Semigroup Forum Vol. 27 (1983) 247–263

RESEARCH ARTICLE

TRANSLATIONAL HULLS AND BLOCK DESIGNS

by

Stuart W. Margolis

Jeffrey H. Dinitz
Communicated by G. Lallement

I. INTRODUCTION

A number of papers have established a connection between combinatorics and completely 0-simple semigroups ([5], [7], [12]). It has also long been known that the automorphisms of a block design provide an important connection between the theory of groups and combinatorics. Here we show more generally that the translational hull $\Omega(D)$ of the Rees matrix semigroup associated with a block design D (see [8]) has a natural interpretation as a semigroup of transformations on both the points and the blocks of D.

We show that placing arithmetic restrictions on the parameters of D results in algebraic restrictions in $\Omega(D)$. For example, the longest chain of ideals in $\Omega(D)$ is bounded by the longest chain of divisors of the block size of D. Conversely, the point image of $f \in \Omega(D)$ is an arc in D ([2], [10]) and thus $\Omega(D)$ reflects some combinatorial properties of D.

In the next section we give some preliminary results including a review of the basic theory of block designs.

Let D be a balanced incomplete block design with parameters

(v, b, r, k, λ). Recall that a semigroup S is a small monoid if S consists of a group of units and a completely 0-simple ideal. In Section 3 we show that any of the following conditions implies that $\Omega(D)$ is a small monoid:

1) $r > \lambda^2$.
2) k is prime.
3) $\gcd(k, r - \lambda) = 1$.

Our main tool here is a lemma which asserts that each $f \in \Omega(D)$ is homogeneous in the following sense:

There exists an integer d such that if $q \in \text{range}(f)$, then $\text{card}(qf^{-1}) = d$. The integer d, called the degree of f, is very useful in studying the structure of $\Omega(D)$.

On the other hand, we show in Section 4 that if S is a semigroup with card(S) = n, then S can be embedded into $\Omega(D)$ where D is a design associated with a finite n dimensional projective space. We do this by generalizing the Fundamental Theorem of Projective Geometry [1].

This paper is a continuation of our work in [3] and [4]. These papers gave applications of these ideas to the theory and construction of block designs. Here our main emphasis is on the structure of the semigroups involved. Our methods here provide a framework for which to study translational hulls and more generally transitive transformation semigroups using combinatorial ideas and results. We believe that a general study of the relationship between combinatorial properties of {0,1} matrices and algebraic properties of their translational hulls will lead to important connections between the theory of semigroups and combinatorial structures.

II. PRELIMINARIES

An incidence system is a pair $D = (V,B)$ where V is a finite set of points and B is a collection of subsets of V called blocks. If D and $D' = (V',B')$ are incidence systems, then a partial function $f: V \to V'$ is continuous if $f^{-1}(b') \in B \cup \{\emptyset\}$ for all $b' \in B'$. Clearly $\Omega(D) = \{f: V \to V \mid f \text{ is continuous}\}$ is a submonoid of the monoid $PF_L(V)$ of all partial functions acting on the left of V. We call $\Omega(D)$ the monoid of continuous maps on D.

For example, if $D = (V,P(V))$, where $P(V)$ is the set of all subsets of V, then $\Omega(D) = PF_L(V)$. If $D = (V,B)$ where $B = \{\{v\} \mid v \in V\}$, then $\Omega(D)$ is the symmetric inverse semigroup on V. If B is a topology on V, then it is easy to show that $f: V \to V$ is continuous in our sense if and only if $\mathrm{dom}(f)$ is an open set and f is continuous in the topological sense on $\mathrm{dom}(f)$.

