-1}$. Thus $e^s \in E(S)$. Also

$$\mathrm{dom}(es) = ess^{-1}e = e$$
$$\mathrm{ran}(es) = s^{-1}ees = s^{-1}es,$$

showing that $e^s \in D$. The reader can verify directly that $e \leq \mathrm{dom}(st)$ if and only if $e \leq \mathrm{dom}(s)$ and $e^s \leq \mathrm{dom}(t)$ and that in this case $(e^s)^t = e^{st}$.

To complete the proof it suffices by Corollary 6.8 to show that S acts transitively and that some element of D acts as a rank one partial transformation. If $e, f \in D$, then there exists $s \in D$ with $\mathrm{dom}(s) = e$, $\mathrm{ran}(s) = f$. Hence

$$e^s = s^{-1}es = s^{-1}ss^{-1}s = s^{-1}s = \mathrm{ran}(s) = f$$

establishing transitivity. If $e, f \in D$ and $e \neq f$, then $efe = ef \notin D$ (as idempotents in a $\mathcal{D}$-class of a finite inverse semigroup are incomparable [18]). Hence we may conclude that e acts as as the rank one partial identity that fixes e and is undefined elsewhere. This completes the proof. □

The above representation is really the right letter mapping representation of Rhodes [16] in disguise and its irreducibility follows from the results of [34]. The direct sum of the above representations over all $\mathcal{D}$-classes is the celebrated Munn representation [18].

The proof of Theorem 6.7 requires some preliminary results that form part of the body of semigroup theory folklore. We begin with the proof of the final statement.

Proposition 6.10. *Let $S \leq \mathfrak{I}_n$ be a transitive inverse semigroup. Let e, f be idempotents of minimal non-zero rank. Then $e \,\mathcal{D}\, f$.*

Proof. If $e = f$, then there is nothing to prove. Assume that $e \neq f$. Choose $x \in \mathrm{dom}(e)$, $y \in \mathrm{dom}(f)$. By transitivity there exists $s \in S$ such that $xs = y$. We claim that $\mathrm{dom}(esf) = e$, $\mathrm{ran}(esf) = f$ and so $e \,\mathcal{D}\, f$. Indeed, $x \in \mathrm{dom}(esf) \subseteq \mathrm{dom}(e)$. Minimality then gives that $\mathrm{dom}(esf) = e$. The argument that $\mathrm{ran}(esf) = \mathrm{ran}(f)$ is similar. □

We shall need that the domains of the minimal rank non-zero idempotents partition $[n]$.

Lemma 6.11. *Let $S \leq \mathfrak{I}_n$ be a transitive inverse semigroup. Then the domains of the minimal rank non-zero idempotents partition $[n]$.*

Proof. We first prove disjointness. Let $i \in \mathrm{dom}(e) \cap \mathrm{dom}(f)$ with e and f minimal rank non-zero idempotents. The $i \in \mathrm{dom}(ef)$ and so $0 < \mathrm{rk}(ef) \leq \mathrm{rk}(e), \mathrm{rk}(f)$. Thus $e = ef = f$, as required. Now we show that any $i \in [n]$ belongs to the domain of some idempotent e of minimal non-zero rank. Let f be any idempotent of minimal non-zero rank and let $j \in \mathrm{dom}(f)$. By transitivity there is an element $s \in S$ with $js = i$. Then $i \in \mathrm{dom}(s^{-1}fs)$ and $0 < \mathrm{rk}(s^{-1}fs) \leq \mathrm{rk}(f)$. It therefore suffices to show that $s^{-1}fs$ is idempotent. Since $j \in \mathrm{dom}(f) \cap \mathrm{dom}(s)$, it follows by the minimality of f that $f \leq \mathrm{dom}(s)$. Thus $s^{-1}fs$ is an idempotent (c.f. Corollary 6.9). □

6.4.1. *Proof of Theorem 6.7.* We shall use the notation from the statement of Theorem 6.7. Define $h : S \to \mathbb{Z}$ by setting

$$h(s) = \sum_{t \leq s} \theta(t)\mu(t, s). \tag{6.11}$$

We need a key technical lemma.

Lemma 6.12. *The function $h : S \to \mathbb{Z}$ is given by the formula:*

$$h(s) = \begin{cases} \theta(s) & \text{if } s \text{ has minimal non-zero rank} \\ 0 & \text{else.} \end{cases}$$

Proof. Let M be the set of elements of S of minimal non-zero rank and let M^c be the complement of M. Recall that $\theta(t) = |\mathrm{Fix}(t)|$ for $t \in S$. In particular $\theta(0) = 0$. If $s \in M$, then from $\theta(0) = 0$ and (6.11) we may deduce $h(s) = \theta(s)$, as desired. So assume $s \notin M$, i.e. $s \in M^c$.

Möbius inversion (Theorem 3.1) gives us

$$\theta(s) = \sum_{t \leq s} h(t) = \sum_{t \in M,\ t \leq s} h(t) + \sum_{t \in M^c,\ t \leq s} h(t) \tag{6.12}$$

By the case already handled, for $t \in M$,

$$h(t) = \theta(t) = |\mathrm{Fix}(t)|.$$

Let $t, t' \in M$ with $t, t' \leq s$ and suppose that $\mathrm{Fix}(t) \cap \mathrm{Fix}(t') \neq \emptyset$. In particular, $\mathrm{dom}(t) \cap \mathrm{dom}(t') \neq \emptyset$. Since $\mathrm{dom}(t)$, $\mathrm{dom}(t')$ are minimal rank non-zero idempotents, Lemma 6.11 tells us that $\mathrm{dom}(t) = \mathrm{dom}(t')$. Since they are both restrictions of s, we conclude that $t = t'$. Hence distinct $t \in M$ with $t \leq s$ have disjoint fixed-points sets implying

$$\sum_{t \in M,\ t \leq s} h(t) = \sum_{t \in M,\ t \leq s} |\mathrm{Fix}(t)| = \left| \bigcup_{t \in M,\ t \leq s} \mathrm{Fix}(t) \right|. \tag{6.13}$$

We claim

$$\bigcup_{t \in M,\ t \leq s} \mathrm{Fix}(t) = \mathrm{Fix}(s).$$

Clearly the left hand side is contained in the right hand side. Conversely, if $x \in \mathrm{Fix}(s)$, then by Lemma 6.11 there is a (unique) minimal non-zero idempotent f such that $x \in \mathrm{dom}(f)$. Hence $x \in \mathrm{Fix}(fs)$, $fs \in M$ and $fs \leq s$. Thus the sum in (6.13) is $|\mathrm{Fix}(s)| = \theta(s)$. Putting this together with (6.12), we obtain

$$0 = \sum_{t \in M^c,\ t \leq s} h(t) \tag{6.14}$$

The formula (6.14) is valid for any element $s \in M^c$. Therefore, Möbius inversion in M^c (with the induced ordering) and (6.14) imply $h(s) = 0$ for any $s \in M^c$, as desired. □

