A SIMPLE PROOF OF BROWN'S DIAGONALIZABILITY THEOREM

BENJAMIN STEINBERG

We present here a simple proof of Brown's diagonalizability theorem for certain elements of the algebra of a left regular band [1, 2], including probability measures. Brown's theorem also provides a uniform explanation for the diagonalizability of certain elements of Solomon's descent algebra, since the descent algebra embeds in a left regular band algebra [1, 2]. Recall that a left regular band is a semigroup satisfying the identities $x^2 = x$ and $xyx = xy$. In this paper all semigroups are assumed finite.

Let S be a left regular band with identity (there is no loss of generality in assuming this) and let L be the lattice of principal left ideals of S ordered by inclusion[1]. We view L as a monoid via its meet, which is just intersection. There is a natural surjective homomorphism $\sigma\colon S \to L$, called the *support map*, given by $\sigma(s) = Ss$. A key fact that we shall exploit is that $\sigma(s) \leq \sigma(t)$ if and only if $st = s$, that is, $s \in St$ if and only if $st = s$. Indeed, let S act on the right of itself. Because t is an idempotent, it acts as the identity on its image; but this is just St.

Let k be a field and let

$$w = \sum_{t\in S} w_t t \in kS. \tag{1}$$

For $X \in L$, define

$$\lambda_X = \sum_{\sigma(t)\geq X} w_t. \tag{2}$$

Brown [1, 2] showed that $k[w]$ is split semisimple provided that $X > Y$ implies $\lambda_X \neq \lambda_Y$. We give a new proof of this by showing that if $\lambda_1, \ldots, \lambda_k$ are the distinct elements of $\{\lambda_X \mid X \in L\}$, then

$$0 = \prod_{i=1}^{k}(w - \lambda_i). \tag{3}$$

This immediately implies that the minimal polynomial of w has distinct roots and hence $k[w]$ is split semisimple.

Everything is based on the following formula for sw.

Date: October 10, 2009.

The author was supported in part by NSERC.

[1] Brown calls the dual of this lattice the support lattice.

Lemma 1. *Let $s \in S$. Then*

$$sw = \lambda_{\sigma(s)}s + \sum_{\sigma(t)\ngeq\sigma(s)} w_t st$$

and moreover, $\sigma(s) > \sigma(st)$ for all t with $\sigma(t) \ngeq \sigma(s)$.

Proof. Using that $\sigma(t) \geq \sigma(s)$ implies $st = s$, we compute

$$\begin{aligned} sw &= \sum_{\sigma(t)\geq\sigma(s)} w_t st + \sum_{\sigma(t)\ngeq\sigma(s)} w_t st \\ &= \sum_{\sigma(t)\geq\sigma(s)} w_t s + \sum_{\sigma(t)\ngeq\sigma(s)} w_t st \\ &= \lambda_{\sigma(s)}s + \sum_{\sigma(t)\ngeq\sigma(s)} w_t st. \end{aligned}$$

It remains to observe that $\sigma(t) \ngeq \sigma(s)$ implies $\sigma(st) = \sigma(s)\sigma(t) < \sigma(s)$. $\square$

The proof of (3) proceeds via an induction on the support. Let us write $\widehat{0}$ for the bottom of L and $\widehat{1}$ for the top. If $X \in L$, put

$$\Lambda_X = \{\lambda_Y \mid Y \leq X\} \text{ and } \Lambda'_X = \{\lambda_Y \mid Y < X\}.$$

Our hypothesis says exactly that $\Lambda_X = \{\lambda_X\} \mathbin{\dot\cup} \Lambda'_X$ (disjoint union). Define polynomials $p_X(z)$ and $q_X(z)$, for $X \in L$, by

$$\begin{aligned} p_X(z) &= \prod_{\lambda_i\in\Lambda_X} (z-\lambda_i) \\ q_X(z) &= \prod_{\lambda_i\in\Lambda'_X} (z-\lambda_i) = \frac{p_X(z)}{z-\lambda_X}. \end{aligned}$$

Notice that, for $X > Y$, we have $\Lambda_Y \subseteq \Lambda'_X$, and hence $p_Y(z)$ divides $q_X(z)$, because $\lambda_X \notin \Lambda_Y$ by assumption. Also observe that

$$p_{\widehat{1}}(z) = \prod_{i=1}^{k}(z-\lambda_i)$$

and hence establishing (3) is equivalent to proving $p_{\widehat{1}}(w) = 0$.

Lemma 2. *If $s \in S$, then $s \cdot p_{\sigma(s)}(w) = 0$.*

Proof. The proof is by induction on $\sigma(s)$ in the lattice L. Suppose first $\sigma(s) = \widehat{0}$; note that $p_{\widehat{0}}(z) = z - \lambda_{\widehat{0}}$. Then since $\sigma(t) \geq \sigma(s)$ for all $t \in S$, Lemma 1 immediately yields $s(w-\lambda_{\sigma(s)}) = 0$. In general, assume the lemma holds for all $s' \in S$ with $\sigma(s') < \sigma(s)$. Then by Lemma 1

$$s \cdot p_{\sigma(s)}(w) = s \cdot (w - \lambda_{\sigma(s)}) \cdot q_{\sigma(s)}(w) = \sum_{\sigma(t)\ngeq\sigma(s)} w_t st \cdot q_{\sigma(s)}(w) = 0.$$

Here the last equality follows because $\sigma(t) \ngeq \sigma(s)$ implies $\sigma(s) > \sigma(st)$ and so $p_{\sigma(st)}(z)$ divides $q_{\sigma(s)}(z)$, whence induction yields $st \cdot q_{\sigma(s)}(w) = 0$. $\square$

Applying the lemma to the identity element of S yields $p_{\widehat{1}}(w) = 0$ and hence we have proved:

Theorem 3. *Let w be as in* (1) *and let λ_X be as in* (2) *for $X \in L$. If $X > Y$ implies $\lambda_X \neq \lambda_Y$, then $k[w]$ is split semisimple.*

If $k = \mathbb{R}$, and w is a probability measure, then $X > Y$ implies $\lambda_X > \lambda_Y$ provided the support of w generates S as a monoid. If this is not the case, then semisimplicity of $\mathbb{R}[w]$ follows by considering $\mathbb{R}[w] \subseteq \mathbb{R}T \subseteq \mathbb{R}S$ where T is the submonoid generated by the support of w.

References

[1] K. S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[2] K. S. Brown. Semigroup and ring theoretical methods in probability. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 3–26. Amer. Math. Soc., Providence, RI, 2004.

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario K1S 5B6, Canada

E-mail address: bsteinbg@math.carleton.ca