Every $f \in \Omega(D)$ induces a partial function $\overline{f}: B \to B$ acting on the right of B, defined by $b\overline{f} = \begin{cases} f^{-1}(b) & \text{if } f^{-1}(b) \neq \emptyset \\ \text{undefined otherwise.} \end{cases}$

We say that f and $\overline{f}$ are linked maps. Clearly $\overline{fg} = \overline{f}\,\overline{g}$, so the assignment $f \to \overline{f}$ is a homomorphism. We will say that an incidence structure $D = (V,B)$ is reduced if for all $v, w \in V$:

$$\{b \in B \mid v \in b\} = \{b \in B \mid w \in b\} \text{ implies } v = w.$$

It is easy to show that D is reduced if and only if the assignment $f \to \overline{f}$ is an isomorphism. See Lemma 2.1 of [3] for example. This allows us to view $\Omega(D)$ as both a transformation monoid acting on the left of V and on the right of B. This duality will be used throughout this paper.

We now show that $\Omega(D)$ is isomorphic to the translational hull of a completely 0-simple semigroup associated with D. We will assume that $\emptyset \notin B$.

Let D = (V,B) be an incidence structure. The incidence matrix of D is the v by b matrix A such that $A(v_i,b_j) = \begin{cases} 1 & \text{if } v_i \in b_j, \\ 0 & \text{otherwise.} \end{cases}$

It is well known that if Q is a set, then $PF_L(Q)$ is isomorphic to the monoid of $|Q| \times |Q|$ column monomial matrices over {0,1} under the assignment $f \to C_f$ where $C_f(v,w) = \begin{cases} 1 & \text{if } f(w) = v \\ 0 & \text{otherwise.} \end{cases}$

There is a dual result for $PF_R(Q)$ and the monoid of $|Q| \times |Q|$ row monomial matrices of {0,1}.

LEMMA 2.1. Let D = (V,B) be an incidence system having incidence matrix A. Then $f \in PF_L(V)$ is continuous on D if and only if there exists a row monomial matrix R such that $RA^t = A^tC_f$. Furthermore, if D is reduced then $R = R_{\overline{f}}$.

PROOF. Let $f \in PF_L(V)$. Direct matrix multiplication gives:

$$A^tC_f(b,v) = \begin{cases} 1 & \text{if } f(v) \in b \\ 0 & \text{otherwise.} \end{cases}$$

Thus row b of A^tC_f is the characteristic vector of $f^{-1}(b)$. Similarly if R_g is the row monomial matrix corresponding to $g\colon B \to B$, then row b of R_gA^t is the characteristic vector of bg. The assertion of the lemma follows immediately. Furthermore, if D is reduced then f uniquely determines $\overline{f}$, so that C_f uniquely determines R.

Let D = (V,B) be an incidence structure. The semigroup of D is $S(D) = (V \times B) \cup \{0\}$ where $(v,b)(v',b') = \begin{cases} (v,b') & \text{if } v' \in b \\ 0 & \text{otherwise} \end{cases}$

and

$$0 \cdot (v,b) = (v,b) \cdot 0 = 0 \cdot 0 = 0.$$

Clearly $S(D) \approx M^0(\{1\}, V, B, A^t)$.

COROLLARY 2.2. $\Omega(D)$ is isomorphic to the translational hull of S(D).

PROOF. It is well known that the translational hull of a completely 0-simple semigroup $M^o(G, I, \Lambda, P)$ is isomorphic to the semigroup of pairs of row monomial $|\Lambda| \times |\Lambda|$ matrices R and $|I| \times |I|$ column monomial matrices C over G^o such that RP = PC. See [11], for example. The corollary now follows from Lemma 2.1.

In the remainder of this paper we will work exclusively with the class of incidence structures called (balanced incomplete) block designs. We include some basic definitions and results. For further details see [6].

An incidence system D = (V,B) is a balanced incomplete block design (BIBD) with parameters (v,b,r,k,λ) if:

1) $|V| = v$, $|B| = b$.
2) Every point in V is on exactly r blocks in B.
3) Every block contains exactly k points of V.
4) Every pair of distinct points in V is on exactly λ blocks in B.

In terms of the incidence matrix A of D, 2) - 4) above translate into:

2') $AJ_{|B|x1} = rJ_{|B|x1}$

3') $J_{1x|V|}A = kJ_{1x|B|}$

4') $AA^t = (r - \lambda)I_{|V|} + \lambda J_{|V|x|V|}$

where J_{mxn} is the m by n matrix all of whose entries equal 1 and I_m is the m by m identity matrix.