We may now complete the proof of Theorem 6.7. One calculates

$$\begin{aligned}
(\chi, \theta)_S &= \sum_{f \leq e} (\chi, \theta_f)_{G_e} \mu(e, f) \\
&= \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) \sum_{f \leq e} \theta(fg) \mu(f, e) \\
&= \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) \sum_{t \leq g} \theta(t) \mu(t, g) \\
&= \frac{1}{|G_e|} \sum_{g \in G_e} \chi(g^{-1}) h(g) \\
&= (\chi, h|_{G_e})_{G_e}.
\end{aligned}$$

But $h|_{G_e} = 0$ if e is not of minimal non-zero rank and is $\theta|_{G_e}$ otherwise by Lemma 6.12. This establishes Theorem 6.7. □

7. The Character Table and Solomon's Approach

In this section we generalize [42, Section 3] to arbitrary finite inverse semigroups. If s is an element of a finite semigroup, then s^ω denotes its unique idempotent power. We set $s^{\omega+1} = ss^\omega$. The element $s^{\omega+1}$ belongs to the unique maximal subgroup of $\langle s\rangle$. It is well known that any character χ of S has the property that $\chi(s) = \chi(s^{\omega+1})$ [20, 34]. Indeed, if $s^n = s^\omega$, then $s^{2n} = s^n$. So if ρ is the representation affording χ, then $\rho(s)$ satisfies the polynomial $x^n(x^n - 1)$. It follows that $\rho(s)$ and $\rho(s)^{n+1}$ have the same trace by considering, say, their Jordan canonical form over the algebraic closure. Thus $\chi(s) = \chi(s^{n+1}) = \chi(s^{\omega+1})$.

Let S be a finite inverse semigroup. Let e and f be isomorphic idempotents. Then $g \in G_e$ and $h \in G_f$ are said to be *conjugate* if there exists $s \in S$ such that $\mathrm{dom}(s) = e$, $\mathrm{ran}(s) = f$ and $g = shs^{-1}$. Note that if $e = f$, this reduces to the usual notion of conjugacy in G_e. Conjugacy is easily verified to be an equivalence relation. Let χ be a character of S and suppose that g and h are conjugate, say $g = shs^{-1}$ with s as above. Then

$$\chi(g) = \chi(shs^{-1}) = \chi(hs^{-1}s) = \chi(h)$$

since $s^{-1}s = \mathrm{ran}(s) = f$. Here we use that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for matrices.

Finally define, for $s, t \in S$, $s \sim t$ if $s^{\omega+1}$ is conjugate to $t^{\omega+1}$. It is easy to see that this is an equivalence relation, which we call *character equivalence*. The discussion above shows that the characters of S are constant on character equivalence classes. Let $e_1, \ldots, e_n$ represent the distinct isomorphism classes of idempotents and set $G_i = G_{e_i}$. Then there is a bijection between character equivalence classes and the (disjoint) union of the conjugacy classes of the G_i. Namely, each character equivalence class intersects one and only one maximal subgroup G_i and it intersects that subgroup in a conjugacy class. We use $\overline{s}$ for the character equivalence class of $s \in S$.

One can the define (c.f. [20, 24, 34]) the *character table* C of a finite inverse semigroup S to have rows indexed by the irreducible characters of S over the complex field $\mathbb{C}$ and columns indexed by the character equivalence classes. The entry in the row of a character χ and the column of a character equivalence class $\overline{s}$ gives the value of χ on $\overline{s}$. The table C is square since both sides are in bijection with the union of the conjugacy classes of the G_i. In order to arrange the table in the most convenient way possible, let us recall that there is a preorder on the idempotents of any inverse semigroup S defined as follows [18]: $e \preceq f$ if there is an idempotent e' with $e \mathrel{\mathcal{D}} e' \leq f$. This is the same as saying there are elements $s, t \in S$ with $e = sft$. For a finite inverse semigroup, $e \preceq f$ and $f \preceq e$ if and only if $e \mathrel{\mathcal{D}} f$ [18]. In particular, if $e_1, \ldots, e_n$ are as above, then $\preceq$ induces a partial order on $\{e_1, \ldots, e_n\}$. Reordering if necessary, we may assume that $e_i \preceq e_j$ implies $i \leq j$. It is easy to check (and well known) that if χ is an irreducible character coming from a maximal subgroup G_i and $g \in G_j$ is such that

$\chi(g) \neq 0$, then $e_i \preceq e_j$. Indeed, for the character not to vanish, g must have a restriction in the $\mathcal{D}$-class of e_i.

Instead of labelling the rows by irreducible characters of S we label them by $\biguplus \mathrm{Irr}(G_i)$ and similarly instead of labelling the columns by character equivalence classes, we label them by the conjugacy classes of the G_i. So if $\chi \in \mathrm{Irr}(G_i)$ and C is a conjugacy class of G_j, then $\mathsf{C}_{\chi,C}$ is the value of χ^* on the character equivalence class containing C. If we group the rows and columns in blocks corresponding to the ordering $G_1, \ldots, G_n$, then the character table becomes block upper triangular. More precisely, we have

$$\mathsf{C} = \begin{pmatrix} \mathsf{X}_1 & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mathsf{X}_{n-1} & * \\ 0 & \cdots & 0 & \mathsf{X}_n \end{pmatrix}$$

where X_i is the character table of G_i. In particular, the matrix C is invertible (as character tables of finite groups are invertible). Notice that Solomon's table [42] differs from ours cosmetically in how the rows and columns are arranged.

Define a block diagonal matrix

$$\mathsf{Y} = \mathrm{diag}(\mathsf{X}_1, \ldots, \mathsf{X}_n) \tag{7.1}$$

Then there are unique block upper unitriangular matrices A and B such that

$$\mathsf{C} = \mathsf{YA} \qquad \text{and} \qquad \mathsf{C} = \mathsf{BY}. \tag{7.2}$$

So to determine the character table of S, one just needs Y (that is the character tables of the maximal subgroups) and A or B. We aim to show that A is determined by combinatorial data associated to S. Solomon explicitly calculated this matrix for $\mathfrak{I}_n$, but it seems to be a daunting task in general. If $g \in G_i$, we use C^i_g to denote the conjugacy class of g in G_i.

Proposition 7.1. *Let $h \in G_i$ and $g \in G_j$. Then $\mathsf{A}_{C^i_h, C^j_g}$ is the number of restrictions of g that are conjugate to h in S.*

Proof. Let $g \in G_j$ and χ be an irreducible character coming from G_i. For each idempotent $f \mathrel{\mathcal{D}} e_i$, choose p_f with $\mathrm{dom}(p_f) = e_i$, $\mathrm{ran}(p_f) = f$. Given $h \in G_i$, let $a_{C^i_h, C^j_g}$ be the number of restrictions of g conjugate to h in S (this number can easily be verified to depend only on C^i_h and C^j_g).

