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# New results on the conjecture of Rhodes and on the topological conjecture

S.W. Margolis

*Department of Computer Science, University of Nebraska – Lincoln, Lincoln, NE 68588, USA*

J.E. Pin

*CNRS and University of Paris 7, LITP, Tour 55–65, 4 Place Jussieu, 75252 Paris Cédex 05, France*

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*Abstract*

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The Conjecture of Rhodes, originally called the 'type II conjecture' by Rhodes, gives an algorithm to compute the kernel of a finite semigroup. This conjecture has numerous important consequences and is one of the most attractive problems on finite semigroups. It was known that the conjecture of Rhodes is a consequence of another conjecture on the finite group topology for the free monoid. In this paper, we show that the topological conjecture and the conjecture of Rhodes are both equivalent to a third conjecture and we prove this third conjecture in a number of significant particular cases.

## 1. The conjecture of Rhodes and the topological conjecture

In this paper, all semigroups (respectively monoids, groups) are finite except in the case of free monoids or free groups. If $M$ is a monoid, $E(M)$ (respectively $\mathrm{Reg}(M)$) denotes the set of idempotents (respectively regular elements) of $M$. If $x \in M$, $x^\omega$ denotes the unique idempotent of the subsemigroup of $M$ generated by $x$.

A *block-group monoid* is a monoid in which every $\mathscr{R}$-class and every $\mathscr{L}$-class contain at most one idempotent. A number of equivalent conditions are given in [7]. For instance, a monoid $M$ is a block-group monoid if and only if, for every regular $\mathscr{J}$-class $D$ of $S$, the semigroup $D^0$ is a Brandt semigroup, or if and only if

the submonoid generated by $E(M)$ is $\mathscr{J}$-trivial. The class of all block-group monoids forms a (pseudo)variety of monoids, denoted by **BG**.

We refer to [13] for an introduction to the conjecture of Rhodes and for all undefined notations. Let $M$ be a finite monoid. Recall that the *kernel* of $M$ is the submonoid

$$K(M) = \bigcap 1\tau^{-1},$$

where the intersection is taken over all relational morphisms $\tau$ from $M$ into a group $G$. $D(M)$ is the smallest submonoid of $M$ closed under 'weak conjugation': for every $s,t \in M$ such that either $sts = s$ or $tst = t$, the condition $u \in D(M)$ implies $sut \in D(M)$. It is sometimes convenient to use the following equivalent definition: $D(M)$ is the subset of $M$ generated by the grammar $\mathscr{G}_M = (\{\xi\}, \xi, P)$ whose productions are

(1) $\xi \to 1$,
(2) $\xi \to \xi\xi$,
(3) for every $s,t \in M$ such that $sts = s$, $\xi \to s\xi t$,
(4) for every $s,t \in M$ such that $tst = t$, $\xi \to s\xi t$.

It is known that $D(M)$ is a submonoid of $K(M)$, and the conjecture of Rhodes states that $K(M) = D(M)$ for every monoid $M$.

We introduce a new monoid $R(M)$, which is the smallest submonoid of $M$ such that, for every $s,t \in M$ and $e \in E(M)$, $st \in R(M)$ implies $set \in R(M)$.

The next proposition makes precise the relations between $D(M)$ and $R(M)$.

**Proposition 1.1.** *For every monoid $M$, $D(M)$ is contained in $R(M)$.*

**Proof.** It suffices to show that, if $u \in R(M)$ and $sts = s$, then $sut, tus \in R(M)$. Since $u \in R(M)$, there exists a sequence $1 = u_0, u_1, u_2, \ldots, u_n = u$ such that $u_{i+1} = s_i e_i t_i$ and $u_i = s_i t_i$ for some $s_i, t_i \in M$ and $e_i \in E(M)$. Now consider the sequence $1 = v_0, v_1 = st, v_{i+1} = su_i t, \ldots, v_{n+1} = sut$. Then, for every $i \geq 0$, $v_{i+2} = su_{i+1}t = (ss_i)e_i(t_i t)$ and $v_{i+1} = ss_i t_i t$. Furthermore, $v_1 = 1.(st).1 \in E(M)$ and $v_0 = 1$. Since $st$ is idempotent, it follows that $sut \in R(M)$ as required. The proof for $tus$ is dual. □

It is useful to know the behaviour of our three submonoids under quotient.

**Proposition 1.2.** *Let $\pi : M \to N$ be a surjective morphism of monoids. Then*
(a) $(K(M))\pi = K(N)$,
(b) $(D(M))\pi = D(N)$,
(c) $(R(M))\pi \subset R(N)$.