It is easy to show that $\det(AA^t) = (r - \lambda)^{v-1}((v - 1)\lambda + r)$ so that (except for degenerate cases not considered here) AA^t is invertible. In particular, rank(A) = v and thus $v \leq b$. This fact is known as Fischer's Inequality. We give a generalization of this inequality for continuous maps.

LEMMA 2.3. Let $D = (V,B)$ be a block design, if $f \in \Omega(D)$, then $\text{rank}(f) \leq \text{rank}(\bar{f})$.

PROOF. By Lemma 2.1, $A^tC_f = R_{\bar{f}}A^t$. Since AA^t is invertible we have $C_f = (AA^t)^{-1}AR_{\bar{f}}A^t$ and the result follows.

Letting f be the identity map gives Fischer's inequality.

By elementary counting techniques it can also be shown that $bk = vr$ and $r(k - 1) = \lambda(v - 1)$ are relations which hold among the parameters of D. Thus any three paramenters determine the other two. We usually give the parameters (v,k,λ) in describing a block design.

We close this section with a lemma which describes the injective maps in $\Omega(D)$. Recall that an injective function is a partial 1 - 1 map.

LEMMA 2.4. Let $D = (V,B)$ be a (v,k,λ) design. If $f: V \to V \in \Omega(D)$ is a nonempty injective map, then Domain(f) = V.

PROOF. Assume that Domain(f) = W, a proper subset of V. Then $\text{card}(f(V)) = \text{card}(W) < v$. Since f is nonempty, there exist $x,y \in V$ with $x \in f(V)$, $y \notin f(V)$. But D is a block design so there is a $b \in B$ with $\{x,y\} \subset b$. Therefore, $1 \leq \text{card}(f^{-1}(b)) = \text{card}(f^{-1}(b - \{y\})) < k$ since f in injective. Therefore $f^{-1}(b) \notin B \cup \{\emptyset\}$ and f is not continuous, a contradiction.

COROLLARY 2.5. Let D be a design. Then $f \neq 0 \in \Omega(D)$ is injective if and only if $f \in \text{Aut}(D)$, the automorphism group of D.

III. IDEAL STRUCTURE OF $\Omega(D)$

In this section $D = (V,B)$ will be a BIBD with parameters (v,b,r,d,λ). We show that various relationships between the parameters of D restrict the ideal structure of $\Omega(D)$. Following [8], we call a monoid M small if M consists of a group of units and a unique completely 0-simple ideal.

LEMMA 3.1. *Let $f \in \Omega(D)$. If $r > \lambda^2$, then $\mathrm{rank}(f) = v$ or $\mathrm{rank}(f) \leq 1$.*

PROOF. See Theorem 3.3 of [3].

COROLLARY 3.2. *If D is a block design with $r > \lambda^2$, then $\Omega(D)$ is a small monoid.*

PROOF. Let $f \in \Omega(D)$. If rank $f = v$, then f is in the group of units of $\Omega(D)$. Otherwise $\mathrm{rank}(f) \leq 1$ and it follows that $f \in S(D)$, the unique completely 0-simple ideal of $\Omega(D)$.

The following example shows that the bound $r > \lambda^2$ in Lemma 3.1 is the best possible.

EXAMPLE 3.3. Let $V = \{0,1,2,3,4,5,6\}$ and let $B = \{b_i | 0 \leq i \leq 6\}$ be the collection of subsets where

$$b_0 = \{0,3,5,6\}$$
$$b_1 = \{1,4,6,0\}$$
$$b_2 = \{2,5,0,1\}$$
$$b_3 = \{3,6,1,2\}$$
$$b_4 = \{4,0,2,3\}$$
$$b_5 = \{5,1,3,4\}$$
$$b_6 = \{6,2,4,5\}$$

Then $D = (V,B)$ is a $(7,4,2)$ design.

