Poset cohomology, Leray numbers and the global dimension of left regular bands

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Outline

The nerve construction

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- A subset $\mathscr{F}' \subseteq \mathscr{F}$ is a simplex iff $\bigcap \mathscr{F}' \neq \emptyset$.
- The nerve of an open cover is fundamental to Čech cohomology.

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- The modern way to formulate his result is via Leray numbers.

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- $L(X) = 0$ iff X is a simplex.

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- Let $\Gamma = (V, E)$ be a graph.
- Then $\mathsf{Flag}(\Gamma)$ is the flag complex with vertex set V and simplices the cliques of Γ (vertices which induce a complete subgraph).

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1. *X is 1-representable;*
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3. *X is the flag complex of a chordal graph.*

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- $R(X)$ being Cohen-Macaulay is a topological invariant.

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- To the best of my knowledge people in this area independently discovered the connection of chordal graphs and Leray number 1.

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- We consider only finite monoids in this talk.

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- Diaconis says the LRB techniques are off only by a factor of two for riffle shuffling cards.

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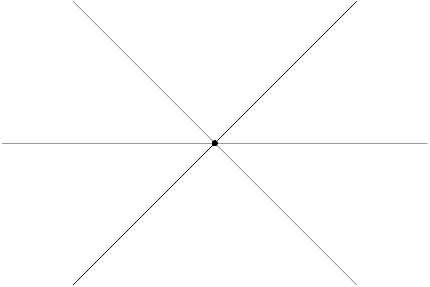
  - long-term behavior: favorite books move to the front

Faces of a hyperplane arrangement

A set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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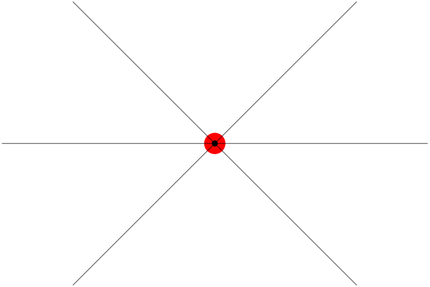

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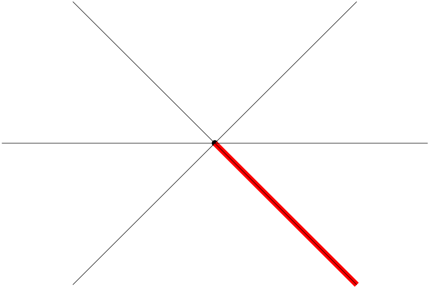

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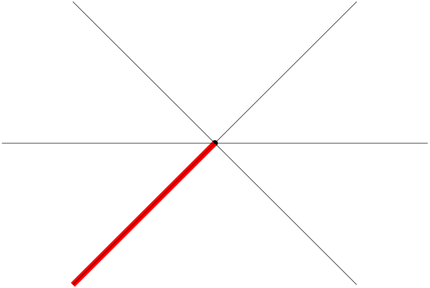

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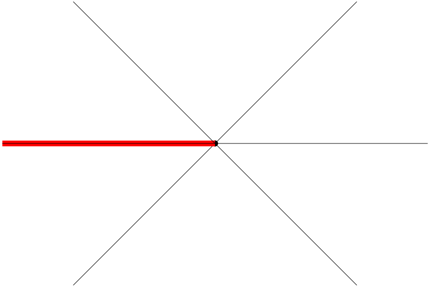

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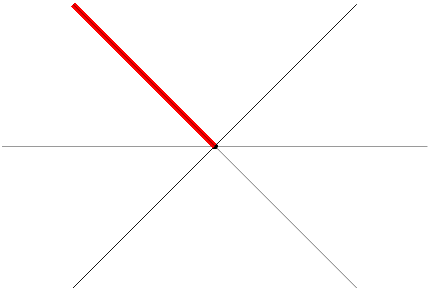