**Proof.** (a) By Proposition 4.1 of [13] there exists a finite group $G$ and a relational morphism $\tau : M \to G$ such that $1\tau^{-1} = K(M)$. Then $\pi^{-1}\tau : N \to G$ is a relational morphism. Thus $1(\pi^{-1}\tau)^{-1} = 1\tau^{-1}\pi = K(M)\pi$ and hence $K(N) \subset K(M)\pi$.

Conversely, let $\tau : N \to G$ be a relational morphism such that $1\tau^{-1} = K(N)$. Then $\pi\tau : M \to G$ is a relational morphism. Thus $K(M) \subset 1(\pi\tau)^{-1} = (1\tau^{-1})\pi^{-1} = K(N)\pi^{-1}$. Therefore, $K(M)\pi \subset K(N)\pi^{-1}\pi = K(N)$.

(b) To every production $\xi \to s\xi t$ (with $sts = s$) of $\mathcal{G}_M$, there corresponds in $\mathcal{G}_N$ the production $\xi \to (s\pi)\xi(t\pi)$ (with $(s\pi)(t\pi)(s\pi) = s\pi$). Therefore, if $\xi \to u$ in $\mathcal{G}_M$, then $\xi \to u\pi$ in $\mathcal{G}_N$, and hence $u\pi \in D(N)$. Conversely, let $\xi \to s'\xi t'$ (with $s't's' = s'$) be a production of $\mathcal{G}_N$, and let $x \in s'\pi^{-1}$ and $y \in t'\pi^{-1}$. Set $s = (xy)^{2\omega-1}x$ and $t = y$. Then $sts = s$, $s\pi = s'$ and $t\pi = t'$. Therefore, $\xi \to s\xi t$ is a production of $\mathcal{G}_M$, and for every $u' \in D(N)$, one can find $u \in D(M)$ such that $u\pi = u'$.

(c) Let $u \in R(M)$. Then there exists a sequence $1 = u_0, u_1, u_2, \ldots, u_n = u$ such that, for $1 \leq i \leq n$, $u_{i+1} = s_i e_i t_i$ and $u_i = s_i t_i$ for some $s_i, t_i \in M$ and $e_i \in E(M)$. It follows $u_{i+1}\pi = (s_i\pi)(e_i\pi)(t_i\pi)$ and $u_i\pi = (s_i\pi)(t_i\pi)$. Since $e_i\pi$ is idempotent, this shows that $u\pi \in R(N)$. □

The topological conjecture refers to the coarsest topology on the free monoid $A^*$ such that every monoid morphism $\varphi$ from $A^*$ into a finite discrete group is continuous (see [10–12] for more details). It states that for every monoid morphism $\pi : A^* \to M$ into a monoid $M$, if $P$ is a subset of $M$ satisfying

(*) for every $s,t \in M$ and $e \in E(M)$, $st \in P$ implies $set \in P$,

then the language $P\pi^{-1}$ is open.

We can now prove the main result of this section.

**Theorem 1.3.** *The following statements are equivalent*:

(1) *for every monoid* $M$, $K(M) = D(M)$ (*the conjecture of Rhodes*),

(2) *for every monoid* $M$, $K(M) \subset R(M)$,

(3) *the topological conjecture is true.*

**Proof.** It is proved in [11] that the topological conjecture implies the conjecture of Rhodes. Thus (3) implies (1) and (1) implies (2) follows from Proposition 1.1. Finally, suppose that $K(M)$ is contained in $R(M)$. Let $\pi : A^* \to M$ be a monoid morphism and let $P$ be a subset of $M$ satisfying

(*) for every $s,t \in M$ and $e \in E(M)$, $st \in P$ implies $set \in P$.

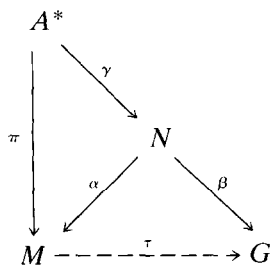
This condition implies that for every $s_1 s_2 \ldots s_k \in P$,

$$R(M)s_1R(M)s_2R(M)\ldots R(M)s_kR(M) \subset P.$$

We have to show that the language $L = P\pi^{-1}$ is open. Let $u = a_1 \ldots a_n \in L$. By Proposition 4.1 of [13] there exists a finite group $G$ and a relational morphism $\tau : M \to G$ such that $1\tau^{-1} = K(M) \subset R(M)$. Let

$$M \xrightarrow{\alpha^{-1}} N \xrightarrow{\beta} G$$

be the canonical factorization of $\tau$. Now, by the universal property of the free monoid, there exists a morphism $\gamma : A^* \to N$ such that the following diagram commutes:




Therefore, $\omega = 1(\gamma\beta)^{-1}$ is an open set and we have

$$\omega\pi = 1\beta^{-1}\gamma^{-1}\gamma\alpha \subset 1\beta^{-1}\alpha = 1\tau^{-1} = K(M) \subset R(M) .$$

It follows that

$$(\omega a_1 \omega \ldots \omega a_n \omega)\pi \subset R(M)(a_1\pi)R(M) \ldots R(M)(a_n\pi)R(M) \subset P ,$$

whence $\omega a_1 \omega \ldots \omega a_n \omega \subset L$. Now, by [12, Corollary 4.3], $\omega a_1 \omega \ldots \omega a_n \omega$ is an open set containing $u = a_1 \ldots a_n$. Therefore $L$ is open, and this proves the topological conjecture. □

## 2. A reduction result

In this section, we give a simple proof of a theorem of Henckell and Rhodes [5] which shows that the conjecture of Rhodes can be reduced to the case that $M$ belongs to the variety **BG**. We first establish an elementary result.

**Proposition 2.1.** *For every monoid $M$, the syntactic monoid of $R(M)$ belongs to* **BG**.

**Proof.** Let $\sim_{R(M)}$ be the syntactic congruence of $R(M)$ in $M$. Recall that $u \sim_{R(M)} v$ if and only if, for every $x, y \in M$,

$$xuy \in R(M) \Leftrightarrow xvy \in R(M) .$$

It suffices to show that, for all $e, f \in E(M)$, $e \mathscr{R} f$ (respectively $e \mathscr{L} f$) implies $e \sim_{R(M)} f$. We treat the case $e \mathscr{R} f$ (the other case being similar). Then we have $ef = f$ and $fe = e$. Suppose that $set \in R(M)$. Then $(se)ft \in R(M)$ and hence

*sft* $\in R(M)$. Conversely, *sft* $\in R(M)$ implies *set* $\in R(M)$. It follows that $e \sim_{R(M)} f$ as required. □

**Theorem 2.2.** *Let $M$ be a monoid, and let $\pi : M \to N$ be the syntactic morphism of $R(M)$. Then $K(M)$ is contained in $R(M)$ if and only if $K(N)$ is contained in $R(N)$.*

**Proof.** Suppose first that $K(M)$ is contained in $R(M)$ and let $n \in K(N)$. Since $(K(M))\pi = K(N)$ by Proposition 1.2, there exists $m \in K(M)$ such that $m\pi = n$. Thus $m \in R(M)$ and $n = m\pi \in R(N)$ by Proposition 1.2. Thus $K(N)$ is contained in $R(N)$.

Conversely, suppose that $K(N)$ is contained in $R(N)$, and let $m \in K(M)$. Then, by Proposition 1.2, $n = m\pi \in K(N)$ and hence $n \in R(N)$. Therefore, there exists a sequence $1 = u_0, u_1, u_2, \ldots, u_k = n$ such that, for $1 \leq i \leq k$, $u_{i+1} = s_i e_i t_i$ and $u_i = s_i t_i$ for some $s_i, t_i \in N$ and $e_i \in E(N)$. We show by induction on $i$ that $u_i \pi^{-1}$ is contained in $R(M)$. Since $N$ is the syntactic monoid of $R(M)$, it is equivalent to show that $u_i \pi^{-1} \cap R(M) \neq \emptyset$. The result is trivial for $i = 0$, since $1 \in 1\pi^{-1}$. Suppose that $u_i \pi^{-1} \cap R(M) \neq \emptyset$, and let $x_i \in s_i \pi^{-1}$, $y_i \in t_i \pi^{-1}$ and let $f_i \in e_i \pi^{-1}$ be an idempotent. Then $x_i y_i \in u_i \pi^{-1}$ and thus $x_i y_i \in R(M)$ by the induction hypothesis. Therefore, $x_i f_i y_i \in R(M)$, $(x_i f_i y_i)\pi = u_{i+1}$ and hence $u_{i+1}\pi^{-1} \cap R(M) \neq \emptyset$. Finally, we have $m \in n\pi^{-1} = u_k \pi^{-1}$ and thus $m \in R(M)$. □