Define $f: V \to V$ by

$$f(1) = f(3) = 0$$
$$f(2) = f(6) = 5$$
$$f(4) = f(5) = 4$$

Then,

$$f^{-1}(b_0) = f^{-1}(b_2) = b_3$$
$$f^{-1}(b_1) = f^{-1}(b_4) = b_5$$
$$f^{-1}(b_5) = f^{-1}(b_6) = b_6$$
$$f^{-1}(b_3) = \emptyset$$

Therefore, $f \in \Omega(D)$ and $1 < \text{rank}(f) < v$.

We remark that the blocks of D in the above example are the complements of the lines in the 7 point projective plane. In the next section we will analyze designs arising in such a way from finite projective spaces. The results there can be used to show that for D as in Example 3.3, $\Omega(D)$ has 3 nonzero ideals.

LEMMA 3.4 (HOMOGENEOUS LEMMA). Let $D = (V,B)$ be a (v,b,r,k,λ) design. If $f \in \Omega(D)$ is nonempty, then there is an integer d dividing k such that $\text{card}(f^{-1}(v)) = d$ for all $v \in f(V)$. Moreover, $r(k - d) = \lambda(m - d)$ where $m = \text{card}(\text{Domain}(f))$.

PROOF. Let A(V) be the abelian group freely generated by V. If $v \in V$, let $d_v = \text{card}(f^{-1}(v))$. Let $F:A(V) \to Z$ be the morphism such that $F(v) = d_v$ for each $v \in V$. If $b \in B$, let $\overline{b} = \sum_{v \in b} v$ and let $\overline{V} = \sum_{v \in V} v$. Note that $F(\overline{V}) = m$, where $m = \text{card}(\text{Domain}(f))$. Furthermore

$$(1)\quad F(\overline{b}) = \begin{cases} k \text{ if } f^{-1}(b) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

since f is continuous.

Now let $v \in f(V)$. It follows from the fact that D is a design, that

$$\sum_{v \in b} \overline{b} = rv + \lambda(\overline{V}-v) = (r-\lambda)v + \lambda\overline{V}$$

Therefore,

$$F(\sum_{v \in b} \overline{b}) = (r-\lambda)F(v) + \lambda F(\overline{V}) = (r-\lambda)d_v + \lambda m.$$

On the otherhand, it follows from (1), that

$$F(\sum_{v \varepsilon b} \overline{b}) = \sum_{v \varepsilon b} F(\overline{b}) = \sum_{v \varepsilon b} k = rk.$$

Therefore,

$$(r-\lambda)d_v + \lambda m = rk$$

and thus,

$$(2) \quad d_v = \frac{rk - \lambda m}{r - \lambda} .$$

Since the right hand side of (2) is independent of $v \varepsilon f(V)$ we have $d = \frac{rk - \lambda m}{r - \lambda} = \text{card}(f^{-1}(v))$ for all $v \varepsilon f(V)$.

We call d the degree of the continuous map f and write $d = \deg(f)$. For example, the map f in Example 3.3 has $\deg(f) = 2$.

COROLLARY 3.5. *Let $D = (V,B)$ be a block design. Then $f \varepsilon \Omega(D)$ has Domain f $= V$ if and only if $f \varepsilon \text{Aut}(D)$.*

PROOF. Assume $\text{Domain}(f) = V$. It suffices to prove that $\deg(f) = 1$ by Corollary 2.5. By the Homogeneous Lemma, $\deg(f) = \frac{rk - \lambda v}{r - \lambda}$. Since D is a design, $r(k - 1) = \lambda(v - 1)$ and thus $rk - \lambda v = r - \lambda$. Therefore, $\deg(f) = 1$.

COROLLARY 3.6. *Let $f \varepsilon \Omega(D) - \text{Aut}(D)$. Then $\text{rank}(f) < \frac{v}{2}$.*

PROOF. By Corollary 3.5 and Corollary 2.5, $\deg(f) \geq 2$. But clearly $m = \text{card}(\text{Domain}(f)) = \deg(f) \cdot \text{rank}(f)$. Therefore, $\text{rank}(f) \leq \frac{m}{2} < \frac{v}{2}$.