Then, by the results of [44] or direct calculation,

$$\begin{aligned} \mathsf{C}_{\chi, C^j_g} &= \sum_{f \leq e_j, f \mathcal{D} e_i, g^{-1}fg = f} \chi(p_f(fg)p_f^{-1}) \\ &= \sum_{C^i_h} \chi(h) a_{C^i_h, C^j_g} \end{aligned}$$

where the last sum is over the conjugacy classes C^i_h of G_i. But

$$\sum_{C^i_h} \chi(h) a_{C^i_h, C^j_g} = \sum_{C^i_h} \mathsf{Y}_{\chi, C^i_h} a_{C^i_h, C^j_g}.$$

As Y is invertible, a quick glance at (7.2) allows us to deduce the equality $\mathsf{A}_{C^i_h, C^j_g} = a_{C^i_h, C^j_g}$, as required. □

Solomon computed A explicitly for $\mathfrak{I}_n$. He showed that if the conjugacy class of $\sigma \in \mathfrak{S}_r$ corresponds to the partition α of r and the conjugacy class of $\tau \in \mathfrak{S}_\ell$ corresponds to the partition β of ℓ where α, β have respectively a_i, b_i parts equal to i, then

$$\mathsf{A}_{C^\ell_\tau, C^r_\sigma} = \binom{\alpha}{\beta} = \prod_{i \geq 1} \binom{a_i}{b_i}.$$

Similarly we can calculate B. If $h \in G_i$, denote by z_h the size of the centralizer of h in G_i. So the equality $|C^i_h| = \frac{|G_i|}{z_h}$ holds.

Proposition 7.2. *If χ, θ are irreducible characters of G_i, respectively G_j, then*

$$\mathsf{B}_{\chi,\theta} = \sum_{C^i_h, C^j_g} z_g^{-1} \mathsf{A}_{C^i_h, C^j_g} \chi(C^i_h) \theta(C^j_g) \tag{7.3}$$

where C^i_h runs over the conjugacy classes of G_i and C^j_g runs over the conjugacy classes of G_j.

Proof. Define for $1 \leq i \leq n$ the matrix $\mathsf{Z}_i = \mathsf{X}_i^T \mathsf{X}_i$. By the second orthogonality relation for group characters, Z_i is a diagonal matrix whose entry in the diagonal position corresponding to a conjugacy class C^i_h of G_i is z_h. Let $\mathsf{W} = \mathrm{diag}(\mathsf{Z}_1, \ldots \mathsf{Z}_n)$. Then $\mathsf{Y}^T \mathsf{Y} = \mathsf{W}$. So from (7.2), we see that

$$\mathsf{B} = \mathsf{Y}\mathsf{A}\mathsf{Y}^{-1} = \mathsf{Y}\mathsf{A}\mathsf{W}^{-1}\mathsf{Y}^T.$$

Comparing the χ, θ entry of the right hand side of the above equation with the right hand side of (7.3) completes the proof. □

For $\mathfrak{I}_n$, Solomon gave a combinatorial interpretation for the entries of B in terms of Ferrer's diagrams [42, Proposition 3.11]. We now turn to the analogue of Solomon's multiplicity formulas for $\mathfrak{I}_n$ [42, Lemma 3.17], in terms of A and B, for the general case. It is not clear how usable these formulas are since one needs quite detailed information about the inverse semigroup S to determine these matrices. Our previous computations with our multiplicity formula (5.2) often just used knowledge of the idempotent set, while Solomon's approach requires much more. It also explains why partitions, Ferrer's diagrams and symmetric functions come into Solomon's approach for the tensor and exterior powers of the rook matrix representation of $\mathfrak{I}_n$, but they play no role in our approach. We retain the notation above.

Theorem 7.3. *Let χ be an irreducible character of G_i and θ a character of S. Then the following two formulas are valid:*

$$(\chi, \theta)_S = \sum_{C^i_h} \chi(h) z_h^{-1} \sum_{C^j_g} \mathsf{A}^{-1}_{C^j_g, C^i_h} \theta(g). \tag{7.4}$$

where the first sum is over the conjugacy classes C^i_h of G_i and the second over all conjugacy classes C^j_g of all the G_j; and

$$(\chi, \theta)_S = \sum_{j=1}^{n} \sum_{\psi \in \mathrm{Irr}(G_j)} \mathsf{B}^{-1}_{\psi,\chi} (\psi, \theta|_{G_j})_{G_j} \tag{7.5}$$

Proof. For an irreducible character ψ of G_j, denote by m_ψ the multiplicity of ψ^* in θ. So $\theta = \sum_\psi m_\psi \psi^*$. Hence, for a conjugacy class C^j_g of G_j,

$$\theta(g) = \sum_\psi m_\psi \psi^*(g) = \sum_\psi m_\psi \mathsf{C}_{\psi, C^j_g}$$

where the sum runs over all the irreducible characters of all the G_j. Using this, we obtain:

$$(\chi, \theta)_S = \sum_\psi m_\psi \delta_{\psi,\chi} = \sum_\psi m_\psi \sum_{C^j_g} \mathsf{C}_{\psi, C^j_g} \mathsf{C}^{-1}_{C^j_g, \chi} = \sum_{C^j_g} \mathsf{C}^{-1}_{C^j_g, \chi} \theta(g) \tag{7.6}$$

where the last sum runs over the conjugacy classes C^j_g of the G_j, $j = 1, \ldots, n$. Setting $\mathsf{W} = \mathsf{Y}^T \mathsf{Y}$ again (as in the proof of Proposition 7.2), we compute:

$$\mathsf{C}^{-1}_{C^j_g, \chi} = (\mathsf{A}^{-1} \mathsf{W}^{-1} \mathsf{Y}^T)_{C^j_g, \chi} = \sum_{C^i_h} \mathsf{A}^{-1}_{C^j_g, C^i_h} z_h^{-1} \chi(h)$$

where the last sum runs over the conjugacy classes C^i_h in G_i. This, in conjunction with (7.6) implies (7.4). Also, for $g \in G_j$,

$$\mathsf{C}^{-1}_{C^j_g, \chi} = (\mathsf{W}^{-1} \mathsf{Y}^T \mathsf{B}^{-1})_{C^j_g, \chi} = z_g^{-1} \sum_{\psi \in \mathrm{Irr}(G_j)} \psi(g) \mathsf{B}^{-1}_{\psi,\chi}. \tag{7.7}$$

This last equality uses that Y^T is block diagonal. Combining (7.7) with (7.6) and the fact that as C^j_g runs over all conjugacy classes of G_j,

$$\sum_{C^j_g} z_g^{-1} \psi(g) \theta(g) = (\overline{\psi}, \theta)_{G_j},$$

where $\overline{\psi}$ is the conjugate character of ψ, gives (7.5). This completes the proof of the theorem. $\square$

8. Semigroups with commuting idempotents and generalizations

There is a large class of finite semigroups whose representation theory is controlled in some sense by an inverse semigroup. For instance, every irreducible representation of an idempotent semigroup factors through a semilattice; for a readable account of this classical fact for non-specialists, see [9]. This section will require a bit more of a semigroup theoretic background than the previous ones; the reader is referred to [10] for basics about semigroups and [16] for results specific to finite semigroups. The book of Almeida [2] contains more modern results.

