# FREE INVERSE MONOIDS AND GRAPH IMMERSIONS

*Dedicated to the memory of A. H. Clifford*


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The relationship between covering spaces of graphs and subgroups of the free group leads to a rapid proof of the Nielsen-Schreier subgroup theorem. We show here that a similar relationship holds between immersions of graphs and closed inverse submonoids of free inverse monoids. We prove using these methods, that a closed inverse submonoid of a free inverse monoid is finitely generated if and only if it has finite index if and only if it is a rational subset of the free inverse monoid in the sense of formal language theory. We solve the word problem for the free inverse category over a graph Γ. We show that immersions over Γ may be classified via conjugacy classes of loop monoids of the free inverse category over Γ. In the case that Γ is a bouquet of $X$ circles, we prove that the category of (connected) immersions over Γ is equivalent to the category of (transitive) representations of the free inverse monoid FIM($X$). Such representations are coded by closed inverse submonoids of FIM($X$). These monoids will be constructed in a natural way from groups acting freely on trees and they admit an idempotent pure retract onto a free inverse monoid. Applications to the classification of finitely generated subgroups of free groups via finite inverse monoids are developed.


## 1. Introduction

The notion of an *immersion*, that is a locally injective graph morphism, has recently been used to prove a number of results about free groups [22], [5]. In this paper we show that inverse monoids play the same role in the theory of immersions that groups play in the theory of coverings. We use this connection to describe the closed inverse submonoids of free inverse monoids. We prove that each such object is uniquely determined by a (free) group acting freely on a tree. Furthermore, we show that a closed inverse submonoid has finite index if and only if it is finitely generated as a closed submonoid. This allows us to lift questions about finitely generated subgroups of a free group to the closed inverse submonoid of the corresponding free inverse monoid that it generates and allows us to canonically associate a finite inverse monoid with any finitely generated subgroup of a free group (in fact with *any* finitely generated


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closed inverse submonoid of a free inverse monoid). This inverse monoid plays the same role that the syntactic monoid plays in classifying recognizable subsets of a free monoid ([4], [10], [16]).

By a *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ we will mean a graph in the sense of [21]. Thus every edge $e: v \to w$, for $v, w \in V(\Gamma)$ comes equipped with an inverse edge $e^{-1}: w \to v$ such that $(e^{-1})^{-1} = e$ and $e^{-1} \neq e$. The initial vertex of an edge $e$ will be denoted by $\alpha(e)$ and the terminal vertex by $\omega(e)$. There is an evident notion of morphism between graphs.

If $v \in V(\Gamma)$ let $\mathrm{star}(\Gamma, v) = \{e \in E(\Gamma) : \alpha(e) = v\}$: a morphism $f$ from a graph $\Gamma$ to a graph $\Gamma'$ induces a map $f_v : \mathrm{star}(\Gamma, v) \to \mathrm{star}(\Gamma', vf)$ for each vertex $v \in V(\Gamma)$. Following Stallings [22], we say that $f$ is a *cover* if each $f_v$ is a bijection and that $f$ is an *immersion* if each $f_v$ is an injection.

It is well-known ([6], [11], [22], [23]) that covers of a connected graph $\Gamma$ may be classified via the fundamental groupoid $\pi_1(\Gamma)$ of $\Gamma$. We briefly review the relevant definitions and terminology.

If $p$ and $q$ are paths in the graph $\Gamma$ we write $p \downarrow q$ if $q$ is obtained from $p$ by removing a pair of consecutive edges of the form $ee^{-1}$: we denote by $\sim$ the equivalence relation on the set $P(\Gamma)$ of paths of $\Gamma$ induced by $\downarrow$. If we view $\sim$ as a congruence on the free category $C(\Gamma)$ over $\Gamma$, then the quotient category $\pi_1(\Gamma) = C(\Gamma)/\sim$ is a groupoid, called the *fundamental groupoid* of $\Gamma$. Denote the $\sim$ equivalence class of the path $p$ by $[p]$ and for each vertex $v \in V(\Gamma)$ let $\pi_1(\Gamma, v) = \{[p] \in \pi_1(\Gamma) : \alpha(p) = \omega(p) = v\}$. Then $\pi_1(\Gamma, v)$ is a group, the *fundamental group* of $\Gamma$ *based at* $v$. The fundamental groups of $\Gamma$ enjoy the following well-known properties.

**Proposition 1.1.** *Let $\Gamma$ be a connected graph. Each group $\pi_1(\Gamma, v)$ is a free group. If $T$ is a spanning tree of $\Gamma$ then the rank of $\pi_1(\Gamma, v)$ is the number of positively oriented edges in $\Gamma - T$. If $v_1$ and $v_2$ are two vertices of $\Gamma$ then $\pi_1(\Gamma, v_1) \cong \pi_1(\Gamma, v_2)$.*

For example, if $B_X$ denotes the bouquet of $|X|$ circles (i.e., the graph with one vertex and $|X|$ positively oriented edges), then $\pi_1(B_X) \cong \mathrm{FG}(X)$, the free group on $X$.

Now let $\Delta$ and $\Gamma$ be connected graphs, let $f: \Delta \to \Gamma$ be a cover of the graph $\Gamma$ and choose $v_1 \in V(\Delta)$ with $v_1 f = v \in V(\Gamma)$. Then $f$ induces an embedding of $\pi_1(\Delta, v_1)$ into $\pi_1(\Gamma, v)$. Conversely, if $H$ is a subgroup of $\pi_1(\Gamma, v)$ then there exists a connected graph $\Delta$, a cover $f: \Delta \to \Gamma$ and a vertex $v_1 \in V(\Delta)$ such that $(\pi_1(\Delta, v_1))f = H$: the graph $\Delta$ is unique (up to graph isomorphism) and the cover $f$ is unique (up to equivalence). Furthermore, if $H$ and $K$ are subgroups of $\pi_1(\Gamma, v) = \mathrm{FG}(X)$ then $H$ and $K$ determine equivalent covers of $\Gamma$ if and only if $H$ is conjugate to $K$ in $\mathrm{FG}(X)$. Thus (connected) covers of a connected graph $\Gamma$ may be classified by conjugacy classes of subgroups of $\pi_1(\Gamma, v)$, for any vertex $v$ of $\Gamma$. In view of Proposition 1.1 we sometimes abuse notation slightly and denote $\pi_1(\Gamma, v)$ by $\pi_1(\Gamma)$ if the vertex $v$ is of no particular concern.

The *universal* cover of a connected graph $\Gamma$ is the cover $f: \tilde{\Gamma} \to \Gamma$ determined by the trivial subgroup of $\pi_1(\Gamma, v)$: this is clearly equivalent to the condition that each fundamental group $\pi_1(\tilde{\Gamma}, v_1)$ is trivial, and hence to the fact that $\tilde{\Gamma}$ is a tree that covers $\Gamma$. For example, if $\Gamma$ is a finite tree then $\tilde{\Gamma} = \Gamma$; if $\Gamma$ is the bouquet of $|X|$ circles, then $\tilde{\Gamma}$ is the Cayley graph of $\mathrm{FG}(X)$ relative to the usual presentation, etc. All of these ideas are classical, and may be found in several standard sources, for example Higgins [6], Lyndon and Schupp [11] or Stillwell [23].

Let $\Gamma$ be an arbitrary graph and let $E_+(\Gamma)$ denote its set of positively oriented edges (see [21]). Let $X$ be a set and $X^{-1}$ a set disjoint from $X$ and in one-one correspondence with $X$ by the map $x \to x^{-1}$ such that:

(1) each edge of $E_+(\Gamma)$ is labelled by an element of $X$: denote the label on the edge $e \in E_+(\Gamma)$ by $l(e)$;

(2) if $e \in E_+(\Gamma)$ and $l(e) = x \in X$, then $e^{-1}$ is labelled by $x^{-1}$ (i.e. $l(e^{-1}) = [l(e)]^{-1}$: thus each edge of $\Gamma$ is labelled by an element of $X \cup X^{-1}$;

(3) if $e, f \in \text{Star}(\Gamma, v)$ with $e \neq f$, then $l(e) \neq l(f)$.

Note that such a labelling is always possible: for example we may choose $X = E_+(\Gamma)$, but usually we would want to make a smaller choice of label set $X$ if possible.

It is easy to see that if $\Gamma$ is labelled over $X \cup X^{-1}$ in accordance with (1)–(3) above and if $\eta$ is the map $\eta : \Gamma \to B_X$ of $\Gamma$ to the bouquet of $|X|$ circles that maps an edge $e \in E(\Gamma)$ onto the loop of $B_X$ labelled by $l(e)$, then $\eta$ is an immersion of $\Gamma$ over $B_X$. Conversely, every immersion $\eta : \Gamma \to B_X$ induces a labelling of the edges of $\Gamma$ in an obvious way. Thus we may view immersions over $B_X$ as graphs whose edges are labelled over $X \cup X^{-1}$ according to (1)–(3). In addition, if $f : \Delta \to \Gamma$ is an immersion over $\Gamma$ then it is possible to label the edges of $\Delta$ and $\Gamma$ over $X \cup X^{-1}$ consistent with immersions of $\Delta$ and $\Gamma$ over $B_X$ such that $f$ is a labelled graph morphism from $\Delta$ to $\Gamma$ (i.e., a morphism that preserves labelling). Conversely, if $\Delta$ and $\Gamma$ are labelled over $X \cup X^{-1}$, such that both labellings are consistent with immersions over $B_X$, then any labelled graph morphism $f : \Delta \to \Gamma$ is an immersion. *We shall consistently assume, in the remainder of the paper, that all graphs are labelled this way and that immersions correspond to labelled graph morphisms, as described above.*

Recall that a semigroup $S$ is an *inverse semigroup* if for every $s \in S$ there is a unique $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. It is well known (see, for example, [3]) that this is equivalent to the condition that there exists such an inverse for each $s \in S$ and that $E(S)$, the set of idempotents of $S$, is a semilattice. That is, idempotents in $S$ commute. Recall that every inverse semigroup $S$ has a multiplicative partial order defined by $s \leq t$ iff $s \in E(S)t$. If $T$ is a subset of $S$, then $T^\omega = \{s | s \geq t \text{ for some } t \in T\}$. We refer the reader to the book of Petrich [15] for the basic definitions and notions concerning inverse semigroups.

