Global dimensions of left-regular bands

Stuart Margolis, Bar-Ilan University
Franco Saliola, Université du Québec à Montréal
Benjamin Steinberg, City College of New York

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Left-regular bands (LRBs)

Definition (LRB)

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Remarks

- *Informally: identities say ignore "repetitions".*
- *We consider only finite monoids here.*

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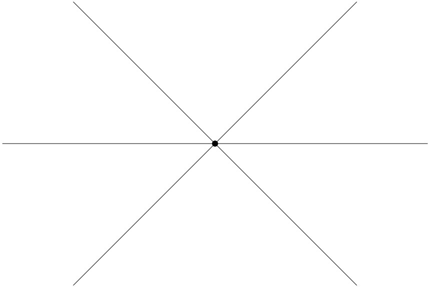
- ordering of the books $\leftrightarrow$ word containing every letter
- move book to the front $\leftrightarrow$ left-multiplication by generator
- long-term behaviour: favourite books move to the front

Faces of a hyperplane arrangement

a set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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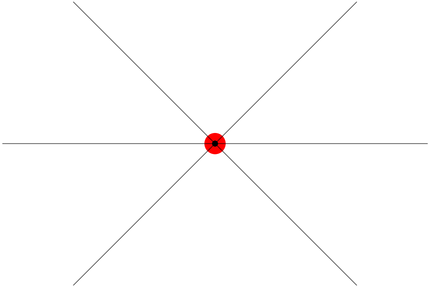

the origin

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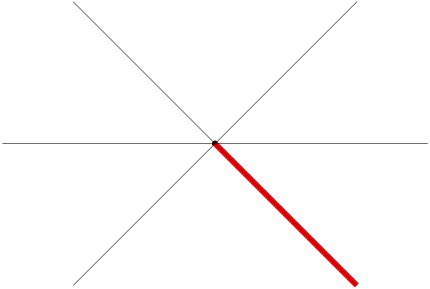

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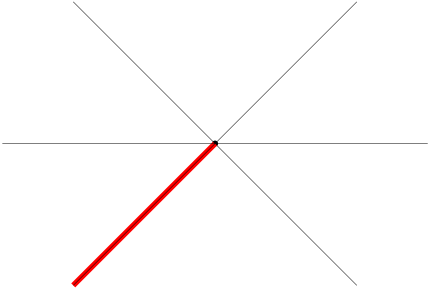

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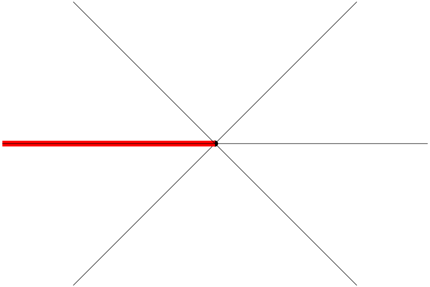

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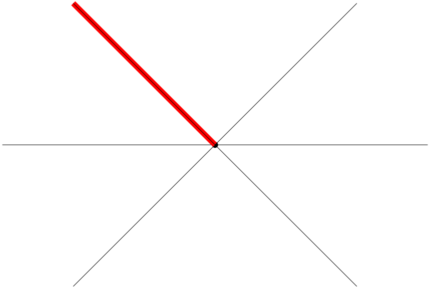

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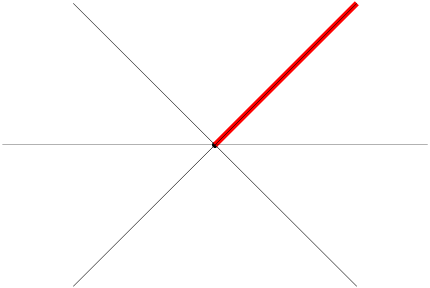

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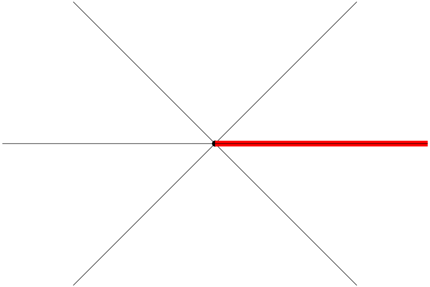