Notice that the map f in Example 3.3 has rank $= \frac{v - 1}{2}$, the extremal case.

An *n-arc* in block design $D = (V,B)$ is a subset W of V such that every $b \varepsilon B$ intersects W in 0 or n points. See [10]. The following corollary is immediate.

COROLLARY 3.7. *Let (V,B) be a (v,k,λ) design. Let f be a nonempty continuous map. Then $f(V)$ is a $\frac{k}{d}$ arc.*

From these results it is possible to obtain many constraints on the parameters of D and the ranks of continuous maps on D.

THEOREM 3.8. Let $D = (V,B)$ be a (v,b,r,k,λ) design. Let $f \in \Omega(D)$ have $\deg(f) = d$. If $k' = \frac{k}{d}$ and $v' = \text{card}(f(V))$, then

1) $\frac{v' - 1}{k' - 1} = \frac{v - 1}{k - 1} = \frac{r}{\lambda}$

2) If $k' < k$, then k' divides $r - \lambda$.

In this case, if $tk' = r - \lambda$, then

a) λk divides $rt(k - k')$.

b) λ divides $r(k' - 1)$.

c) λ divides $r(k - k')$.

d) λ divides $r(r - t)$.

PROOF. See [3], Theorem 4.6.

COROLLARY 3.9. Let D be a block design. If either k is prime or $\text{g.c.d.}(k, r - \lambda) = 1$, then $\Omega(D)$ is a small monoid.

PROOF. It suffices to prove that if $f \in \Omega(D)$, then $\text{rank}(f) = v$ or $\text{rank}(f) \leq 1$. Assume then that $1 < \text{rank}(f) < v$. It follows from Corollary 2.5 that $d = \deg(f) \geq 2$. Thus d divides k and $k' = \frac{k}{d}$ divides $r - \lambda$. In either case, it follows that $d = k$. Since $r(k - d) = \lambda(m - d)$ we have $m = \text{card}(\text{Domain}(f)) = k$. Hence f must map a block onto a point and thus $\text{rank}(f) = 1$.

We now relate the notion of degree of a map to the Green relations on $\Omega(D)$.

LEMMA 3.10. Let D be a design. If $s \leq_L t$ with $s,t \in \Omega(D) - \{0\}$, then $\deg(t)$ divides $\deg(s)$.

PROOF. Let $y \in \Omega(D)$ be such that $s = yt$. Since $s \neq 0$, there is a $p \in \text{range}(s)$. Then $\deg(s) = \text{card}(s^{-1}(p))$. Let $e = \text{card}(\text{range}(t) \cap y^{-1}(p))$. It follows easily from the fact that

$s^{-1}(p) = t^{-1}(y^{-1}(p))$ that $\deg(s) = \deg(t) \cdot e$.

LEMMA 3.11. If $s\mathcal{J}t$ in $\Omega(D) - \{0\}$, then $\deg(s) = \deg(t)$.

PROOF. By Lemma 3.10 we can assume $s\mathcal{R}t$. However if $s\mathcal{R}t$, then range(s) = range(t) (since $s, t \in PF_L(V)$) and it follows from Corollary 3.7 that $\frac{k}{\deg(s)} = \frac{k}{\deg(t)}$ and thus $\deg(s) = \deg(t)$.

If J is a $\mathcal{J}$ class of $\Omega(D)$ let $\deg(J) = \deg(f)$ for some $f \in J$.

Let S be a finite semigroup. The depth $S\delta$ of S is defined to be the length of the longest chain of $\mathcal{J}$ classes containing non-trivial groups. See ([13]).

THEOREM 3.12. $\Omega(D)\delta$ is less than or equal to the longest chain of proper divisors of k.

