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# Representations of the rook monoid

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## 1. Introduction

Let $n$ be a positive integer and let $\mathbf{n} = \{1, \ldots, n\}$. Let $R$ be the set of all one-to-one maps $\sigma$ with domain $I(\sigma) \subseteq \mathbf{n}$ and range $J(\sigma) \subseteq \mathbf{n}$. If $i \in I(\sigma)$ let $i\sigma$ denote the image of $i$ under $\sigma$. There is an associative product $(\sigma, \tau) \mapsto \sigma\tau$ on $R$ defined by composition of maps: $i(\sigma\tau) = (i\sigma)\tau$ if $i \in I(\sigma)$ and $i\sigma \in I(\tau)$. Thus the domain $I(\sigma\tau)$ consists of all $i \in I(\sigma)$ such that $i\sigma \in I(\tau)$. The set $R$, with this product, is a monoid (semigroup with identity) called the *symmetric inverse semigroup*. We agree that $R$ contains a map with empty domain and range which behaves as a zero element. Let $F$ be a field. Let $\mathbf{M}_n(F)$ denote the algebra of $n \times n$ matrices over $F$. There is a one-to-one map $R \to \mathbf{M}_n(F)$ defined by

$$\sigma \mapsto [\sigma] = \sum_{i \in I(\sigma)} E_{i,i\sigma} \tag{1.1}$$

where $E_{ij}$ is a matrix unit with an entry 1 in the $(i, j)$ position and 0's elsewhere. The corresponding set $\mathcal{R}$ of matrices consists of those zero-one matrices which have at most one entry equal to 1 in each row and column. In particular, $E_{ij}$ corresponds to the map $\sigma$ with $I(\sigma) = \{i\}$, $J(\sigma) = \{j\}$ which takes $i$ to $j$. Since $[\sigma\tau] = [\sigma][\tau]$ for $\sigma, \tau \in R$, the set $\mathcal{R}$ is a monoid under matrix multiplication which is isomorphic to $R$. Since the elements of $\mathcal{R}$ are in one-to-one correspondence with placements of nonattacking rooks on an $n \times n$ chessboard, we call $\mathcal{R}$ the *rook monoid*. The author used the name "rook monoid" in the title of this paper to (perhaps) increase the marketability of a paper on the symmetric inverse semigroup to those who are interested in combinatorics and representation theory.

If $\sigma \in R$, define the *rank* of $\sigma$ by $\mathrm{rk}(\sigma) = |I(\sigma)|$. Thus $\mathrm{rk}(\sigma)$ is equal to the rank of the matrix $[\sigma]$. For $0 \leqslant r \leqslant n$ let $R^r = \{\sigma \in R \mid \mathrm{rk}(\sigma) = r\}$. Then


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$$\left|R^r\right| = \binom{n}{r}^2 r!, \quad \text{so} \quad |R| = \sum_{r=0}^{n} \binom{n}{r}^2 r!. \tag{1.2}$$

To see the first equality in terms of rooks, note that there are $\binom{n}{r}$ ways to choose the rows, $\binom{n}{r}$ ways to choose the columns and $r!$ ways to place $r$ nonattacking rooks, once the rows and columns containing the rooks are chosen. For $1 \leqslant r \leqslant n$ let $S_r \subseteq R$ be the symmetric group on $\mathbf{r} = \{1, \ldots, r\}$. Note that $R^r \supseteq S_r$ and that $R^n = S_n$. The restriction of the map $\sigma \mapsto [\sigma]$ to $S_n$ is the natural representation of $S_n$ by permutation matrices. For convenience and uniformity of statement define $S_0$ by $S_0 = R^0$; this is a group whose unique element is the map with empty domain and range.

In this paper we consider various aspects of the representation theory of $R$ over a field $F$ of characteristic zero. It is understood that representations are finite-dimensional, although we sometimes allow graded modules of infinite dimension in which the homogeneous components are of finite dimension. We identify a representation of $R$ with its $F$-linear extension to a representation of the monoid algebra $FR = \bigoplus_{\sigma \in R} F\sigma$ and make a similar convention for representations of $S_r$ and $FS_r$. The main concerns in this paper are Munn's representation theory and character formula, character multiplicities, the representation of $R$ on the polynomial algebra $F[x_1, \ldots, x_n]$ and the representation of $R$ on tensors by "place permutations." We do not assume any facts from semigroup theory. We do assume some facts about symmetric functions and the representation theory of the symmetric group [7, Chapter I].

In Section 2 we describe the irreducible representations of $R$. The ideas and results in this section are due to W.D. Munn [10] who proved that $FR$ is semisimple and found its irreducible representations in terms of the irreducible representations of the symmetric groups $S_r$ for $0 \leqslant r \leqslant n$. Munn also defined a character table for $R$. The irreducible characters $\zeta^\lambda$ are indexed by partitions $\lambda$ of integers $r$ with $0 \leqslant r \leqslant n$; $\zeta^{(1)}$ is the character of the representation $\sigma \mapsto [\sigma]$ by rook matrices. The main new feature in this section is an explicit formula for certain central idempotents of $FR$ which were introduced [14] in the context of the Möbius algebra of a lattice.

In Section 3 we define two square matrices $\mathsf{A}$ and $\mathsf{B}$ either of which, together with the character tables of the $S_r$, is sufficient to determine Munn's character table. Both $\mathsf{A}$ and $\mathsf{B}$ may be described in combinatorial terms. See Proposition 3.5 which gives $\mathsf{A}$ in terms of binomial coefficients and Proposition 3.11 which gives $\mathsf{B}$ in terms of Ferrers boards. Since $R$ is not a group, we do not have the usual orthogonality relations for irreducible characters to help compute character multiplicities. Lemma 3.17 shows how to compute multiplicities in terms of $\mathsf{A}$ or $\mathsf{B}$. In Example 3.18 we use $\mathsf{A}$ to decompose the character of the $p$th tensor power of the representation $\sigma \mapsto [\sigma]$: if $\lambda$ is a partition of $r$ then the multiplicity of $\zeta^\lambda$ in the $p$th tensor power is $S(p,r) f^\lambda$ where $S(p,r)$ is a Stirling number of the second kind and $f^\lambda$ is the degree of the corresponding character of $S_r$. In Example 3.22 we use $\mathsf{B}$ to show that the $p$th exterior power of the representation $\sigma \mapsto [\sigma]$ is an irreducible representation with character $\zeta^{(1^p)}$.

In Section 4 we study the action of $R$ on the polynomial algebra $F[x_1, \ldots, x_n]$. We decompose the $R$-module $F[x_1, \ldots, x_n]$ into its isotypic components, in terms of analogous (known) data for the symmetric groups $S_r$ for $0 \leqslant r \leqslant n$.

In Section 5 we study the action of $R$ on tensors by "place permutations." If $V$ is a vector space over $F$ then $S_n$ acts on $V^{\otimes n}$ by place permutations: $w(v_1 \otimes \cdots \otimes v_n) = v_{1w} \otimes \cdots \otimes v_{nw}$, for $w \in S_n$. According to Schur and Weyl, the centralizer algebra for this action is the algebra of endomorphisms of $V^{\otimes n}$ provided by the natural action of $\mathbf{GL}(V)$ on $V^{\otimes n}$. If $\sigma \in R$, we cannot define $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{1\sigma} \otimes \cdots \otimes v_{n\sigma}$ because the domain of $\sigma$ need not be all of $\mathbf{n}$. We try to approximate the last formula as best we can: replace $V$ by $U = F \oplus V$ and use the field $F$ as a wastebasket for the undefined $i\sigma$. We prove that the centralizer algebra for this action of $R$ is the algebra of endomorphisms of $U^{\otimes n}$ provided by the natural action of $\mathbf{GL}(V)$ on $U^{\otimes n}$, where $\mathbf{GL}(V)$ acts on $U = F \oplus V$ by fixing $F$.

In Section 6 we give a presentation for $R$ in terms of the Moore–Coxeter generators for $S_n$ and the element $\nu \in R$ which corresponds to a nilpotent Jordan block in $\mathcal{R}$. This section is not about representation theory. However, the argument given here is the $q = 1$ version of an argument which will be used in a representation-theoretic context [16]; see the second paragraph of Section 6 for some brief remarks about a $q$-analogue of $FR$. The representation theory of this $q$-analogue has been studied by Tom Halverson [3].

Cheryl Grood [2] defined the notion of a $\lambda_r^n$-tableau where $\lambda$ is a partition of an integer $r$ with $1 \leqslant r \leqslant n$. This is a Ferrers board of shape $\lambda$ filled with distinct elements of $\mathbf{n}$. She has used the standard $\lambda_r^n$-tableaux to construct $R$-modules which are analogous to the Specht modules in the theory of the symmetric group and has shown that they furnish a complete set of irreducible $R$-modules.

The work in this article, except for the examples at the end of Section 3, was outlined in a talk at the Centre de Recherches Mathématiques, Université de Montréal in June 1997. I would like to thank Ira Gessel and Glenn Tesler who heard the talk and settled two points which were left open. Both Gessel and Tesler gave (independently) an explicit formula for the inverse of the matrix $\mathsf{B}$; see Remark 3.27. Gessel gave a direct proof of the Schur function identity in Corollary 4.10, which follows in this paper from facts about the representation of $R$ on $F[x_1, \ldots, x_n]$. In July 1997 I learned from Grant Walker that he has studied the representation of $R$ on $F[x_1, \ldots, x_n]$ in case $F = \mathbf{F}_p$ is the field of $p$ elements; see [18].

## 2. Munn's representation theory and character formula

The main ideas and the results in section are due to W.D. Munn [10], who found the irreducible representations of $R$ and gave a formula for its irreducible characters in terms of the irreducible characters of the symmetric groups $S_r$ for $0 \leqslant r \leqslant n$. This was a special but interesting case of his general theory [9] of representations of finite semigroups. Let $A = FR = \bigoplus_{\sigma \in R} F\sigma$ be the monoid algebra. The identity element $1_A$ is the identity map of $\mathbf{n}$. It is understood that an algebra homomorphism $A \to B$ maps $1_A$ to $1_B$. For $0 \leqslant r \leqslant n$ let

$$I^{(r)} = \sum_{\mathrm{rk}(\sigma) \leqslant r} F\sigma. \tag{2.1}$$

Since $\mathrm{rk}(\sigma\tau) \leqslant \mathrm{rk}(\sigma)$ and $\mathrm{rk}(\tau\sigma) \leqslant \mathrm{rk}(\sigma)$ for all $\sigma, \tau \in R$, it follows that $I^{(r)}$ is a two sided ideal of $A$. Thus we have an ascending chain $F \simeq I^{(0)} \subset \cdots \subset I^{(r-1)} \subset I^{(r)} \subset \cdots \subset I^{(n)} = A$ of two sided ideals. Munn proved [10, Theorem 3.1] that $A$ is semisimple. Thus there exists for each $0 \leqslant r \leqslant n$ a uniquely determined central idempotent $\eta_r$ of $A$ such that

$$I^{(r)} = I^{(r-1)} \oplus A\eta_r \quad \text{for } 1 \leqslant r \leqslant n. \tag{2.2}$$

The alternating sum formula for $\eta_r$ given by (2.4) and (2.8) is the new feature in the present (self-contained) exposition of Munn's work. The splitting (2.2) is proved directly in Corollary 2.14 without assuming semisimplicity. The formula for $\eta_r$ has some antecedents. R. Penrose [9, p. 11] gave a formula for the identity element of $A$, which amounts to $1_A = \sum_{r=0}^{n} \eta_r$ in our notation. The pairwise orthogonal idempotents $\eta_K$ defined in (2.4) were introduced in the context of the Möbius algebra of a lattice; see [14, p. 605] or [17, p. 124]. In fact, the subalgebra of $A$ generated by the idempotents of $R$ is isomorphic to the Möbius algebra of the lattice of subsets of $\mathbf{n}$ and, under this isomorphism, the $\eta_K$ correspond to the primitive idempotents of the Möbius algebra. The idempotents $\eta_K$ also occur naturally as projections in the action of $R$ on tensors; see (5.12) and (5.15).

If $K \subseteq \mathbf{n}$ is nonempty let $\varepsilon_K \in R$ be the identity map of $K$. If $K = \emptyset$ let $\varepsilon_\emptyset$ be the zero element of $R$. Then $\varepsilon_K$ is idempotent and

$$\varepsilon_J \varepsilon_K = \varepsilon_{J \cap K} = \varepsilon_K \varepsilon_J \tag{2.3}$$

for $J, K \subseteq \mathbf{n}$. Thus $E = \{\varepsilon_K \mid K \subseteq \mathbf{n}\}$ is a commutative submonoid of $R$. The element $\varepsilon_\emptyset$ is the zero element of $R$ but is not the zero element of $A$. Similarly, $[\varepsilon_\emptyset]$ is the zero matrix but is not the zero element of $F\mathcal{R}$. If $K \subseteq \mathbf{n}$ define $\eta_K \in FE = \bigoplus_{K \subseteq \mathbf{n}} F\varepsilon_K$ by

$$\eta_K = \sum_{J \subseteq K} (-1)^{|K-J|} \varepsilon_J. \tag{2.4}$$

It follows by inclusion–exclusion that

$$\varepsilon_K = \sum_{J \subseteq K} \eta_J. \tag{2.5}$$

Thus $FE = \bigoplus_{K \subseteq \mathbf{n}} F\eta_K$.

**Lemma 2.6.** *If $J, K \subseteq \mathbf{n}$ then $\eta_K \eta_J = \delta_{K,J}\eta_K$. Thus the $\eta_K$ are pairwise orthogonal idempotents of $A$.*

**Proof.** We show first that

$$\varepsilon_K \eta_J = \begin{cases} \eta_J & \text{if } J \subseteq K, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Fix $J$ and $K$. Let $L = J \cap K$ and let $M = J \cap (\mathbf{n} - K)$. Any subset of $J$ has the form $X \cup Y$ where $X \subseteq L$ and $Y \subseteq M$. Since $K \cap (X \cup Y) = X$ we have $\varepsilon_K \varepsilon_{X \cup Y} = \varepsilon_X$ by (2.3). Thus

$$\begin{aligned} \varepsilon_K \eta_J &= \varepsilon_K \sum_{X \subseteq L} \sum_{Y \subseteq M} (-1)^{|L \cup M - X \cup Y|} \varepsilon_{X \cup Y} \\ &= \sum_{X \subseteq L} \left( \sum_{Y \subseteq M} (-1)^{|M-Y|} \right) (-1)^{|L-X|} \varepsilon_X. \end{aligned}$$

If $K \supseteq J$ then $M$ is empty and $L = J$ so the inner sum is 1 and we get $\varepsilon_K \eta_J = \eta_J$ by (2.4). If $K$ does not include $J$ then $M$ is nonempty so the inner sum is 0 and we get $\varepsilon_K \eta_J = 0$. This proves (2.7). By (2.4) and (2.7),

$$\eta_K \eta_J = \sum_{L \subseteq K} (-1)^{|K-L|} \varepsilon_L \eta_J = \sum_{J \subseteq L \subseteq K} (-1)^{|K-L|} \eta_J.$$

The last sum is zero unless $K = J$ when it is 1. □

We may use the $\eta_K$ to construct some pairwise orthogonal central idempotents of $A$. If $0 \leqslant r \leqslant n$ define

$$\eta_r = \sum_{|K|=r} \eta_K. \tag{2.8}$$

It follows from Lemma 2.6 that

$$\eta_j \eta_k = \delta_{jk} \eta_k \quad \text{for } 0 \leqslant j,\ k \leqslant n. \tag{2.9}$$

Thus the $\eta_r$ are pairwise orthogonal idempotents of $A$. If $w \in S_n$ and $K \subseteq \mathbf{n}$ then both $\varepsilon_K w$ and $w \varepsilon_{Kw}$ have domain $K$. If $i \in K$ then $i \varepsilon_K w = iw = iw\varepsilon_{Kw}$. Thus $\varepsilon_K w = w \varepsilon_{Kw}$ so

$$w^{-1} \varepsilon_K w = \varepsilon_{Kw} \quad \text{and hence} \quad w^{-1} \eta_K w = \eta_{Kw}. \tag{2.10}$$

Let $\sigma \in R$ and let $K = I(\sigma)$. Choose $w \in S_n$ so that $iw = i\sigma$ for $i \in K$. Then $\sigma = \varepsilon_K w$. Thus

$$R = ES_n = S_n E. \tag{2.11}$$

It follows from (2.8) and (2.10) that $\eta_r$ centralizes $S_n$. Since $FE$ is a commutative algebra, $\eta_r$ also centralizes $E$. Thus $\eta_r$ centralizes $R$ and hence lies in the center of $A$. Thus $A\eta_r$ is a two sided ideal of $A$. Since $1_A = \varepsilon_{\mathbf{n}}$, it follows from (2.5) with $K = \mathbf{n}$ that $\varepsilon_{\mathbf{n}} = \sum_{r=0}^{n} \eta_r$. Thus

$$A = \bigoplus_{r=0}^{n} A\eta_r \tag{2.12}$$

a direct sum of two sided ideals.

**Lemma 2.13.** *Suppose* $1 \leqslant r \leqslant n$.

(i) *If* $\sigma \in R$ *and* $\mathrm{rk}(\sigma) < r$ *then* $\sigma\eta_r = 0$.
(ii) *The set* $\{\sigma\eta_r \mid \sigma \in R^r\}$ *is an* $F$*-basis for* $A\eta_r$.
(iii) $I^{(r)} = \bigoplus_{j=0}^{r} A\eta_j$.