We briefly review here the transitive representation theory of inverse monoids by injective maps. If $Q$ is a set, then $I(Q)$ is the set of injective functions on $Q$, that is partial one to one functions on $Q$. $I(Q)$ becomes an inverse monoid under the usual composition of partial functions and inverse of injective functions. The Preston-Wagner Theorem states that every inverse semigroup $S$ has a faithful representation by injective functions on $S$. We note that for $f, g \in I(Q)$, $f \leq g$ if and only if $f$ is the restriction of $g$ to the Domain of $f$.

An inverse monoid $M$ acts (on the right by injective functions) on a set $Q$ if there is a morphism from $M$ to $I(Q)$. If $q \in Q$ and $m \in M$ then we denote by $qm$ the image of $q$ under the action of $m$ if $q \in \text{Domain}(m)$. An action is transitive if for all $p, q \in Q$ there is an $m \in M$ such that $qm = p$. Notice that this implies that $pm^{-1} = q$.

We say that an inverse submonoid $N$ of an inverse monoid $M$ is *closed* if $N = N^\omega$. For example, any subgroup of a group is closed and if an inverse monoid $M$ acts on $Q$, then for every $q \in Q$, $\text{Stab}(q) = \{m | qm = q\}$ is a closed inverse submonoid of $M$.

Conversely given a closed inverse submonoid $N$ of an inverse monoid $M$ we can construct a transitive representation of $M$ as follows. Let $m \in M$ be such that $mm^{-1} \in N$. A subset of $M$ of the form $(Nm)^{\omega} = \{s | s \geq nm$ for some $n \in N\}$ is called a *right $\omega$-coset* of $N$. Notice that $N = (N1)^{\omega}$ is a right $\omega$-coset of itself. Let $X_N$ be the set of right $\omega$-cosets of $N$. If $m \in M$, define an action on $X_N$ by $Y \cdot m = (Ym)^{\omega}$ if $(Ym)^{\omega} \in X_N$ and undefined otherwise. It can be checked that this does define a transitive action of $M$ on $X_N$. Conversely, if $M$ acts transitively on $Q$, then this action is equivalent, in the obvious sense, to the action of $M$ on the right $\omega$-cosets of Stab($q$) in $M$ for any $q \in Q$. For details see [19] or [15]. Note that if $M$ is a group, then the above construction is just the usual coset representation of $M$ on some subgroup $N$. Thus closed inverse submonoids of an inverse monoid play the same role in the transitive representation theory of inverse monoids by injective functions as subgroups do in the representation theory of groups by permutations.

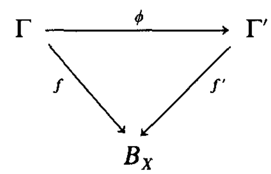
Let $N$ be a closed inverse submonoid of $M$. Define the *index* $[M : N]$ of $N$ in $M$ to be the cardinality of the set $X_N$ of $\omega$-cosets of $N$. Given a subset $Y$ of $M$, let $\langle Y \rangle$ be the closed inverse submonoid of $M$ generated by $Y$. Thus $\langle Y \rangle = \{m | m \geq y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n}$ for some $n \geq 0, y_i \in Y, \varepsilon_i = \pm 1, 1 \leq i \leq n\}$. A closed inverse submonoid is *finitely generated* if $N = \langle Y \rangle$ for some finite set $Y$. One of our main results will show that if $M$ is a free inverse monoid, then a closed inverse submonoid of $M$ is finitely generated if and only if it has finite index.

We shall classify (connected) immersions over a bouquet of cricles. We show that the category of connected immersions over a bouquet of $X$ circles is naturally equivalent to the category of transitive representations of the free inverse monoid on $X$. First recall that inverse monoids can be considered as algebras of type $\langle 2, 1, 0 \rangle$ consisting of multiplication, inversion and the identity. Inverse monoids are then defined by the associative law, the identity law and the following: $((x)^{-1})^{-1} = x$, $(xy)^{-1} = y^{-1}x^{-1}$, $xx^{-1}x = x$ and $(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$. The last law states that idempotents commute. Therefore there is a free inverse monoid FIM($X$) over any set $X$.

There are many beautiful results concerning free inverse monoids. Here we briefly review Munn's solution [14] to the word problem for FIM($X$). We clearly have that FIM$(X) \cong (X \cup X^{-1})^*/\rho$ where $\rho$ is the Wagner congruence on $(X \cup X^{-1})^*$ (i.e. the congruence generated by requiring that the laws defining inverse monoids hold). Let $\Gamma(X)$ be the Cayley graph of the free group on $X$ relative to the standard presentation. Then $\Gamma(X)$ is a tree. For each word $w \in (X \cup X^{-1})^*$ we let MT($w$) be the finite subtree of $\Gamma(X)$ traversed by reading the path in $\Gamma(X)$ labelled by $w$, starting at the vertex 1 and ending at $r(w)$ (the reduced form of $w$). Munn's theorem [14] then states that, for all words $u, v \in (X \cup X^{-1})^*$, $u\rho v$ (i.e. $u = v$ in FIM($X$)) if and only if $(\text{MT}(u), r(u)) = (\text{MT}(v), r(v))$. The tree MT($w$) is referred to as the *Munn tree* of $w$. Let $M(X) = \{(\Gamma', g)$: $\Gamma'$ is a finite connected subtree of $\Gamma(X)$ such that 1 and $g$ are vertices of $\Gamma'\}$. Under the product $(\Gamma_1, g_1)(\Gamma_2, g_2) = (\Gamma_1 \cup g_1\Gamma_2, g_1 g_2)$, $M(X)$ is isomporhic to the free inverse monoid on $X$ (see [15] or [12] for details and extensions of this result). Here $g_1\Gamma_2$ is the left translation of $\Gamma_2$ by $g_1$.

Let $X$ be a set. We define Im($X$) to be the category whose objects are *connected* immersions $f : \Gamma \to B_X$ over the bouquet of $X$ circles. A morphism between $f : \Gamma \to B_X$

and $f' : \Gamma' \to B_X$ is given by an immersion $\phi : \Gamma \to \Gamma'$ such that the diagram:




commutes.

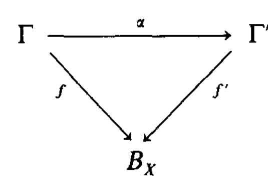
Let $\mathrm{Inj}(X)$ be the category whose objects are *transitive* representations $\eta : \mathrm{FIM}(X) \to I(Q)$ of the free inverse monoid by injective functions on some set $Q$. A morphism between $\eta : \mathrm{FIM}(X) \to I(Q)$ and $\xi : \mathrm{FIM}(X) \to I(P)$ is a function $f : Q \to P$ such that $q(w\eta)f = qf(w\xi)$ for all $q \in Q$ and $w \in \mathrm{FIM}(X)$.

**Theorem 1.2.** *The categories* $\mathrm{Im}(X)$ *and* $\mathrm{Inj}(X)$ *are naturally equivalent.*

**Proof.** Let $f : \Gamma \to B_X$ be a connected immersion where $\Gamma = (V, E)$. For each $x \in X \cup X^{-1}$, define an action on $V$ by:

$$vx = w \text{ if there is an edge } e : v \to w \text{ such that } l(e) = x$$

and $vx$ is undefined otherwise. Since $f$ is an immersion it follows that this action is an injective function on $V$ with inverse given by the action of $x^{-1}$ on $V$. Thus $f$ induces a (unique) representation $\Phi(f) : \mathrm{FIM}(X) \to I(V)$. $\Phi$ is transitive because $\Gamma$ is connected. Let




be a morphism in $\mathrm{Im}(X)$. Let $\Phi(\alpha)$ be the vertex map of $\alpha$. It is clear that the pair $(\Phi(f), \Phi(\alpha))$ defines a functor $\Phi : \mathrm{Im}(X) \to \mathrm{Inj}(X)$.

Conversely, if $\eta : \mathrm{FIM}(X) \to I(Q)$ is an object of $\mathrm{Inj}(X)$, let $\Psi(\eta)$ be the graph $\Psi(\eta) = (Q, E)$ where $E = \{(p, x, q) | px = q, x \in X \cup X^{-1}\}$. If $e = (p, x, q)$, let $e\alpha = p$, $e\omega = q$ and $e^{-1} = (q, x^{-1}, p)$. The transitivity of $\eta$ ensures that $\Psi(\eta)$ is connected. Furthermore, the assignment $(p, x, q) \to x$ yields an immersion $\Psi(\eta) \to B_X$. If $\tau : \mathrm{FIM}(X) \to P$ is an object of $\mathrm{Inj}(X)$ and $f : Q \to P$ is a morphism define $\Psi(f) : \Psi(\eta) \to \Psi(\tau)$ by

$$q\Psi(f) = qf \quad \text{for all } q \in Q.$$

$$(p, x, q)\Psi(f) = (pf, x, qf).$$

It is clear that $\Psi : \mathrm{Inj}(X) \to \mathrm{Im}(X)$ is a functor. Furthermore, $\Phi\Psi$ is the identity functor on $\mathrm{Inj}(X)$ and $\Psi\Phi$ is the functor that takes an immersion $f : \Gamma \to B_X$ and labels each edge $e$ by the letter $l(e) = ef \in X \cup X^{-1}$. Thus $(\Phi, \Psi)$ is an equivalence between $\mathrm{Inj}(X)$ and $\mathrm{Im}(X)$.