PROOF. Let $J_1 < J_2 < \dots < J_n$ be a chain of $\mathcal{J}$ classes of $\Omega(D)$ containing non-trivial groups. It is an easy exercise to show that there exist idempotents $e_i \in J_i$ $i = 1,\dots,n$ such that $e_1 < e_2 < \dots < e_n$ in the usual idempotent ordering. It follows from Lemma 3.10, that $\deg(e_{i+1})$ divides $\deg(e_i)$ for $i = 1,\dots,n-1$. Furthermore, $\deg(e_{i+1}) < \deg(e_i)$ $i = 1,\dots,n-1$. For if $\deg(e_i) = \deg(e_{i+1})$, the Homogeneous Lemma implies that $\mathrm{card}(\mathrm{Domain}(e_i)) = \mathrm{card}(\mathrm{Domain}(e_{i+1}))$ and thus $\mathrm{rank}(e_i) = \mathrm{rank}(e_{i+1})$, a contradiction. Finally $\deg(e_1) < k$, for the only $\mathcal{J}$ class J with $\deg(J) = k$ is the $\mathcal{J}$ class of elements of rank 1, which of course does not contain a nontrivial group. Thus n is bounded by the length of the longest chain of proper divisors of k and the result follows.

COROLLARY 3.13. The complexity of $\Omega(D)$ is bounded by the length of the longest chain of proper divisors of k.

PROOF. The Depth Decomposition Theorem [13] insures that the complexity of S is bounded by $S\delta$.

See [14] for an exposition of complexity theory. Our examples in the next section show that these bounds can be obtained, for each $n > 0$.

IV. FINITE PROJECTIVE SPACES

The results of the last section may indicate that the class of semigroups of the form $\Omega(D)$ for D a design is rather restricted. However, in this section we show that any finite semigroup S can be faithfully represented by continuous maps on a design D associated with a finite projective space of dimension card(S). We characterize $\Omega(D)$ by obtaining a generalization of the Fundamental Theorem of Projective Geometry [1]. We begin with some terminology.

Let $F = GF(q)$ be the finite field of order q and let V be an $n + 1$ dimensional vector space over F. Define an equivalence relation $\equiv$ on $V - \{0\}$ by $v \equiv w \iff \alpha \in F - \{0\}$ such that $v = \alpha w$. Let PG (V) be the set of equivalence classes of V modulo $\equiv$. Then PG(V) is the Desarguesian projective geometry of dimension n over F. If K is a subspace of V, let [K] be the set of equivalence classes of elements of $K - \{0\}$. Subsets of the form [K] are called subspace of PG(V). The dimension of a subspace [K] of PG(V) is one less than the dimension of K over GF(q).

There are many block designs that can be derived from PG(V). Here P will denote the design with point set PG(V) and blocks $\{[H] | H$ is a hyperplane in $V\}$. It is easy to show that P is a $(\frac{q^{n+1} - 1}{q - 1}, \frac{q^n - 1}{q - 1}, \frac{q^{n-1} - 1}{q - 1})$ design. P^c, the complement of P, is the design with points PG(V) and blocks $\{PG(V) - [H] | H$ is a hyperplane of $V\}$. P^c is a $(\frac{q^{n+1} - 1}{q - 1}, q^n, q^{n-1}(q - 1))$ design.

For example, the design in Example 3.3 is the complement of PG(V) where V is a 3 dimensional vector space over GF(2).

LEMMA 4.1. $\Omega(P)$ is a small monoid.

PROOF. Since q is a power of a prime, it follows that k and $r - \lambda$ are relatively prime. Corollary 3.9 implies that $\Omega(P)$ is small.

We now show that $\Omega(P^c)$ has a much richer structure. Recall that a semilinear map is a total function $f: V \to V$ such that

1) $f(v + w) = f(v) + f(w)$ for all $v, w \in V$.
2) $f(\alpha v) = \sigma(\alpha)\, f(v)$ for all $\alpha \in F$, $v \in V$ and some automorphism $\sigma : F \to F$.

Classically, a collineation on PG(V) is a total function $g: PG(V) \to PG(V)$ such that $g(\ell)$ is a line for every line ℓ of PG(V). The proof of the following is straightforward and is omitted.

LEMMA 4.2. The group of collineations on PG(V) is isomorphic to the group of units of both $\Omega(P)$ and $\Omega(P^c)$.