**Proof.** Suppose $\sigma \in R$ and $\mathrm{rk}(\sigma) < r$. Let $K = I(\sigma)$. Since $|K| = \mathrm{rk}(\sigma) < r$, it follows from (2.7) that if $J \subseteq \mathbf{n}$ and $|J| = r$ then $\varepsilon_K \eta_J = 0$. Thus $\varepsilon_K \eta_r = 0$. Write $\sigma = \varepsilon_K w$, where $w \in S_n$. Since $\eta_r$ is central, $\sigma\eta_r = \varepsilon_K \eta_r w = 0$. This proves (i). Suppose $K \subseteq \mathbf{n}$ and $|K| = r$. If $J \subseteq \mathbf{n}$ and $|J| \leqslant r$ then $\varepsilon_J \in I^{(r)}$. Thus $\eta_K \in I^{(r)}$ by (2.4), so $\eta_r \in I^{(r)}$ and thus $A\eta_r \subseteq I^{(r)}$. Suppose $\alpha \in A\eta_r$. We may write $\alpha = \sum_{\mathrm{rk}(\sigma)\leqslant r} c_\sigma \sigma$ with $c_\sigma \in F$ so $\alpha = \alpha\eta_r = \sum_{\mathrm{rk}(\sigma)=r} c_\sigma \sigma\eta_r$ by (i). Thus the set $\{\sigma\eta_r \mid \sigma \in R^r\}$ spans $A\eta_r$. It follows that $\dim A = \sum_{r=0}^{n} |R^r| \geqslant \sum_{r=0}^{n} \dim A\eta_r = \dim A$ where the last equality comes from (2.12). Thus $\dim A\eta_r = |R^r|$. This proves (ii). To prove (iii) let $A^{(r)} = \sum_{j=0}^{r} A\eta_j$. If $0 \leqslant j \leqslant r$ then $\eta_j \in I^{(j)} \subseteq I^{(r)}$ by (2.1) and (2.8) so $A^{(r)} \subseteq I^{(r)}$. To prove the reverse inclusion it suffices to show that if $\sigma \in R^0 \cup \cdots \cup R^r$ then $\sigma \in A^{(r)}$. Write $\sigma = \varepsilon_K w$ where $w \in S_n$ and $K = I(\sigma)$. Then $|K| = \mathrm{rk}(\sigma) \leqslant r$. If $|J| > r$ then $\varepsilon_K \eta_J = 0$ by (2.7). Thus $\varepsilon_K \eta_j = 0$ for $j > r$. Since $1_A = \varepsilon_{\mathbf{n}} = \sum_{j=0}^{n} \eta_j$ we have $\varepsilon_K = \sum_{j=0}^{n} \varepsilon_K \eta_j = \sum_{j=0}^{r} \varepsilon_K \eta_j \in A^{(r)}$ so $\sigma = \varepsilon_K w \in A^{(r)}$. $\square$

**Corollary 2.14.** $I^{(r)} = I^{(r-1)} \oplus A\eta_r$ *for* $0 \leqslant r \leqslant n$.

Corollary 2.14 gives the splitting promised in (2.2). If $r = 0$ then $\eta_0 = \varepsilon_\emptyset$ so $I^{(0)} = F\varepsilon_\emptyset = A\varepsilon_\emptyset = A\eta_0$. Thus (ii) and (iii) in Lemma 2.13 hold for $r = 0$.

Next we describe the structure of $A\eta_r$. Suppose $r \geqslant 1$. Choose, for each $K \subseteq \mathbf{n}$ with $|K| = r$, an element $\mu_K \in R$ which maps $\mathbf{r} = \{1, \ldots, r\}$ to $K$. If $K = \mathbf{r}$, choose $\mu_K = \varepsilon_K$. If $\sigma \in R$ define $\sigma^- \in R$ by $I(\sigma^-) = J(\sigma)$, $J(\sigma^-) = I(\sigma)$ and $j\sigma^- = i$ if $i\sigma = j$. Here $i \in I(\sigma)$ and $j \in J(\sigma)$. Thus $\sigma\sigma^-$ is the identity map of $I(\sigma)$ and $\sigma^-\sigma$ is the identity map of $J(\sigma)$. Note that $\mu_K^- \mu_K = \varepsilon_K$ is the identity map of $K$. If $\sigma \in R^r$ and $I = I(\sigma)$ and $J = J(\sigma)$ then $\sigma = \varepsilon_I \sigma \varepsilon_J = \mu_I^- \mu_I \sigma \mu_J^- \mu_J$. Define $p(\sigma) \in S_r$ by

$$p(\sigma) = \mu_I \sigma \mu_J^- \tag{2.15}$$

where $I = I(\sigma)$ and $J = J(\sigma)$. Then

$$\sigma = \mu_I^- p(\sigma)\mu_J. \tag{2.16}$$

This expression is unique: if $I, J$ are $r$-subsets of $\mathbf{n}$ and $\mu_I^- p\mu_J = \mu_I^- q\mu_J$ with $p, q \in S_r$ then $p = q$.

**Lemma 2.17.** *For $1 \leqslant r \leqslant n$ let $A_r = \mathbf{M}_{\binom{n}{r}}(FS_r)$ be the $F$-algebra of all matrices with rows and columns indexed by $r$-subsets $I, J$ of $\mathbf{n}$ and entries in $FS_r$. Let $E_{IJ} \in A_r$ denote the natural basis of matrix units. Define an $F$-linear map $\psi_r : A\eta_r \to A_r$ by*

$$\psi_r(\sigma\eta_r) = p(\sigma)E_{IJ} \quad \textit{where } \sigma \in R^r,\ I = I(\sigma),\ J = J(\sigma). \tag{2.18}$$

*For $r = 0$ we agree that $A_0 = F$ and that $\psi_0 : A\eta_0 = F\varepsilon_\emptyset \to F$ is defined by $\psi_0(\varepsilon_\emptyset) = 1$. If $0 \leqslant r \leqslant n$ then $\psi_r$ is an isomorphism of $F$-algebras.*

**Proof.** We may assume that $r \geqslant 1$. The map $\psi_r$ is well defined by Lemma 2.13(ii). Write $\psi = \psi_r$ and $\eta = \eta_r$. We show first that $\psi$ is a homomorphism of algebras. It suffices to show that $\psi((\sigma\eta)(\tau\eta)) = \psi(\sigma\eta)\psi(\tau\eta)$ for $\sigma, \tau \in R^r$. Let $I = I(\sigma)$ and $J = J(\sigma)$. Let $p = p(\sigma)$, $q = p(\tau)$ and let $L = I(\tau)$, $K = J(\tau)$. Then $\psi(\sigma\eta)\psi(\tau\eta) = (pE_{IJ})(qE_{LK}) = pqE_{IK}$ if $J = L$ and $\psi(\sigma\eta)\psi(\tau\eta) = 0$ otherwise. Note that $J = L$ if and only if $\mathrm{rk}(\sigma\tau) = r$, in which case $\sigma\tau = (\mu_I^- p\mu_J)(\mu_J^- q\mu_K) = \mu_I^- pq\mu_K$. Thus, if $\mathrm{rk}(\sigma\tau) = r$ then $\psi((\sigma\eta)(\tau\eta)) = \psi(\sigma\tau\eta) = pqE_{IK} = \psi(\sigma\eta)\psi(\tau\eta)$. If $\mathrm{rk}(\sigma\tau) < r$ then $\sigma\tau \in I^{(r-1)} = A\eta_0 + \cdots + A\eta_{r-1}$ so $\sigma\tau\eta = 0$ because $\eta_j\eta = 0$ for $0 \leqslant j \leqslant r-1$ by (2.9). Thus $\psi((\sigma\eta)(\tau\eta)) = 0 = \psi(\sigma\eta)\psi(\tau\eta)$. Thus $\psi$ is a homomorphism of algebras. Suppose $a \in \ker\psi$. By Lemma 2.13(ii) and (2.16) we may write $a = \sum c_{IJ}(p)\mu_I^- p\mu_J\eta_r$ where $c_{IJ}(p) \in F$; the sum is over all $r$-subsets $I, J$ of $\mathbf{n}$ and all $p \in S_r$. Apply $\psi$. This gives $0 = \sum_{I,J}\sum_p c_{IJ}(p)pE_{IJ}$. Thus $\sum_p c_{IJ}(p)p = 0$ for all $I, J$ so $c_{IJ}(p) = 0$ for all $I, J$ and $p$. Thus $\psi$ is one-to-one. It follows from Lemma 2.13(ii) and (1.2) that $\dim A\eta = |R^r| = \binom{n}{r}^2 r! = \dim A_r$. Thus $\psi$ is an isomorphism. $\square$

**Corollary 2.19** (Munn). *$A \simeq \bigoplus_{r=0}^n \mathbf{M}_{\binom{n}{r}}(FS_r)$. In particular, $A$ is a semisimple algebra.*

**Proof.** The first assertion follows from (2.12) and Lemma 2.17. The second assertion follows from the first since $F$ has characteristic zero. $\square$

It is convenient, for the moment, to let $A$ be any associative $F$-algebra with identity and let $\eta \in A$ be a central idempotent. Let $B$ be an associative $F$-algebra with identity, let $d$ be a positive integer and let $\psi : A\eta \to \mathbf{M}_d(B)$ be an algebra homomorphism. If $a \in A$ define $\beta_{ij}(a) \in B$ for $1 \leqslant i, j \leqslant d$ by

$$\psi(a\eta) = \sum_{i,j=1}^{d} \beta_{ij}(a) E_{ij}. \tag{2.20}$$

If $a, b \in A$ then $\beta_{ij}(ab) = \sum_{k=1}^{d} \beta_{ik}(a)\beta_{kj}(b)$. It is understood in what follows that representations of $A$ or $B$ are matrix representations with coefficients in $F$. If $\rho$ is a representation of $B$ then we may define a representation $\rho^*$ of $A$ by

$$\rho^*(a) = \sum_{i,j=1}^{d} \rho\bigl(\beta_{ij}(a)\bigr) E_{ij}; \tag{2.21}$$

to get the matrix $\rho^*(a)$ we apply $\rho$ to the matrix entries of $\psi(a\eta)$.

**Lemma 2.22.** *Let $A$ be an associative algebra over $F$. Suppose $A$ contains pairwise orthogonal central idempotents $\eta_0, \eta_1, \ldots, \eta_n$ such that $A = \bigoplus_{r=0}^{n} A\eta_r$. Suppose for each $0 \leqslant r \leqslant n$ that there exists an integer $d_r$, a semisimple algebra $B_r$ and an algebra isomorphism $\psi_r : A\eta_r \to \mathbf{M}_{d_r}(B_r)$. Let $\widehat{B_r}$ be a full set of inequivalent irreducible representations of $B_r$. Then $\{\rho^* \mid 0 \leqslant r \leqslant n,\ \rho \in \widehat{B_r}\}$ is a full set of inequivalent irreducible representations of $A$.*

**Proof.** The hypotheses in Lemma 2.22 insure that the algebra $A$ is semisimple. If $A$ is simple then it has a unique irreducible representation, up to equivalence, and $B$ is also simple, so the assertion is clear. The case $n = 0$ may be reduced to the case $A$ simple. The general case may be reduced to the case $n = 0$. □

Apply Lemma 2.22 with $A = FR$, with $B_r = FS_r$ and $\psi_r$ as in Lemma 2.17. If $\rho$ is an irreducible representation of $S_r$ and hence of $B_r$ we say that $\rho^*$ is an irreducible representation of $A$ or $R$ of rank $r$. Note that

$$\deg \rho^* = \binom{n}{r} \deg \rho. \tag{2.23}$$

For $1 \leqslant r \leqslant n$ let $\mathcal{P}_r$ denote the set of partitions of $r$. The equivalence classes of irreducible representations of $S_r$ are indexed by $\mathcal{P}_r$. Choose, for each $1 \leqslant r \leqslant n$ and $\lambda \in \mathcal{P}_r$, an irreducible representation $\rho^\lambda$ of $S_r$ indexed by $\lambda$. We agree that $\mathcal{P}_0$ consists of the empty partition written $\lambda = (0)$ and that the corresponding irreducible representation of $S_0$ is given by $\rho^{(0)}(\varepsilon_\emptyset) = 1 \in F$. Since $B_0 = F\varepsilon_\emptyset$ and $\psi_0(\sigma\eta_0) = \psi_0(\sigma\varepsilon_\emptyset) = \psi_0(\varepsilon_\emptyset) = 1$ we have $\rho^{(0)*}(\sigma) = 1$ for all $\sigma \in R$. This "trivial representation" $\rho^{(0)*}$ is the unique irreducible representation of $R$ which has rank zero.

**Theorem 2.24** (Munn). *Let $\mathcal{Q} = \bigcup_{r=0}^{n} \mathcal{P}_r$. The set $\{\rho^{\lambda*} \mid \lambda \in \mathcal{Q}\}$ is a full set of inequivalent irreducible representations of $R$.*

**Proof.** This follows from (2.12), Lemmas 2.17 and 2.22. □

To compute the value of the character of $\rho^* = \rho^{\lambda *}$ on an element of $R$ we need a formula for $\rho^*(\sigma)$ with $\sigma \in R$.

**Proposition 2.25.** *Suppose* $1 \leqslant r \leqslant n$. *If* $\rho$ *is a representation of* $S_r$ *and* $\sigma \in R$ *then*

$$\rho^*(\sigma) = \sum_{\substack{|K|=r \\ \operatorname{rk}(\varepsilon_K\sigma)=r}} \rho\big(p(\varepsilon_K\sigma)\big) E_{I(\varepsilon_K\sigma), J(\varepsilon_K\sigma)}. \tag{2.26}$$

**Proof.** If $\operatorname{rk}(\sigma) < r$ then we cannot have $|K| = r$ and $K \subseteq I(\sigma)$ so the right-hand side of side of (2.26) is zero. On the other hand, $\sigma \in I^{(r-1)} = \bigoplus_{j=0}^{r-1} A\eta_j$ by Lemma 2.13(iii), so $\sigma\eta_r = 0$ by (2.9). Thus

$$\rho^*(\sigma) = 0 \quad \text{if } \operatorname{rk}(\sigma) < r \tag{2.27}$$

by (2.20) and (2.21). This proves (2.26) if $\operatorname{rk}(\sigma) < r$. If $\operatorname{rk}(\sigma) = r$ then the sum on the right-hand side of (2.26) consists of a single term with $K = I(\sigma)$ in which case $\varepsilon_K\sigma = \sigma$. Thus the right-hand side of (2.26) is $\rho(p(\sigma))E_{IJ}$ where $I = I(\sigma)$ and $J = J(\sigma)$. On the other hand, $\psi_r(\sigma\eta_r) = p(\sigma)E_{IJ}$ by (2.18). Thus

$$\rho^*(\sigma) = \rho\big(p(\sigma)\big)E_{IJ} \quad \text{if } \operatorname{rk}(\sigma) = r \tag{2.28}$$

by (2.20) and (2.21). This proves (2.26) if $\operatorname{rk}(\sigma) = r$. Finally suppose $\operatorname{rk}(\sigma) > r$. Let $\equiv$ denote congruence mod $I^{(r-1)}$. Then

$$\eta_r = \sum_{|K|=r} \eta_K = \sum_{|K|=r} \sum_{J \subseteq K} (-1)^{|K-J|} \varepsilon_J \equiv \sum_{|K|=r} \varepsilon_K = \varepsilon_r.$$

Since $\eta_r$ is a central idempotent, we get

$$\sigma\eta_r = \eta_r\sigma\eta_r \equiv \varepsilon_r\sigma\eta_r = \sum_{|K|=r} \varepsilon_K\sigma\eta_r.$$

But $\rho^*(a) = \rho^*(a\eta_r)$ for all $a \in FR$ by (2.20) and (2.21). Thus $\rho^*(\sigma) = \sum_{|K|=r,\, \operatorname{rk}(\varepsilon_K\sigma)=r} \rho^*(\varepsilon_K\sigma)$ by (2.27). Now (2.26) follows from (2.28). □

**Example 2.29.** If $r = 1$ then $\lambda = (1)$ and $\rho = \rho^{(1)} : S_r \to F$ is defined by $\rho(\varepsilon_{\{1\}}) = 1 \in F$. The conditions on $K$ in (2.26) are $K = \{i\}$ and $i \in I(\sigma)$. Since $p(\varepsilon_K\sigma) = 1$ and hence $\rho(p(\varepsilon_K\sigma)) = 1$, it follows that $\rho^{(1)*}(\sigma) = \sum_{i \in I(\sigma)} E_{i,i\sigma}$. Thus $\rho^{(1)*}$ is the representation (1.1) of $R$ by rook matrices. Suppose $r = n$. Then $K = \mathbf{n}$ and $\varepsilon_K\sigma = \sigma$ in (2.26). If $\operatorname{rk}(\sigma) < n$ then $\rho^*(\sigma) = 0$ by (2.27). If $\operatorname{rk}(\sigma) = n$ then $\sigma \in S_n$ and $p(\sigma) = \sigma$ in (2.26) so $\rho^*(\sigma) = \rho(\sigma)$. Thus the representations of maximal rank $n$ have the shape $\rho^* = \rho \circ \pi$ where $\pi : FR \to FS_n$ is the $F$-linear map, in fact homomorphism of algebras, defined by $\pi(\sigma) = \sigma$ if $\sigma \in S_n$ and $\pi(\sigma) = 0$ if $\sigma \in R - S_n$.