8.1. **Semigroups with commuting idempotents.**

Let us first begin with a class that is very related to inverse semigroups: semigroups with commuting idempotents. Every inverse semigroup has commuting idempotents, as does every subsemigroup of an inverse semigroup. However, not every finite semigroup with commuting idempotents is a subsemigroup of an inverse semigroup. It is a very deep result of Ash [5, 6] that every finite semigroup with commuting idempotents is a quotient of a subsemigroup of a finite inverse semigroup (Ash's original proof uses Ramsey theory in an extremely clever way [5]; this result can also be proved using a theorem of Ribes and Zalesskiĭ about the profinite topology on a free group [35, 15]).

Elements s, t of a semigroup S are said to be *inverses* if $sts = s$ and $tst = t$. Elements with an inverse are said to be (von Neumann) *regular*. Denote by $R(S)$ the set of regular elements of S. A semigroup in which all elements are regular is called, not surprisingly, *regular*. Regular semigroups are very important: a connected algebraic monoid with zero has a reductive group of units if and only if it is regular [26, 32]; the semigroup algebra of a finite regular semigroup is quasi-hereditary [29]. It is known that a semigroup S is an inverse semigroup if and only if it is regular and has commuting idempotents [18, 10]. More generally it is known that if the idempotents of a semigroup commute, then the regular elements have unique inverses [10]. We give a proof for completeness.

Proposition 8.1. *Suppose S has commuting idempotents and t, t' are inverses of s. Then $t = t'$.*

Proof. Using that $st, ts, st', t's$ are idempotents we obtain:

$$\begin{aligned} t &= tst = ts(t'st')st = t'stst'st \\ &= t's(tst)st' = t'st' = t'. \end{aligned}$$

This establishes the uniqueness of the inverse. □

If S is a semigroup with commuting idempotents and $u \in R(S)$, then we denote by u^{-1} the (unique) inverse of u. The following is a standard fact about semigroups with commuting idempotents.

Proposition 8.2. *Let S be a finite semigroup with commuting idempotents. Then the set $R(S)$ of regular elements of S is an inverse semigroup.*

Proof. The key point is that $R(S)$ is a subsemigroup of S. Indeed, if $a, b \in R(S)$, then we claim $(ab)^{-1} = b^{-1}a^{-1}$. Observing that aa^{-1}, $a^{-1}a$, bb^{-1} and $b^{-1}b$ are idempotents, we compute:

$$ab(b^{-1}a^{-1})ab = a(bb^{-1})(a^{-1}a)b = a(a^{-1}a)(bb^{-1})b = ab.$$

Similarly one verifies $b^{-1}a^{-1}(ab)b^{-1}a^{-1} = b^{-1}a^{-1}$. Since the inverse of any regular element is regular, $R(S)$ is closed under taking inverses and hence is a regular semigroup with commuting idempotents and therefore an inverse semigroup. □

Our next goal is to show that if S has commuting idempotents and K is a field of characteristic zero, then $KS/\mathrm{Rad}(KS) \cong KR(S)$ and the isomorphism is the identity on $KR(S)$. This means that we can use our results for inverse semigroups to obtain character formulas for multiplicities of irreducible constituents in representations of S.

Proposition 8.3. *Let S be a semigroup with commuting idempotents. Let $u \in R(S)$ and $s \in S$. Then the following are equivalent:*

(1) $uu^{-1}s = u$;
(2) $u = es$ *with* $e \in E(S)$;
(3) $su^{-1}u = u$;
(4) $u = sf$ *with* $f \in E(S)$.

Proof. Clearly (1) implies (2). For (2) implies (1), suppose that $u = es$ with $e \in E(S)$. Then $eu = ees = es = u$, so

$$u = uu^{-1}u = uu^{-1}es = euu^{-1}s = uu^{-1}s.$$

The equivalence of (3) and (4) is dual.

To prove that (1) implies (3), assume $uu^{-1}s = u$. We show that $su^{-1}u$ is an inverse for u^{-1}. Then the equality $su^{-1}u = u$ will follow by uniqueness of inverses in $R(S)$ (Proposition 8.2). Indeed

$$(su^{-1}u)u^{-1}(su^{-1}u) = su^{-1}(uu^{-1}s)u^{-1}u = su^{-1}uu^{-1}u = su^{-1}u.$$

Also, using $u^{-1} = u^{-1}uu^{-1}$,

$$u^{-1}(su^{-1}u)u^{-1} = u^{-1}(uu^{-1}s)u^{-1} = u^{-1}uu^{-1} = u^{-1}.$$

The implication (3) implies (1) is proved similarly. □

Let S be a semigroup with commuting idempotents. Define, for $s \in S$,

$$s^{\downarrow} = \{u \in R(S) \mid uu^{-1}s = u\}.$$

In other words, $s^{\downarrow}$ is the set of elements for which the equivalent conditions of Proposition 8.3 hold. Notice that if s is regular, then $s^{\downarrow}$ just consists of all elements below s in the natural partial order on $R(S)$, whence the notation. We can now establish an analogue of Lemma 4.1.

Lemma 8.4. *Let $\nu : S \to KR(S)$ be given by $\nu(s) = \sum_{t \in s^{\downarrow}} \lfloor t \rfloor$. Then ν is an onto homomorphism that restricts to the identity on $R(S)$.*

Proof. The observation before the proof shows that if $s \in R(S)$, then

$$\nu(s) = \sum_{t \leq s} \lfloor t \rfloor = s$$

via Möbius inversion and Theorem 4.2.