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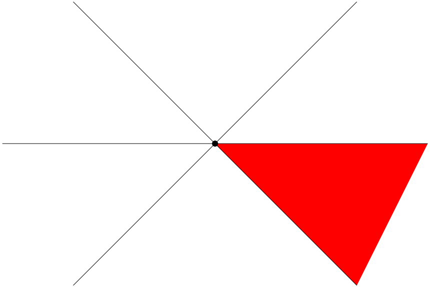

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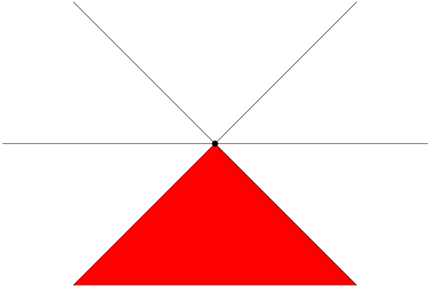

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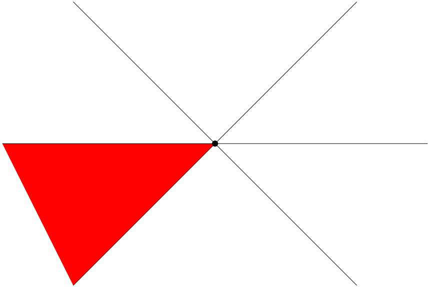

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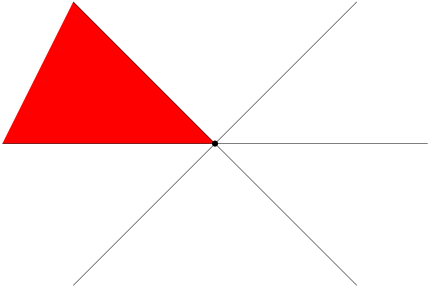

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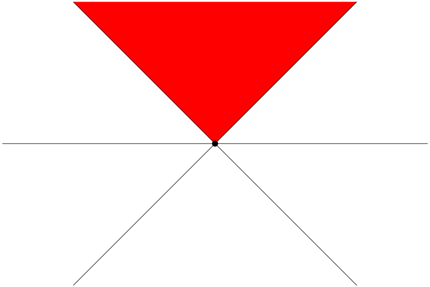

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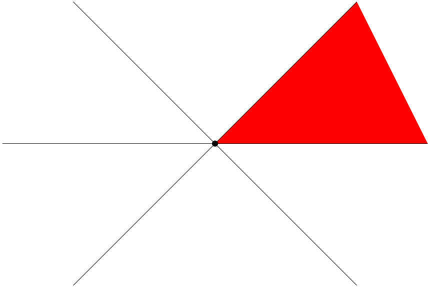

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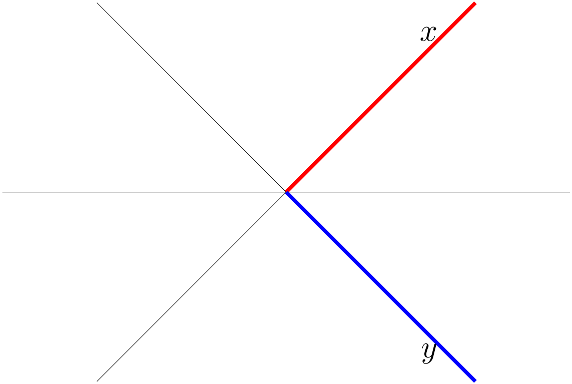
Product of faces

$$xy := \begin{cases} \text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y \end{cases}$$



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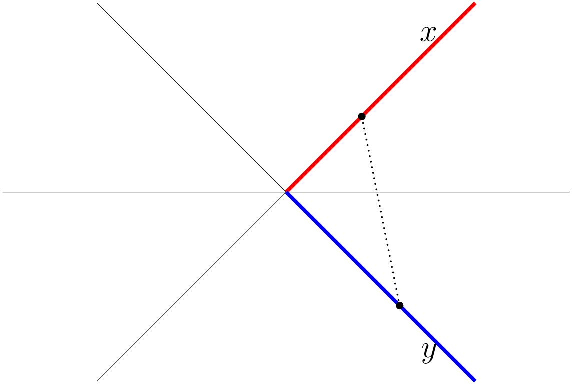