If $\chi$ is the character of a representation $\rho$ of $S_r$ let $\chi^*$ denote the character of $\rho^*$. We identify $\chi$ with its $F$-linear extension to a character of $FS_r$ and identify $\chi^*$ with its $F$-linear extension to a character of $A = FR$. Since $A$ is semisimple, two representations of $A$ are equivalent if and only if they have the same character. The following theorem of Munn [10, Theorem 3.5] gives a formula for $\chi^*(\sigma)$ when $\sigma \in R$.

**Theorem 2.30** (Munn). *Suppose $1 \leqslant r \leqslant n$. If $\chi$ is a character of $S_r$ and $\chi^*$ is the corresponding character of $R$ then*

$$\chi^*(\sigma) = \sum_{\substack{K \subseteq I(\sigma),\, |K|=r \\ K\sigma = K}} \chi\left(\mu_K \sigma \mu_K^-\right). \tag{2.31}$$

**Proof.** It follows from Proposition 2.25 that

$$\chi^*(\sigma) = \sum_{\substack{|K|=r,\ \mathrm{rk}(\varepsilon_K\sigma)=r \\ I(\varepsilon_K\sigma)=J(\varepsilon_K\sigma)}} \chi\big(p(\varepsilon_K\sigma)\big).$$

The simultaneous occurrence of $\mathrm{rk}(\varepsilon_K\sigma) = |K|$ and $I(\varepsilon_K\sigma) = J(\varepsilon_K\sigma)$ is equivalent to the simultaneous occurrence of $K \subseteq I(\sigma)$ and $K\sigma = K$. If $K \subseteq I(\sigma)$ then $p(\varepsilon_K\sigma) = \mu_K\varepsilon_K\sigma\mu_K^-$ by (2.15). Now (2.31) follows since $\mu_K\varepsilon_K = \mu_K$. $\square$

**Example 2.32.** Suppose that $n = 5$ and $r = 3$. Suppose that $I(\sigma) = \{1, 2, 3, 5\}$ and that $\sigma : 1 \mapsto 3 \mapsto 5 \mapsto 1$ and $\sigma : 2 \mapsto 4$ with $4\sigma$ undefined. There are four sets $K$ with $|K| = 3$ and $K \subseteq I(\sigma)$. The action of $\sigma$ on these sets is

| $K$ | $K\sigma$ |
|---|---|
| $\{1, 2, 3\}$ | $\{3, 4, 5\}$ |
| $\{1, 2, 5\}$ | $\{1, 3, 4\}$ |
| $\{1, 3, 5\}$ | $\{1, 3, 5\}$ |
| $\{2, 3, 5\}$ | $\{1, 4, 5\}$ |

Thus $K\sigma = K$ only for $K = \{1, 3, 5\}$. Choose $\mu_K : \{1, 2, 3\} \to \{1, 3, 5\}$ so that $\mu_K : 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5$. Then $\mu_K\sigma\mu_K^- \in S_3$ has domain $\{1, 2, 3\}$ and maps $1 \mapsto 2 \mapsto 3 \mapsto 1$. Thus $\mu_K\sigma\mu_K^- = (123)$ in the usual cycle notation for permutations so $\chi^*(\sigma) = \chi((123))$.

For $0 \leqslant r \leqslant n$ and $\alpha, \lambda \in \mathcal{P}_r$ let $\chi_\alpha^\lambda$ be the value which the irreducible character $\chi^\lambda$ of $S_r$ assumes on elements of the conjugacy class of $S_r$ indexed by $\alpha$. The character table of $S_r$ is the square matrix $\mathsf{X}_r$ of size $|\mathcal{P}_r|$ with $(\alpha, \lambda)$ entry equal to $\chi_\alpha^\lambda$. Note that $\chi^{(0)}(\varepsilon_\emptyset) = 1$ so $\chi_{(0)}^{(0)} = 1$ and $\mathsf{X}_0$ is an identity matrix of size 1. In [10] Munn defined a character table for $R$. This is a square matrix $\mathsf{M}$ of size $|\mathcal{Q}|$. To define it we introduce an equivalence relation on $R$ as follows. If $\sigma \in R$ let $I^\circ(\sigma)$ denote the set of $i \in \mathbf{n}$ such $i\sigma^k$ is defined for all $k \geqslant 1$. Then $I^\circ(\sigma) \subseteq I(\sigma)$ and $I^\circ(\sigma)$ is stable under $\sigma$. Define $\sigma^\circ \in R$ to have domain $I^\circ(\sigma)$ and let $\sigma^\circ$ act on its domain as $\sigma$ does. For example, if $\sigma$ is as in Example 2.32 then $I^\circ(\sigma) = \{1, 3, 5\}$ and $\sigma^\circ : 1 \mapsto 3 \mapsto 5 \mapsto 1$. Note that $I(\varepsilon_\emptyset)$ and $I^{(0)}(\varepsilon_\emptyset)$ are empty, so $\varepsilon_\emptyset^\circ = \varepsilon_\emptyset$. Say that $\sigma, \tau \in R$ are Munn equivalent and write $\sigma \approx \tau$ if there exists $w \in S_n$ with $\tau^\circ = w^{-1}\sigma^\circ w$. Munn introduced this equivalence relation in [10] and called $\mathrm{rk}(\sigma^\circ)$ the subrank of $\sigma$. Any Munn equivalence class meets a unique group $S_r$ where $r$ is the common subrank of all elements in the class. The Munn classes of $R$ which meet $S_r$ are indexed by conjugacy classes of $S_r$ and hence by $\mathcal{P}_r$. Thus the Munn classes of $R$ are indexed by $\mathcal{Q}$.

**Proposition 2.33.** *If $\sigma, \tau \in R$ are Munn equivalent and $\zeta$ is the character of a representation of $FR$ then $\zeta(\sigma) = \zeta(\tau)$.*

**Proof.** Since there exists $w \in S_n$ with $\tau^\circ = w^{-1}\sigma^\circ w$ we have $\zeta(\sigma^\circ) = \zeta(\tau^\circ)$. It thus suffices to show that $\zeta(\sigma) = \zeta(\sigma^\circ)$. We may assume that $\zeta$ is the character of an irreducible representation and apply (2.31) with $\zeta = \chi^*$. The simultaneous occurrence of $K \subseteq I(\sigma)$ and $K\sigma = K$ is equivalent to the simultaneous occurrence of $K \subseteq I(\sigma^\circ)$ and $K\sigma^\circ = K$. Furthermore, if these conditions hold, then $\mu_K\sigma = \mu_K\sigma^\circ$. The assertion $\zeta(\sigma) = \zeta(\sigma^\circ)$ thus follows from (2.31) applied to both $\sigma$ and $\sigma^\circ$. $\square$

In the rest of this paper we let $\zeta^\lambda = \chi^{\lambda *}$ denote the irreducible character of $R$ which corresponds to the irreducible character $\chi^\lambda$ of $S_r$. From (2.23) we get

$$\zeta^\lambda(1) = \binom{n}{r} f^\lambda, \tag{2.34}$$

where $f^\lambda = \chi^\lambda(1)$. If $\alpha, \lambda \in \mathcal{Q}$ let $\zeta_\alpha^\lambda$ be the value which $\zeta^\lambda$ assumes on elements of the Munn class indexed by $\alpha$. This is well defined by Proposition 2.33. Munn's character table is the square matrix $\mathsf{M}$ of size $|\mathcal{Q}|$ with $(\alpha, \lambda)$ entry $\mathsf{M}_{\alpha\lambda} = \zeta_\alpha^\lambda$.

## 3. Character table and character multiplicities

In this section we use various matrices $\mathsf{T}$ with rows and columns indexed by $\mathcal{Q}$. If $\lambda \in \mathcal{P}_r$, write $|\lambda| = r$. To label the rows and columns of $\mathsf{T}$ we linearly order $\mathcal{Q}$: if $\lambda, \mu \in \mathcal{Q}$ say that $\lambda$ precedes $\mu$ if $|\lambda| > |\mu|$, or $|\lambda| = |\mu| = r$ and $\lambda$ precedes $\mu$ in the reverse lexicographic order on $\mathcal{P}_r$. Let $\mathsf{T}_{\alpha\lambda}$ denote the $(\alpha, \lambda)$ entry of $\mathsf{T}$. Say that $\mathsf{T}$ is block upper triangular if $\mathsf{T}_{\alpha\lambda} = 0$ for $|\lambda| > |\alpha|$ and block upper unitriangular if, in addition, $\mathsf{T}_{\alpha\lambda} = \delta_{\alpha\lambda}$ when $|\lambda| = |\alpha|$.

**Lemma 3.1.** *Suppose $\alpha \in \mathcal{P}_m$ and $\lambda \in \mathcal{P}_r$ where $0 \leqslant m \leqslant r \leqslant n$. If $m < r$ then $\zeta_\alpha^\lambda = 0$. If $m = r$ then $\zeta_\alpha^\lambda = \chi_\alpha^\lambda$.*

**Proof.** Choose $\sigma \in R$ with $\zeta^\lambda(\sigma) = \zeta_\alpha^\lambda$. Since the Munn class of $\sigma$ meets $S_m$ we may assume by Proposition 2.33 that $\sigma \in S_m$. If $m < r$ then (2.27) gives $\zeta^\lambda(\sigma) = 0$. If $m = r = 0$ then $\zeta_{(0)}^{(0)} = 1 = \chi_{(0)}^{(0)}$. If $m = r > 0$, apply Theorem 2.30. Since $I(\sigma) = \mathbf{r}$, the sum in (2.31) consists of a single term corresponding to $K = \mathbf{r}$, in which case $\mu_K = \varepsilon_K$ by our choice of $\mu_K$. Thus $\mu_K \sigma \mu_K^- = \sigma$, so $\zeta^\lambda(\sigma) = \chi^\lambda(\sigma)$. □

It follows from Lemma 3.1 and the definition of $\mathsf{M}$ that

$$\mathsf{M} = \begin{bmatrix} \mathsf{X}_n & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & \mathsf{X}_1 & * \\ 0 & \cdots & 0 & \mathsf{X}_0 \end{bmatrix} \tag{3.2}$$

is block upper triangular where $\mathsf{X}_r$ is the character table of $S_r$. Define a block diagonal matrix $\mathsf{Y}$ by

$$\mathsf{Y} = \operatorname{diag}[\mathsf{X}_n, \ldots, \mathsf{X}_1, \mathsf{X}_0]. \tag{3.3}$$

Since the matrices $\mathsf{X}_r$ are invertible, $\mathsf{Y}$ is invertible. Thus there are unique block upper unitriangular matrices $\mathsf{A}, \mathsf{B}$ with rows and columns indexed by $\mathcal{Q}$ such that

$$\mathsf{M} = \mathsf{AY} \quad \text{and} \quad \mathsf{M} = \mathsf{YB}. \tag{3.4}$$

Thus either $\mathsf{A}$ or $\mathsf{B}$ and the character tables of the $S_r$ determine the character table of $R$. If $\alpha, \beta \in \mathcal{Q}$ have $a_i, b_i$ parts equal to $i$ define $\binom{\alpha}{\beta} = \prod_{i \geqslant 1} \binom{a_i}{b_i}$ where $\binom{a_i}{b_i}$ is the binomial coefficient. We agree that $\binom{0}{0} = 1$.

**Proposition 3.5.** *If $\alpha, \beta \in \mathcal{Q}$ then $\mathsf{A}_{\alpha\beta} = \binom{\alpha}{\beta}$.*

**Proof.** Let $r = |\lambda|$ and $m = |\alpha|$. Choose $\sigma \in R$ such that $\zeta_\alpha^\lambda = \zeta^\lambda(\sigma)$. We may assume, as in the proof of Proposition 3.1, that $\sigma \in S_m$. Thus $I(\sigma) = \{1, \ldots, m\}$. It follows from (2.31) that

$$\zeta_\alpha^\lambda = \sum_{\substack{K \subseteq \{1, \ldots, m\},\, |K| = r \\ K\sigma = K}} \chi^\lambda(\mu_K \sigma \mu_K^-). \tag{3.6}$$

A set $K$ which appears in (3.6) is a union of $\sigma$-orbits. Let $\sigma|_K$ denote the restriction of $\sigma$ to $K$. Since $|K| = r$ the permutation $\mu_K \sigma \mu_K^-$ of $\mathbf{r}$ has the same cycle pattern as the permutation $\sigma|_K$ of $K$. Thus if $\sigma|_K$ has $b_i$ cycles of length $i$ then $\chi^\lambda(\mu_K \sigma \mu_K^-) = \chi_\beta^\lambda$ where $\beta \in \mathcal{P}_r$ has $b_i$ parts equal to $i$. Since $\sigma$ has $a_i$

orbits of size $i$, there are $\binom{\alpha}{\beta}$ ways to choose these orbits in such a way that $\sigma|_K$ has cycle pattern $\beta$. Thus

$$\zeta_\alpha^\lambda = \sum_{\beta\in\mathcal{P}_r} \binom{\alpha}{\beta}\chi_\beta^\lambda = \sum_{\beta\in\mathcal{Q}} \binom{\alpha}{\beta}\mathsf{Y}_{\beta\lambda}.$$

The assertion $\mathsf{A}_{\alpha\beta} = \binom{\alpha}{\beta}$ follows since $\mathsf{Y}$ is invertible. □

**Corollary 3.7.** *If $\lambda, \mu \in \mathcal{Q}$ and $n = |\lambda|$, $m = |\mu|$ then*

$$\mathsf{B}_{\lambda\mu} = \sum_{\alpha\in\mathcal{P}_n,\ \beta\in\mathcal{P}_m} z_\alpha^{-1}\binom{\alpha}{\beta}\chi_\alpha^\lambda\chi_\beta^\mu,$$

*where $z_\alpha = \prod_{i\geqslant 0} a_i!i^{a_i}$ if $\alpha$ has $a_i$ parts equal to $i$.*

**Proof.** For $0 \leqslant r \leqslant n$ define a diagonal matrix $\mathsf{Z}_r$ of size $|\mathcal{P}_r|$ by $(\mathsf{Z}_r)_{\alpha\beta} = \delta_{\alpha\beta}z_\alpha$ for $\alpha, \beta \in \mathcal{P}_r$. Let $\mathsf{W} = \operatorname{diag}[\mathsf{Z}_n, \ldots, \mathsf{Z}_1, \mathsf{Z}_0]$. The second orthogonality relation for the characters of $S_r$ gives $\mathsf{X}_r\mathsf{X}_r^\top = \mathsf{Z}_r$ where $^\top$ means transpose. Thus $\mathsf{Y}\mathsf{Y}^\top = \mathsf{W}$. From (3.4) we get $\mathsf{B} = \mathsf{Y}^{-1}\mathsf{A}\mathsf{Y} = \mathsf{Y}^\top\mathsf{W}^{-1}\mathsf{A}\mathsf{Y}$. Now compare $(\lambda, \mu)$ entries on both sides of the last equation. □

We may also compute the matrix entries $\mathsf{B}_{\lambda\mu}$ in terms of Ferrers diagrams. To do this, recall some facts about symmetric functions and characters of $S_n$ [7, Chapter I]. Let $\Lambda$ be the **Q**-algebra of symmetric functions in a sequence of indeterminates. For $n = 1, 2, 3, \ldots$ let $h_n \in \Lambda$ be the complete homogeneous symmetric function of degree $n$ and let $p_n \in \Lambda$ be the power sum of degree $n$. We agree that $h_0 = 1$ and that $h_n = 0$ for $n < 0$. If $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}_n$ let $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. The Schur function $s_\lambda$ may be defined by [7, Chapter I, (3.4)]

$$s_\lambda = \det[h_{\lambda_i - i + j}], \tag{3.8}$$

where the matrix has size equal to the number of parts in $\lambda$. Let $C_n$ be the space of **Q**-valued functions on $S_n$ which are constant on conjugacy classes. The characteristic map $\mathrm{ch}: \bigoplus_{n\geqslant 0} C_n \to \Lambda$ is defined by [7, Chapter I, (7.2)]

$$\mathrm{ch}(f) = \sum_{\alpha\in\mathcal{P}_n} z_\alpha^{-1} f_\alpha p_\alpha, \tag{3.9}$$

where $f \in C_n$ and $f_\alpha$ is the value which $f$ assumes on the conjugacy class indexed by $\alpha$. It is bijective. Let $\eta_n$ be the principal character of $S_n$. Then [7, Chapter I, (7.3) and (7.4)]

$$\mathrm{ch}(\eta_n) = h_n \quad \text{and} \quad \mathrm{ch}(\chi^\lambda) = s_\lambda. \tag{3.10}$$

Identify $\lambda \in \mathcal{P}_n$ with its Ferrers diagram. If $\lambda, \mu \in \mathcal{Q}$, say that the set theoretic difference $\lambda - \mu$ is a horizontal strip if it has at most one node in each column. For $\lambda = (0)$ we agree that the empty Ferrers diagram is a horizontal strip.