Let $f: V \to V$ be a semilinear map. The induced map $\overline{f}: PG(V) \to PG(V)$ is the partial function with $\text{Domain}(\overline{f}) = PG(V) - [\ker(f)]$ and such that $\overline{f}([v])_{\equiv}) = [f(v)]_{\equiv}$. It is clear that $\overline{f}$ is well defined. Also, if f is 1 - 1, then $\overline{f}$ is a collineation of PG(V). The Fundamental Theorem of Projective Geometry says conversely that every collineation of PG(V) is of the form $\overline{f}$ for some 1 - 1 semilinear map $f: V \to V$. More generally we have

THEOREM 4.3. If $f: V \to V$ is a semilinear map, then $\overline{f} \in \Omega(P^c)$. Conversely, if $g \in \Omega(P^c)$, then $g = \overline{f}$ for some semilinear map $f: V \to V$.

PROOF. Let $f: V \to V$ be a semilinear map. Let $H \subseteq V$ be a hyperplane and let R = Range(f) and K = Kernel(f). Then,

$$\dim(f^{-1}(H)) = \dim(H \cap R) + \dim(K).$$

Since, $\dim(R) - 1 \leq \dim(H \cap R) \leq \dim(R)$, it follows that

$\dim(V) - 1 \leq f^{-1}(H) \leq \dim(V)$. Thus, $f^{-1}(V - H)$ is either the complement of a hyperplane or empty. Therefore, $\overline{f} \in \Omega(P^C)$.

For the converse, see [4].

Let S(V) be the monoid of semilinear maps on V. Let PS(V) be the quotient of S(V) by the congruence $\theta = \{(f,g) | f = \alpha g$ for some $\alpha \in F - \{0\}\}$ PS(V) is called the monoid of projective transformations on PG(V). It is easy to verify that $(f,g) \in \theta$ iff $\overline{f} = \overline{g}$.

COROLLARY 4.4. $\Omega(P^C)$ is isomorphic to PS(V).

COROLLARY 4.5. If S is a semigroup of order n, then S is isomorphic to a subsemigroup of $\Omega(PG(V)^C)$ where V is an n + 1 dimension vector space over F.

PROOF. Clearly PS(V) contains a copy of the monoid C_{n+1} of n + 1 by n + 1 column monomial matrices over {0,1} and thus a copy of S.

We now determine local and global parameters of $\Omega(P^C)$.

COROLLARY 4.6. Let $f \in \Omega(P^C)$ have Domain(f) = D and Range(f) = R. Then:

1) D^C and R are subspaces of PG(V)

2) $\dim(PG(V)) = \dim(D^C) + \dim(R) + 1$.

PROOF. Let $f = \overline{g}$ where $g \in S(V)$. Then D^C = [ker(g)] and R = [Range(g)]. The results follow.

COROLLARY 4.7. Let $f, g \in \Omega(P^C)$. Then

1) $f \leq_{\mathcal{L}} g \;=\; (\text{Domain}(g))^C \subseteq (\text{Domain}(f))^C$

2) $f \leq_{\mathcal{R}} g \;=\; \text{Range}(f) \subseteq \text{Range}(g)$.

3) $f \leq_{\mathcal{J}} g \;=\; \dim(\text{Range}(f)) \leq \dim(\text{Range}(g))$.

PROOF. Follows from Corollary 4.4 familiar facts about S(V) and the fact that the congruence θ is contained in $\mathcal{H}$.

COROLLARY 4.8. Let $f = f^2 \in \Omega(P^c)$. The maximal subgroup containing f is isomorphic to the group of collineations of Range(f).

COROLLARY 4.9. The complexity of $\Omega(P^c)$ is equal to dim(PG(V)).

PROOF. It follows from Corollaries 4.7 and 4.8 that the depth of $\Omega(P^c)$ equals dim(PG(V)). Therefore the complexity of $\Omega(P^c) \leq$ dim(PG(V)). On the other hand, it is known that the complexity of C_{n+1} the monoid of (n + 1) x (n + 1) column monomial matrices over {0,1} is n. Since C_{n+1} is isomorphic to a submonoid of PS(V) $\approx \Omega(P^c)$, it follows that also the complexity of $\Omega(P^c) \geq$ dim(PG(V)).