For arbitrary $s, t \in S$, we have that

$$\nu(s)\nu(t) = \sum_{u \in s^{\downarrow}, v \in t^{\downarrow},\ \mathrm{ran}(u) = \mathrm{dom}(v)} \lfloor uv \rfloor.$$

First we show that if $u \in s^{\downarrow}$, $v \in t^{\downarrow}$ and $\mathrm{ran}(u) = \mathrm{dom}(v)$ then $uv \in (st)^{\downarrow}$. Indeed $\mathrm{dom}(uv) = \mathrm{dom}(u)$. Hence $uv(uv)^{-1} = uu^{-1}$. Also $u^{-1}u = vv^{-1}$ so

$$(uv)(uv)^{-1}st = uu^{-1}st = ut = u(u^{-1}u)t = uvv^{-1}t = uv.$$

To complete the proof, we must show that every element u of $(st)^{\downarrow}$ can uniquely be written in the form $s't'$ with $s' \in s^{\downarrow}$, $t' \in t^{\downarrow}$ and $\mathrm{ran}(s') = \mathrm{dom}(t')$. The proof proceeds exactly along the lines of that of Lemma 4.1. Namely, to obtain such a factorization $u = s't'$ we must have $\mathrm{dom}(s') = \mathrm{dom}(u)$ and $\mathrm{ran}(t') = \mathrm{ran}(u)$, that is we are forced to set $s' = uu^{-1}s$ and $t' = tu^{-1}u$. Then one shows, just as in the proof of Lemma 4.1, that s' is regular with inverse tu^{-1} and t' is regular with inverse $u^{-1}s$. Hence $s' \in s^{\downarrow}$ and $t' \in t^{\downarrow}$ (the latter requires Proposition 8.3). The proofs that $s't' = u$ and $\mathrm{ran}(s') = \mathrm{dom}(t')$ also proceed along the lines of the proof of Lemma 4.1 and so we omit them. □

Our current aim is to show that the induced map $\nu : KS \to KR(S)$ has a nilpotent kernel. In any event ν splits as a K-vector space map and so $KS = KR(S) \oplus \ker \nu$ as K-vector spaces. Clearly then $\ker \nu$ has basis $B = \{s - \nu(s) \mid s \in S \setminus R(S)\}$. Indeed, the number of elements in B is the dimension of $\ker \nu$ and these elements are clearly linearly independent since the unique non-regular element in the support of $s - \nu(s)$ is s itself. Notice that via ν any irreducible representation of $R(S)$ extends to S. We shall prove the converse. Our method of proof will show that $\ker \nu$ is contained in the radical of KS. First we need some definitions.

An *ideal* of a semigroup S is a subset I such that $SI, IS \subseteq I$. We then place a preorder on S by ordering elements in terms of the principal ideal they generate. That is, if $s \in S$, let $J(s)$ be the principal ideal generated by s. Define $s \leq_{\mathcal{J}} t$ if $J(s) \subseteq J(t)$. We write $s \mathrel{\mathcal{J}} t$ if $J(s) = J(t)$. This is an example of one of Green's relations [13]. There are similar relations, denoted $\mathcal{R}$ and $\mathcal{L}$, corresponding to right and left principal ideals, respectively. The $\mathcal{J}$-relation on a finite inverse semigroup S coincides with what we called $\mathcal{D}$ earlier [18]. That is for $s, t \in S$, we have $s \mathrel{\mathcal{J}} t$ if and only if $\mathrm{dom}(s)$ is isomorphic to $\mathrm{dom}(t)$. It is known that in a finite semigroup the following are equivalent for a $\mathcal{J}$-class J [10, 16]:

- J contains an idempotent;
- J contains a regular element;
- Every element of J is regular.

Such a $\mathcal{J}$-class is called a *regular $\mathcal{J}$-class*. Any inverse of a regular element belongs to its $\mathcal{J}$-class. If s, t are $\mathcal{J}$-equivalent regular elements, then there are (regular) elements x, y, u, v ($\mathcal{J}$-equivalent to s and t) such that $xsy = t$ and $utv = s$ [10, 16]. Hence if S has commuting idempotents, then the $\mathcal{J}$-classes of $R(S)$ are precisely the regular $\mathcal{J}$-classes of S. The following proof uses a result of Munn [23] that is exposited in [10]. See [34, 33, 3] for refinements.

Theorem 8.5. *Let S be a finite semigroup with commuting idempotents and let $\rho : S \to M_n(K)$ be an irreducible representation. Then:*

(1) $\rho|_{R(S)}$ *is an irreducible representation;*
(2) $\ker \nu \subseteq \ker \rho$.

Proof. Let J be a $\leq_{\mathcal{J}}$-minimal $\mathcal{J}$-class of S on which ρ does not vanish. It is a result of Munn [10, Theorem 5.33] (see also [3]) that J must be a regular $\mathcal{J}$-class, say it is the $\mathcal{J}$-class of an idempotent e and that ρ must vanish on any $\mathcal{J}$-class that is not $\leq_{\mathcal{J}}$-above J [10, 34, 3]. Let $I = J(e) \setminus J$. Then I is an ideal of $J(e)$ on which ρ vanishes and so $\rho|_{J(e)}$ factors through a representation $\overline{\rho}$ of the quotient $J^0 = J(e)/I$. Munn proved [10, Theorem 5.33] that $\overline{\rho}$ is an irreducible representation of J^0. Since J is a regular $\mathcal{J}$-class and $J^0 = J(e)/I = (J(e) \cap R(S))/(I \cap R(S))$, it follows that $\overline{\rho}$ is also induced by $\rho|_{J(e)\cap R(S)}$. It is then immediate that $\rho|_{R(S)}$ is an irreducible representation since any $R(S)$-invariant subspace is $(J(e) \cap R(S))$-invariant. Moreover, since $\rho|_{R(S)}$ is induced by $\overline{\rho}$, it must be an irreducible representation of $KR(S)$ associated to the direct summand of $KR(S)$ spanned by $\{\lfloor s \rfloor \mid s \in J\}$, which we denote KJ (recall that J is a $\mathcal{D}$-class).

Now let $s \in S$. We show that $\rho(s) = \rho(\nu(s))$. Since elements of the form $s - \nu(s)$ span $\ker \nu$, this will show that $\ker \nu \subseteq \ker \rho$. If s is not $\leq_{\mathcal{J}}$-above J, then $\rho(s)$ is zero. Since each element of $s^{\downarrow}$ is $\leq_{\mathcal{J}}$-below s, $\rho(\nu(s)) = 0$. Suppose that $s \geq_{\mathcal{J}} J$. Consider $1_J = \sum_{f \in E(J)} \lfloor f \rfloor$. This is the identity element of KJ and hence is sent to the identity matrix under ρ, as $\rho|_{R(S)}$ is induced by first projecting to KJ. Therefore $\rho(s) = \rho(1_J s)$. It thus suffices to show that $\rho(s1_J) = \rho(\nu(s))$. Since, for $t \in J$, every summand but t of $\lfloor t \rfloor$ is strictly $\leq_{\mathcal{J}}$-below J, we see that $\rho(\lfloor t \rfloor) = \rho(t)$, for $t \in J$. Now for $f \in E(J)$ either: $fs <_{\mathcal{J}} J$, and hence $\rho(\lfloor f \rfloor s) = 0$; or $fs \in J$, and so $fs \in s^{\downarrow}$. Conversely, if $t \in J \cap s^{\downarrow}$, then $t = tt^{-1}s$ and $tt^{-1} \in E(J)$. Thus

$$\rho(1_J s) = \sum_{f \in E(J), fs \in J} \rho(\lfloor f \rfloor s) = \sum_{f \in E(J), fs \in J} \rho(fs) = \sum_{t \in s^{\downarrow} \cap J} \rho(t). \tag{8.1}$$