**Proposition 3.11.** *If $\lambda, \mu \in \mathcal{Q}$ then*

$$\mathsf{B}_{\lambda\mu} = \begin{cases} 1 & \textit{if } \lambda \supseteq \mu \textit{ and } \lambda - \mu \textit{ is a horizontal strip,} \\ 0 & \textit{otherwise.} \end{cases} \tag{3.12}$$

**Proof.** Let $\mathsf{C}$ be the square matrix of size $|\mathcal{Q}|$ with entries $\mathsf{C}_{\lambda\mu}$ given by the right-hand side of (3.12). We must prove that $\mathsf{B} = \mathsf{C}$. Since $\mathsf{Y}$ is invertible it suffices to show that $\mathsf{M} = \mathsf{YC}$. Argue by induction on $n$. For $n = 1$

$$\mathsf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \mathsf{YC}.$$

Suppose $n \geqslant 2$. Let $R' \subset R$ be the semigroup of all one-to-one maps with domain and range included in $\{1, \dots, n-1\}$. The irreducible representations of $R'$ are indexed by $\mathcal{Q}' = \bigcup_{r=0}^{n-1} \mathcal{P}_r = \mathcal{Q} - \mathcal{P}_n$. We have used $R$ to define square matrices $\mathsf{M}, \mathsf{A}, \mathsf{C}, \mathsf{Y}$. Let $\mathsf{M}', \mathsf{A}', \mathsf{C}', \mathsf{Y}'$ be the corresponding matrices for $R'$. Let $\mathsf{I}$ be an identity matrix of size $|\mathcal{P}_n|$ and let $\mathsf{0}$ be a zero matrix of appropriate size. Since $\mathsf{M} = \mathsf{AY}$ and $\mathsf{M}' = \mathsf{A}'\mathsf{Y}'$ we have

$$\mathsf{M} = \mathsf{AY} = \begin{bmatrix} \mathsf{I} & * \\ \mathsf{0} & \mathsf{A}' \end{bmatrix}\begin{bmatrix} \mathsf{X}_n & \mathsf{0} \\ \mathsf{0} & \mathsf{Y}' \end{bmatrix} = \begin{bmatrix} \mathsf{X}_n & * \\ \mathsf{0} & \mathsf{A}'\mathsf{Y}' \end{bmatrix} = \begin{bmatrix} \mathsf{X}_n & * \\ \mathsf{0} & \mathsf{M}' \end{bmatrix}$$

by (3.3) and Proposition 3.5 and the definition of $\mathsf{Y}$. On the other hand, by definition of $\mathsf{C}$ and $\mathsf{C}'$ and induction, we have

$$\mathsf{YC} = \begin{bmatrix} \mathsf{X}_n & \mathsf{0} \\ \mathsf{0} & \mathsf{Y}' \end{bmatrix}\begin{bmatrix} \mathsf{I} & * \\ \mathsf{0} & \mathsf{C}' \end{bmatrix} = \begin{bmatrix} \mathsf{X}_n & * \\ \mathsf{0} & \mathsf{Y}'\mathsf{C}' \end{bmatrix} = \begin{bmatrix} \mathsf{X}_n & * \\ \mathsf{0} & \mathsf{M}' \end{bmatrix}.$$

To prove $\mathsf{M} = \mathsf{YC}$ it thus suffices to show that $\mathsf{M}_{\alpha\mu} = (\mathsf{YC})_{\alpha\mu}$ for $\alpha \in \mathcal{P}_n$ and $\mu \in \mathcal{Q}'$. This amounts to $\zeta_\alpha^\mu = \sum_{\lambda \in \mathcal{P}_n} \chi_\alpha^\lambda \mathsf{C}_{\lambda\mu}$ for $\alpha \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_r$ where $0 \leqslant r \leqslant n-1$. This is clear for $r = 0$ since $\zeta_{(0)}^{(0)} = 1 = \chi_{(0)}^{(0)}$ and $\mathsf{C}_{(0)(0)} = 1$. Suppose $1 \leqslant r \leqslant n-1$. The restriction of $\zeta^\mu$ to $S_n \subseteq R$ is a character of $S_n$ which we write as $\zeta^\mu|_{S_n}$. We must prove that

$$\zeta^\mu|_{S_n} = \sum_{\lambda \in \mathcal{P}_n} \mathsf{C}_{\lambda\mu}\chi^\lambda \quad \text{for } \mu \in \mathcal{P}_r \text{ and } 1 \leqslant r \leqslant n-1. \tag{3.13}$$

To do this we use Pieri's formula [7, Chapter I, (5.16)]. This states, in terms of the matrix $\mathsf{C}$, that $s_\mu h_{n-r} = \sum_{\lambda \in \mathcal{P}_n} \mathsf{C}_{\lambda\mu} s_\lambda$. Apply the characteristic map. Fix $r$ and let $P = \{w \in S_n \mid \mathbf{r}w \subseteq \mathbf{r}\} \simeq S_r \times S_{n-r}$ be the stabilizer of $\mathbf{r}$. Then $\mathrm{ch}(\mathrm{ind}_P^{S_n}(\chi^\mu \times \eta_{n-r})) = s_\mu h_{n-r}$ as in [7, Chapter I, (7.1)] where ind means induction of characters. Thus Pieri's formula amounts to the character formula $\mathrm{ind}_P^{S_n}(\chi^\mu \times \eta_{n-r}) = \sum_{\lambda \in \mathcal{P}_n} \mathsf{C}_{\lambda\mu}\chi^\lambda$. It remains to prove that

$$\zeta^\mu|_{S_n} = \mathrm{ind}_P^{S_n}\left(\chi^\mu \times \eta_{n-r}\right) \quad \text{for } \mu \in \mathcal{P}_r \text{ and } 1 \leqslant r \leqslant n-1.$$

Apply (3.6) with $m = n$ and $\sigma = w \in S_n$. This gives $\zeta^\mu(w) = \sum \chi^\mu(\mu_K w \mu_K^-)$ where the sum is over all subsets $K$ of $\mathbf{n}$ with $|K| = r$ and $Kw = K$. Extend

$\mu_K : \mathbf{r} \to K$ to an element $w_K \in S_n$. The condition $Kw = K$ is equivalent to $w_K w w_K^{-1} \in P$. If $Kw = K$ then $\mu_K w \mu_K^{-}$ is the restriction of $w_K w w_K^{-1}$ to $\mathbf{r}$ so $\chi^\mu(\mu_K w \mu_K^{-1}) = (\chi^\mu \times \eta_{n-r})(w_K w w_K^{-1})$. Thus

$$\zeta^\mu(w) = \sum \bigl(\chi^\mu \times \eta_{n-r}\bigr)\bigl(w_K w w_K^{-1}\bigr),$$

where the sum is over all subsets $K$ of $\mathbf{n}$ with $|K| = r$ and $w_K w w_K^{-1} \in P$. Since the elements $w_K$ with $|K| = r$ are a set of coset representatives for $S_n$ mod $P$ the last sum is equal to $\operatorname{ind}_P^{S_n}(\chi^\mu \times \eta_{n-r})(w)$. $\square$

**Corollary 3.14.** *Let $n \geqslant m$ be positive integers. Suppose $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_m$. Then*

$$\sum_{\alpha \in \mathcal{P}_n,\ \beta \in \mathcal{P}_m} z_\alpha^{-1} \binom{\alpha}{\beta} \chi_\alpha^\lambda \chi_\beta^\mu = \begin{cases} 1 & \textit{if } \lambda \supseteq \mu \textit{ and } \lambda - \mu \textit{ is a horizontal strip}, \\ 0 & \textit{otherwise}. \end{cases}$$

This is a statement about characters of symmetric groups which does not involve the monoid $R$. If $\lambda = (n)$ and $\mu = (m)$, we get $\sum_{\alpha \in \mathcal{P}_n,\, \beta \in \mathcal{P}_m} z_\alpha^{-1} \binom{\alpha}{\beta} = 1$ for any positive integers $n \geqslant m$. The case $n = m$ is Cauchy's formula $\sum_{\alpha \in \mathcal{P}_n} z_\alpha^{-1} = 1$.

**Corollary 3.15.** *If $1 \leqslant r \leqslant n$ and $\mu \in \mathcal{P}_r$ then $\zeta^\mu|_{S_n} = \sum_{\lambda \in \mathcal{P}_n} \mathsf{B}_{\lambda\mu} \chi^\lambda$.*

**Proof.** If $1 \leqslant r \leqslant n - 1$ this follows from (3.13) since $\mathsf{C} = \mathsf{B}$. If $r = n$ then $\lambda \supseteq \mu$ only for $\lambda = \mu$ so $\mathsf{B}_{\lambda\mu} = \mathsf{C}_{\lambda\mu} = \delta_{\lambda\mu}$. $\square$

**Example 3.16.** Suppose $n = 3$. The partitions which index the rows and columns of our matrices are written in the order (3), (21), $(1^3)$, (2), $(1^2)$, (1), (0) and $\mathsf{M} = \mathsf{AY} = \mathsf{YB}$ where $\mathsf{Y}$ is block diagonal with diagonal blocks equal to the character tables $\mathsf{X}_3$, $\mathsf{X}_2$, $\mathsf{X}_1$, $\mathsf{X}_0$. We compute $\mathsf{A}$ with Proposition 3.5, compute $\mathsf{B}$ with Proposition 3.11 and find

$$\begin{array}{ccc}
\mathsf{A} & \mathsf{M} & \mathsf{B} \\
\begin{bmatrix}
1&0&0&0&0&0&1\\
0&1&0&1&0&1&1\\
0&0&1&0&3&3&1\\
0&0&0&1&0&0&1\\
0&0&0&0&1&2&1\\
0&0&0&0&0&1&1\\
0&0&0&0&0&0&1
\end{bmatrix}
&
\begin{bmatrix}
1&-1&1&0&0&0&1\\
1&0&-1&1&-1&1&1\\
1&2&1&3&3&3&1\\
0&0&0&1&-1&0&1\\
0&0&0&1&1&2&1\\
0&0&0&0&0&1&1\\
0&0&0&0&0&0&1
\end{bmatrix}
&
\begin{bmatrix}
1&0&0&1&0&1&1\\
0&1&0&1&1&1&0\\
0&0&1&0&1&0&0\\
0&0&0&1&0&1&1\\
0&0&0&0&1&1&0\\
0&0&0&0&0&1&1\\
0&0&0&0&0&0&1
\end{bmatrix}
\end{array}$$

Now consider the decomposition of characters into irreducible characters. Let $\psi$ be a character of $R$. For $\lambda \in \mathcal{Q}$ let $(\psi : \zeta^\lambda)_R \in \mathbf{Z}$ denote the multiplicity of the irreducible character $\zeta^\lambda$ as a constituent of $\psi$. Thus

$$\psi = \sum_{\lambda \in \mathcal{Q}} \left(\psi : \zeta^\lambda\right)_R \zeta^\lambda.$$

Since $R$ is not a group we do not have the usual orthogonality relations for irreducible characters to help compute multiplicities. The following lemma shows how to compute multiplicities in terms of either (i) the matrix $\mathsf{A}^{-1}$ and the values of the character $\psi$ or (ii) the matrix $\mathsf{B}^{-1}$ and the decomposition into irreducible characters of the restriction of $\psi$ to $S_r$ for $0 \leqslant r \leqslant n$. After the proof we give examples to illustrate (i) and (ii).

**Lemma 3.17.** *Let $\psi$ be a character of $R$. Let $\psi_\alpha$ be the value which $\psi$ assumes on elements of the Munn class indexed by $\alpha$. Let $\psi|_{S_r}$ be the restriction of $\psi$ to $S_r$. For $\mu \in \mathcal{P}_r$ let $(\psi|_{S_r} : \chi^\mu)_{S_r}$ be the multiplicity of the irreducible character $\chi^\mu$ as a constituent of $\psi|_{S_r}$.*

(i) *If $\lambda \in \mathcal{P}_r$ then* $$\left(\psi : \zeta^\lambda\right)_R = \sum_{\beta \in \mathcal{P}_r} \chi^\lambda_\beta z_\beta^{-1} \sum_{\alpha \in \mathcal{Q}} \mathsf{A}^{-1}_{\beta\alpha} \psi_\alpha.$$

(ii) *If $\lambda \in \mathcal{Q}$ then* $$\left(\psi : \zeta^\lambda\right)_R = \sum_{r=0}^{n} \sum_{\mu \in \mathcal{P}_r} \mathsf{B}^{-1}_{\lambda\mu} \left(\psi|_{S_r} : \chi^\mu\right)_{S_r}.$$

**Proof.** For $\mu \in \mathcal{Q}$ let $c_\mu = (\psi : \zeta^\mu)_R$. Then $\psi = \sum_{\mu \in \mathcal{Q}} c_\mu \zeta^\mu$ so $\psi_\alpha = \sum_{\mu \in \mathcal{Q}} c_\mu \zeta^\mu_\alpha = \sum_{\mu \in \mathcal{Q}} c_\mu \mathsf{M}_{\alpha\mu}$. If $\lambda \in \mathcal{Q}$ then

$$\left(\psi : \zeta^\lambda\right)_R = \sum_{\mu \in \mathcal{Q}} c_\mu \delta_{\lambda\mu} = \sum_{\mu \in \mathcal{Q}} c_\mu \sum_{\alpha \in \mathcal{Q}} \mathsf{M}^{-1}_{\lambda\alpha} \mathsf{M}_{\alpha\mu} = \sum_{\alpha \in \mathcal{Q}} \mathsf{M}^{-1}_{\lambda\alpha} \psi_\alpha.$$

Let $\mathsf{W}$ be as in the proof of Corollary 3.7. Then $\mathsf{Y}\mathsf{Y}^\top = \mathsf{W}$. If $\lambda \in \mathcal{P}_r$ then $\mathsf{Y}_{\alpha\lambda} = \chi^\lambda_\alpha$ for $|\alpha| = r$ and $\mathsf{Y}_{\alpha\lambda} = 0$ otherwise. Thus

$$\mathsf{M}^{-1}_{\lambda\alpha} = \left(\mathsf{Y}^\top \mathsf{W}^{-1} \mathsf{A}^{-1}\right)_{\lambda\alpha} = \sum_{\beta \in \mathcal{P}_r} \chi^\lambda_\beta z_\beta^{-1} \mathsf{A}^{-1}_{\beta\alpha}.$$

This implies (i). Similarly

$$\mathsf{M}^{-1}_{\lambda\alpha} = \left(\mathsf{B}^{-1} \mathsf{Y}^\top \mathsf{W}^{-1}\right)_{\lambda\alpha} = \sum_{\mu \in \mathcal{Q}} \mathsf{B}^{-1}_{\lambda\mu} \mathsf{Y}_{\alpha\mu} z_\alpha^{-1}$$

and

$$\sum_{\alpha \in \mathcal{P}_r} \chi^\mu_\alpha z_\alpha^{-1} \psi_\alpha = \left(\psi|_{S_r} : \chi^\mu\right)_{S_r}$$

for any $\lambda \in \mathcal{Q}$. This implies (ii). $\square$

**Example 3.18.** We know from Example 2.29 that $\zeta^{(1)}$ is the character of the representation $\sigma \mapsto [\sigma]$ of $R$ by rook matrices. Let $\psi = (\zeta^{(1)})^p$ be the character of the $p$th tensor power of this representation. We use Lemma 3.17(i) to show for $p \geqslant 1$ and $\lambda \in \mathcal{Q}$ that

$$\bigl(\psi : \zeta^\lambda\bigr)_R = S(p,r) f^\lambda, \tag{3.19}$$

where $r = |\lambda|$ and $S(p,r)$ is a Stirling number of the second kind [17, p. 34]. To do this, use the formula [17, p. 34, (24a)]

$$S(p,r) = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^p.$$

We may sum here over $0 \leqslant k \leqslant n$ since $\binom{r}{k} = 0$ for $k > r$. Define a column vector $\Psi$ with components $\Psi_\alpha$ for $\alpha \in \mathcal{Q}$ by $\Psi_\alpha = \psi(\sigma)$ if $\sigma$ lies in the Munn class corresponding to $\alpha$. Define a column vector $\Theta$ with components $\Theta_\alpha$ for $\alpha \in \mathcal{Q}$ by

$$\Theta_\alpha = \begin{cases} r!S(p,r) & \text{if } \alpha = (1^r) \text{ for some } 1 \leqslant r \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\alpha \in \mathcal{Q}$ and let $a$ be the number of parts of $\alpha$ which are equal to 1. Then $\mathsf{A}_{\alpha,(1^r)} = \binom{a}{r}$ so

$$\begin{aligned} (\mathsf{A}\Theta)_\alpha &= \sum_{\beta \in \mathcal{Q}} \mathsf{A}_{\alpha\beta} \Theta_\beta = \sum_{r=1}^{n} \mathsf{A}_{\alpha,(1^r)} \sum_{k=0}^{n} (-1)^{r-k} \binom{r}{k} k^p \\ &= \sum_{k=0}^{n} \left( \sum_{r=1}^{n} (-1)^{r-k} \binom{a}{r} \binom{r}{k} \right) k^p. \end{aligned}$$