**Corollary 2.3** (Henckell and Rhodes [5]). *If the conjecture of Rhodes is true for every monoid $M \in \mathbf{BG}$, then it is true for every monoid.*

**Proof.** Suppose that $K(N) = D(N)$ for every monoid $N \in \mathbf{BG}$. Then, by Proposition 1.1, $D(N)$ is contained in $R(N)$ for every monoid $N \in \mathbf{BG}$. Now let $M$ be a monoid, and let $N$ be the syntactic monoid of $R(M)$. Then $N \in \mathbf{BG}$ by Proposition 2.1, and thus $K(N)$ is contained in $R(N)$. By Theorem 2.2, this implies that $K(M)$ is contained in $R(M)$. By Theorem 1.3, this proves the conjecture of Rhodes. □

## 3. Some particular cases

In this section, we prove the conjecture '$K(M)$ is contained in $R(M)$' in some significant particular cases. We first recall the proof of an important result of Henckell and Rhodes [6].

**Theorem 3.1.** *If $M$ is a $\mathcal{J}$-trivial monoid, then $K(M)$ is contained in $R(M)$.*

**Proof.** Denote by $\leq$ the $\mathcal{R}$-order on $M$. Thus $x < y$ means that $x \leq y$ and that $x$ and $y$ are not $\mathcal{R}$-related. Let $E$ be the set of all sequences $(s_1, \ldots, s_n)$ such that

$$1 > s_1 > \cdots > s_1 \ldots s_n .$$

In particular, ( ) denotes the empty sequence. For each $s \in M$, we define a transformation $\hat{s}$ on $E$ by setting

$$(s_1, \ldots, s_n)\hat{s} = \begin{cases} (s_1, \ldots, s_n, s) & \text{if } s_1 \ldots s_n > s_1 \ldots s_n s\,, \\ (s_1, \ldots, s_i) & \text{if } s_1 \ldots s_n = s_1 \ldots s_n s, \text{ where } i \text{ is} \\ & \text{the smallest index such that } s_i \neq s \\ & \text{and } s_{i+1} = \cdots = s_n = s\,. \end{cases}$$

$\hat{s}$ is actually a permutation on $E$ for

$$(s_1, \ldots, s_{n-1}, s_n) = (s_1, \ldots, s_{n-1})\hat{s}$$
$$\text{if } s_n = s \text{ and } s_1 \ldots s_n < s_1 \ldots s_{n-1}\,,$$

and

$$(s_1, \ldots, s_n)$$
$$= (s_1, \ldots, s_n, s_{n+1}, \ldots, s_{n+k})\hat{s} \quad \text{with } s_{n+1} = \cdots = s_{n+k} = s$$
$$\text{if } s_n \neq s \text{ and } s_1 \ldots s_n > s_1 \ldots s_n s > \cdots$$
$$> s_1 \ldots s_n s^k = s_1 \ldots s_n s^{k+1}\,.$$

Let $S(E)$ be the symmetric group on $E$ and let $\tau : M \to S(E)$ be the relational morphism defined by

$$m\tau = \{\hat{s}_1 \ldots \hat{s}_n \mid s_1 \ldots s_n = m\}\,.$$

Let $m \in K(M)$. Then, by definition of $K(M)$, $m \in 1\tau^{-1}$ and there exist $s_1, \ldots, s_n \in M$ such that $s_1 \ldots s_n = m$ and $\hat{s}_1 \ldots \hat{s}_n$ is the identity on $E$. Set, for every $(s_1, \ldots, s_n) \in E$, $(s_1, \ldots, s_n)\pi = s_1 \ldots s_n$. We claim that, for $0 \leq i \leq n$,

$$u_i = (( )\hat{s}_1 \ldots \hat{s}_{n-i})\pi s_{n-i+1} \ldots s_n \in R(M)\,.$$

If $i = 0$, since $\hat{s}_1 \ldots \hat{s}_n$ is the identity on $E$, we have

$$u_0 = (( )\hat{s}_1 \ldots \hat{s}_n)\pi = ( )\pi = 1 \in R(M)\,.$$

By induction, suppose that the claim holds for $i$, and put $( )\hat{s}_1 \ldots \hat{s}_{n-(i+1)} = (r_1, \ldots, r_k)$. Thus, by induction $((r_1, \ldots, r_k)\hat{s}_{n-i})\pi s_{n-i+1} \ldots s_n \in R(M)$. Two cases arise.