Suppose $t \in s^{\downarrow}$. If $t \notin J$, then $\lfloor t \rfloor$ is not in KJ and so $\rho(\lfloor t \rfloor) = 0$. Thus

$$\rho(\nu(s)) = \sum_{t \in s^{\downarrow} \cap J} \rho(\lfloor t \rfloor) = \sum_{t \in s^{\downarrow} \cap J} \rho(t). \tag{8.2}$$

Comparing (8.1) and (8.2) shows that $\rho(1_J s) = \rho(\nu(s))$, establishing that $\ker \nu \subseteq \ker \rho$. $\square$

Corollary 8.6. *Let S be a finite semigroup with commuting idempotents and K a field. Define $\nu : KS \to KR(S)$ on $s \in S$ by*

$$\nu(s) = \sum_{t \in s^{\downarrow}} \lfloor t \rfloor.$$

Then ν is a retraction and with nilpotent kernel. Hence $KS/\mathrm{Rad(KS)} = KR(S)/\mathrm{Rad}(KR(S))$.

In particular, if the characteristic of K is 0 (or more generally if the characteristic of K does not divide the order of any maximal subgroup of S), then $\ker\nu = \mathrm{Rad}(KS)$. Hence $\dim(\mathrm{Rad}(KS)) = |S \setminus R(S)|$ and a basis for $\mathrm{Rad}(KS)$ is given by the set $\{s - \nu(s) \mid s \in S \setminus R(S)\}$.

Proof. Theorem 8.5 shows that $\ker\nu$ is contained in the kernel of every irreducible representation of KS and hence $\ker\nu$ is a nilpotent ideal. From this the first paragraph follows. In the context of the second paragraph, we have that $KR(S)$ is semisimple and so has no nilpotent ideals. Thus $\ker\nu$ is the largest nilpotent ideal of KS and hence is the radical. The remaining statements are clear. □

It follows that the irreducible representations of S are in bijection with the irreducible representations of its maximal subgroups up to $\mathscr{J}$-equivalence (actually this is true for any finite semigroup [10, 34]). We can use our multiplicity formulas for inverse semigroups verbatim for semigroups S with commuting idempotents.

Theorem 8.7. *Let S be a finite semigroup with commuting idempotents and K a field of characteristic zero. Let χ be an irreducible character of a maximal subgroup G with identity e and let d be the dimension of the associated endomorphism division algebra. Then if θ is a character of S and m is the multiplicity of the irreducible representation of S associated to χ as a constituent in θ, then*

$$md = \sum_{f \leq e} (\chi, \theta_f)_G \mu(f, e)$$

where μ is the Möbius function of $E(S)$.

It follows that a completely reducible representation of a finite semigroup with commuting idempotents is determined by its character. In general the irreducible constituents are determined by the character.

8.2. **Multiplicities for some more general classes of semigroups.** We now want to consider a wider class of semigroups whose irreducible representations are controlled by inverse semigroups. First we recall the notion of an **LI**-morphism. A finite semigroup S is said to be *locally trivial* if, for each idempotent $e \in S$, $eSe = e$. A homomorphism $\varphi : S \to T$ is said to be an **LI**-*morphism* if, for each locally trivial subsemigroup U of T, the semigroup $\varphi^{-1}(U)$ is again locally trivial. The following result, showing that **LI**-morphisms correspond to algebra morphisms with nilpotent kernel, was proved in [3].

Theorem 8.8 ([3])**.** *Let K be a field and let $\varphi : S \to T$ be a homomorphism of finite semigroups. If φ is an* **LI***-morphism, then the induced map $\overline{\varphi} : KS \to KT$ has nilpotent kernel. The converse holds if the characteristic of K is zero.*

Therefore, if S is a finite semigroup with an **LI**-morphism $\varphi : S \to T$ to a semigroup T with commuting idempotents, then we can conclude that $KS/\mathrm{Rad}(KS) = KR(T)/\mathrm{Rad}(KR(T))$ (equals $KR(T)$ if $\mathrm{char}(K) = 0$). In particular we can use our multiplicity formula for inverse semigroups to calculate multiplicities for irreducible constituents for representations of S in characteristic zero. For instance, if S is an idempotent semigroup, then one can find such a map φ with T a semilattice. This is what underlies some of the work of Brown [8, 9], as well as some more general work of Putcha [29]. See also [44].

Let us describe those semigroups with such a map φ. This class is well known to semigroup theorists and it would go too far afield to give a complete proof here, so we restrict ourselves to just describing the members of the class. First we describe the semigroups with an **LI**-morphism to a semilattice L. This class was first introduced by Schützenberger [38] in the context of formal language theory. It consists precisely of those finite semigroups S such that $R(S) = E(S)$, that is those finite semigroups all of whose regular elements are idempotents. This includes of course all idempotent semigroups. The semilattice L is in fact the set $\mathcal{U}(J)$ of regular $\mathcal{J}$-classes ordered by $\leq_{\mathcal{J}}$. The map sends $s \in S$ to the $\mathcal{J}$-class of its unique idempotent power. See [44] for details. The class of such semigroups is usually denoted **DA** in the semigroup literature (meaning that regular $\mathcal{D}$-classes are aperiodic subsemigroups). It was shown in [3] that this class consists precisely of those finite semigroups whose semigroup algebra in characteristic zero is basic.

Now if $\varphi : S \to T$ is an **LI**-morphism, then it is not to hard to show that if $U \leq S$ is a subsemigroup, then $\varphi|_U$ is again an **LI**-morphism. Suppose that $\varphi : S \to T$ is an **LI**-morphism of finite semigroups where T is a semigroup with commuting idempotents. Then $E(T)$ is a semilattice and $\langle E(S) \rangle$ maps onto $E(T)$ via φ. Hence $\langle E(S) \rangle \in$ **DA**. Let us denote by **EDA** the collection of all finite semigroups S such that $E(S)$ generates a semigroup in **DA**; this includes all semigroups whose idempotents form a subsemigroup (in particular the class of so-called orthodox semigroups [10]). It is well known to semigroup theorists that if $S \in$ **EDA**, then S admits an **LI**-morphism $\varphi : S \to T$ to a semigroup T of partial permutations. In particular, T has commuting idempotents. The transitive components of T correspond to the action of S on the right of a regular $\mathcal{R}$-class of S modulo a certain equivalence relation corresponding to identifying elements that differ by right multiplication by an element of the idempotent-generated subsemigroup; the reader can look at [45] to infer details. Alternatively, one can easily verify that each generalized group mapping image of S corresponding to a regular J-class acts by partial permutations on its minimal ideal [16]. The key point is that φ is explicitly constructible and hence the multiplicity formulas for calculating irreducible constituents for representations of T can be transported back to S. The congruence giving rise to φ is defined as follows. Let $S \in \mathbf{EDA}$ and $s, t \in S$. Define $s \equiv t$ if, for each regular $\mathcal{J}$-class J of S and each $x, y \in J$, one has either $xsy = xty$ or both $xsy, xty \notin J$ [16, 3]. The details are left to the reader.