**Remark.** If $\eta : \mathrm{FIM}(X) \to I(Q)$ is an object of $\mathrm{Inj}(X)$, then the graph $\Psi(\eta)$ constructed in the proof of Theorem 1.2 may be identified with the graph $\Gamma_N$ of right $\omega$-cosets of $N = \mathrm{stab}(q)$ in $\mathrm{FIM}(X)$, for any $q \in Q$. The set of vertices of $\Gamma_N$ is $X_N$ (the set of right $\omega$-cosets of $N$ in $\mathrm{FIM}(X)$): there is an edge labelled by $x \in X \cup X^{-1}$ in $\Gamma_N$ from $(Nm_1)^\omega$ to $(Nm_2)^\omega$ if $(Nm_2)^\omega = (Nm_1 x)^\omega$.

## 2. Structure of Closed Inverse Submonoids of FIM($X$)

The well-known Nielsen-Schreier theorem asserts that a subgroup of a free group is free. Standard proofs using graph covers or free actions of groups on trees may be found, for example in the books of Lyndon and Schupp [11] or Serre [21] respectively. In view of the fact that closed inverse submonoids play the same role in the theory of representations of inverse monoids by partial one-one transformations as subgroups play in the theory of representations of groups by permutations, one is led to expect that some sort of analogue of the Nielsen-Schreier theorem should hold for closed inverse submonoids of the free inverse monoid. It is immediately obvious that a closed inverse submonoid of $\mathrm{FIM}(X)$ is not necessarily free—for example the semilattice of idempotents is a closed inverse submonoid of $\mathrm{FIM}(X)$ but is not a free inverse monoid. On the other hand, closed inverse submonoids of $\mathrm{FIM}(X)$ share many of the properties of free inverse monoids and are structurally "close" to free, as we indicate in this section.

It is well-known [21] that a group $G$ is free if and only if $G$ acts freely on a tree. Recall that a group $G$ is said to act *freely* on a tree $T$ (on the left) if, for all vertices $v \in V(T)$, $\mathrm{stab}(v) = \{g \in G : g \cdot v = v\}$ is the trivial subgroup $\{1\}$ of $G$. We indicate below how all closed inverse submonoids of a free inverse monoid may be obtained from free actions of groups on trees.

Let $G$ be a group that acts freely on a tree $T$ (so that $G$ is a free group). Fix a root $v_0 \in V(T)$. Let $M(T, G, v_0) = \{(t, g)$: $t$ is a finite subtree of $T$, $g \in G$ and $v_0, g \cdot v_0 \in V(t)\}$ and define a multiplication on $M(T, G, v_0)$ by

$$(t_1, g_1)(t_2, g_2) = (t_1 \cup g_1 \cdot t_2, g_1 g_2).$$

Here $g_1 \cdot t_2$ denotes the translate of the tree $t_2$ by the action of $g_1$ and $t_1 \cup g_1 \cdot t_2$ is the graph whose set of vertices (edges) is the union of those of the graphs $t_1$ and $g_1 \cdot t_2$.

**Example.** Let $G = \mathrm{FG}(X)$ and $T = \Gamma(X)$ (the Cayley graph of $\mathrm{FG}(X)$ relative to the usual presentation); then $T$ is a tree, $G$ acts freely on $T$ by left multiplication and $M(T, G, 1) \approx \mathrm{FIM}(X)$ by Munn's Theorem [14] or [12].

An inverse monoid $M$ is called *E-unitary* if the natural morphism $\sigma : M \to G$ of $M$ onto its maximal group image $G$ is "idempotent-pure": i.e. $1\sigma^{-1} = E(M)$.

**Lemma 2.1.** *$M(T, G, v_0)$ is an E-unitary inverse monoid with maximal group image $G$.*

**Proof.** Let $M = M(T, G, v_0)$. It is clear that $M$ is a monoid with identity $(\{v_0\}, 1)$. Furthermore, $(t, g) \in E(M)$ if and only if $g = 1$. From this it follows that $E(M)$ is isomorphic to the semilattice of finite subtrees of $T$ under union. A direct calculation shows that $(g^{-1}t, g^{-1})$ is a semigroup inverse of $(t, g)$ so that $M$ is a regular monoid with commuting idempotents, that is, an inverse monoid. It follows easily that the map $\sigma : M \to G$ sending $(t, g)$ to $g$ gives the maximal group image and that $M$ is $E$-unitary.

We will show that monoids of the form $M(T, G, v_0)$ are exactly the closed inverse submonoids of free inverse monoids. Let $N$ be a closed inverse submonoid of $\mathrm{FIM}(X)$ and let $\Gamma_N$ be the immersion over $B_X$ corresponding to $N$, i.e. $\Gamma_N$ is the graph of right $\omega$-cosets of $N$ (see remark after Theorem 1.2).

**Lemma 2.2.** *The maximal group image of $N$ is isomorphic to $\pi_1(\Gamma_N)$, the fundamental group of $\Gamma_N$.*

**Proof.** Let $v$ be the vertex (right $\omega$-coset) $N$ of $\Gamma_N$. It is well known that $\pi_1(\Gamma_N)$ can be identified with the group of reduced paths at $v$ under the usual product of reduced words. If $n \in N$, then $n$ labels a path from $v$ to $v$ in $\Gamma_N$. It is clear that the geodesic in $\Gamma(X)$ from 1 to the reduced form of $n$ also labels a path from $v$ to $v$ in $\Gamma_N$, since every subpath of the form $ee^{-1}$ is a loop in $\Gamma_N$. It follows that the map $\sigma : N \to \pi_1(\Gamma_N)$ that sends $n$ to the $\sim$-class of its geodesic induces the minimal group congruence on $N$.

Now let $\tilde{\Gamma}_N$ be the universal cover of $\Gamma_N$. It is easy to see that $\tilde{\Gamma}_N$ is isomorphic to the subtree of $\Gamma(X)$ whose vertices consist of all (reduced) words labelling paths in $\Gamma_N$. We can assume that 1 covers $v$. It is a topological fact that $\pi_1(\Gamma_N)$ acts (by "deck transformations") on $\tilde{\Gamma}_N$ without fixed points and that the quotient graph $\pi_1(\Gamma_N)\backslash\tilde{\Gamma}_N$ is isomorphic to $\Gamma_N$. That is, if $v \in V(\tilde{\Gamma}_N)$ and $g \in \pi_1(\Gamma_N)$, then $gv \in V(\tilde{\Gamma}_N)$ and the assignment $v \mapsto gv$ is a well defined action of $\pi_1(\Gamma_N)$ on $\tilde{\Gamma}_N$ without fixed point. Furthermore the quotient graph, that is the graph of orbits of $V(\tilde{\Gamma}_N)$ and $E(\tilde{\Gamma}_N)$, is isomorphic to $\Gamma_N$.

**Lemma 2.3.** *$N$ is isomorphic to $M(\tilde{\Gamma}_N, \pi_1(\Gamma_N), 1)$.*

**Proof.** Let $M = M(\tilde{\Gamma}_N, \pi_1(\Gamma_N), 1)$. Then $M$ consists of pairs $(t, g)$ where $g \in \pi_1(\Gamma_N)$ and $t$ is a finite birooted subtree of $\tilde{\Gamma}_N$ with roots 1 and $g \cdot 1$. But $g \in \pi_1(\Gamma_N)$ means precisely that 1 and $g \cdot 1$ cover the vertex $v$ of $\Gamma_N$ such that $N$ is the stabilizer of $v$. Thus considering $(t, g)$ as a Munn tree and therefore a member of $\mathrm{FIM}(X)$, we have $v \cdot (t, g) = v$, that is $(t, g) \in N$. Conversely any element of $N$ stabilizes the vertex $v$ and so can be considered to be an element of $M$. Therefore, $N$ is isomorphic to $M$.

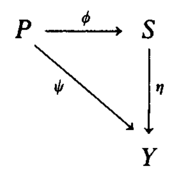
Thus we see that every closed inverse submonoid $N$ of a free inverse monoid can be naturally constructed from the topological invariants $\tilde{\Gamma}_N$ and $\Gamma_N \approx \pi_1(\Gamma_N)\backslash\tilde{\Gamma}_N$. We now show conversely that every monoid constructed this way is isomorphic to a closed inverse submonoid of an appropriate free inverse monoid.

**Lemma 2.4.** *Let $G$ be a group acting freely on a tree $T$ with root $v$. Let $\Gamma = G\backslash T$ and let $X = E_+(\Gamma)$ be an orientation of $\Gamma$. Then $M(T, G, v)$ is isomorphic to a closed inverse submonoid of* $\mathrm{FIM}(X)$.

**Proof.** Since $G$ acts freely on $T$, $G \cong \pi_1(\Gamma)$ and $T$ is the universal covering space of $\Gamma$. Let $f: \Gamma \to B_X$ be any immersion where $X = E_+(\Gamma)$. Thus $f$ induces a labelling of $E(\Gamma)$ as described in Sec. 1. Let $v'$ be the image of $v$ in $\Gamma$. Then the closed inverse monoid $N = \mathrm{Stab}_\Gamma(v') \subseteq \mathrm{FIM}(X)$ consists of all Munn trees that traverse a path in $\Gamma$ from $v'$ to $v'$. Every such path lifts uniquely to a finite subtree of $T$ beginning at $v$ and ending at $g \cdot v$ for a unique $g \in \pi_1(\Gamma) = G$. It is clear that this assignment induces an isomorphism $N \approx M(T, G, v)$.

**Remark.** We can easily give a $P$-representation (in the sense of [15]) of the monoid $M(T, G, v)$. Indeed, let $\mathscr{X}$ be the poset of rooted finite subtrees $t$ of $T$ with root $r(t) = g \cdot v$ for some $g \in G$. $\mathscr{X}$ is partially ordered by defining $t_1 \leq t_2$ iff $V(t_2) \subseteq V(t_1)$ and $r(t_1) = r(t_2)$. The action of $G$ on $T$ extends to an action of $G$ on $\mathscr{X}$ by order automorphisms. Let $\mathscr{Y}$ be the subsemilattice of $\mathscr{X}$ consisting of all $t$ with $r(t) = v$. It is straightforward to verify that $(G, \mathscr{X}, \mathscr{Y})$ is a McAlister triple [15] and that $M(T, G, v) \approx P(G, \mathscr{X}, \mathscr{Y})$.