The inner sum is, by a known identity for binomial coefficients, equal to $\delta_{a,k}$. Thus $(\mathsf{A}\Theta)_\alpha = a^p$. Choose $\sigma \in R$ in the Munn class corresponding to $\alpha$. Then $\zeta_\alpha^{(1)} = \operatorname{trace}[\sigma] = a$ so $\Psi_\alpha = a^p = (\mathsf{A}\Theta)_\alpha$. This is true for all $\alpha \in \mathcal{Q}$ so $\Psi = \mathsf{A}\Theta$. Thus $\sum_{\alpha \in \mathcal{Q}} \mathsf{A}^{-1}_{\beta\alpha} \Psi_\alpha = (\mathsf{A}^{-1}\Psi)_\beta = \Theta_\beta$ for $\beta \in \mathcal{Q}$. If $\lambda \in \mathcal{Q}$ then by Lemma 3.17(i) and Proposition 3.5

$$\bigl(\psi : \zeta^\lambda\bigr)_R = \sum_{\beta \in \mathcal{P}_r} \chi_\beta^\lambda z_\beta^{-1} \Theta_\beta = \chi_{(1^r)}^\lambda z_{(1^r)}^{-1} r! S(p,r).$$

This proves (3.19) since $z_{(1^r)} = r!$ and $\chi_{(1^r)}^\lambda = f^\lambda$. It follows from (3.19) that

$$\bigl(\zeta^{(1)}\bigr)^p = \sum_{\lambda \in \mathcal{Q}} S\bigl(p, |\lambda|\bigr) f^\lambda \zeta^\lambda. \tag{3.20}$$

Since $\zeta^\lambda(1) = \binom{n}{r} f^\lambda$ and $\sum_{\lambda \in \mathcal{P}_r} (f^\lambda)^2 = r!$, the last formula, specialized at $\sigma = 1 \in R$, is the known identity [17, p. 34, (24d)]

$$n^p = \sum_{r=1}^{n} S(p,r) r! \binom{n}{r} = \sum_{r=1}^{n} S(p,r) n(n-1)\cdots(n-r+1).$$

We may also restrict the characters in (3.20) to the group $S_n \subseteq R$. This gives us a formula for the multiplicity of $\chi^\lambda$ in the character of the $p$th tensor power of the representation $w \mapsto [w]$ of $S_n$ by permutation matrices, a statement about characters of symmetric groups which does not involve the monoid $R$:

**Corollary 3.21.** *Let $\varphi$ be the character of the representation $w \mapsto [w]$ of $S_n$ by permutation matrices. If $p \geqslant 1$ and $\lambda \in \mathcal{P}_n$, then the multiplicity of $\chi^\lambda$ as an irreducible constituent of $\varphi^p$ is equal to $\sum S(p,|\mu|) f^\mu$, where $S(p,r)$ is a Stirling number of the second kind and the sum is over all partitions $\mu$ such that $\lambda \supseteq \mu$ and $\lambda - \mu$ is a horizontal strip.*

**Proof.** By (3.20) and Corollary 3.15, $\varphi^p = \sum_{\lambda \in \mathcal{P}_n} \sum_{\mu \in \mathcal{Q}} S(p,|\mu|) f^\mu \mathsf{B}_{\lambda\mu} \chi^\lambda$. The assertion follows from Proposition 3.12. □

**Example 3.22.** Let $F^n$ be the space of row vectors over $F$. Let $x_1, \ldots, x_n$ be the standard basis for $F^n$. Make $F^n$ a right $R$-module by defining

$$x_i\sigma = \begin{cases} x_{i\sigma} & \text{if } i \in I(\sigma), \\ 0 & \text{otherwise,} \end{cases} \tag{3.23}$$

for $\sigma \in R$. Then $x_i\sigma = x_i[\sigma]$ so the $R$-module $F^n$ has character $\zeta^{(1)}$. If $K \subseteq \mathbf{n}$ and $|K| = p$ write $K = \{i_1, \ldots, i_p\}$ where $i_1 < \cdots < i_p$ and let $x_K = x_{i_1} \wedge \cdots \wedge x_{i_p}$. The elements $x_K$ with $|K| = p$ are an $F$-basis for $\wedge^p F^n$. Make $\wedge^p F^n$ a right $R$-module by defining

$$x_K\sigma = \begin{cases} x_{i_1\sigma} \wedge \cdots \wedge x_{i_p\sigma} & \text{if } K \subseteq I(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

For $p = 0$ we agree that $1\sigma = 1$. Let $\psi_p$ be the character of the $R$-module $\wedge^p F^n$. We will show that $\psi_p = \zeta^{(1^p)}$. Thus, in particular, $\psi_p$ is an irreducible character. The main effort is to show for $0 \leqslant p \leqslant n$ and $0 \leqslant r \leqslant n$ and $\lambda \in \mathcal{P}_r$ that

$$\big((\psi_p)|_{S_r} : \chi^\lambda\big)_{S_r} = \mathsf{B}_{\lambda,(1^p)}. \tag{3.24}$$

In doing this we use (3.12) to compute $\mathsf{B}_{\lambda\mu}$. If $r = 0$ and $p = 0$ then both sides of (3.24) are equal to 1 because $\mathsf{B}_{(0),(0)} = 1$ and $(\psi_0)|_{S_0} = \chi^{(0)}$. If $r = 0$ and $p \geqslant 1$ then both sides of (3.24) are equal to 0, the right side because $(0) \supseteq (1^p)$ is impossible and the left side because $\wedge^p[\varepsilon_\emptyset]$ is the zero matrix. Assume from now on that $r \geqslant 1$. Say that a partition $\lambda$ is a hook if $\lambda = (r-m, 1^m)$ for some $0 \leqslant m \leqslant r-1$. If $\lambda$ is not a hook then $\lambda - (1^p)$ cannot be a horizontal strip so $\mathsf{B}_{\lambda,(1^p)} = 0$. If $\lambda = (r-m, 1^m)$ is a hook and $\lambda - (1^p)$ is a horizontal strip then $p = m$ or $p = m+1$. Thus

$$\sum_{p=0}^{n} \mathsf{B}_{\lambda,(1^p)} t^p = \begin{cases} t^m + t^{m+1} & \text{if } \lambda = (r-m, 1^m) \text{ with } 0 \leqslant m \leqslant r-1, \\ 0 & \text{if } \lambda \text{ is not a hook.} \end{cases} \tag{3.25}$$

If $\lambda \in \mathcal{P}_r$ define a polynomial $F^\lambda(t)$ in an indeterminate $t$ by

$$F^\lambda(t) = \sum_{p=0}^{n} \big((\psi_p)|_{S_r}, \chi^\lambda\big)_{S_r} t^p.$$

We will show that

$$F^\lambda(t) = \begin{cases} t^m + t^{m+1} & \text{if } \lambda = (r-m, 1^m) \text{ with } 0 \leqslant m \leqslant r-1, \\ 0 & \text{if } \lambda \text{ is not a hook.} \end{cases} \tag{3.26}$$

Then (3.24) follows by equating coefficients of $t^p$. Identify $F^r$ with $Fx_1 \oplus \cdots \oplus Fx_r \subseteq F^n$ and identify $\wedge^p F^r$ with a subspace of $\wedge^p F^n$. For $0 \leqslant p \leqslant n$ let $\varphi_{p,r}$ be the character of the $S_r$-module $\wedge^p F^r$. Thus $\varphi_{p,r} = 0$ if $p > r$. If $\gamma \in S_r$ then $I(\gamma) = \mathbf{r}$ so $\gamma$ annihilates $x_K$ if $K$ contains at least one of $r+1, \ldots, n$. Thus the trace of $\gamma$ in its action on $\wedge^p F^n$ is equal to the trace of $\gamma$ in its action on $\wedge^p F^r$ so $(\psi_p)|_{S_r} = \varphi_{p,r}$.[1] If $\gamma \in S_r$ then $\sum_{p=0}^n \varphi_{p,r}(\gamma)t^p = \det(1+\gamma t)$ where, on the right side, 1 is an identity matrix of size $r$ and we view $\gamma$ as a permutation matrix of size $r$. Thus

$$F^\lambda(t) = \sum_{p=0}^{n} \big(\varphi_{p,r} : \chi^\lambda\big)_{S_r} t^p = \frac{1}{r!} \sum_{\gamma \in S_r} \chi^\lambda(\gamma) \det(1+\gamma t).$$

If $\gamma$ has cycle type $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{P}_r$ then $\det(1+\gamma t) = (1-(-t)^{\alpha_1}) \times (1-(-t)^{\alpha_2}) \cdots$. As before let $\Lambda$ be the $\mathbf{Q}$-algebra of symmetric functions in a sequence of indeterminates. Then $\Lambda = \mathbf{Q}[p_1, p_2, \ldots]$ where $p_k$ is the $k$th power sum. Define a $\mathbf{Q}$-algebra homomorphism $\Phi : \Lambda \to \mathbf{Q}[t]$ by $\Phi(p_k) = 1-(-t)^k$. Then $\det(1+\gamma t) = \Phi(p_\alpha)$ where $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$. By (3.9), (3.10), and (3.8)

$$F^\lambda(t) = \Phi\left(\sum_{\alpha \in \mathcal{P}_r} z_\alpha^{-1} \chi_\alpha^\lambda p_\alpha\right) = \Phi(s_\lambda) = \det\big[\Phi(h_{\lambda_i - i + j})\big].$$

Let $A^\lambda = [\Phi(h_{\lambda_i - i + j})]$. Since $kh_k = \sum_{j=1}^k p_j h_{k-j}$, we conclude, by induction on $k$, that $\Phi(h_k) = 1+t$ for $k \geqslant 1$. If $\lambda$ is not a hook then $\lambda_2 \geqslant 1$ so all entries in the first two rows of $A^\lambda$ are equal to $1+t$. Thus $F^\lambda(t) = \det A^\lambda = 0$. If $\lambda = (r-m, 1^m)$ for some $0 \leqslant m \leqslant r-1$ then the matrix $A^\lambda$ has size $m+1$ with $(i,j)$ entry equal to $1+t$ if $j \geqslant i$, equal to 1 if $j = i-1$ and equal to 0 if $j \leqslant i-2$. Subtract the second row of $A^\lambda$ from the first row and use induction on $m$ to conclude that $F^\lambda(t) = \det A^\lambda = t^m + t^{m+1}$. This proves (3.26) and thus proves (3.24). If $\nu \in \mathcal{Q}$ then, by Lemma 3.17(ii) and (3.24)

$$\big(\psi_p : \zeta^\nu\big)_R = \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_r} \mathsf{B}_{\nu\lambda}^{-1} \big((\psi_p)|_{S_r} : \chi^\mu\big)_{S_r} = \sum_{\lambda \in \mathcal{Q}} \sum_{\mu \in \mathcal{Q}} \mathsf{B}_{\nu\lambda}^{-1} \mathsf{B}_{\lambda\mu} \delta_{\mu,(1^p)}$$

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[1] This is a slippery spot. In the representation theory of $S_n$ we usually view $S_r$ as a subgroup of $S_n$ when $r \leqslant n$ so an element of $S_r$ fixes $x_{r+1}, \ldots, x_n$. In our present context the elements of $S_r$ have domain $\mathbf{r}$ so $S_r$ is not a subgroup of $S_n$ for $r < n$. An element of $S_r$ annihilates $x_{r+1}, \ldots, x_n$.

$$= \sum_{\mu \in \mathcal{Q}} \delta_{\nu\mu} \delta_{\mu,(1^p)} = \delta_{\nu,(1^p)}.$$

Thus $\psi_p = \zeta^{(1^p)}$. $\square$

**Remark 3.27.** In view of Lemma 3.17 one would like formulas for $\mathsf{A}^{-1}$ and $\mathsf{B}^{-1}$. L.C. Hsu [5, p. 176] showed that

$$\mathsf{A}^{-1}_{\alpha\beta} = (-1)^{l(\alpha)+l(\beta)} \binom{\alpha}{\beta},$$

where $l(\alpha)$ is the number of parts in $\alpha$. The author observed for small $n$ that the entries of $\mathsf{B}^{-1}$ are 0 or $\pm 1$. I. Gessel and G. Tesler proved, independently, that if $\lambda, \mu \in \mathcal{Q}$ then

$$\mathsf{B}^{-1}_{\lambda\mu} = \begin{cases} (-1)^{|\lambda-\mu|} & \text{if } \lambda \supseteq \mu \text{ and } \lambda - \mu \text{ is a vertical strip,} \\ 0 & \text{otherwise.} \end{cases}$$

## 4. The representation of $R$ on $F[x_1, \ldots, x_n]$

In this section all $R$-modules and $S_r$-modules have action on the right. Let $F^n$ be the space of row vectors over $F$. Let $x_1, \ldots, x_n$ be the standard basis for $F^n$. Make $F^n$ an $R$-module as in (3.23). Let $F[x] = F[x_1, \ldots, x_n]$ be the polynomial algebra in commuting indeterminates $x_1, \ldots, x_n$. For $p = 0, 1, 2, \ldots$ let $F[x]_p$ be the space of homogeneous polynomials of degree $p$. We agree that $F[x]_p = 0$ for $p < 0$. Make $F[x]$ an $R$-module by defining $1\sigma = 1$ and

$$(x_{i_1} \cdots x_{i_p})\sigma = (x_{i_1}\sigma) \cdots (x_{i_p}\sigma) \tag{4.1}$$

for $1 \leqslant i_1, \ldots, i_p \leqslant n$. In this section we determine the graded $R$-module structure of $F[x]$; this is the content of Theorem 4.7 and Corollary 4.9.

If $M$ is an $S_r$-module which affords a representation $\rho$ of $S_r$ let $M^*$ denote an $R$-module which affords the representation $\rho^*$ of $A$ defined in (2.21). Thus $M^*$ is determined by $M$ up to isomorphism. It follows from (2.23) that $\dim M^* = \binom{n}{r} \dim M$. If $\lambda \in \mathcal{P}_r$ choose an $S_r$-module $M^\lambda$ which affords the irreducible representation $\rho^\lambda$ with character $\chi^\lambda$ and an $R$-module $N^\lambda$ which affords the irreducible representation $\rho^{\lambda *}$ with character $\zeta^\lambda$. Let $J^\lambda$ be the isotypic component of $F[x]$ of type $\lambda$. This is by definition the sum of all simple $R$-submodules of $F[x]$ which are isomorphic to $N^\lambda$. For $p = 0, 1, 2, \ldots$ let $J^\lambda_p = J^\lambda \cap F[x]_p$. Then

$$F[x] = \bigoplus_{\lambda \in \mathcal{Q}} J^\lambda \quad \text{and} \quad F[x]_p = \bigoplus_{\lambda \in \mathcal{Q}} J^\lambda_p.$$

**Example 4.2.** (i) Suppose $\lambda = (0)$. Then $\rho^{\lambda *}$ is the trivial representation. We may choose $N^\lambda = F$ with action $1\sigma = 1$ for $\sigma \in R$. Suppose $f \in J^\lambda$ is homogeneous. Then $f\sigma = f$ for $\sigma \in R$. Define $\nu \in R$ by $I(\nu) = \{1, \dots, n-1\}$ and $k\nu = k+1$ for $1 \leqslant k \leqslant n-1$. Then $x_k\nu = x_{k+1}$ for $1 \leqslant k \leqslant n-1$ and $x_n\nu = 0$. If $\deg f > 0$ then $f = f\nu^n = 0$. Thus $J^\lambda = F$; the only $R$-invariant polynomials are constants. The $S_n$-invariant polynomials in $x_1, \dots, x_n$ occur in (iii) below.

(ii) Suppose $\lambda = (1)$. By Example 2.29 we may choose $N^\lambda = Fx_1 + \cdots + Fx_n$ with $R$-action as in (3.23). Let $\theta : N^\lambda \to F[x]_p$ be a nonzero $R$-module homomorphism, where $p \geqslant 1$. Let $\nu \in R$ be as in (i). Write $\theta(x_1) = \sum c_{i_1,\dots,c_{i_n}} x_1^{i_1} \cdots x_n^{i_n}$ where $c_{i_1,\dots,i_n} \in F$ and the sum is over all $(i_1, \dots, i_n)$ with $i_1 + \cdots + i_n = p$. Since $x_1\nu^{n-1} = x_n$ and $x_i\nu^{n-1} = 0$ for $2 \leqslant i \leqslant n$ we have $\theta(x_n) = \theta(x_1\nu^{n-1}) = \theta(x_1)\nu^{n-1} = c_{p,0,\dots,0}x_n^p$. Since $N^\lambda$ is a simple module, $\theta$ is one-to-one so $c_{p,0,\dots,0} \neq 0$. By replacing $\theta$ by a nonzero constant multiple we may assume that $\theta(x_n) = x_n^p$. Now apply powers of $\nu^-$ to get $x_k^p \in \theta(N^\lambda)$ for $1 \leqslant k \leqslant n$. Thus $\theta(N^\lambda) \supseteq Fx_1^p + \cdots + Fx_n^p$ so $\theta(N^\lambda) = Fx_1^p + \cdots + Fx_n^p$ by simplicity of $N^\lambda$. Since this is true for all $\theta$, we have $J_p^\lambda = Fx_1^p + \cdots + Fx_n^p$.