Just to give a sample computation, let $S \in \mathbf{DA}$ and suppose that the map $\varphi : S \to M_n(K)$ is an irreducible representation. The maximal subgroups of S are trivial. If J is a regular $\mathcal{J}$-class, then the (unique) irreducible representation of S associated to J is given by

$$\rho_J(s) = \begin{cases} 1 & s \geq_{\mathcal{J}} J \\ 0 & \text{else} \end{cases}$$

(see [44]). To obtain a formula for the multiplicity of ρ_J in φ, we must choose an idempotent e_J for each regular $\mathcal{J}$-class J. Then the multiplicity of ρ_J in φ is given by

$$\sum_{J' \leq_{\mathcal{J}} J,\ J' \in \mathcal{U}(J)} \operatorname{rk}(\varphi(e_{J'} e_J e_{J'}))\mu(J', J) \tag{8.3}$$

where μ is the Möbius function of the semilattice $\mathcal{U}(J)$. This generalizes the multiplicity results in [8, 9, 44] for random walks on minimal left ideals of semigroups in $\mathbf{DA}$.

It follows that a completely reducible representation of a semigroup from $\mathbf{EDA}$ is determined by its character and that in general the irreducible constituents are determined by the character.

8.3. **Random walks on triangularizable finite semigroups.** We can now answer a question that remained unsettled in [44]. In that paper we calculated, for a certain class of finite semigroups, namely those admitting an $\mathbf{LI}$-morphism to a commutative inverse semigroup, the eigenvalues for a random walk on a minimal left ideal. This generalized the work of Bidigaire *et al.* [7] and Brown [8, 9]. We showed that there was an eigenvalue corresponding to each irreducible representation of the semigroup (and gave a formula for the idempotent) but at the time we could only prove that the multiplicity of the eigenvalue was the same as the multiplicity of the corresponding irreducible representation as a constituent in the linear representation induced by the left action on the minimal left ideal. We could only calculate the multiplicities explicitly if the semigroup belonged to $\mathbf{DA}$. With our new tools we can now handle the general case.

Let us call a finite semigroup S *triangularizable* if it can be represented faithfully by upper triangular matrices over $\mathbb{C}$. These were characterized in [3] as precisely those semigroups admitting an **LI**-morphism to a commutative inverse semigroup. Equivalently, they were shown to be those finite semigroups in which all maximal subgroups are abelian and such that each regular element satisfies an identity of the form $x^m = x$. Moreover, it was shown that every complex irreducible representation of such a semigroup is of degree one [3]. See also [44] for more. The most important examples are abelian groups and idempotent semigroups, including the face semigroup of a hyperplane arrangement [8, 9].

Let S be a fixed finite triangularizable semigroup. We assume that S has a left identity for convenience (one could always adjoin an identity). Let $\varphi : S \to T$ be its **LI**-morphism to a commutative inverse semigroup. The semigroup T has a unique idempotent in each $\mathcal{D}$-class and the corresponding maximal subgroup is abelian. The lattice of idempotents of T is isomorphic to the set $\mathcal{U}(S)$ of regular $\mathcal{J}$-classes of S. Fix an idempotent e_J for each $J \in \mathcal{U}(S)$. The maximal subgroup G_{e_J} will be denoted G_J. We recall the description of the irreducible characters of S. Suppose G_J is a maximal subgroup and χ is an irreducible character of G_J. Then the associated irreducible character $\chi^* : S \to \mathbb{C}$ is given by

$$\chi^*(s) = \begin{cases} \chi(e_J s e_J) & s \geq_{\mathcal{J}} J \\ 0 & \text{else.} \end{cases}$$

(c.f. [44]).

Suppose one puts a probability distribution p on S. That is we assign probabilities p_s to each $s \in S$ such that $\sum_{s\in S} p_s = 1$. We view p as the element $p = \sum_{s\in S} p_s s \in \mathbb{C}S$. Let L be a minimal left ideal of S (what follows is independent of the choice of L [44]). The associated random walk on L is then the Markov chain with transition operator the $|L| \times |L|$-matrix M that has in entry ℓ_1, ℓ_2 the probability that if one chooses $s \in S$ according to the probability distribution p, then $s\ell_1 = \ell_2$. It is easy to see that if one takes the representation ρ afforded by $\mathbb{C}L$ (viewed as a left $\mathbb{C}S$-module), then M is the transpose of the matrix of $\rho(p)$ [8, 9]. Now since our semigroup has only degree one irreducible representations, a composition series for $\mathbb{C}L$ puts ρ in upper triangular form with the characters of S on the diagonal, appearing with multiplicities according to their multiplicities as constituents of ρ. Hence there is an eigenvalue λ_χ of M associated to each irreducible character χ of a maximal subgroup G_J of S (where J runs over $\mathcal{U}(S)$), given by the character sum

$$\lambda_\chi = \sum_{s\in S} p_s \chi^*(s) = \sum_{s \geq_{\mathcal{J}} J} p_s \chi(e_J s e_J).$$

Of course, depending on p it could happen that different characters will give the same eigenvalue.

For $s \in S$, let $\mathrm{Fix}_L(s)$ be the set of fixed-points of s acting on L. Then the character χ_ρ of ρ simply counts the cardinality of $\mathrm{Fix}_L(s)$. It is now a straightforward exercise in using (5.2) to verify that the multiplicity of χ^*

in ρ, and hence the multiplicity of λ_χ as an eigenvalue of M, is

$$\frac{1}{|G_J|}\sum_{g\in G_J}\chi(g)\sum_{J'\leq J,\ J'\in\mathcal{U}(S)}|\mathrm{Fix}_L(e_{J'}ge_{J'})|\mu(J',J) \tag{8.4}$$

where μ is the Möbius function of $\mathcal{U}(S)$.

The case where S is an abelian group, (8.3) can be found in the work of Diaconis [11]. In this situation, $L = S$ and the multiplicities are all one. On the other hand, if the semigroup is idempotent, then (8.3) and (8.4) reduce to the results of Brown [8, 9]. If the maximal subgroups are trivial, one obtains the results of [44]. Compare also with (8.3).

Acknowledgments

This paper greatly benefitted from several e-mail conversations with Mohan Putcha.