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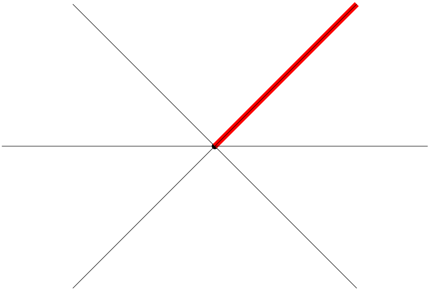

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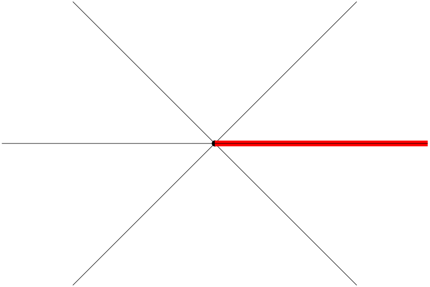

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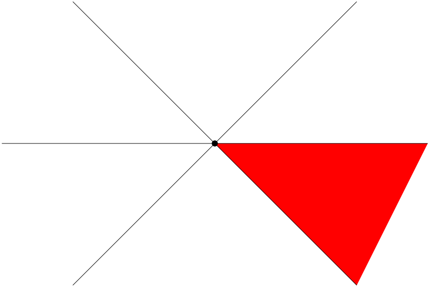

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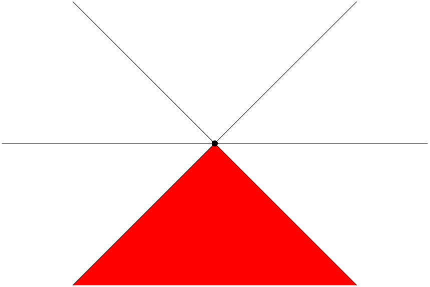

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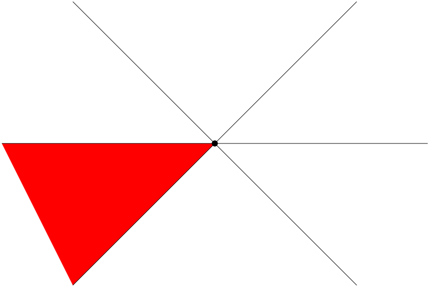

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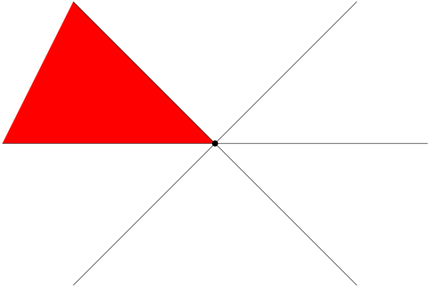

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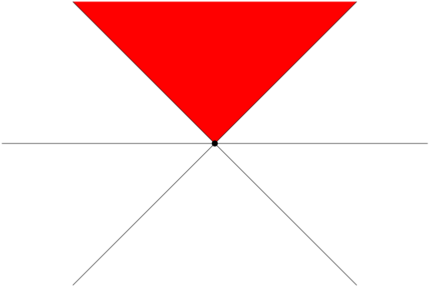

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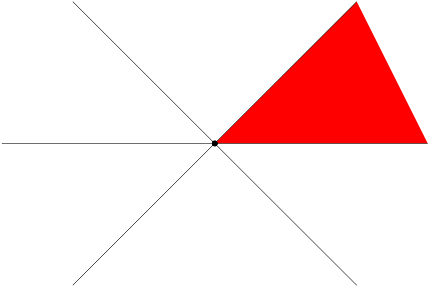

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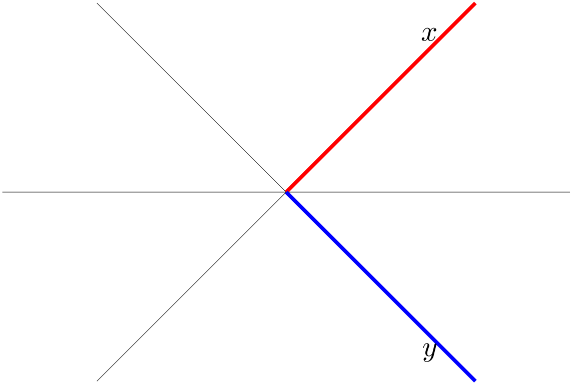