(iii) Suppose $\lambda = (n)$. By Example 2.29 we may choose $N^\lambda = F$ with action $1\sigma = 1$ for $\sigma \in S_n$ and $1\sigma = 0$ for $\sigma \in R - S_n$. Let $\Lambda^n$ be the $F$-algebra of symmetric polynomials in $x_1, \dots, x_n$. Suppose $f \in J^\lambda$ is homogeneous. Then $f\sigma = f$ for $\sigma \in S_n$ and $f\sigma = 0$ for $\sigma \in R - S_n$. In particular, $f \in \Lambda$ and $f$ is not constant. Fix $k \in \mathbf{n}$ and let $K = \{1, \dots, k-1, k+1, \dots, n\}$. Write $f = x_k f' + f''$ where $f''$ does not involve $x_k$. Then $0 = f\varepsilon_K = (x_k\varepsilon_K)(f'\varepsilon_K) + f''\varepsilon_K$. But $x_k\varepsilon_K = 0$ and $f''\varepsilon_K = f''$. Thus $f'' = 0$. Thus $x_k$ divides $f$ for all $k$ so $f \in x_1 \cdots x_n \Lambda^n$. Thus $J^\lambda = x_1 \cdots x_n \Lambda^n$.

We will compute the $J^\lambda$ in terms of corresponding data for the symmetric groups $S_r$ with $0 \leqslant r \leqslant n$. Let $F[x;r] = F[x_1, \dots, x_r]$. We agree that $F[x;0] = F$. Then $F[x;r]$ is an $S_r$-submodule of $F[x]$. The action of $S_r$ defined by (4.1) with $r$ in place of $n$ is the natural action of $S_r$ by automorphisms of $F[x;r]$. Let $F[x;r]_p = F[x;r] \cap F[x]_p$. If $\lambda \in \mathcal{P}_r$ let $I^\lambda$ be the isotypic component of $F[x;r]$ of type $\lambda$, the sum of all simple $S_r$-submodules of $F[x;r]$ which are isomorphic to $M^\lambda$. Let $I_p^\lambda = I^\lambda \cap F[x;r]_p$. Then

$$F[x;r] = \bigoplus_{\lambda \in \mathcal{P}_r} I^\lambda \quad \text{and} \quad F[x;r]_p = \bigoplus_{\lambda \in \mathcal{P}_r} I_p^\lambda.$$

To proceed further we construct for each $S_r$-submodule $M$ of $F[x;r]$ an $R$-submodule $M^\star$ of $F[x]$. Lemma 4.4 states various properties of the correspondence $M \rightsquigarrow M^\star$, among them a module isomorphism $M^\star \simeq M^*$. To construct $M^\star$ recall from Section 2 that we have chosen for each $K \subseteq \mathbf{n}$ with $|K| = r$, an element $\mu_K \in R$ such that $\mu_K$ maps $\mathbf{r}$ to $K$. The element $\mu_K$ is not uniquely determined by $K$, but it is determined by $K$ up to replacement by $\gamma\mu_K$ with $\gamma \in S_r$. Since $M$ is an $S_r$-module, the space $M\mu_K$ is thus uniquely determined by $M$ and $K$. Define $x_K \in F[x]$ by $x_K = \prod_{k \in K} x_k$. Define a subspace $M^\star$ of $F[x]$ by

$$M^\star = \sum_{|K|=r} x_K(M\mu_K). \tag{4.3}$$

For example, suppose that $\lambda = (n)$ and $M = I^{(n)} = \Lambda^n$ is the $F$-algebra of symmetric polynomials in $x_1, \ldots, x_n$. Since $\mu_{\mathbf{n}}$ is the identity of $S_n$, Example 4.2(iii) shows that $M^\star = x_{\mathbf{n}}\Lambda^n = x_1 \cdots x_n \Lambda^n = J^{(n)}$.

If $\dim M$ is finite let $\chi^\star$ be the character of the $R$-module $M^\star$. If $M$ affords a matrix representation $\rho$ of $S_r$ then, by definition, $M^*$ affords the representation $\rho^*$ of $R$. As in Section 2, if $\chi$ is the character of $\rho$ let $\chi^*$ denote the character of $\rho^*$.

**Lemma 4.4.** *Let $M$ be an $S_r$-submodule of $F[x; r]$. Then*

(i) *$M^\star$ is an $R$-submodule of $F[x]$.*
(ii) *If $\dim M$ is finite then $\dim M^\star = \binom{n}{r} \dim M$.*
(iii) *If $M = M_1 + \cdots + M_h$ is a sum of submodules, then $M^\star = M_1^\star + \cdots + M_h^\star$.*
(iv) *If $\dim M$ is finite and $\chi$ is its character then $\chi^\star = \chi^*$ and thus $M^\star \simeq M^*$.*
(v) *If $\lambda \in \mathcal{P}_r$ then $(M^\lambda)^\star \simeq N^\lambda$.*

**Proof.** Let $\sigma \in R$ and $K \subseteq \mathbf{n}$ with $|K| = r$. If $K \subseteq I(\sigma)$ then $\mu_K\sigma : \mathbf{r} \to K\sigma$ is a one-to-one map. Thus there exists $\gamma_{K,\sigma} \in S_r$ such that

$$\mu_K\sigma = \gamma_{K,\sigma}\mu_{K\sigma}. \tag{4.5}$$

Then $(M\mu_K)\sigma = M\gamma_{K,\sigma}\mu_{K\sigma} = M\mu_{K\sigma}$. If $K \subseteq I(\sigma)$ then $x_K\sigma = x_{K\sigma}$. On the other hand, if $K$ is not included in $I(\sigma)$ then $x_K\sigma = 0$. Thus

$$\big(x_K(M\mu_K)\big)\sigma = (x_K\sigma)(M\mu_K\sigma) = \begin{cases} x_{K\sigma}(M\mu_{K\sigma}) & \text{if } K \subseteq I(\sigma), \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

In particular, $(x_K(M\mu_K))\sigma \subseteq M^\star$. This proves (i). To prove (ii) we show first that the sum in (4.3) is direct. Suppose that $\sum_{|K|=r} x_K(m_K\mu_K) = 0$ where $m_K \in M$. It follows from (3.23) that if $K \subseteq \mathbf{n}$ then $x_i\varepsilon_K = x_i$ if $i \in K$ and $x_i\varepsilon_K = 0$ otherwise. Choose $L \subseteq \mathbf{n}$ with $|L| = r$. Then $x_K\varepsilon_L = 0$ if $L \neq K$ and $x_L\varepsilon_L = x_L$. Thus $x_L(m_L\mu_L\varepsilon_L) = 0$ so $m_L\mu_L\varepsilon_L = 0$. But $\mu_L\varepsilon_L = \mu_L$ so $m_L\mu_L = 0$ because $F[x]$ is an integral domain. Since $\mu_L\mu_L^-$ is the identity map of $L$ it follows that $m_L = 0$. Thus the sum is direct. The same argument shows that if $x_K m_K \mu_K = 0$ then $m_K = 0$. Thus $M^\star$ is a direct sum of the $\binom{n}{r}$ subspaces $x_K(M\mu_K)$ of dimension equal to $\dim M$. This proves (ii). Assertion (iii) is clear from the definition of $M^\star$. Suppose that $\dim M$ is finite. Let $\sigma^\star$ be the endomorphism of $M^\star$ which corresponds to $\sigma$. Choose a basis for $M^\star$ adapted to the direct sum decomposition (4.3). The matrix for $\sigma^\star$ is decomposed into blocks of size $\binom{n}{r}$. The diagonal blocks are in one-to-one correspondence with $r$-subsets $K$ of $\mathbf{n}$.

It follows from (4.6) that the $K$th diagonal block is zero unless $K \subseteq I(\sigma)$ and $K\sigma = K$. Assume in the rest of this argument that $K \subseteq I(\sigma)$ and $K\sigma = K$. Then (4.5) gives $\mu_K \sigma = \gamma_{K,\sigma}\mu_K$. Thus the trace of the $K$th diagonal block is $\chi(\gamma_{K,\sigma})$ so

$$\chi^\star(\sigma) = \sum_{\substack{K \subseteq I(\sigma),\, |K|=r \\ K\sigma = K}} \chi(\gamma_{K,\sigma}).$$

Now $\varepsilon_K \sigma = \mu_K^- \mu_K \sigma = \mu_K^- \gamma_{K,\sigma}\mu_{K\sigma}$. Since $I(\varepsilon_K\sigma) = K = J(\varepsilon_K\sigma)$ it follows from the uniqueness in (2.16) and from (2.15) that $\gamma_{K,\sigma} = p(\varepsilon_K\sigma) = \mu_K\sigma\mu_K^-$. Thus

$$\chi^\star(\sigma) = \sum_{\substack{K \subseteq I(\sigma),\, |K|=r \\ K\sigma = K}} \chi(\mu_K\sigma\mu_K^-).$$

Now (iv) follows from (2.31). Finally (v) follows from (iv) since both $(M^\lambda)^\star$ and $N^\lambda$ have the same character $\chi^{\lambda *}$. $\square$

**Theorem 4.7.** *If $\lambda \in \mathcal{Q}$ then $J^\lambda = (I^\lambda)^\star$.*

**Proof.** If $\lambda \in \mathcal{P}_r$ then $(I^\lambda_{p-r})^\star \subseteq F[x]_p$ for all integers $p$ by (4.3); if $p < r$ then $I^\lambda_{p-r} = 0$. Write $I^\lambda_{p-r}$ as a sum of $S_r$-modules isomorphic to $M^\lambda$. By Lemma 4.4(iii) and (v), $(I^\lambda_{p-r})^\star$ is a sum of $R$-modules isomorphic to $N^\lambda$ and is thus included in $J^\lambda$. Thus $J^\lambda_p = J^\lambda \cap F[x]_p \supseteq (I^\lambda_{p-r})^\star$. In particular, $\dim J^\lambda_p \geqslant \dim(I^\lambda_{p-r})^\star$. To complete the proof we show that $\dim J^\lambda_p = \dim(I^\lambda_{p-r})^\star$ whence $J^\lambda_p = (I^\lambda_{p-r})^\star$ and thus $J^\lambda = (I^\lambda)^\star$. Let $t$ be an indeterminate. Since

$$F[x]_p = \bigoplus_{\lambda \in \mathcal{Q}} J^\lambda_p \quad \text{and} \quad \sum_{p \geqslant 0} \dim F[x]_p t^p = \frac{1}{(1-t)^n}$$

we have

$$\sum_{p \geqslant 0} \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_r} \bigl(\dim J^\lambda_p\bigr) t^p = \frac{1}{(1-t)^n}.$$

Similarly $\sum_{p \geqslant 0} \sum_{\lambda \in \mathcal{P}_r} (\dim I^\lambda_p) t^p = 1/(1-t)^r$. If $\lambda \in \mathcal{P}_r$ then $\dim(I^\lambda_{p-r})^\star = \binom{n}{r} \dim I^\lambda_{p-r}$ by Lemma 4.4(ii). Since $I^\lambda_{p-r} = 0$ for $p < r$ we get

$$\begin{aligned}
\sum_{p \geqslant 0} \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_r} \bigl(\dim I^\lambda_{p-r}\bigr)^\star t^p &= \sum_{r=0}^{n} \binom{n}{r} t^r \sum_{p \geqslant 0} \sum_{\lambda \in \mathcal{P}_r} \bigl(\dim I^\lambda_{p-r}\bigr) t^{p-r} \\
&= \sum_{r=0}^{n} \binom{n}{r} t^r \sum_{p \geqslant 0} \sum_{\lambda \in \mathcal{P}_r} \bigl(\dim I^\lambda_p\bigr) t^p
\end{aligned}$$

$$= \sum_{r=0}^{n} \binom{n}{r} t^r \frac{1}{(1-t)^r} = \frac{1}{(1-t)^n}.$$

Now compare coefficients of $t^p$ to get

$$\sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_r} \dim J_p^\lambda = \sum_{r=0}^{n} \sum_{\lambda \in \mathcal{P}_r} \left(\dim I_{p-r}^\lambda\right)^\star.$$

Since $\dim J_p^\lambda \geqslant \dim(I_{p-r}^\lambda)^\star$ for all $p \geqslant 0$, we have $\dim J_p^\lambda = \dim(I_{p-r}^\lambda)^\star$, so $J^\lambda = (I^\lambda)^\star$. $\square$

The following corollary gives the multiplicity of the $R$-module $N^\lambda$ in $F[x]_p$, in terms of analogous data for the symmetric group $S_r$.

**Corollary 4.8.** *Let $\psi_p$ be the character of the $R$-module $F[x]_p$ and let $\varphi_{p,r}$ be the character of the $S_r$-module $F[x;r]_p$. If $\lambda \in \mathcal{P}_r$ then $(\psi_p : \zeta^\lambda)_R = (\varphi_{p-r,r} : \chi^\lambda)_{S_r}$.*

**Proof.** Since $J_p^\lambda$ is isomorphic to a direct sum of $(\psi_p : \zeta^\lambda)_R$ copies of $N^\lambda$ we have $\dim J_p^\lambda = (\psi_p : \zeta^\lambda)_R \dim N^\lambda = (\psi_p : \zeta^\lambda)_R \binom{n}{r} f^\lambda$. Since $I_{p-r}^\lambda$ is isomorphic to a direct sum of $(\varphi_{p-r,r} : \chi^\lambda)_{S_r}$ copies of $M^\lambda$, we have

$$\dim\left(I_{p-r}^\lambda\right)^\star = \binom{n}{r} \dim I_{p-r}^\lambda = \binom{n}{r} \left(\varphi_{p-r,r} : \chi^\lambda\right)_{S_r} f^\lambda.$$

The assertion follows since $J_p^\lambda = (I_{p-r}^\lambda)^\star$. $\square$

The next corollary gives a generating function for the multiplicities $(\psi_p : \zeta^\lambda)_R$.

**Corollary 4.9.** *Suppose $\lambda \in \mathcal{P}_r$. Let $G^\lambda(t) = \sum_{p \geqslant 0} (\psi_p : \zeta^\lambda)_R t^p$. Then*

$$G^\lambda(t) = t^{n(\lambda)+r} \prod_{x \in \lambda} \left(1 - t^{h(x)}\right)^{-1},$$

*where $h(x)$ is the hook length at the node $x$ of the Ferrers diagram and $n(\lambda) = \sum_{i \geqslant 0} (i-1)\lambda_i$.*

**Proof.** By Corollary 4.8 we have $G^\lambda(t) = t^r F^\lambda(t)$ where

$$F^\lambda(t) = \sum_{p \geqslant 0} \left(\varphi_{p,r} : \chi^\lambda\right)_{S_r} t^p.$$

The series $F^\lambda(t)$ are known[2] and may be computed as follows. If $\gamma \in S_r$ then

$$\sum_{p\geqslant 0} \varphi_{p,r}(\gamma)t^p = \det(1-\gamma t)^{-1},$$

where, on the right side, 1 is an identity matrix of size $r$ and we view $\gamma$ as a permutation matrix of size $r$. If $\gamma$ has cycle type $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{P}_r$ then

$$\det(1-\gamma t)^{-1} = \left(1-t^{\alpha_1}\right)^{-1}\left(1-t^{\alpha_2}\right)^{-1}\cdots = p_\alpha\left(1, t, t^2, \ldots\right).$$

Thus by (3.9), (3.10) and [7, p. 45, Example 2]

$$\begin{aligned} F^\lambda(t) &= \sum_{\alpha\in\mathcal{P}_r} z_\alpha^{-1}\chi_\alpha^\lambda p_\alpha\left(1, t, t^2, \ldots\right) = s_\lambda\left(1, t, t^2, \ldots\right) \\ &= t^{n(\lambda)}\prod_{x\in\lambda}\left(1-t^{h(x)}\right)^{-1}. \qquad \square \end{aligned}$$

The next corollary is a statement about Schur functions which does not involve the monoid $R$.

**Corollary 4.10.** *If $\lambda \in \mathcal{P}_r$ then*

$$s_\lambda\left(1, t, t^2, \ldots\right) = \sum t^{|\mu|} s_\mu\left(1, t, t^2, \ldots\right)$$

*where the sum is over all partitions $\mu$ such that $\lambda \supseteq \mu$ and $\lambda - \mu$ is a horizontal strip.*

**Proof.** As in Example 3.22 the restriction of $\psi_p$ to $S_r$ is $\varphi_{p,r}$.[3] It follows from Lemma 3.17(ii) and the definition of $G^\lambda(t)$ and $F^\mu(t)$ that $G^\lambda(t) = \sum_{\mu\in\mathcal{Q}} \mathsf{B}_{\lambda\mu}^{-1} F^\mu(t)$. Thus $F^\lambda(t) = \sum_{\mu\in\mathcal{Q}} \mathsf{B}_{\lambda\mu} G^\mu(t)$. The assertion follows now from Proposition 3.11 since, as in the proof of Corollary 4.9,

$$F^\lambda(t) = s_\lambda\left(1, t, t^2, \ldots\right) \quad \text{and} \quad \mu(t) = t^{|\mu|}F^\mu(t). \qquad \square$$

## 5. The representation of *R* on tensors

Let $V$ be a vector space of finite dimension over $F$. Let $G = \mathbf{GL}(V)$ be the general linear group. Then $V^{\otimes n}$ is a $G$-module with the action

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n \tag{5.1}$$

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[2] See, for example, (2.2.1) in [G. Lusztig, Irreducible representations of finite classical groups, Invent. Math. 43 (1977)], where the formula for $F^\lambda(t)$ is deduced from work of R. Steinberg on characters of $\mathbf{GL}_n(\mathbf{F}_q)$.