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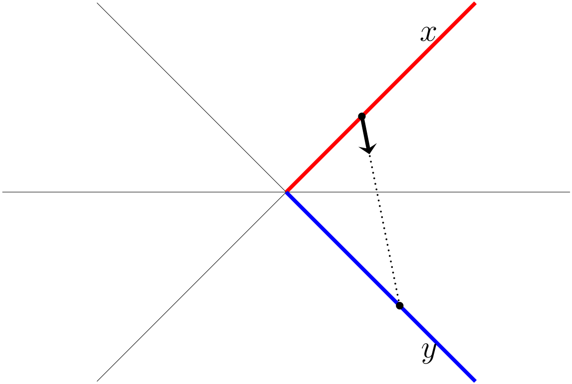

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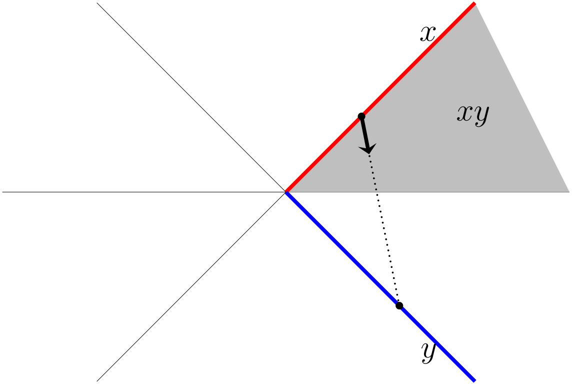

Example: Braid Arrangement

hyperplanes: $H_{i,j} = \{\vec{v} \in \mathbb{R}^n : v_i = v_j\}$

faces: *ordered set partitions* of $\{1, \ldots, n\}$

examples: $\big[\{2,3\},\{4\},\{1,5\}\big]$
$\neq \big[\{4\},\{1,5\},\{2,3\}\big]$

chambers: compositions into singleton blocks

example: $\big[\{2\},\{3\},\{4\},\{1\},\{5\}\big]$

Product of set compositions

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$$\Big[\{2,5\}\{1,3,4,6\}\Big]\cdot\Big[\{4\}\{1\}\{5\}\{6\}\{3\}\{2\}\Big]$$

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$$\left[\underline{\overline{\{2,5\}}}\{1,3,4,6\}\right] \cdot \left[\underline{\overline{\{4\}}}\{1\}\{5\}\{6\}\{3\}\{2\}\right]$$

$$= \Big[\{2,5\} \cap \{4\}$$

Product of set compositions

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

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Product of set compositions

$$\left[\overline{\underline{\color{blue}\{2,5\}}}\{1,3,4,6\}\right]\cdot\left[\{4\}\{1\}\{5\}\overline{\underline{\color{red}\{6\}}}\{3\}\{2\}\right]$$

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starting from an element $\boldsymbol{c}$,
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Others:

Björner, Athanasiadis-Diaconis, Chung-Graham, ...

Free Partitially-Commutative LRB

The *free partially-commutative LRB* $F(G)$ on a graph $G = (V, E)$ is the LRB with presentation:

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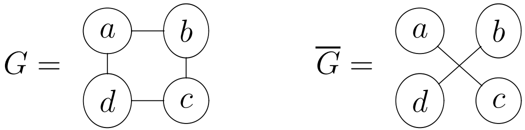
- $F(K_n) =$ free commutative LRB on n generators.
- LRB-version of the Cartier-Foata *free partially-commutative monoid* (aka *trace monoids*).

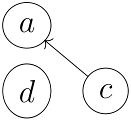
Acyclic orientations

Elements of $F(G)$ correspond to acyclic orientations of induced subgraphs of the complement $\overline{G}$.

Example



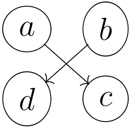
Acyclic orientation on induced subgraph on vertices $\{a, d, c\}$:



In $F(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $F(G)$

States: acyclic orientations of the complement $\overline{G}$



Step: left-multiplication by a generator (vertex) reorients all the edges incident to the vertex away from it

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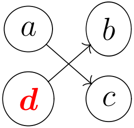

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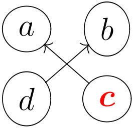

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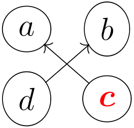

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of G)

LRBs from acyclic quivers

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Theorem (Steinberg)

$B_Q := \{\ell(p) : p$ is a path of $Q\}$ is a LRB and $\mathbb{K}B_Q \cong \mathbb{K}Q$.