Here is another interesting description of $M(T, G, v)$ generalizing a result of Schein [20]. Let $P$ be a poset. A semilattice is free on $P$ if there is an order preserving map $\phi: P \to S$ such that for any order preserving map $\psi: P \to Y$ where $Y$ is a semilattice, there exists a unique semilattice morphism $\eta: S \to Y$ such that the diagram




commutes. Obviously $S$ is an initial object in an appropriate category, so $S$ is unique up to isomorphism.

Let $T$ be a tree with root $v$. If $w_1, w_2 \in V(T)$ define $w_1 \leq w_2$ iff $w_1$ is on the geodesic from $v$ to $w_2$. Then $\leq$ is a partial order on $T$. The following is a generalization of a result of Schein who looked at the case $T = \Gamma(X)$. We omit the proof.

**Lemma 2.5.** *The free semilattice on $T$ is isomorphic to the semilattice of finite subtrees of $T$ ordered by reverse inclusion.*

Putting this all together we have the following Theorem.

**Theorem 2.5.** *Let $M$ be an inverse monoid. Then $M$ is isomorphic to a closed inverse submonoid of a free inverse monoid iff $M \cong M(T, G, v)$ for some free action of a (free) group $G$ on a tree $T$. In this case $E(M)$ is the free semilattice on $T$ and $G$ is the maximal group image of $M$.*

It is known that if a group $G$ acts freely on a tree $T$, then $G$ is free on $X = \{g \neq 1 \in G \mid$ there is an oriented edge $y$ with $y\alpha \in T'$ and $y\omega \in gT'\}$ where $T'$ is a lift of a spanning tree in $G \backslash T$ ([21]). That is, $T'$ is a subtree of $T$ that contains one vertex of each orbit of $V(T)$ under the action of $G$. We use this idea to show that every monoid of the form $M(T, G, v)$ has a retraction onto a free inverse submonoid.

We first recall some standard facts. Let $T$ be a graph and let $F = \bigcup_{i \in I} T_i$ be a disjoint union of subtrees $\{T_i \mid i \in I\}$ of $T$. Define an equivalence relation $\sim_F$ on $V(T)$ by $v \sim_F w$ iff $v = w$ or there is an $i \in I$ such that $\{v, w\} \subseteq V(T_i)$. Define a graph $T/F$ with $V(T/F) =$

$T/{\sim_F}$ and $E(T/F) = E(T) - E(F)$. The maps $\alpha, \omega : E(T/F) \to V(T/F)$ and involution $e \to e^{-1}$ are induced by those in $T$ by passing to quotients. We also have a natural morphism of graphs $T \to T/F$.

Intuitively $T/F$ is obtained from $T$ by contracting the forest $F$ to a point. This can be made precise via the notion of geometric realization (see [21]).

Let $T$ be a graph with a root $v \in V(T)$. The map $T \to T/F$ induces a morphism $(\pi_1(T), v) \to (\pi_1(T/F), [v]{\sim_F})$ that is well known to be an isomorphism by [21]. Now let $T$ be a tree acted on freely by a group $G$. Let $T'$ be a lift of a spanning tree in the quotient graph $G\backslash T$.

Let $F = G \cdot T' = \{g \cdot T' | g \in G\}$ be the set of translates of $T'$. It follows from the fact that $T$ is a tree and that $G$ acts freely on $T$ that $F$ is a disjoint union of subtrees of $T$ such that $V(T) = V(F)$. It follows that $V(T/F)$ can be identified with $G$ and that the induced action of $G$ on $T/F$ is free. Furthermore, $E(T/F) = E(T) - E(F) = \{e | e\alpha \in gT', e\omega \in hT', g \neq h\}$. It follows easily that if $X = \{g \neq 1 |$ there is $y \in E(T)$, $y\alpha \in V(T')$ and $y\omega \in V(gT')\}$, then the map sending $(gT') \to g$ and $e \mapsto (g, g^{-1} \cdot h, h)$ if $e \in E(T/F)$ with $e\alpha \in gT'$ and $e\omega \in hT'$, $g \neq h$, is a graph isomorphism $T/F \to \Gamma(X)$. Since $T/F$ is a tree, $G$ is free on $X$.

Moreover, the free action of $G$ on $T$ commutes with the map $T \to T/F$ so that there is an induced morphism $M(T, G, v) \to M(T/F, G, [v]{\sim_F})$. In particular, if $T$ is a tree and $F$ is as above, then there is an induced map $M(T, G, v) \to M(\Gamma(X), \mathrm{FG}(X), 1) \approx \mathrm{FIM}(X)$. It is clear that $X$ is in bijection with $E_+((G\backslash T)/T')$.

**Theorem 2.6.** *Let $G$ act freely on a tree $T$ with root $v$. Let $T'$ be a spanning tree of $\Gamma = G\backslash T$ and let $F = G \cdot T'$. Then there is an idempotent pure retraction $f : M(T, G, v) \to \mathrm{FIM}(X)$ where $X = E_+((G\backslash T)/T')$.*

**Proof.** We need only prove that the induced map $f : M(T, G, v) \to M(T/F, G, vf) \approx \mathrm{FIM}(X)$ is idempotent pure. Let $\sigma_M : M(T, G, v) \to G$ and $\sigma_{M'} : M(T/F, G, vf) \to G$ be the maximal group morphisms.

It is clear that the following diagram commutes:

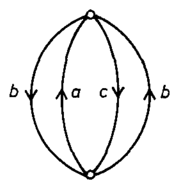
$$\begin{array}{ccc} M(T, G, v) & \xrightarrow{\ f\ } & M(T/F, G, vf) \approx \mathrm{FIM}(X) \\ \Big\downarrow{\scriptstyle \sigma_M} & & \Big\downarrow{\scriptstyle \sigma_{M'}} \\ G \approx \pi_1(T, v) & \approx & \pi_1(T/F, vf) \approx G \end{array}$$

Since $M(T, G, v)$ and $\mathrm{FIM}(X)$ are $E$-unitary, both $\sigma_M$ and $\sigma_{M'}$ are idempotent pure and it follows that $f$ is idempotent pure.

**Remark.** The above gives a sufficient condition for a set $Y \subseteq \mathrm{FIM}(X)$ to generate a free inverse monoid. Namely, let $\Gamma_Y$ be the immersion corresponding to the closed inverse monoid $\langle Y \rangle$. Suppose there is a spanning tree $T'$ of $\Gamma_Y$ such that $Y$ is the set of words constructed in the usual way as a basis for the group generated by $Y$ considered as a subset of $\mathrm{FG}(X)$. Then $Y$ generates a free inverse monoid. That is, for each edge $e$ in $E_+(\Gamma_Y) - E_+(T')$, let $y_e = \alpha_v e \alpha_w^{-1}$ where $\alpha_v(\alpha_w)$ is the geodesic from the root of $\Gamma_Y$ to the initial (terminal) vertex $v(w)$ of $e$.

If $Y' = \{y_e | e \in E_+(\Gamma_Y) - E_+(T')\}$, then the inverse monoid generated by $Y'$ is free on $Y'$. For in this case, the retraction from $\langle Y \rangle$ constructed in Theorem 2.6 maps $Y$ bijectively onto $Y'$. It follows that the inverse monoid $N$ generated by $Y$ maps onto the inverse monoid generated by $Y'$ which is isomorphic to $\mathrm{FIM}(Y')$ by Theorem 2.6. Since $N$ is generated by $Y$, it follows that $N \cong \mathrm{FIM}(Y)$ as well.

One may easily construct examples to show that this condition is not necessary, however. For example, if $Y = \{ab, ac, bc\} \subseteq \mathrm{FIM}(a, b, c)$, then $Y$ generates a free inverse monoid. (This can be proved by checking the criterion given by Reilly [17] for deciding when a subset of an inverse monoid generates a free inverse submonoid, but we shall omit these details). However the graph $\Gamma_Y$ is the graph shown below, and no choice of spanning tree of this graph yields $Y$ as the associated free basis.




$\Gamma_Y$

## 3. Finiteness Conditions

We will need some basic notions from the theory of automata. We refer the reader to [7], [4], or [16] for details. We define an automaton $A$ over $X \cup X^{-1}$ to be *inverse* if every $x \in X \cup X^{-1}$ induces an injective function on the state set of $A$ such that $x^{-1}$ induces the inverse function. It follows that the transition monoid $M(A)$ of an inverse automaton is an inverse monoid. Thus, the natural morphism from $(X \cup X^{-1})^*$ to $M(A)$ factors through the morphism from $(X \cup X^{-1})^*$ to $\mathrm{FIM}(X)$. It is clear from Theorem 1.2 that every immersion is essentially an inverse automaton over $X \cup X^{-1}$.

The following is a special case of a result in [18].

**Lemma 3.1.** *Let $A$ be a connected inverse automaton with one initial state and one terminal state. Then $A$ is a minimal automaton.*

Let $N$ be a closed inverse submonoid of $\mathrm{FIM}(X)$, the free inverse monoid on $X$. Let $X_N$ be the set of right $\omega$-cosets of $N$. By the remarks in the introductory section, this determines a transitive representation of $\mathrm{FIM}(X)$ on $X_N$. Let $\Gamma_N \to B_X$ be the connected immersion corresponding to this representation as given in Theorem 1.2. Let $N$ be chosen as the unique initial and terminal state. We obtain a minimal inverse automaton $A_N$ by Lemma 3.1. It is clear that $A_N$ accepts the language $\{w \in (X \cup X^{-1})^* | w\rho \in N)$. Here $\rho : (X \cup X^{-1})^* \to \mathrm{FIM}(X)$ is the natural map. $A_N$ is the minimal automaton of the language of words that represent elements in $N$. We record this in the following lemma.