[3] In Example 3.22 the $\psi_p$ and $\varphi_{p,r}$ are characters of representations on the exterior algebra rather than the polynomial algebra, but the reasoning for this statement is the same in both cases.

for $v_1, \dots, v_n \in V$ and $g \in G$. The space $V^{\otimes n}$ also has the structure of $S_n$-module in which $S_n$ acts, on the left, by place permutations:

$$w(v_1 \otimes \cdots \otimes v_n) = v_{1w} \otimes \cdots \otimes v_{nw} \tag{5.2}$$

for $v_1, \dots, v_n \in V$ and $w \in S_n$. In his thesis [12] Schur constructed the representations of $\mathbf{GL}(V)$ which are rational and homogeneous of degree $n$ and found their characters in terms of the characters of $S_n$. Schur [13, Hilfsätze V, VI] and Weyl [19, Satz 10] reworked the theory in terms of the actions (5.1) and (5.2) on tensors. It is clear from the definitions that these actions centralize each another. Thus (5.2) defines an algebra homomorphism $\rho : FS_n \to \operatorname{Hom}_G(V^{\otimes n}, V^{\otimes n})$. For $\lambda \in \mathcal{P}_n$, let $V^\lambda$ be the simple $G$-module which corresponds to $\lambda$. Schur and Weyl showed that $V^{\otimes n} \simeq \bigoplus_{\lambda \in \mathcal{P}_n} f^\lambda V^\lambda$, an isomorphism of $G$-modules. It follows that

$$\dim \operatorname{Hom}_G\bigl(V^{\otimes n}, V^{\otimes n}\bigr) = \sum_{\lambda \in \mathcal{P}_n} \bigl(f^\lambda\bigr)^2 = n!,$$

so $\rho : FS_n \to \operatorname{Hom}_G(V^{\otimes n}, V^{\otimes n})$ is an isomorphism of algebras. This isomorphism is often called Schur–Weyl duality.

In Lemma 5.4 we will construct an analogous isomorphism for $FR$. To do this we need the analogue of a place permutation for an element $\sigma \in R$. We cannot define $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{1\sigma} \otimes \cdots \otimes v_{n\sigma}$ as in (5.2) because $\sigma$ need not have domain $\mathbf{n}$. We try to approximate the formula (5.2) as best we can by using the field $F$ as a wastebasket for the undefined $i\sigma$. To this end, let $U = F \oplus V$. View both $F$ and $V$ as subspaces of $U$. Then $U$ is a $G$-module via $g(c + v) = c + gv$ for $g \in G$, $c \in F$, and $v \in V$. We give $U^{\otimes n}$ a $G$-module structure analogous to that in (5.1), namely,

$$g(u_1 \otimes \cdots \otimes u_n) = gu_1 \otimes \cdots \otimes gu_n \tag{5.3}$$

for $u_1, \dots, u_n \in U$ and $g \in G$.

**Lemma 5.4.** *Let $V$ be a vector space over $F$. Let $U = F \oplus V$. Let $u \mapsto u^0$ be the projection of $U$ on $F$ with kernel $V$. Let $u \mapsto u^1$ be the identity map of $U$. For $K \subseteq \mathbf{n}$ and $i \in \mathbf{n}$ define $\delta(i, K) = 1$ if $i \in K$, and $\delta(i, K) = 0$ if $i \notin K$. Thus if $u \in U$ then $u^{\delta(i,K)}$ is defined and is in $U$. If $\sigma \in R$ write $\sigma = \varepsilon_K w$ for some $K \subseteq \mathbf{n}$ and $w \in S_n$, as in* (2.11). *Define*

$$\sigma(u_1 \otimes \cdots \otimes u_n) = u_{1w}^{\delta(1,K)} \otimes \cdots \otimes u_{nw}^{\delta(n,K)}. \tag{5.5}$$

*Then* (5.5) *depends only on $\sigma$ and not on the particular representation $\sigma = \varepsilon_K w$, and gives $U^{\otimes n}$ an $R$-module structure for which the action of $R$ centralizes the action of $G$.*

**Example 5.6.** We precede the proof of Lemma 5.4 with an example to illustrate (5.5). Suppose that $n = 3$ and that $\sigma$ has domain $I(\sigma) = \{1, 2\}$ with $1\sigma = 2$

and $2\sigma = 3$. The corresponding rook matrix $[\sigma]$ is $E_{12} + E_{23}$. Heuristically we want $\sigma(u_1 \otimes u_2 \otimes u_3) = u_2 \otimes u_3 \otimes ?$ where ? lies in the wastebasket $F$. Write $\sigma = \varepsilon_{\{1,2\}} w$ where $w \in S_3$ is the permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$. Then (5.5) says $\sigma(u_1 \otimes u_2 \otimes u_3) = u_2 \otimes u_3 \otimes u_1^0$. Note that if $\sigma = w \in S_n$ then $K = \mathbf{n}$, so $\delta(i, K) = 1$ for all $i \in \mathbf{n}$ and $w(u_1 \otimes \cdots \otimes u_n) = u_{1w} \otimes \cdots \otimes u_{nw}$. Thus the action of $S_n$ on $U^{\otimes n}$ is the usual action of $S_n$ on tensors by place permutations.

**Proof.** We show first that the right-hand side of (5.5) does not depend on the chosen $K$ and $w$. The set $K$ is uniquely determined by $\sigma$ as $K = I(\sigma)$. Thus we must show that if $w, x \in S_n$ and $\varepsilon_K w = \varepsilon_K x$ then

$$u_{1w}^{\delta(1,K)} \otimes \cdots \otimes u_{nw}^{\delta(n,K)} = u_{1x}^{\delta(1,K)} \otimes \cdots \otimes u_{nx}^{\delta(n,K)}.$$

If $i \in K$ then $iw = ix$ so $u_{iw}^{\delta(i,K)} = u_{iw} = u_{ix} = u_{ix}^{\delta(i,K)}$. Write $\mathbf{n} - K = \{j_1, \ldots, j_r\}$. We must show that

$$u_{j_1 w}^0 \otimes \cdots \otimes u_{j_r w}^0 = u_{j_1 x}^0 \otimes \cdots \otimes u_{j_r x}^0.$$

Suppose for simplicity of notation that $\{j_1, \ldots, j_r\} = \mathbf{r}$. We must show that

$$u_{1w}^0 \otimes \cdots \otimes u_{rw}^0 = u_{1x}^0 \otimes \cdots \otimes u_{rx}^0. \tag{5.7}$$

Since $Kw = Kx$ we have $\{1w, \ldots, rw\} = \{1x, \ldots, rx\}$. Choose an $F$-basis $B$ for $V$. We may assume by linearity that the $u_j$ lie in $\{1\} \cup B$ where $1 \in F$ is the unit element. If $1 \leqslant i \leqslant r$ define $1 \leqslant i' \leqslant r$ by $iw = i'x$. If $u_{iw} \in B$ for some $1 \leqslant i \leqslant r$ then $u_{iw}^0 = 0$ and $u_{i'x}^0 = 0$, so both sides of (5.7) are zero. If $u_{iw} = 1$ for all $1 \leqslant i \leqslant r$ then both sides of (5.7) are equal to $1 \otimes \cdots \otimes 1$. Thus the right-hand side of (5.5) does not depend on the choice of $w$.

To show that (5.5) is an $R$-module action we must check that $\sigma(\tau t) = (\sigma\tau)t$ for $\sigma, \tau \in R$ and $t = u_1 \otimes \cdots \otimes u_n$ with $u_j \in U$. Write $\sigma = \varepsilon_K w$ and $\tau = \varepsilon_L x$ where $K, L \subseteq \mathbf{n}$ and $w, x \in S_n$. Then

$$\begin{aligned} \sigma(\tau t) &= \sigma\bigl(u_{1x}^{\delta(1,L)} \otimes \cdots \otimes u_{nx}^{\delta(n,L)}\bigr) \\ &= \bigl(u_{1wx}^{\delta(1w,L)}\bigr)^{\delta(1,K)} \otimes \cdots \otimes \bigl(u_{nwx}^{\delta(nw,L)}\bigr)^{\delta(n,K)}. \end{aligned} \tag{5.8}$$

View 0, 1 in the definition of $u^0$ and $u^1$ as elements of $\mathbf{Z}/2\mathbf{Z}$. Then $(u^a)^b = u^{ab}$ for $a, b \in \mathbf{Z}/2\mathbf{Z}$ and $u \in U$. Also, $\delta(iw, L) = \delta(i, Lw^{-1})$ and $\delta(i, J)\delta(i, K) = \delta(i, J \cap K)$ for all subsets $J, K$ of $\mathbf{n}$. Thus (5.8) may be written as

$$\sigma(\tau t) = u_{1wx}^{\delta(1, Lw^{-1} \cap K)} \otimes \cdots \otimes u_{nwx}^{\delta(n, Lw^{-1} \cap K)}. \tag{5.9}$$

Since $w\varepsilon_L = \varepsilon_{Lw^{-1}} w$, we have $(\varepsilon_K w)(\varepsilon_L x) = \varepsilon_K \varepsilon_{Lw^{-1}} wx = \varepsilon_{Lw^{-1} \cap K} wx$. Thus the right-hand side of (5.9) is $(\sigma\tau)t$. To show that $\sigma g t = g \sigma t$ for $g \in \mathbf{GL}(V)$ we may assume, since $\sigma = \varepsilon_K w$, that $\sigma = \varepsilon_K$. The assertion follows since $g(u^a) = (gu)^a$ for $u \in U$ and $a \in \{0, 1\}$ and thus $g(u^{\delta(i,K)}) = (gu)^{\delta(i,K)}$. $\square$

The following theorem is an analogue, for $R$ and $\mathbf{GL}(V)$, of Schur–Weyl duality for $S_n$ and $\mathbf{GL}(V)$.

**Theorem 5.10.** *Let $V$ be a vector space of finite dimension over a field $F$ of characteristic zero. Let $U = F \oplus V$. Let $G = \mathbf{GL}(V)$ act on $U$ and hence on $U^{\otimes n}$ by fixing $F$. Let $\rho : R \to \mathbf{GL}(U^{\otimes n})$ be the representation of $R$ defined in* (5.5). *If* $\dim V \geqslant n$ *then* $\rho : FR \to \mathrm{Hom}_G(U^{\otimes n}, U^{\otimes n})$ *is an isomorphism of algebras.*

**Proof.** Let $A = FR$. It suffices to show that $\dim \mathrm{Hom}_G(U^{\otimes n}, U^{\otimes n}) = \dim A$ and that $\rho$ is one-to-one. Suppose $0 \leqslant r \leqslant n$. Let $S_r$ be the symmetric group on $\mathbf{r}$. Then $V^{\otimes r}$ is a $G$-module and an $S_r$-module where the action is given by (5.1) and (5.2) with $n$ replaced by $r$. If $r = 0$ we agree that $V^{\otimes 0} = F$ with trivial $G$ action and trivial $S_0$ action. If $r \neq r'$ then [20, Theorem 4.4.F]

$$\mathrm{Hom}_G\bigl(V^{\otimes r}, V^{\otimes r'}\bigr) = 0. \tag{5.11}$$

Expand $V^{\otimes n} = (F \oplus V)^{\otimes n}$ using distributivity of the tensor product. To be precise, let $U_0 = F$ and let $U_1 = V$. For $i = 0, 1$ let $p_i$ be the projection of $U$ on $U_i$ which annihilates $U_j$ for $j \neq i$. If $K \subseteq \mathbf{n}$ define $\pi_K : U^{\otimes n} \to U^{\otimes n}$ by

$$\pi_K = p_{\delta(1,K)} \otimes \cdots \otimes p_{\delta(n,K)}. \tag{5.12}$$

Let $T_K = \pi_K U^{\otimes n}$. For example, if $n = 3$ and $K = \{1, 3\}$ then $\pi_K = p_1 \otimes p_0 \otimes p_1$ and $T_K = V \otimes F \otimes V$. The $\pi_K$ are pairwise orthogonal idempotents with sum equal to the identity map of $U^{\otimes n}$. Thus $U^{\otimes n} = \bigoplus_{K \subseteq \mathbf{n}} T_K$. If $|K| = r$ then $T_K$ is a $G$-submodule of $U^{\otimes n}$ which is isomorphic to $V^{\otimes r}$. Thus there is an isomorphism

$$U^{\otimes n} \simeq \bigoplus_{r=0}^{n} \binom{n}{r} V^{\otimes r} \tag{5.13}$$

of $G$-modules where $\binom{n}{r} V^{\otimes r}$ means a direct sum of $\binom{n}{r}$ copies of $V^{\otimes r}$. It follows from (5.11), (5.13) and Schur–Weyl duality that

$$\begin{aligned}\mathrm{Hom}_G\bigl(U^{\otimes n}, U^{\otimes n}\bigr) &\simeq \mathrm{Hom}_G\left(\bigoplus_{r=0}^{n} \binom{n}{r} V^{\otimes r}, \bigoplus_{r=0}^{n} \binom{n}{r} V^{\otimes r}\right) \\ &\simeq \bigoplus_{r=0}^{n} \mathbf{M}_{\binom{n}{r}}\bigl(\mathrm{Hom}_G\bigl(V^{\otimes r}, V^{\otimes r}\bigr)\bigr) \simeq \bigoplus_{r=0}^{n} \mathbf{M}_{\binom{n}{r}}(FS_r).\end{aligned}$$

Thus, by (1.2),

$$\dim \mathrm{Hom}_G\bigl(U^{\otimes n}, U^{\otimes n}\bigr) = \sum_{r=0}^{n} \binom{n}{r}^2 r! = \dim A.$$

To prove that $\rho$ is one-to-one we first construct an $F$-basis for $A$ in terms of the idempotents $\eta_K$ defined in (2.4). For $K \subseteq \mathbf{n}$ let $S_K = \{w \in S_n \mid iw = i$ for all $i \in K\}$ be the fixer of $K$. Suppose $w \in S_K$. If $J \subseteq K$ then $w \in S_J$ so $\varepsilon_J w = \varepsilon_J$. It follows from (2.4) that $\eta_K w = \eta_K$. Write $S_n = S_K X_K$ where $X_K$ is a set of coset representatives. By (2.4) and (2.5) we have

$$FE = \bigoplus_{K \subseteq \mathbf{n}} F\varepsilon_K = \bigoplus_{K \subseteq \mathbf{n}} F\eta_K.$$

Thus

$$A = FR = FES_n = \bigoplus_{K \subseteq \mathbf{n}} F\eta_K S_n = \bigoplus_{K \subseteq \mathbf{n}} F\eta_K S_K X_K = \bigoplus_{K \subseteq \mathbf{n}} F\eta_K X_K.$$

Thus

$$Q = \{\eta_K w \mid K \subseteq \mathbf{n} \text{ and } w \in X_K\}$$

spans $A$ as an $F$ vector space. If $|K| = r$ then $|S_K| = (n-r)!$, so

$$|Q| \leqslant \sum_{r=0}^{n} \binom{n}{r} \frac{n!}{(n-r)!} = \sum_{r=0}^{n} \binom{n}{r}^2 r! = \dim A.$$

Thus $Q$ is an $F$-basis for $A$. Suppose that $a \in A$ and $\rho(a) = 0$. Write

$$a = \sum_{K \subseteq \mathbf{n}} \sum_{w \in X_K} c_{K,w} \eta_K w$$

for uniquely determined $c_{K,w} \in F$. To complete the proof we show that $c_{K,w} = 0$ for all $K \subseteq \mathbf{n}$ and $w \in X_K$. Fix $K \subseteq \mathbf{n}$. Then

$$0 = \rho(\eta_K)\rho(a) = \sum_{J \subseteq \mathbf{n}} \sum_{w \in X_J} c_{J,w} \rho(\eta_K \eta_J)\rho(w).$$