Idempotent derivations

Theorem (Lawvere)

If A *is an algebra over a field* $\mathbb{K}$ *with* $\mathrm{char}(\mathbb{K}) \neq 2$*,*

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- Lawvere calls them "graphic monoids"; the identity $xyx = xy$ is called the "Schützenberger-Kimura" identity.
- "graphic topos": a topos which is generated by objects whose endomorphism monoid is a finite LRB.

Simple $\mathbb{K}B$-modules

Let $\Lambda(B)$ denote the lattice of principal left ideals of B, ordered by inclusion:

$$\Lambda(B) = \{Bb : b \in B\} \qquad\qquad Ba \cap Bb = B(ab)$$

Monoid surjection:

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$$\ker(\sigma) \quad = \quad \operatorname{rad}(\mathbb{K}B)$$

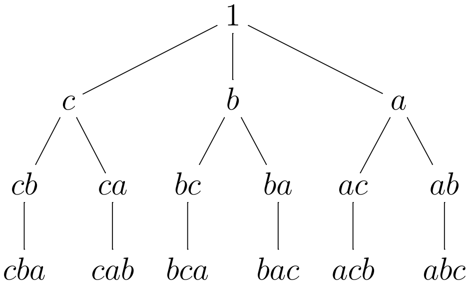
So the simple $\mathbb{K}B$-modules S_X are indexed by $X \in \Lambda(B)$.

Poset of a LRB

B is a partially-ordered set via

$$a \leq b \quad \Leftrightarrow \quad ba = a$$

Example: $F(\{a, b, c\})$

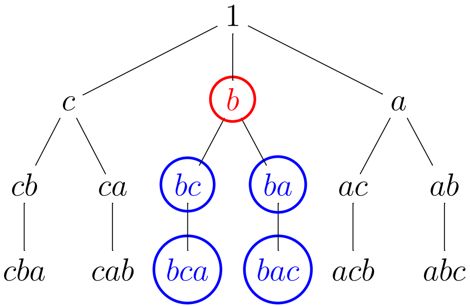


Certain subposets of a LRB

For $Ba \subseteq Bb$, consider the subposet of B:

$$B_{[Ba,Bb)} = \left\{ x \in B : x < b \text{ and } Ba \le Bx \right\}$$

Example: $B(abc) \subseteq Bb$



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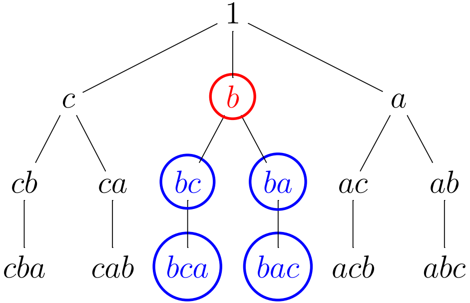

$$B_{[Babc,Bb)} = \{bc, ba, bca, bac\}$$

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where $\Delta B_{[X,Y)}$ is the *order complex* of the subposet $B_{[X,Y)}$.

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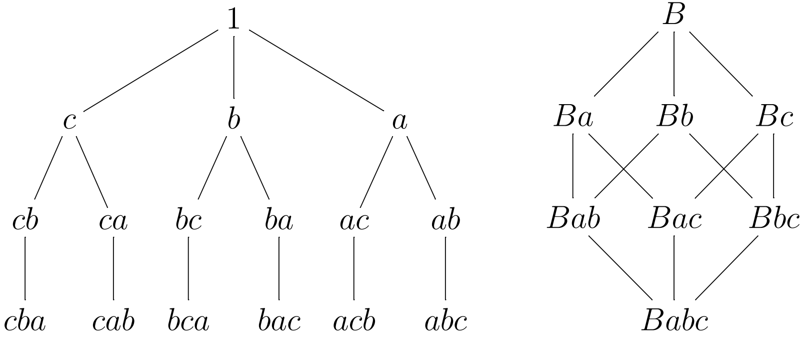
Poset and $\Lambda(B)$ for $B = F(\{a, b, c\})$



Quiver of $\mathbb{K}B$

Corollary. The quiver of $\mathbb{K}B$ has vertex set $\Lambda(B)$. The number of arrows $X \to Y$ is 0 if $X \not< Y$; otherwise, it is one less than the number of connected components of $\Delta B_{[X,Y)}$.