(a) If $(r_1, \ldots, r_k)\hat{s}_{n-i} = (r_1, \ldots, r_k, s_{n-i})$, then

$$u_{i+1} = (r_1, \ldots, r_k)\pi s_{n-i} \ldots s_n$$
$$= (r_1, \ldots, r_k, s_{n-i})\pi s_{n-i+1} \ldots s_n$$
$$= ((r_1, \ldots, r_k)\hat{s}_{n-i})\pi s_{n-i+1} \ldots s_n \in R(M)\,.$$

(b) If $(r_1, \ldots, r_k)\hat{s}_{n-i} = (r_1, \ldots, r_i)$ with $r_{i+1} = \cdots = r_k = s_{n-i}$ and $r_1 \ldots r_k s_{n-i} = r_1 \ldots r_k$, then $r_1 \ldots r_k s_{n-i} = r_1 \ldots r_i s_{n-i}^{\omega}$, and hence, since $r_1 \ldots r_k s_{n-i+1} \ldots s_n \in R(M)$, one also has $r_1 \ldots r_i s_{n-i}^{\omega} s_{n-i+1} \ldots s_n \in R(M)$, whence $r_1 \ldots r_i s_{n-i} s_{n-i+1} \ldots s_n \in R(M)$, that is, $u_{i+1} \in R(M)$.

Thus the claim holds, and in particular, $u_n = s_1 \ldots s_n \in R(M)$. □

Another important particular case follows from the theorem of Ash [1].

**Theorem 3.2.** *If the idempotents of $M$ commute, then $D(M) = K(M) = E(M)$ is contained in $R(M)$.* □

These two results have the following consequences, which should be compared with the results of [2, 3].

**Corollary 3.3.** *If $D(M)$ contains the regular elements of $M$, then $K(M)$ is contained in $R(M)$.*

**Proof.** Let $\pi : M \to N$ be the syntactic morphism of $R(M)$. Then, by Proposition 1.2,

$$(\mathrm{Reg}(M))\pi = \mathrm{Reg}(N) \subset D(M)\pi = D(N).$$

But $N \in \mathbf{BG}$ by Proposition 2.1, and it is shown in [7] that, for every block-group monoid $N$, there exists a group $G$ and a relational morphism $\tau : N \to G$ such that $1\tau^{-1}$ is $\mathcal{J}$-trivial. In particular, $K(N)$ is $\mathcal{J}$-trivial. Finally, since $D(N)$ is contained in $K(N)$, $D(N)$ is also $\mathcal{J}$-trivial. Therefore, $\mathrm{Reg}(N) = E(N)$, and thus $N$ itself is $\mathcal{J}$-trivial. Now by Theorem 3.1, $K(N)$ is contained in $R(N)$, and by Theorem 2.2, $K(M)$ is contained in $R(M)$. □

**Corollary 3.4.** *If the idempotents of $M$ form a subsemigroup, then $K(M)$ is contained in $R(M)$.*

**Proof.** Let $\pi : M \to N$ be the syntactic morphism of $R(M)$. Then $N \in \mathbf{BG}$ by Proposition 2.1, and, by Proposition 2.3 of [7], the idempotents of $N$ generate a $\mathcal{J}$-trivial monoid $T$. But since the idempotents of $M$ form a subsemigroup, $T$ is also an idempotent semigroup (or band). Now a semigroup that is both $\mathcal{J}$-trivial and idempotent is commutative. In other words, the idempotents of $N$ commute, and $K(N)$ is contained in $R(N)$ by Theorem 3.2. The corollary now follows from Theorem 2.2. □

## Added in proof

Since the time that this paper was submitted for publication in February 1989, a number of important results related to this paper including two proofs of the Rhodes conjecture have appeared. The first proof due to Ash [17,18] uses Ramsey Theory and the Theory of Inverse Semigroups. The second proof due to Ribes and Zaleskii [22] uses the theory of profinite groups acting on profinite graphs. This last approach was motivated by a conjecture of Pin and Reutenauer [21]: "The product of a finite collection of finitely generated subgroups of a free group is closed in the profinite topology." The main result of [21] showed that this conjecture implies the Rhodes conjecture.

The main result of the present paper now immediately gives that the topological conjecture in the free monoid is true. Soon after hearing the proof of Ash, Margolis proved that the Rhodes conjecture also implied the conjecture of Pin and Reutenauer cited above [20]. Thus, the Rhodes conjecture is equivalent to both the topological conjecture on the free monoid and the Pin–Reutenauer conjecture on the profinite topology of the free group. These and many other nontrivial consequences of these important occurrences have appeared in [19].

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