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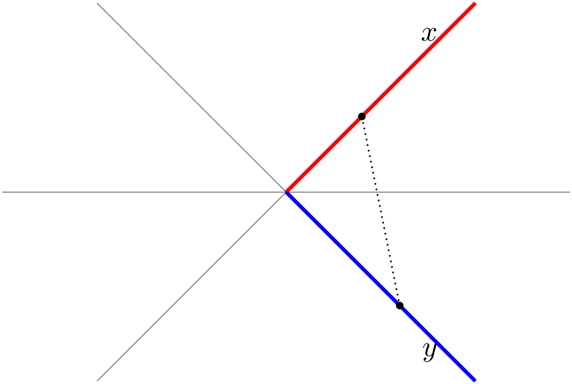

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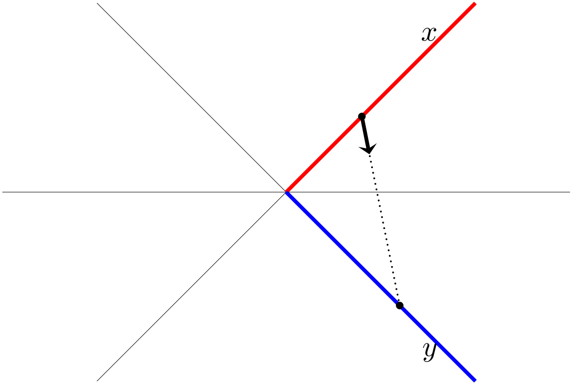

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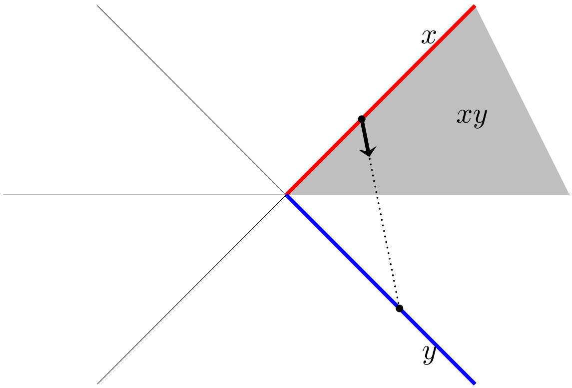

Right-angled Artin LRBs

- The right-angled Artin LRB $B(\Gamma)$ on a graph $\Gamma = (V, E)$ is the LRB with presentation:

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- LRB-version of right-angled Artin groups or trace monoids.

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- Word problem: same as in the right-angled Artin group.

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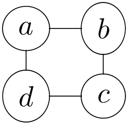
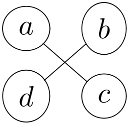
$\Gamma =$ 

$\overline{\Gamma} =$ 

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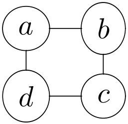

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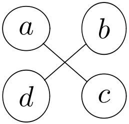

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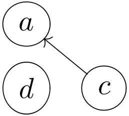

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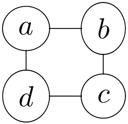
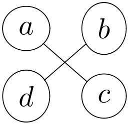
$\Gamma =$ 

$\overline{\Gamma} =$ 

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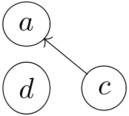

In $B(\Gamma)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $B(\Gamma)$

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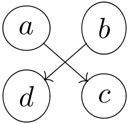

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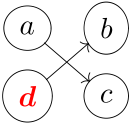

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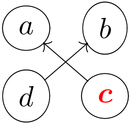

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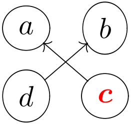

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of Γ)

Global dimension

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- The proof offers no real insight.

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- The main techniques are a Shapiro lemma, classifying spaces of small categories and Quillen's theorem A.
- For right-angled Artin LRBs we also use Rota's cross-cut theorem.

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- Usual proof uses the Kunneth theorem.