References

1. M. Aguiar and R. Orellana, *The Hopf algebra of uniform block permutations*, preprint 2004.
2. J. Almeida, "Finite Semigroups and Universal Algebra", World Scientific, Singapore, 1995.
3. J. Almeida, S. W. Margolis, B. Steinberg and M. V. Volkov, *Representation theory of finite semigroups, semigroup radicals and formal language theory*, preprint 2004.
4. F. Arnold and B. Steinberg, *Synchronizing groups and automata*, Theor. Comp. Sci., to appear.
5. C. J. Ash, *Finite semigroups with commuting idempotents* J. Austral. Math. Soc. Ser. A **43** (1987), 81–90.
6. C. J. Ash, *Inevitable graphs: A proof of the type II conjecture and some related decision procedures*, Internat. J. Algebra Comput. **1** (1991), 127–146.
7. P. Bidigare, P. Hanlon and D. Rockmore, *A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements*, Duke Math. J. **99** (1999), 135–174.
8. K. Brown, *Semigroups, rings, and Markov chains*, J. Theoret. Probab. **13** (2000), 871–938.
9. K. Brown, *Semigroup and ring theoretical methods in probability*, in: "Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry", 3–26, (V. Dlab, ed.) Fields Inst. Commun., **40**, Amer. Math. Soc., Providence, RI, 2004.
10. A. H. Clifford and G. B. Preston, "The Algebraic Theory of Semigroups", Mathematical Surveys No. 7, AMS, Providence, RI, Vol. 1, 1961.
11. P. Diaconis, "Group representations in probability and statistics", Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11. Institute of Mathematical Statistics, Hayward, CA, 1988.
12. S. Eilenberg, "Automata, Languages and Machines", Academic Press, New York, Vol A, 1974; Vol B, 1976.
13. J. A. Green, *On the structure of semigroups*, Annals Math. **54**, (1951), 163–172.
14. M. Hall, "Combinatorial theory", Reprint of the 1986 second edition. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1998.
15. K. Henckell, S. W. Margolis, J.-E. Pin, and J. Rhodes, *Ash's type II theorem, profinite topology and Mal'cev products, Part I*, Internat. J. Algebra Comput. **1** (1991), 411–436.

16. K. Krohn, J. Rhodes and B. Tilson, *Lectures on the algebraic theory of finite semigroups and finite-state machines*, Chapters 1, 5–9 (Chapter 6 with M. A. Arbib) of "Algebraic Theory of Machines, Languages, and Semigroups", ed. M. A. Arbib, Academic Press, New York, 1968.
17. G. Lallement and M. Petrich, *Irreducible matrix representations of finite semigroups* Trans. Amer. Math. Soc. **139** (1969), 393–412.
18. M. V. Lawson, "Inverse Semigroups: The theory of partial symmetries", World Scientific, Singapore, 1998.
19. S. MacLane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971.
20. D. B. McAlister, *Characters of finite semigroups*, J. Algebra **22** (1972), 183–200.
21. D. B. McAlister, *Representations of semigroups by linear transformations I, II*, Semigroup Forum **2** (1971), 189–263; ibid. **2** (1971), 283–320.
22. W. D. Munn, *On semigroup algebras*, Proc. Cambridge Philos. Soc. **51**, (1955). 1–15.
23. W. D. Munn, *Matrix representations of semigroups*, Proc. Cambridge Philos. Soc. **53** (1957), 5–12.
24. W. D. Munn, *The characters of the symmetric inverse semigroup*, Proc. Cambridge Philos. Soc. **53** (1957), 13–18.
25. A. L. T. Paterson, "Groupoids, Inverse Semigroups, and their Operator Algebras", Birkhäuser, Boston, 1999.
26. M. S. Putcha, "Linear Algebraic Monoids", London Mathematical Society Lecture Note Series, **133**. Cambridge University Press, Cambridge, 1988.
27. M. S. Putcha, *Complex representations of finite monoids*, Proc. London Math. Soc. **73** (1996), 623–641.
28. M. S. Putcha, *Monoid Hecke algebras*, Trans. Amer. Math. Soc. **349** (1997), 3517–3534.
29. M. S. Putcha, *Complex representations of finite monoids. II. Highest weight categories and quivers*, J. Algebra **205** (1998), 53–76.
30. M. S. Putcha, *Semigroups and weights for group representations*, Proc. Amer. Math. Soc. **128** (2000), 2835–2842.
31. L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*, J. Algebra **101** (1986), 303–338.
32. L. E. Renner, "Linear algebraic monoids", Encyclopaedia of Mathematical Sciences, **134**, Invariant Theory and Algebraic Transformation Groups, V. Springer-Verlag, Berlin, 2005.
33. J. Rhodes, *Characters and complexity of finite semigroups*, J. Combinatorial Theory **6** (1969), 67–85.
34. J. Rhodes and Y. Zalcstein, *Elementary representation and character theory of finite semigroups and its application* in: "Monoids and Semigroups with Applications" (J. Rhodes, ed.), Berkeley, CA, 1989, 334–367, World Sci. Publishing, River Edge, NJ, 1991.
35. L. Ribes and P. A. Zalesskiĭ, *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993), 37–43.
36. B. E. Sagan, "The symmetric group. Representations, combinatorial algorithms, and symmetric functions," Second edition, Graduate Texts in Mathematics, **203**, Springer-Verlag, New York, 2001.
37. M.-P. Schützenberger, $\overline{\mathcal{D}}$ *représentation des demi-groupes*, C. R. Acad. Sci. Paris **244** (1957), 1994–1996.
38. M. P. Schützenberger, *Sur le produit de concatenation non ambigu*, Semigroup Forum **13** (1976), 47–75.
39. J.-P. Serre, "Linear representations of finite groups." Translated by Leonard L. Scott. Graduate Texts in Mathematics, Vol. **42**. Springer-Verlag, New York-Heidelberg, 1977.

40. L. Solomon, *The Burnside algebra of a finite group*, J. Combinatorial Theory **2** (1967), 603–615.
41. L. Solomon, *The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field*, Geom. Dedicata **36** (1990), 15–49.
42. L. Solomon, *Representations of the rook monoid*, J. Algebra **256** (2002), 309–342.
43. R. Stanley, "Enumerative Combinatorics. Vol. 1", Cambridge Studies in Advanced Mathematics, vol.49, Cambridge University Press, 1997, with a foreword by Gian-Carlo Rota, corrected reprint of the 1986 original.
44. B. Steinberg, *Möbius functions and semigroup representation theory*, J. Combinatorial Theory A, to appear.
45. H. Straubing and D. Thérien, *Regular languages defined by generalized first-order formulas with a bounded number of bound variables*, Theory Comput. Syst. **36** (2003), 29–69.
46. Y. Zalcstein, *Studies in the representation theory of finite semigroups*, Trans. Amer. Math. Soc. **161** (1971), 71–87.

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario K1S 5B6, Canada

E-mail address: `bsteinbg@math.carleton.ca`