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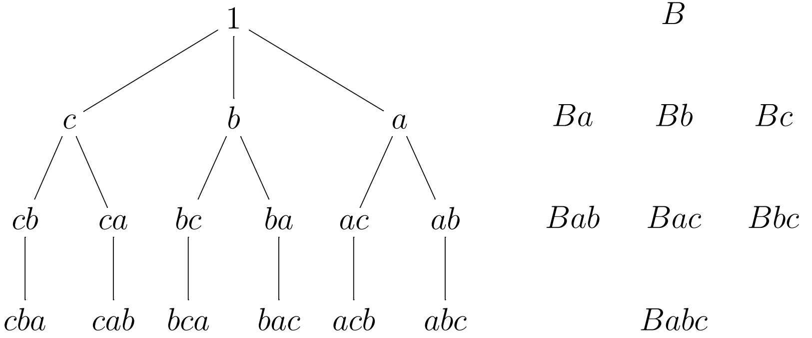
Proof. For $X < Y$:

$$\operatorname{Ext}^1_{\mathbb{K}B}(S_X, S_Y) = \widetilde{H}^0\big(\Delta B_{[X,Y)}, \mathbb{K}\big)$$

Computing the quiver of $B = F(\{a, b, c\})$



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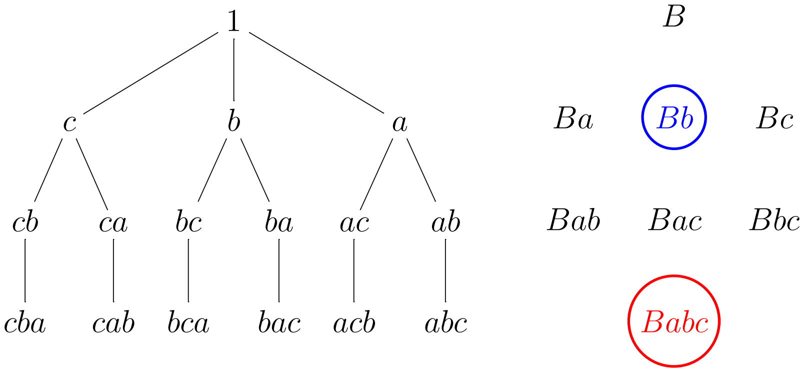

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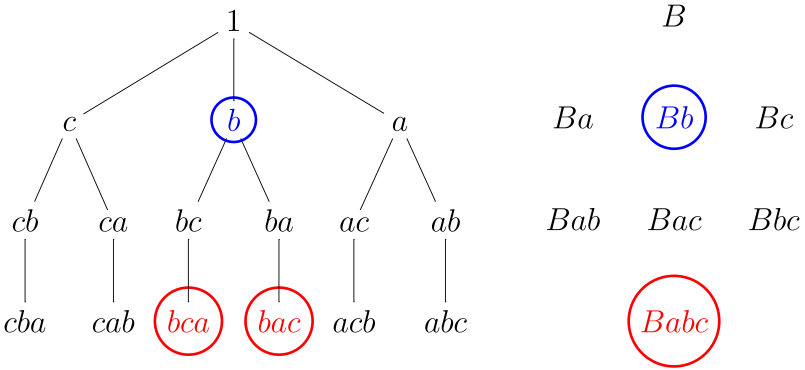

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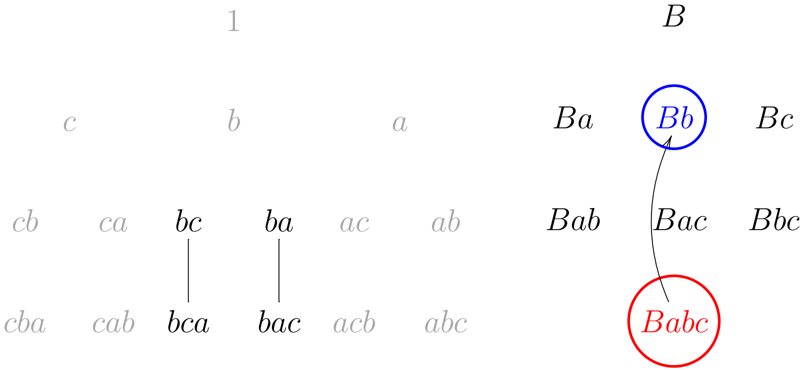

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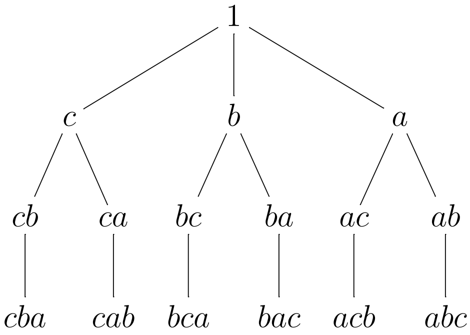