**Lemma 3.2.** *Let $N$ be a closed inverse submonoid of* $\mathrm{FIM}(X)$. *Then the automaton $A_N = (X_N, N, \{N\})$ is an inverse automaton and is the minimal automaton of $N\rho^{-1}$.*

Since $A_N$ is an inverse automaton, we can also consider $A_N$ to be the minimal automaton of $N$ considered as a subset of FIM($X$). We define the *syntactic monoid* of $N$, Synt($N$) to be the transition monoid of $A_N$. Of course Synt($N$) is an inverse monoid. The main theorem of this secton shows that $A_N$ is a finite state automaton if and only if $N$ is a finitely generated closed inverse submonoid of FIM($X$). We will also show that these conditions are equivalent to the condition that $N$ be a rational subset of FIM($X$).

Recall that the set of rational subsets of a monoid $M$ is the smallest collection of subsets of $M$ containing the singletons and closed under finite union, product of subsets and submonoid generation. This last operation is usually called "star". A subset $S$ of $M$ is recognizable, if there is a finite monoid $N$ and a morphism $f: M \to N$ and a subset $P$ of $N$ such that $S = Pf^{-1}$. See [2] for details. Let Rat($M$) be the set of rational subsets of $M$ and let Rec($M$) be the set of recognizable subsets of $M$. We have the following important theorems.

**Theorem 3.3.** (Kleene) *If $M$ is a finitely generated free monoid, then* Rec($M$) = Rat($M$).

**Theorem 3.4.** (Anissimov and Seifert) [1] *Let $G$ be a finitely generated group and let $H$ be a subgroup. Then $H \in$ Rec($G$) if and only if $[G : H]$ is finite. $H \in$ Rat($G$) if and only if $H$ is finitely generated.*

It follows from Theorem 3.4 that if $G$ is any infinite group, then the trivial group is rational, but not recognizable. We also list the following consequence of Kleene's Theorem due to McKnight.

**Theorem 3.5.** *Let $M$ be a finitely generated monoid. Then* Rec($M$) *is contained in* Rat($M$).

The following lemma is an adaptation of the result of Anissimov and Seifert [1] to the case of inverse monoids. Recall, that if $Y$ is a subset of an inverse monoid, then $\langle Y \rangle$ denotes the closed inverse submonoid of $M$ generated by $Y$ and that a closed inverse submonoid of $M$ is finitely generated if $N = \langle Y \rangle$ for some finite set $Y$.

**Lemma 3.6.** *Let $M$ be an inverse monoid and let $N$ be a closed inverse submonoid of $M$. Then $N = \langle Y \rangle$ for some rational subset $Y$ of $M$ if and only if $N$ is finitely generated.*

**Proof.** If $N = \langle Y \rangle$ for some finite set $Y$, then certainly $N$ is generated by a rational set, since every finite set is rational. To prove the converse, we first recall the notion of star-height of a rational set.

Let $M$ be a monoid. Define a sequence of subsets $\mathrm{Rat}_h(M)$, $h \geq 0$, recursively as follows:

$$\mathrm{Rat}_0(M) = \{X \subseteq M \mid X \text{ is finite}\}$$

$\mathrm{Rat}_{h+1}(M)$ = finite unions of sets of the form $B_1 \dots B_m$ where each $B_i$ is either a singleton or $B_i = C_i^*$ for some $C_i \in \mathrm{Rat}_h(M)$.

It is well known that $\mathrm{Rat}(M) = \bigcup_{h \geq 0} \mathrm{Rat}_h(M)$. We now prove that if $N = \langle Y \rangle$ for some set $Y \in \mathrm{Rat}_h(M)$ for $h > 0$, then $N = \langle Y' \rangle$ for some $Y' \in \mathrm{Rat}_{h-1}(M)$. It follows that $N$ is finitely generated.

First consider a subset of the form:

(**) $L = x_1 T_1{}^* x_2 \ldots T_n{}^* x_{n+1}$ where $x_i \in M$, $1 \le i \le n+1$, and $T_j \subseteq M$, $1 \le j \le n$.

Let $y_i = x_1 \ldots x_i$ for $i = 1, \ldots, n+1$ and $S_i = y_i T_i y_i^{-1}$ for $i = 1, \ldots, n$. We claim that $\langle L \rangle = \langle L' \rangle$ where $L' = y_{n+1} \cup S_1 \cup \cdots \cup S_n$. First note that $y_{n+1}, y_{n+1}{}^{-1} \in \langle L \rangle$. Also, if $m \in S_i$, then $m = y_i t y_i^{-1}$ for some $t \in T_i$. But $y_{n+1}^{-1} = x_{n+1}{}^{-1} \ldots x_{i+1}{}^{-1} y_i^{-1}$, so that $y_i^{-1} \ge x_{i+1} \ldots x_{n+1} y_{n+1}{}^{-1}$.

Therefore $m = y_i t y_i^{-1} \ge (y_i t x_{i+1} \ldots x_{n+1}) y_{n+1}{}^{-1} \in \langle L \rangle$. Thus $S_i \subseteq \langle L \rangle$ for $i = 1, \ldots, n$ and it follows that $\langle L' \rangle \subseteq \langle L \rangle$.

Conversely, note that $x_1 = y_1$ and that $x_i \ge y_{i-1}{}^{-1} y_i$ for $i = 2, \ldots, n+1$. Thus, if $m \in L$, $m = x_1 t_1 x_2 \ldots t_n x_{n+1}$, $t_i \in T_i^*$, $1 \le i \le n$, then $m \ge y_1 t_1 y_1{}^{-1} y_2 t_2 y_2{}^{-1} y_3 \ldots y_n t_n y_n{}^{-1} y_{n+1}$. Furthermore, if $t_i = t_{i1} \ldots t_{ik}$, $t_{ij} \in T_i$, $1 \le j \le k$, then $y_i t_i y_i^{-1} \ge y_i t_{i1} y_i^{-1} \ldots y_i t_{ik} y_i^{-1}$ and it follows that $m \in \langle S_1^* \ldots S_n^* y_{n+1} \rangle$ and thus $m \in \langle L' \rangle$. Therefore $\langle L \rangle = \langle L' \rangle$ as claimed.

Now assume that $N = \langle Y \rangle$ with $Y \in \mathrm{Rat}_h(M)$ for $h > 0$. Then $Y = L_1 \cup \cdots \cup L_k$ where each $L_i$ has the form (**) and $L_i \in \mathrm{Rat}_h(M)$ for $1 \le i \le k$. Let $Y' = L_1{}' \cup \cdots \cup L_k{}'$ where each $L_i'$ is derived from $L_i$ as above. It follows that $\langle Y' \rangle = \langle Y \rangle = N$ and that $Y' \in \mathrm{Rat}_{h-1}(M)$. Therefore the minimal star height of a set of generators for $N$ is 0 and thus $N$ is finitely generated.

We are now ready to prove the main theorem of this section.

**Theorem 3.7.** *Let $M = \mathrm{FIM}(X)$ and $N$ be a closed inverse submonoid of $M$. Then the following conditions are equivalent:*

a) *$N$ is recognized by a finite inverse automaton.*

b) *$N$ has finite index in $M$.*

c) *$N$ corresponds to a finite immersion over the bouquet of $|X|$ circles.*

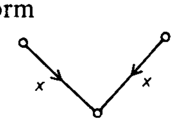
d) *$N$ is a recognizable subset of $M$.*

e) *$N$ is a rational subset of $M$.*

f) *$N$ is finitely generated.*

**Proof.** We have seen in Lemma 3.2 that the automaton of right $\omega$-cosets is the minimal automaton of $N$ considered as a subset of $M$. This remark along with Theorem 1.2 and Theorem 3.5 give us the implications a) implies b) implies c) implies d) implies e). If $N$ is a rational subset of $M$, then $N$ has a rational set of generators (i.e. $N$) and it follows from Lemma 3.6 that $N$ is finitely generated. Thus we need only prove that f) implies a).

Let $N = \langle Y \rangle$ where $Y = \{w_i \mid i = 1, \ldots, n\} \subseteq (X \cup X^{-1})^*$. We first build the "flower automaton", $F(Y)$. $F(Y)$ has a distinguished vertex $v$ and one "petal" for each $w_i \in Y$. If $w_i = x_{i1} \ldots x_{in_i}$, then there is one edge for each $x_{ij}$ so that we spell out $w_i$ in a loop from $v$ to $v$. $F(Y)$ is connected and there is an obvious graph map from $F(Y)$ to $B_X$ that may fail to be an immersion. To rectify this situation we successively fold [22] or collapse [24] edges of the form




$x \in X \cup X^{-1}$

In this way we obtain a sequence of graph maps:

$$F(Y) = \Gamma_0 \xrightarrow{\alpha_1} \Gamma_1 \xrightarrow{\alpha_2} \Gamma_2 \ldots \Gamma_n = \Gamma(Y) \xrightarrow{\phi} B_X$$

where each $\alpha_i$ is a fold and $\phi$ is an immersion. It is proved in [22] that $\Gamma(Y)$ is independent of the order of edges chosen to fold. By choosing the image of $v$ as initial and terminal vertex we obtain an inverse automaton $A(Y)$. By Lemma 3.1 and Lemma 3.2 $A(Y)$ (considered as a representation of FIM($X$)) is the minimal automaton of a closed inverse submonoid $N'$ of FIM($X$). We will show that $N = N'$.

Let $A_i$ be the automaton whose underlying graph is $\Gamma_i$ and whose initial and terminal vertex is the image of $v$. It is clear that as subsets of $(X \cup X^{-1})^*$, $Y \subseteq L(F(Y)) \subseteq L(A_1) \subseteq \cdots \subseteq L(A_n) \subseteq L(A(Y))$ where $L(A)$ is the language accepted by the automaton $A$. It follows that $N = \langle Y \rangle \subseteq N'$, since $N'$ is a closed inverse submonoid of FIM($X$).