It follows from Lemma 2.6 that

$$\sum_{w \in X_K} c_{K,w} \rho(\eta_K)\rho(w) = 0. \tag{5.14}$$

Next we compute $\rho(\eta_K)$. If $u_1, \ldots, u_n \in U$ then $\varepsilon_K(u_1 \otimes \cdots \otimes u_n) = u_1^{\delta(1,K)} \otimes \cdots \otimes u_n^{\delta(n,K)}$. If $u \in U$ then $u^{\delta(i,K)} = p_0 u + p_1 u$ if $i \in K$ and $u^{\delta(i,K)} = p_0 u$ if $i \notin K$. Thus

$$\begin{aligned} \varepsilon_K(u_1 \otimes \cdots \otimes u_n) &= \sum_{J \subseteq K} p_{\delta(1,J)} u_1 \otimes \cdots \otimes p_{\delta(n,J)} u_n \\ &= \sum_{J \subseteq K} \pi_J(u_1 \otimes \cdots \otimes u_n) \end{aligned}$$

by (5.5). For example, if $n = 3$ and $K = \{1, 3\}$ then

$$\varepsilon_K(u_1 \otimes u_2 \otimes u_3) = (p_0 u_1 + p_1 u_1) \otimes p_0 u_2 \otimes (p_0 u_3 + p_1 u_3)$$

and expansion of the right-hand side gives a sum indexed by subsets of $\{1, 3\}$. Thus $\rho(\varepsilon_K) = \sum_{J \subseteq K} \pi_J$. It follows from (2.5) by induction on $|K|$ that

$$\rho(\eta_K) = \pi_K. \tag{5.15}$$

Thus, by (5.14) and (5.12),

$$\sum_{w \in X_K} c_{K,w}(p_{\delta(1,K)}u_{1w} \otimes \cdots \otimes p_{\delta(n,K)}u_{nw}) = 0 \tag{5.16}$$

for all $u_1, \ldots, u_n \in U$. Let $m = \dim V$. Let $\{b_1, \ldots, b_m\}$ be a basis for $V$. Choose a subset $L$ of $\mathbf{n}$ with $|L| = |K|$ and hold it fixed until further notice. Let $Y$ be the set of all $w \in X_K$ such that $Kw = L$. Since, by hypothesis, $|L| \leqslant n \leqslant \dim V = m$ we may define $u_1, \ldots, u_n \in U$ by $u_j = b_j$ for $j \in L$, and $u_j = 1$ for $j \in \mathbf{n} - L$. Let $t_{K,w} = p_{\delta(1,K)}u_{1w} \otimes \cdots \otimes p_{\delta(n,K)}u_{nw}$. Suppose $w \in X_K - Y$. Then $iw \in L$ for some $i \in \mathbf{n} - K$. For this $i$ we have $p_{\delta(i,K)}u_{iw} = p_0 b_{iw} = 0$. Thus if $w \in X_K - Y$ then $t_{K,w} = 0$. Thus (5.16) implies $\sum_{w \in Y} c_{K,w} t_{K,w} = 0$. Suppose $w \in Y$. If $i \in K$ then $iw \in L$, so $p_{\delta(i,K)}u_{iw} = p_1 u_{iw} = b_{iw}$. If $i \in \mathbf{n} - K$ then $iw \in \mathbf{n} - L$, so $p_{\delta(i,K)}u_{iw} = p_0 1 = 1$. If $w, w' \in Y$ and $b_{iw} = b_{iw'}$ for all $i \in K$ then $iw = iw'$ for all $i \in K$ so $w'w^{-1} \in X_K$ and thus $w = w'$. Since $\{1, b_1, \ldots, b_m\}$ is a basis for $U$, the tensors $t_{K,w}$ with $w \in Y$ are thus distinct elements of a basis for $U^{\otimes n}$. It follows that $c_{K,w} = 0$ for all $w \in Y$. Now let $L$ range over all subsets of $\mathbf{n}$ for which $|L| = |K|$ to conclude that $c_{K,w} = 0$ for all $w \in X_K$. □

**Remark 5.17.** Let $G = \mathbf{GL}(V)$ where $\dim V \geqslant n$. If $\lambda \in \mathcal{Q}$, let $V^\lambda$ be an irreducible rational $G$-module which corresponds to $\lambda$. By Schur–Weyl duality, $V^{\otimes r} \simeq \bigoplus_{\lambda \in \mathcal{P}_r} f^\lambda V^\lambda$, an isomorphism of $G$-modules. Thus, by (5.13), $U^{\otimes n} \simeq \bigoplus_{r=0}^n \bigoplus_{\lambda \in \mathcal{P}_r} \binom{n}{r} f^\lambda V^\lambda$. Since $\binom{n}{r} f^\lambda = \zeta^\lambda(1)$, there is a $G$-module isomorphism

$$U^{\otimes n} \simeq \bigoplus_{\lambda \in \mathcal{Q}} \zeta^\lambda(1) V^\lambda.$$

**Corollary 5.18.** *If* $\dim V \geqslant n$ *and* $U = F \oplus V$ *then* $\mathrm{Hom}_{FR}(U^{\otimes n}, U^{\otimes n})$ *is the subalgebra of* $\mathrm{Hom}(U^{\otimes n}, U^{\otimes n})$ *generated by all endomorphisms* $u_1 \otimes \cdots \otimes u_n \mapsto gu_1 \otimes \cdots \otimes gu_n$ *with* $u_1, \ldots, u_n \in U$ *and* $g \in \mathbf{GL}(V)$.

**Proof.** Since $FR$ is semisimple this follows from Theorem 5.10 and double centralizer theory. □

## 6. A presentation for $R$

For $1 \leqslant i \leqslant n-1$ let $s_i \in S_n$ be the transposition of $i$ and $i+1$. E.H. Moore [8] found the now familiar presentation

$$\begin{aligned} &\text{(i)} && s_i^2 = 1, \\ &\text{(ii)} && s_i s_j = s_j s_i, \quad \text{if } |i-j| \geqslant 2, \\ &\text{(iii)} && s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \end{aligned} \tag{6.1}$$

for $S_n$. There are several known presentations for $R$, found by L.M. Popova [11], D. Easdown [1], and S. Lipscomb [6, Chapter 9]. These presentations adjoin an idempotent $\varepsilon$ of rank $n-1$ to the generating set $\{s_1, \ldots, s_{n-1}\}$ for $S_n$ and use the relations (6.1) together with further relations which involve $\varepsilon$. In this section we give a presentation for $R$ in terms of $\{s_1, \ldots, s_{n-1}\}$ and the nilpotent element $\nu$ defined by $I(\nu) = \{1, 2, \ldots, n-1\}$ and $i\nu = i+1$ for $1 \leqslant i \leqslant n-1$. The rook matrix $[\nu] \in \mathcal{R}$ corresponding to $\nu$ is the Jordan block $E_{12} + \cdots + E_{n-1,n}$.

The incentive to look for a presentation of $R$ which involves the element $\nu$ rather than an idempotent $\varepsilon$ lies in [15], which concerns an algebra—call it $\mathcal{I}(q)$ here—with a basis $\{T_\sigma \mid \sigma \in R\}$. The algebra $\mathcal{I}(q)$ is a $q$-analogue of the monoid algebra $\mathcal{I}(1) \simeq FR$. It contains the Iwahori Hecke algebra with basis $\{T_w \mid w \in S_n\}$ just as $R$ contains $S_n$. In [16] we will define a representation of $\mathcal{I}(q)$ on tensors which is a $q$-analogue of the representation of $R$ on tensors defined in Lemma 5.4. To define the representation of $\mathcal{I}(q)$ we use a presentation for $\mathcal{I}(q)$ in terms of Iwahori generators $T_{s_1}, \ldots, T_{s_{n-1}}$ and $T_\nu$, which is $q$-analogous to the presentation for $R$ in Theorem 6.2. It seems that there is no simple presentation for $\mathcal{I}(q)$ in terms of $T_{s_1}, \ldots, T_{s_{n-1}}$ and an element $T_\varepsilon$ which corresponds to an idempotent $\varepsilon$ of rank $n-1$.

**Theorem 6.2.** *The monoid $R$ has a presentation with generators $s_1, \ldots, s_{n-1}, \nu$ and defining relations*

$$\begin{array}{rl}
\text{(i)} & s_i^2 = 1, \\
\text{(ii)} & s_i s_j = s_j s_i, \quad \textit{if } |i-j| \geqslant 2, \\
\text{(iii)} & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\
\text{(iv)} & \nu^{i+1} s_i = \nu^{i+1}, \\
\text{(v)} & s_i \nu^{n-i+1} = \nu^{n-i+1}, \\
\text{(vi)} & s_i \nu = \nu s_{i+1}, \\
\text{(vii)} & \nu s_1 s_2 \cdots s_{n-1} \nu = \nu,
\end{array} \tag{6.3}$$

*where $1 \leqslant i \leqslant n-1$ and $1 \leqslant i \leqslant n-2$ in* (iii) *and* (vii).

**Proof.** To start, let $s_1, \ldots, s_{n-1}, \nu$ be as in the first paragraph of this section. We show that $s_1, \ldots, s_{n-1}, \nu$ satisfy (iv)–(vii) and generate $R$. For this it is convenient to use the isomorphism $R \simeq \mathcal{R}$ defined by $\sigma \mapsto [\sigma]$. Left multiplication of $[\sigma]$ by $[s_i]$ permutes rows $i$ and $i+1$. Right multiplication by $[s_i]$ permutes columns $i$ and $i+1$. Thus (iv) holds in $\mathcal{R}$ since the first $i+1$ columns of $\nu^{i+1}$ are zero and (v) holds since the last $i+1$ rows of $\nu^{i+1}$ are zero. Relation (vii) holds in $\mathcal{R}$ since $[\nu][s_1][s_2]\cdots[s_{n-1}] = E_{11} + \cdots + E_{n-1,n-1}$ is idempotent. To check (vi) in $\mathcal{R}$ examine the matrices on both sides of the formula. Thus $s_1, \ldots, s_{n-1}, \nu$ satisfy the relations (6.3).

Let $M$ be the submonoid of $R$ generated by $\{s_1, \ldots, s_{n-1}, \nu\}$. For $K \subseteq n$ let $\varepsilon_K$ be as in (2.3) and let $E = \{\varepsilon_K \mid K \subseteq \mathbf{n}\}$. Thus $[\varepsilon_K]$ is the diagonal idempotent matrix with nonzero entries in the rows indexed by $K$. Let $J = \{1, \ldots, n-1\}$.

Since $[\varepsilon_J] = [\nu][s_1] \cdots [s_{n-1}]$ we have $\varepsilon_J \in M$. Conjugation by a suitable element of $S_n$ shows that $M$ contains all $\varepsilon_K$ with $|K| = n-1$ and hence, by (2.3), $M$ contains all $\varepsilon_K$ with $K \subseteq \mathbf{n}$. It follows from (2.11) that $M = R$. Thus $s_1, \ldots, s_{n-1}, \nu$ generate $R$.

Let $R'$ be the monoid generated by elements $s'_1, \ldots, s'_{n-1}, \nu'$ subject to defining relations (6.3), with $s_i$ replaced by $s'_i$ and $\nu$ replaced by $\nu'$; see [4, p. 10] for the definition of a monoid presentation by generators and relations. We must show that $R' \simeq R$. Since $R$ satisfies (6.3), there is a surjective monoid homomorphism $\vartheta : R' \to R$ such that $\vartheta(s'_i) = s_i$ and $\vartheta(\nu') = \nu$. Let $S'_n = \langle s'_1, \ldots, s'_n \rangle \subseteq R'$. By (6.1) and (6.3), (i)–(iii), there is a group homomorphism $\psi : S_n \to S'_n$ such that $\psi(s_i) = s'_i$. Since $\psi\vartheta$ is the identity map of $S'_n$ the restriction of $\vartheta$ to $S'_n$ is an isomorphism $S'_n \simeq S_n$. Thus $S'_n$ acts on $\mathbf{n}$ and $s'_i$ acts as the transposition $(i, i+1)$. To show that $\vartheta : R' \to R$ is an isomorphism of monoids it suffices to show that $|R'| \leqslant |R|$ where $|R|$ is given by (1.2).

To avoid cluttered notation using “primed” letters, we replace the letters $\nu', s'_i$ by $\nu, s_i$ and write $R, S_n$ in place of $R', S'_n$. There is no danger in this provided we are careful with what we know about the current $R, S_n$. We know that $R$ is a monoid generated by elements $s_1, \ldots, s_{n-1}, \nu$ which satisfy the relations (6.3), that $S_n = \langle s_1, \ldots, s_{n-1} \rangle$ acts on $\mathbf{n}$ and that $s_i$ acts as the transposition $(i, i+1)$. It suffices to show using these properties of $R, S_n$ that

$$|R| \leqslant \sum_{r=0}^{n} \binom{n}{r}^2 r!. \tag{6.4}$$

For $0 \leqslant j \leqslant n-1$ let $w_j = s_1 s_2 \cdots s_j$ where we agree that $w_0 = 1$. Argue by descending induction on $j$ that $\nu^n = \nu w_j \nu^n$ for $0 \leqslant j \leqslant n-1$. For $j = n-1$, (6.3)(vii) gives

$$\nu^n = \nu\nu^{n-1} = \nu w_{n-1} \nu \nu^{n-1} = \nu w_{n-1} \nu^n.$$

Suppose that $\nu^n = \nu w_j \nu^n$ for some $1 \leqslant j \leqslant n-1$. Since $w_j = w_{j-1} s_j$, (6.3)(v) implies

$$\nu = \nu w_{j-1} s_j \nu^{n-j+1} \nu^{j-1} = \nu w_{j-1} \nu^{n-j+1} \nu^{j-1} = \nu w_{j-1} \nu^n.$$

This completes the induction. For $j = 0$ we get $\nu^{n+1} = \nu^n$. Agree to write $\nu^0 = 1 \in R$. We show next for $r \geqslant 0$ and $0 \leqslant j \leqslant n-1$ that

$$\nu w_j \nu^r = \begin{cases} \nu^r & \text{if } r + j \geqslant n, \\ \nu^{r+1} s_{r+1} \cdots s_{r+j} & \text{if } r + j \leqslant n-1. \end{cases} \tag{6.5}$$

This is clear for $r = 0$, so assume that $r \geqslant 1$. Suppose that $r + j \geqslant n$. If $j + 1 \leqslant i \leqslant n-1$ then $r \geqslant n - i + 1$, so $s_i \nu^r = \nu^r$ by (6.3)(v). Thus, since $w_j = w_{n-1} s_{n-1} \cdots s_{j+1}$, we have

$$\nu w_j \nu^r = \nu w_{n-1} s_{n-1} \cdots s_{j+1} \nu^r = \nu w_{n-1} \nu^r = \nu w_{n-1} \nu \nu^{r-1} = \nu \nu^{r-1} = \nu^r$$

by (6.3)(vii). Suppose that $r + j \leqslant n - 1$. If $1 \leqslant i \leqslant j$ then $r + i \leqslant n - 1$, so $s_i \nu^r = \nu^r s_{r+i}$ by $r$ applications of (6.3)(vi). Thus $\nu w_j \nu^r = \nu s_1 \cdots s_j \nu^r = \nu \nu^r s_{r+1} \cdots s_{r+j}$. This proves (6.5).

Recall that $s_i$ acts on $\mathbf{n}$ as the transposition $(i, i+1)$. Since $\langle s_2, \ldots, s_{n-1}\rangle$ is the stabilizer of $1 \in \mathbf{n}$ and $1w_j = j + 1$ for $0 \leqslant j \leqslant n - 1$, we have $S_n = \bigcup_{j=0}^{n-1} \langle s_2, \ldots, s_{n-1}\rangle w_j$. It follows from (6.3)(vi) that $\nu\langle s_2, \ldots, s_{n-1}\rangle \subseteq S_n \nu$. Thus $\nu S_n \subseteq \bigcup_{j=0}^{n-1} S_n \nu w_j$. It follows from (6.5) that

$$\nu w_j \nu^r \subseteq \nu^r S_n \cup \nu^{r+1} S_n \quad \text{for } 0 \leqslant j \leqslant n-1 \text{ and } r \geqslant 0,$$

and thus

$$\nu S_n \nu^r \subseteq \bigcup_{j=0}^{n-1} S_n \nu w_j \nu^r \subseteq S_n \nu^r S_n \cup S_n \nu^{r+1} S_n \quad \text{for } r \geqslant 0.$$

Thus the set $\bigcup_{r \geqslant 0} S_n \nu^r S_n$ is stable under left multiplication by $\nu$. Since the same set is clearly stable under left multiplication by $S_n$ and contains $\nu^0 = 1$, we have $\bigcup_{r \geqslant 0} S_n \nu^r S_n = R$. Since $\nu^{n+1} = \nu^n$ and hence $\nu^r = \nu^n$ for $r \geqslant n$, we conclude that

$$R = \bigcup_{r=0}^{n} S_n \nu^r S_n. \tag{6.6}$$

To get an upper bound on $|S_n \nu^r S_n|$, suppose first that $1 \leqslant r \leqslant n - 1$. Let

$$S_{r,n-r} = \langle s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{n-1}\rangle \simeq S_r \times S_{n-r}.$$

Write $S_n = S_{r,n-r} X_r$ where $X_r$ is a set of coset representatives. If $1 \leqslant i \leqslant r - 1$ then $\nu^r s_i = \nu^r$ by (6.3)(iv). If $r + 1 \leqslant i \leqslant n - 1$ then $\nu^r s_i = s_{i-r} \nu^r$ by $r$ applications of (6.3)(vi). Thus $\nu^r S_n = \nu^r S_{r,n-r} X_r \subseteq S_n \nu^r X_r$, so

$$\left|S_n \nu^r S_n\right| \leqslant \left|S_n \nu^r X_r\right| \leqslant \left|S_n \nu^r\right| |X_r| = \binom{n}{r} \left|S_n \nu^r\right|.$$

If $0 \leqslant r \leqslant n$ and $n - r + 1 \leqslant i \leqslant n - 1$ then $s_i \nu^r = \nu^r$ by (6.3)(v). Since $\langle s_{n-r+1}, \ldots, s_{n-1}\rangle \simeq S_r$, it follows that $|S_n \nu^r| \leqslant |S_n : S_r|$. Thus

$$\left|S_n \nu^r S_n\right| \leqslant \binom{n}{r} \frac{n!}{r!} = \binom{n}{r}^2 (n-r)! \tag{6.7}$$

for $1 \leqslant r \leqslant n - 1$. In fact (6.7) holds for $r = 0$ and $r = n$ as well. This is clear for $r = 0$. It follows from (6.3), (iv) and (v), that $s_i \nu^n = \nu^n = \nu^n s_i$ for $1 \leqslant i \leqslant n - 1$, so $S_n \nu^n S_n = \{\nu^n\}$. Thus (6.7) holds for $r = n$. The desired inequality (6.4) follows from (6.6), (6.7), and the symmetry $\binom{n}{r} = \binom{n}{n-r}$. $\square$

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