Although all the examples of the form $\Omega(D)$ treated in this paper are regular semigroups, this is not true in the general case. We have constructed an example of a design D with parameters (16,24,9,6,3) such that $\Omega(D)$ is not regular.

V. CONCLUSIONS AND FURTHER QUESTIONS

It is shown here that there are interesting connections between combinatorial properties of the structure matrix C of a completely 0-simple semigroup S and algebraic properties of $\Omega(S)$, in the case that C^t is the incidence matrix of a balanced incomplete block design. Conversely, it is shown in [3] that certain combinatorial properties of C are reflected by the structure of $\Omega(S)$.

More generally, if C is a $|\Lambda| \times |I|$ regular matrix over {0,1} let $\Omega(C)$ be the translational hull of the corresponding Rees matrix semigroup. It would be interesting to find relationships between properties of C and $\Omega(C)$. For example, in [9] it is shown that if every column of C has sum $\leq k$ for some $k \in \mathbb{N}$, then the complexity of $\Omega(C)$ is also bounded by k. Lemma 3.1 indicates that the complexity of $\Omega(C)$ may be bounded by the maximum value of the inner products $c \cdot d$, where c and d are distinct columns of C. We ask whether this is true.

Let $D = (V,B)$ be a block design. We ask whether $\overline{f}$ satisfies the conclusion of the Homogeneous Lemma if $f \in \Omega(D)$. This is easily seen to be true for symmetric designs (that is, $v = b$). This would imply that $f(D) = (f(V), \overline{f}(B))$ is a design where $\overline{f}(B) = \{b' \mid \emptyset \neq b' = b\overline{f} \cap f(V)\}$.

REFERENCES

[1] Artin, E., Geometric Algebra, New York-London: Interscience, 1957.

[2] Denniston, R.H.F., On biplanes with 56 points, Ars Combinatoria, (9) 1980, 167-180.

[3] Dinitz, J.H. & S.W. Margolis, Continuous maps on block designs, to appear, Ars Combinatoria.

[4] Dinitz, J.H. & S.W. Margolis, Continuous maps in finite projective spaces, to appear, Congressus Numerantium

[5] Graham, R., On Finite O-simple semigroups and graph theory, Math. Syst. Theory (2) 1968, 325-339.

[6] Hall M., Combinatorial Theory, Ginn/Blaisdell, Waltham, Massachusetts, 1967.

[7] Houghton, C.H. Completely O-simple semigroups and their associated graphs and groups, Semigroup Forum, (14) 1977, 41-68.

[8] Lallement, G., Semigroups and Combinatorial Applications, Wiley-Interscience, New York, 1979.

[9] Margolis, S.W., K-Transformation semigroups and a conjecture of Tilson, J. Pure & App. Alg. (15) 1980, 313-322.

[10] Morgan, E.J., Arcs in block designs, Ars Combinatoria 4 (1977), 3-16.

[11] Petrich, M., The translational hull of a completely O-simple semigroup, Glasgow Math J. (9) 1968, 1-11.

[12] Pollatchek, A., Relationships between combinatorics and O-simple semigroups, J. Pure Appl. Alg. (9) 1977, 301-334.

[13] Tilson, B., Depth Decomposition Theorem, Ch. 11 in S. Eilenberg, Automata, Languages & Machines, Vol. B, Academic Press, New York, 1976.

[14] Tilson, B., Complexity of Morphisms and Semigroups, Ch. 12 in S. Eilenberg, Automata, Languages & Machines, Vol. B, Academic Press, New York, 1976.

Stuart W. Margolis
Department of Computer Science
University of Vermont
Burlington, Vermont 05405

Jeffrey H. Dinitz
Department of Mathematics
University of Vermont
Burlington, Vermont 05405

Received July 1, 1982 and October 22, 1982 in revised and final form