Conversely, let $w \in N'$. Then there is a $z \in (X \cup X^{-1})^*$ such that $z \in L(A_i)$ for some $0 \leq i \leq n$ and $z\rho = w$. We prove by induction on $i$ that $w \in N$. If $i = 0$, then there is a path $p$ from $v$ to $v$ reading $z$ in $F(Y)$. We can factor $p = \beta_1 \ldots \beta_m$, $m \geq 1$, uniquely so that each $\beta_j$ goes from $v$ to $v$ and never passes through $v$ except at its first and last vertex. Thus each $\beta_j$ reads a word $z_j$ that travels along a petal of $F(Y)$ corresponding to some $y_j \in Y$. It is easy to see that $z_j$ labels a path in the Munn tree of $y_j$ from the initial to the terminal vertex and thus $z_j\rho \geq y_j\rho$ in the natural partial order of FIM($X$). Therefore, $w = z\rho = z_1\rho \ldots z_m\rho \geq y_1\rho \ldots y_m\rho \in \langle Y \rangle = N$.

Now assume that $L(A_i) \subseteq N$ for some $i$, $0 \leq i < n$. Then $\Gamma_{i+1}$ is obtained from $\Gamma_i$ by folding some pair of edges $p \xrightarrow{x} r$, $q \xrightarrow{x} r$ in $\Gamma_i$ to the edge $e: \{p, q\} \xrightarrow{x} r$ in $\Gamma_{i+1}$. If $z \in L(A_{i+1})$, then a straightforward induction on the number of times an accepting path passes through $\{p, q\}$ shows that $z\rho \geq t\rho$ for some $t \in L(A_i)$. By induction, $t\rho \in N$ and thus $z\rho \in N$ as desired.

Recall that Howson's Theorem states that if $H_1$ and $H_2$ are finitely generated subgroups of FG($X$), then so is $H_1 \cap H_2$ [8]. Stallings [22] gave a proof of this theorem using immersions. By interpreting this result in FIM($X$) we get the following version of Howson's Theorem for closed inverse submonoids of FIM($X$).

**Corollary 3.8.** *Let $N_1$ and $N_2$ be finitely generated closed inverse submonoids of* FIM($X$). *Then $N_1 \cap N_2$ is finitely generated.*

**Proof.** Let $\Gamma_i$ be the immersion corresponding to $N_i$, $i = 1, 2$. Since $N_i$ is finitely generated, $V(\Gamma_i)$ is finite for $i = 1, 2$. Let $\Gamma_1 \times \Gamma_2$ be the immersion with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge $(v, w) \xrightarrow{x} (v', w')$ iff $v \xrightarrow{x} v'$, $w \xrightarrow{x} w'$ are edges in $\Gamma_1$, $\Gamma_2$ respectively.

Let $i_1$ and $i_2$ be the roots of $\Gamma_1$ and $\Gamma_2$ respectively and let $\Gamma_1 \wedge \Gamma_2$ be the connected component of $\Gamma_1 \times \Gamma_2$ containing $(i_1, i_2)$. It is clear that with root $(i_1, i_2)$, $\Gamma_1 \wedge \Gamma_2$ recognizes $N_1 \cap N_2$. Therefore $N_1 \cap N_2$ has finite index and thus is finitely generated by Theorem 3.7.

We remark that Corollary 3.8 does not remain true if we assume only that $N_1$ and $N_2$ are finitely generated inverse submonoids of FIM($X$). See [9] for an example.

## 4. Free Inverse Categories: Classification of Immersions

In order to classify immersions over a graph $\Gamma$ we shall make use of the free inverse category over $\Gamma$. Recall ([12]) that a category $C$ is *inverse* if for each morphism $p$ of $C$, there is a unique morphism $p^{-1}$ of $C$ such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$. Denote the loop monoid at a vertex (object) $v$ of $C$ by Mor $(v, v)$; that is, Mor $(v, v)$ is the set of morphisms $p$ from $v$ to $v$, together with the multiplication induced by $C$. It is clear that if a category $C$ is inverse, then each loop monoid Mor $(v, v)$ is an inverse monoid.

We define the *free inverse category* FIC($\Gamma$) of a graph $\Gamma$ to be quotient of the free category on $\Gamma$ by the congruence $\sim_i$ induced by all relations of the form $p = pp^{-1}p$, $p^{-1} = p^{-1}pp^{-1}$, and $pp^{-1}qq^{-1} = qq^{-1}pp^{-1}$ if $\alpha(p) = \alpha(q)$, for paths $p$, $q$ in $\Gamma$. (Here, as usual, $p^{-1}$ is the path $e_n^{-1}e_{n-1}^{-1}\dots e_2^{-1}e_1^{-1}$ if $p$ is the path $e_1e_2\dots e_n$ where the $e_i$ are edges of $\Gamma$: note that $(p^{-1})^{-1} = p$.) Denote the $\sim_i$ class of $p \in P(\Gamma)$ by $[[p]]$. It is easy to see that FIC($\Gamma$) is an inverse category. If $\Gamma = B_X$, the bouquet of $|X|$ circles, then FIC($\Gamma$) = FIM($X$), the free inverse monoid on $X$.

The first problem of interest to us at present is the *word problem* for the free inverse category of a connected graph $\Gamma$: namely, find an algorithm that will decide, given any two paths $p$, $q$ in $\Gamma$, whether $p$ and $q$ represent the same morphism in FIC($\Gamma$). This may be solved in a manner very similar to the way in which Munn [14] solved the corresponding problem for FIM($X$).

Let $f: \tilde{\Gamma} \to \Gamma$ be the universal cover of the graph $\Gamma$. For each vertex $v$ of $\Gamma$ choose (and fix) a vertex $\tilde{v}$ of $\tilde{\Gamma}$ such that $\tilde{v}f = v$. Since $\tilde{\Gamma}$ covers $\Gamma$, each path $p$ of $\Gamma$ starting at $v$ lifts to a unique path $\tilde{p}$ of $\tilde{\Gamma}$ starting at $\tilde{v}$. Let $M(p)$ be the (finite) subtree of $\tilde{\Gamma}$ obtained by traversing the path $\tilde{p}$ in $\tilde{\Gamma}$ (starting at $\tilde{v}$). The following result provides a solution to the word problem for FIC($\Gamma$).

**Theorem 4.1.** *Let $\Gamma$ be a graph and $p$, $q$ paths in $\Gamma$. Then $p$ and $q$ represent the same morphism in* FIC($\Gamma$) *if and only if $\alpha(p) = \alpha(q)$, $\omega(p) = \omega(q)$ and $M(p) = M(q)$.*

**Proof.** Let $E = E(\Gamma)$ and identify FIM($E$) with the one-object category FIC($B_E$) in the usual way. There is a natural functor $F$ : FIC($\Gamma$) $\to$ FIM($E$). It is obvious how to define $F$ on objects of FIC($\Gamma$) (since FIM($E$) has just one object): if $p = e_1 \dots e_n$ is a path in $\Gamma$ then define $F([[p]]) = \mathrm{MT}(e_1e_2\dots e_n)$, the Munn tree of the word $e_1e_2\dots e_n$. The fact that $F$ is well-defined follows easily by looking at elementary transitions relative to the congruence $\sim_i$ on the free category on $\Gamma$. If $p = e_1e_2\dots e_n$ is a path in $\Gamma$ then it is clear that $\mathrm{MT}(p) = \mathrm{MT}(pp^{-1}p)$; if $p$ and $q$ are two paths in $\Gamma$ with $\alpha(p) = \alpha(q)$ then clearly $\mathrm{MT}(pp^{-1}qq^{-1}) = \mathrm{MT}(qq^{-1}pp^{-1})$: hence $F$ is well-defined. It is not difficult to see that $F$ is a functor from FIC($\Gamma$) to FIM($E$).

We claim that $F$ is in fact a faithful functor (i.e. $F$ is one-one on each Hom set.) Suppose that $p = e_1 \dots e_n$ and $q = f_1 \dots f_m$ are paths in $\Gamma$ with $\alpha(p) = \alpha(q)$ and $\omega(p) = \omega(q)$. If $\mathrm{MT}(p) = \mathrm{MT}(q)$ then there is a finite sequence of elementary transitions (of the form $u \to uu^{-1}u$ or $uu^{-1}u \to u$ or $uu^{-1}vv^{-1} \to vv^{-1}uu^{-1}$) leading from the word $p$ to the word $q$. Each such elementary transition corresponds in the obvious way to an elementary transition relative to the congruence $\sim_i$ on the free category on $\Gamma$: it follows that $p \sim_i q$, i.e. $[[p]] = [[q]]$ in FIC($\Gamma$). Hence $F$ is a faithful functor.

Thus $[[p]] = [[q]]$ in FIC($\Gamma$) if and only if $\alpha(p) = \alpha(q)$, $\omega(p) = \omega(q)$ and $\mathrm{MT}(p) = \mathrm{MT}(q)$. It is clear that $\tilde{\Gamma}$ is a subtree of the Cayley graph of FG($E$) and that for each path $p$ in $\Gamma$, $M(p)$ is a birooted tree isomorphic to MT($p$). This completes the proof of the theorem.

**Example 4.2.** Let $\Gamma$ be the graph (labelled over $\{x, y, x^{-1}, y^{-1}\}$ consistent with an immersion over $B_{\{x,y\}}$) as indicated in the diagram 1 below. (Here only the positively oriented edges are shown.)