B

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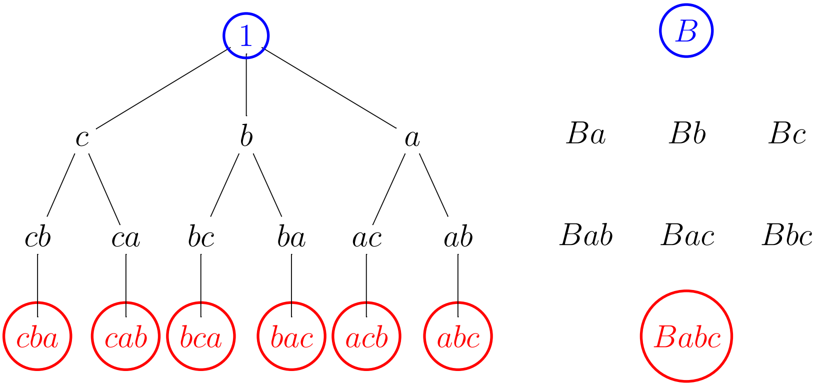

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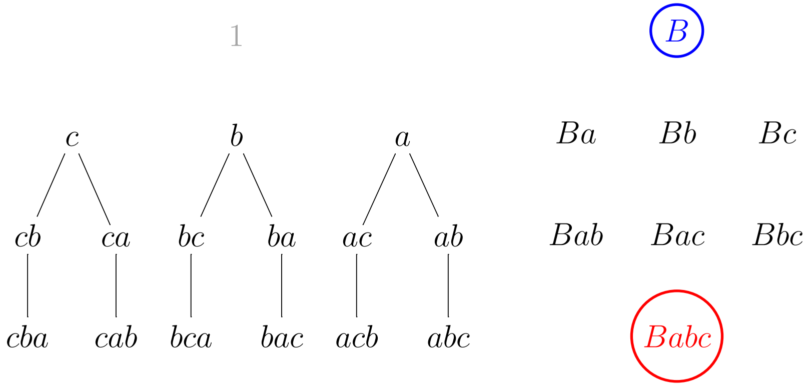

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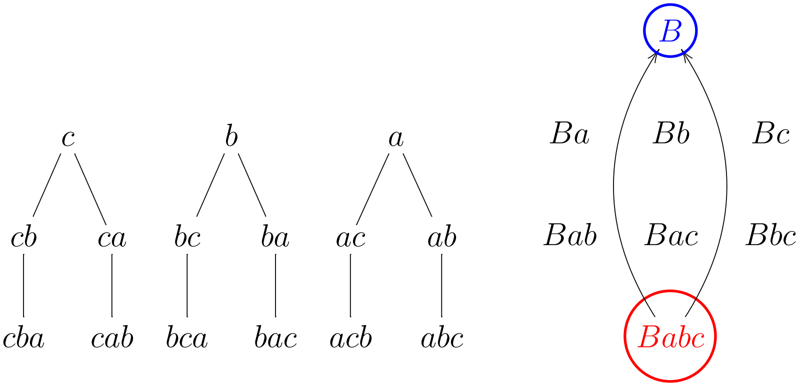

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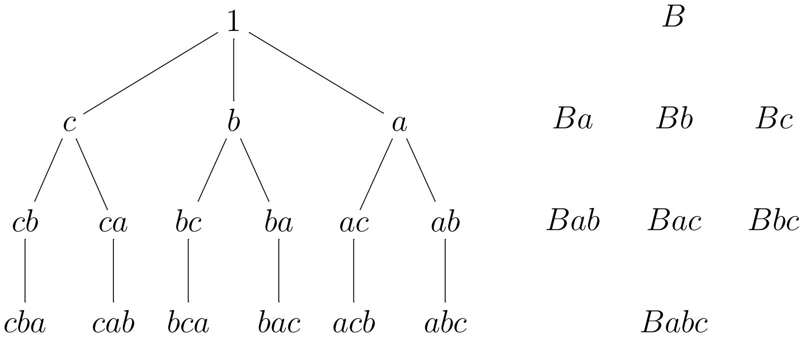