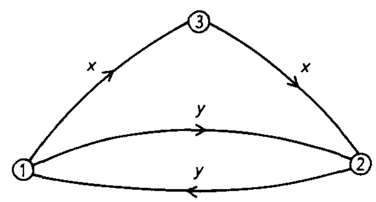



Diagram 1

A quick sketch of the part of the universal cover of $\Gamma$ will convince the reader that if $p$, $q$ and $r$ are the paths in $\Gamma$ starting at 1 and labelled as follows:

$$p = yx^{-2}yy^{-2}yx^2y^2y^{-1}xx^{-1}y^{-1}x^{-2}x$$

$$q = y^3y^{-1}xx^{-1}y^{-2}yx^{-2}y^{-1}y^2y^{-1}x$$

$$r = y^2x^2y^{-2}x^{-2}x,$$

then $p = q \neq r$ in FIC($\Gamma$).

We turn now to a study of the loop monoids of the free inverse category of a graph $\Gamma$. Denote the loop monoid of FIC($\Gamma$) at $v$ by $L(\Gamma, v)$. We first make the following observation.

**Proposition 4.3.** *Let $\Gamma$ be a connected graph labelled over $X \cup X^{-1}$ consistent with an immersion over the bouquet of $|X|$ circles. Then each loop monoid of* FIC($\Gamma$) *is a closed inverse submonoid of* FIM($X$).

**Proof.** This follows immediately from the proof of Theorem 1.2 since for each $v \in V(\Gamma)$ the loop monoid $L(\Gamma, v)$ is the stabilizer of $v$ with respect to the action of FIM($X$) on $V(\Gamma)$ defined in the proof of that theorem.

Let $H, K$ be two closed inverse submonoids of a free inverse monoid FIM($X$). Define $H \approx K$ ("$H$ is conjugate to $K$") if there exists $m \in$ FIM($X$) such that $m^{-1}Hm \subseteq K$ and $mKm^{-1} \subseteq H$. It is easy to see that $\approx$ is an equivalence relation (called "conjugation") on the set of closed inverse submonoids of FIM($X$). We refer to the equivalence classes of $\approx$ as *conjugacy classes.* We remark that conjugate closed inverse submonoids of $\mathrm{FIM}(X)$ are not necessarily isomorphic. For example, if $\Gamma$ denotes the graph of Diagram 1 and if $v_1$ (respectively $v_3$) denotes the vertex labelled 1 (respectively 3), then $L(\Gamma, v_1)$ has three maximal (non-identity) idempotents (in the natural partial order) and $L(\Gamma, v_3)$ has two maximal idempotents, so $L(\Gamma, v_1)$ is not isomorphic to $L(\Gamma, v_3)$.

Immersions over $B_X$ are classified by conjugacy classes of closed inverse submonoids of $\mathrm{FIM}(X)$, as indicated in the following theorem.

**Theorem 4.4.** *Let $\Gamma$ be a connected graph, labelled over a set $X \cup X^{-1}$ consistent with an immersion over $B_X$. Then each loop monoid is a closed inverse submonoid of* $\mathrm{FIM}(X)$ *and the set of all loop monoids $L(\Gamma, v)$ for $v \in V(\Gamma)$ is a conjugacy class of the set of closed inverse submonoids of* $\mathrm{FIM}(X)$. *Conversely, if $H$ is any closed inverse submonoid of a free inverse monoid* $\mathrm{FIM}(X)$ *then there is some graph $\Gamma$ and an immersion $\eta : \Gamma \to B_X$ such that $H$ is a loop monoid of* $\mathrm{FIC}(\Gamma)$: *furthermore, $\Gamma$ is unique (up to graph isomorphism) and $\eta$ is unique (up to equivalence).*

**Proof.** We already know from Proposition 4.3 that each $L(\Gamma, v)$ is a closed inverse submonoid of $\mathrm{FIM}(X)$. Suppose that $v, w \in V(\Gamma)$. Since $\Gamma$ is connected there is a path $p$ in $\Gamma$ with $\alpha(p) = v$ and $\omega(p) = w$: let $m \in (X \cup X^{-1})^*$ be the label of the path $p$ and regard $m$ as an element of $\mathrm{FIM}(X)$. If $n \in L(\Gamma, v)$ then $n$ labels some path in $\Gamma$ from $v$ to $v$. It follows that $m^{-1}nm$ labels a path in $\Gamma$ from $w$ to $w$, so $m^{-1}nm \in L(\Gamma, w)$. Hence $m^{-1}L(\Gamma, v)m \subseteq L(\Gamma, w)$ and similarly $mL(\Gamma, w)m^{-1} \subseteq L(\Gamma, v)$, so $L(\Gamma, v) \approx L(\Gamma, w)$ in $\mathrm{FIM}(X)$. Now suppose that $v \in V(\Gamma)$ and $H$ is a closed inverse submonoid of $\mathrm{FIM}(X)$ that is conjugate to $L(\Gamma, v)$. There is some $m \in \mathrm{FIM}(X)$ such that $m^{-1}L(\Gamma, v)m \subseteq H$ and $mHm^{-1} \subseteq L(\Gamma, v)$. In particular $mm^{-1} \in L(\Gamma, v)$ so there is some path $p$ in $\Gamma$ labelled by $m$ with $\alpha(p) = v$: let $w = \omega(p)$. If $h \in H$ then $mhm^{-1} \in L(\Gamma, v)$, so $mhm^{-1}$ labels a path in $\Gamma$ from $v$ to $v$, whence $h$ labels a path in $\Gamma$ from $w$ to $w$, i.e. $h \in L(\Gamma, w)$. Thus $H \subseteq L(\Gamma, w)$. On the other hand if $n_1 \in L(\Gamma, w)$ then $mn_1m^{-1} \in L(\Gamma, v)$ so $m^{-1}mn_1m^{-1}m \in m^{-1}L(\Gamma, v)m \subseteq H$. Since $H$ is a closed inverse submonoid of $\mathrm{FIM}(X)$ and $m^{-1}m$ is an idempotent of $\mathrm{FIM}(X)$ it follows that $n_1 \in H$: thus $L(\Gamma, w) \subseteq H$, whence $L(\Gamma, w) = H$. Hence the set of all loop monoids $L(\Gamma, v)$ for $v \in V(\Gamma)$ is a conjugacy class of the set of all closed inverse submonoids of $\mathrm{FIM}(X)$.

Suppose now that $H$ is any closed inverse submonoid of a free inverse monoid $\mathrm{FIM}(X)$. We construct a graph $\Gamma$ and an immersion $\eta : \Gamma \to B_X$ as in Sec. 1: $H$ determines a transitive representation of $\mathrm{FIM}(X)$ by partial one-one transformations on the set of right $\omega$-cosets of $H$, there is a natural immersion $\eta$ from the graph $\Gamma$ of this representation to $B_X$ and $H$ is the stabilizer (loop monoid) corresponding to the vertex $H$ of $\Gamma$. The uniqueness of $\Gamma$ and $\eta$ follows by a routine argument.

The results of Theorem 4.4 can be extended somewhat so as to yield a classification of connected immersions over an arbitrary connected graph $\Gamma$.

**Theorem 4.5.** *Let $f : \Delta \to \Gamma$ be an immersion over $\Gamma$, where $\Delta$ and $\Gamma$ are connected graphs labelled over a set $X \cup X^{-1}$ consistent with immersions over $B_X$ (so $f$ is a labelled graph morphism from $\Delta$ to $\Gamma$). If $v \in V(\Gamma)$ and $v_1 \in V(\Delta)$ such that $v_1 f = v$, then $f$ induces an embedding of $L(\Delta, v_1)$ into $L(\Gamma, v)$. Conversely, let $\Gamma$ be a graph labelled over $X \cup X^{-1}$*

*as usual and let $H$ be a closed inverse submonoid of* $\mathrm{FIM}(X)$ *such that* $H \subseteq L(\Gamma, v)$ *for some vertex* $v \in V(\Gamma)$. *Then there exist a graph* $\Delta$, *an immersion* $f: \Delta \to \Gamma$ *and a vertex* $v_1 \in V(\Delta)$ *such that* $v_1 f = v$ *and* $L(\Delta, v_1)f = H$. *Furthermore* $\Delta$ *is unique (up to graph isomorphism) and* $f$ *is unique (up to equivalence). If* $H$, $K$ *are two closed inverse submonoids of* $\mathrm{FIM}(X)$ *with* $H, K \subseteq L(\Gamma, v)$ *then the corresponding immersions* $f: \Delta \to \Gamma$ *and* $g: \Delta' \to \Gamma$ *are equivalent if and only if* $H \approx K$ *in* $\mathrm{FIM}(X)$.

**Proof.** The proof is just an adaptation of the standard proof of the corresponding theorem classifying covers of a graph $\Gamma$ via subgroups of $\pi_1(\Gamma)$, but couched in a somewhat simpler form as a consequence of our convention about labelling graphs consistent with immersions over $B_X$.

To prove the first part of the theorem (the embedding of $L(\Delta, v_1)$ into $L(\Gamma, v)$) note that if $m \in L(\Delta, v_1)$ then $m$ labels a path $p$ from $v_1$ to $v_1$ in $\Delta$, so on application of the labelled graph morphism $f$ we see that $m$ also labels a path $p'$ from $v$ to $v$ in $\Gamma$, so we may regard $m$ as an element of $L(\Gamma, v)$. Since $f$ is an immersion, $p$ is the only path in $\Delta$ that maps under $f$ to $p'$, so the map that sends $m \in L(\Delta, v_1)$ to $mf \in L(\Gamma, v)$ is injective, whence $L(\Delta, v_1)$ embeds into $L(\Gamma, v)$.

For the converse, suppose that $H$ is a closed inverse submonoid of $\mathrm{FIM}(X)$ such that $H \subseteq L(\Gamma, v)$. Construct the graph $\Delta$ of right $\omega$-cosets of $H$ and the immersion $\eta: \Delta \to B_X$ as in the proof of Theorem 4.4 (i.e. Theorem 1.2). Since $H \subseteq L(\Gamma, v)$ it follows that if $Hm$ is a right $\omega$-coset of $H$ then $mm^{-1} \in H \subseteq L(\Gamma, v)$, so $m$ labels a path in $\Gamma$ starting at $v$. From this it is easy to see that $\eta$ factors as $f \circ \eta_1$ where $\eta_1$ is the natural immersion from $\Gamma$ to $B_X$ and $f$ is an immersion from $\Delta$ to $\Gamma$. Clearly there is a vertex $v_1 \in V(\Delta)$ (in fact $v_1$ is the right $\omega$-coset $H$) such that $v_1 f = v$ and $L(\Delta, v_1) = H$. The uniqueness of $\Delta$ and $f$ and the last statement of the theorem follow in a routine fashion.