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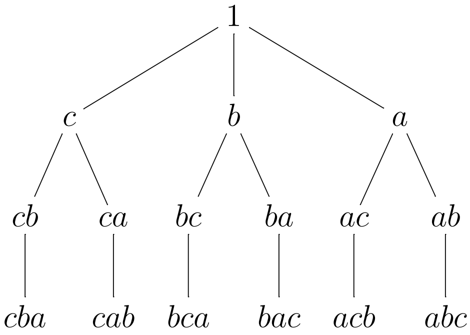

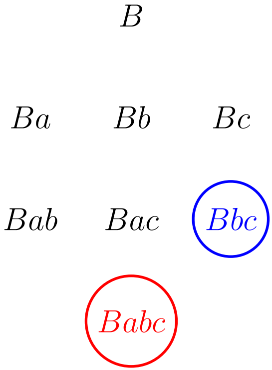

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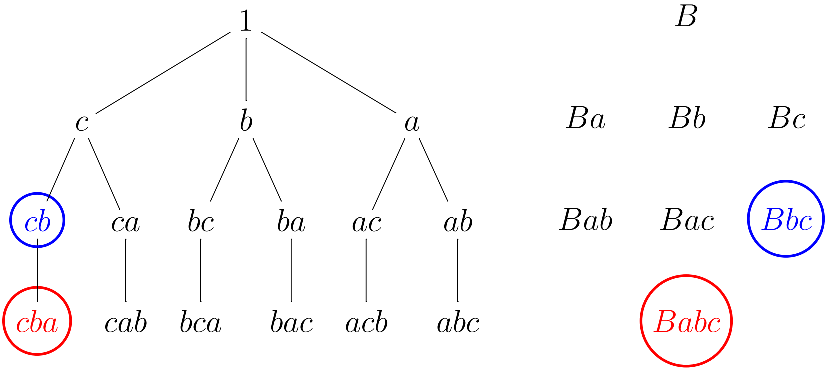

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1

c b a

cb ca bc ba ac ab

cba cab bca bac acb abc

B

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Bab Bac Bbc

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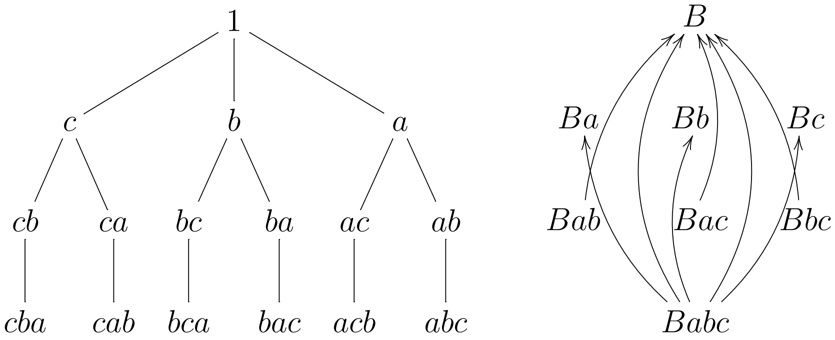

Global dimension and Leray numbers

$$\mathrm{gl.\,dim}\,\mathbb{K}B = \sup\left\{n : \widetilde{H}^{n-1}\left(\Delta B_{[X,Y)}, \mathbb{K}\right) \neq 0 \text{ for all } X < Y\right\}$$

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4. For G triangle-free and *not* a forest: $\mathrm{gl.dim}\,\mathbb{K}F(G) = 2$

Outline of Proof

An Eckmann-Shapiro–type lemma reduces to the case:

$$\begin{aligned}
& \operatorname{Ext}^n_{\mathbb{K}B}(S_{\widehat{0}}, S_{\widehat{1}}) & \\
= \ & H^n(B, S_{\widehat{1}}) & \text{(monoid cohomology)} \\
= \ & H^{n-1}(B, \mathbb{K}^{B_{[\widehat{0},\widehat{1})}}) & \text{(dimension shift)} \\
= \ & H^{n-1}(B \ltimes B_{[\widehat{0},\widehat{1})}, \mathbb{K}) & \text{(Eckmann-Shapiro)} \\
= \ & H^{n-1}(|B \ltimes B_{[\widehat{0},\widehat{1})}|, \mathbb{K}) & \text{(classifying space)} \\
= \ & H^{n-1}(\Delta(B_{[\widehat{0},\widehat{1})}), \mathbb{K}) & \text{(Quillen's Theorem A)}
\end{aligned}$$