## 5. Finitely Generated Subgroups of Free Groups and Finite Inverse Monoids

In Sec. 3 we canonically associated a connected finite immersion with every finitely generated closed inverse submonoid of $\mathrm{FIM}(X)$. We also have seen that (connected) immersions are essentially (transitive) inverse automata. In this section we associate a canonical finite connected immersion with every finitely generated subgroup $H$ of the free group, $\mathrm{FG}(X)$. We then show that the transition inverse monoid of the corresponding inverse automaton can be used to algorithmically check properties of $H$. We give an introduction to these ideas here. Further examples and results will appear in future papers.

Let $H$ be a subgroup of $\mathrm{FG}(X)$. Let $T_H$ be the subtree of $\Gamma = \Gamma(X)$ spanned by $H$. That is, $T_H$ is the smallest subtree of $\Gamma$ containing the vertices of $H$. Then $H$ acts by left multiplication on $T_H$. We then have the closed inverse submonoid $\overline{H} = M(T_H, H, 1)$ and the immersion $\eta: I_H \to B_X$ where $I_H = H \backslash T_H$. Clearly $\pi_1(I_H, [\{1\}]) = H$. Thus a reduced word is in $H$ if and only if it labels a loop from $[\{1\}]$ to $[\{1\}]$ in $I_H$. Considered as an inverse automaton, $I_H$ is the minimal automaton of $\overline{H}$. It follows easily that $H$ is the maximal group image of $\overline{H}$ and that $\overline{H}$ is finitely generated if and only if $H$ is finitely generated if and only if $I_H$ is finite. We define the syntactic monoid $\mathrm{Synt}(H)$ of $H$ to be the transition monoid of $I_H$. Thus $\mathrm{Synt}(H)$ is an inverse monoid and $\mathrm{Synt}(H)$

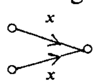
is finite if and only if $H$ is finitely generated. In the terminology of [22], $I_H$ is the core immersion associated with $H$.

The following algorithm computes $I_H$ from a finite set $Y$ of generators of $H$. This is essentially the construction outlined in Theorem 3.7 of Sec. 3.

*Algorithm* Input: Finite set $Y \subseteq \mathrm{FG}(X)$.

Output: $I_H$, the canonical immersion of $H = \langle Y \rangle$ (the subgroup generated by $Y$).

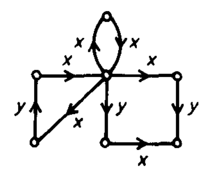
*Step* 1. Compute $F(Y)$, the flower automaton of $Y$. $F(Y)$ has a distinguished vertex $v$ and a loop reading $y$ for each $y \in Y$.

*Step* 2. Fold edges of $F(Y)$ to obtain an immersion. That is iteratively identify edges of the form  for some $x \in X$ until an immersion $I_Y$ is obtained.

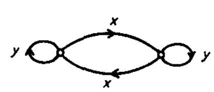
*Step* 3. Iteratively remove vertices of degree 1 (and the edges connecting such vertices to the rest of the graph) from $I_Y$ to obtain $I_H$.

**Example.** 1) Let $Y = \{x^2, xyx^{-1}y^{-1}, xyx\}$.

Then $F(Y)$ is



After performing all folds we obtain



Thus $\mathrm{Synt}(H) \approx Z_2$. It follows that $H = \langle Y \rangle$ is a subgroup of index 2 in $\mathrm{FG}(X)$ and $I_H$ is a cover of $B_X$. More generally, we have the following result, which shows how $\mathrm{Synt}(H)$ can be used to detect algorithmically whether or not $H$ has finite index in $\mathrm{FG}(X)$.

**Theorem 5.1.** *Let $H$ be a finitely generated subgroup of the free group* $\mathrm{FG}(X)$. *The following are equivalent*:

a) $\mathrm{Synt}(H)$ *is a group.*

b) *The immersion* $\eta : I_H \to B_X$ *is a cover.*

c) *$H$ has finite index in* $\mathrm{FG}(X)$.

d) *$\overline{H}$ is a full (closed) inverse submonoid of* $\mathrm{FIM}(X)$ [*i.e. $\overline{H}$ contains all the idempotents of* $\mathrm{FIM}(X)$].

**Proof.** a) $\to$ b). Suppose that $\mathrm{Synt}(H)$ is a group. Then every letter of $X$ must induce a permutation of $V(I_H)$, since the identity transformation is in $\mathrm{Synt}(H)$. Thus, for every $v \in V(I_H)$, and every $x \in X \cup X^{-1}$, there is exactly one edge labelled by $x$ starting at $v$ so $\eta : I_H \to B_X$ is a cover.

b) → c). If $\eta : I_H \to B_X$ is a cover, then it is clear that $I_H$ is the coset graph of FG($X$) on the cosets of $H$. Thus $H$ has finite index in FG($X$). (The equivalence of b) and c) was noted by Stallings [22].)

c) → d). If $H$ has finite index in FG($X$), then clearly $I_H$ is the coset graph of $H$ in $FG(X)$. Thus every word in $(X \cup X^{-1})^*$ labels a path in $I_H$ from any vertex. In particular, every Dyck word (i.e. every word whose reduced form is 1) labels a path from $H$ to $H$. Since every idempotent in FIM($X$) is the image of a Dyck word, it follows that $\overline{H}$ is full.

d) → a). Suppose $\overline{H}$ is full. Let $p$ be a vertex of $I_H$. Then there is a word $w \in (X \cup X^{-1})^*$ labelling a path from $v$ to $p$ where $v$ is the vertex of $I_H$ corresponding to $\overline{H}$. Let $x \in X \cup X^{-1}$. Then $wxx^{-1}w^{-1}$ labels a path from $v$ to $v$, since $\overline{H}$ is full. It follows that there must be an edge of $I_H$ with initial vertex $p$ and labelled by $x$. Thus every $x$ induces a permutation of $V(I_H)$ and Synt($H$) is a group.

**Corollary 5.2.** ([22]) *Let Y be a finite subset of* FG($X$). *Then it is decidable whether the subgroup generated by Y has finite index.*

**Proof.** Let $H = \langle Y \rangle$. Given $Y$, by the algorithm above we can effectively compute $I_H$ and then check whether every letter of $X$ induces a permutation.

More generally, properties of $H$ can be translated into algebraic properties of Synt($H$). Since Synt($H$) can be effectively computed from a set of generators for $H$, this leads to an algorithm to test the desired property. As a second example we have the following simple results.

**Theorem 5.3.** *Let* $Y \subseteq$ FG($X$) *and let* $H = \langle Y \rangle$. *Then* $H =$ FG($X$) *if and only if* Synt($H$) $= \{1\}$.

**Proof.** If $H =$ FG($X$), then cleary $I_H = B_X$, the bouquet of $X$-circles and thus Synt($H$) $= \{1\}$. Conversely, if Synt($H$) $= \{1\}$, then every $x \in X$ induces the identity function on $I_H$. Since $I_H$ is connected, it follows that $I_H$ has exactly 1 vertex, so $I_H = B_X$ and $H =$ FG($X$).

**Theorem 5.4.** *Let* $Y \subseteq$ FG($X$) *be a finite set and let* $H = \langle Y \rangle$. *The following are equivalent*:

a) $H =$ FG($Z$) *for some* $Z \subseteq X$.

b) Synt($H$) *is a semilattice, that is an idempotent and commutative semigroup.*

**Proof.** a) → b) If $H =$ FG($Z$) for some $Z \subseteq X$, then every $z \in Z$ labels a loop at the distinguished vertex of $I_H$. Since $I_H$ is a connected immersion without edges of degree 1, no letter in $X/Z$ can label an edge of $I_H$. It follows that $I_H = B_Z$, the bouquet of $Z$-circles. Thus every letter in $Z$ acts as the identity transformation and every letter of $X/Z$ acts as the empty transformation. Thus Synt($H$) is either $\{0, 1\}$ under the usual multiplication if $Z \neq X$, or Synt($H$) $= \{1\}$, if $Z = X$. In either case, Synt($H$) is a semilattice.

Conversely, if Synt($H$) is a semilattice, then every $x \in X$ induces a partial identity function on $I_H$. Since $I_H$ is connected it follows that $I_H$ has exactly one vertex. Thus $H =$ FG($Z$), where $Z$ is the set of letters labelling an edge in $I_H$.

We close the paper by indicating without proof some additional results of this type which will be developed in a future paper.

There is an obvious analogy between the results listed above and the well developed theory of varieties of rational languages and varieties of finite monoids ([4], [10], [16]). Eilenberg's variety theorem sets up a bijection between the set of varieties of finite monoids and the set of varieties of rational languages. This allows combinatorial properties about rational languages to be translated into algebraic properties of their syntactic monoids.

In some recent (as yet unpublished) work, R. Ruyle develops an analogue of the Eilenberg Theorem for varieties of finite inverse monoids and relates these to suitable collections of rational (that is, finitely generated) subgroups of free groups and closed inverse submonoids of free inverse monoids. Along these lines, the authors, in collaboration with J.-C. Birget and P. Weil, have recently proved that a finitely generated subgroup $H$ of $FG(X)$ is a pure subgroup (i.e. $x^n \in H$ for some $n > 1$ implies $x \in H$) if and only if $Synt(H)$ has only trivial subgroups. This provides an algorithm for deciding whether the subgroup of $FG(X)$ generated by a finite subset $Y$ is pure or not.

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