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Maximal equivariant compactifications ☆

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A R T I C L E   I N F O

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Dedicated to the centennial of Yu.M. Smirnov's birth

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A B S T R A C T

Let G be a locally compact group. Then for every G-space X the maximal G-proximity β_G can be characterized by the maximal topological proximity β as follows:

$$A\ \overline{\beta_G}\ B \Leftrightarrow \exists V \in N_e \quad VA\ \overline{\beta}\ VB.$$

Here, $\beta_G \colon X \to \beta_G X$ is the maximal G-compactification of X (which is an embedding for locally compact G by a classical result of J. de Vries), V is a neighbourhood of e and $A\ \overline{\beta_G}\ B$ means that the closures of A and B do not meet in $\beta_G X$.

Note that the local compactness of G is essential. This theorem comes as a corollary of a general result about maximal $\mathcal{U}$-uniform G-compactifications for a useful wide class of uniform structures $\mathcal{U}$ on G-spaces for not necessarily locally compact groups G. It helps, in particular, to derive the following result. Let $(\mathbb{U}_1, d)$ be the Urysohn sphere and $G = \mathrm{Iso}\,(\mathbb{U}_1, d)$ is its isometry group with the pointwise topology. Then for every pair of subsets A, B in $\mathbb{U}_1$, we have

$$A\ \overline{\beta_G}\ B \Leftrightarrow \exists V \in N_e \quad d(VA, VB) > 0.$$

More generally, the same is true for any $\aleph_0$-categorical metric G-structure (M, d), where $G := \mathrm{Aut}\,(M)$ is its automorphism group.

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1. Introduction

A *topological transformation group* (*G-space*) is a continuous action of a topological group G on a topological space X. Compactifiability of Tychonoff topological spaces means the existence of topological embeddings into compact Hausdorff spaces. For the compactifiability of G-spaces we require, in addition, the continuous extendability of the original action. Compactifiable G-spaces are known also as *G-Tychonoff spaces*.

Compactifications of G-spaces is a quite an active research field. We do not intend here to give a comprehensive bibliography but try to refer the interested readers to some publications, where G-compactifications play a major role. See, for example, R. Brook [8], J. de Vries [52–57], Yu.M. Smirnov [43–46], Antonyan–Smirnov [4], Smirnov–Stoyanov [47], L. Stoyanov [49,50], M. Megrelishvili [26–30,32–34], Dikranjan-Prodanov-Stoyanov [10], Megrelishvili–Scarr [36], V. Uspenskij [51], S. Antonyan [2], Gonzalez–Sanchis [15], V. Pestov [39,40], J. van Mill [37], A. Sokolovskaya [48], Google–Megrelishvili [16], Kozlov–Chatyrko [24], N. Antonyan, S. Antonyan and M. Sanchis [3], K. Kozlov [19–22], N. Antonyan [1], Karasev–Kozlov [18], Ibarlucia–Megrelishvili [17] (and many additional references in these publications).

Compactifications of a Tychonoff space X can be described in several ways:

- Banach subalgebras of $C^b(X)$ (Gelfand-Kolmogoroff 1-1 correspondence));
- Completion of totally bounded uniformities on X (Samuel compactifications);
- Proximities on X (Smirnov compactifications).

It is well known (see for example [8,4,54,55,27]) that the first two correspondences admit dynamical generalizations in the category of G-spaces. Instead of continuous bounded functions, we should use special subalgebras of generalized right uniformly continuous functions (in other terminology, *π-uniform functions*) and instead of precompact uniformities, we need now precompact *equiuniformities* (Definition 3.2).

For every Tychonoff G-space X, the algebra $\mathrm{RUC}_G(X)$ of all right uniformly continuous bounded functions on X induces the corresponding Gelfand (maximal ideal) space $\beta_G X \subset \mathrm{RUC}_G(X)^*$ and the *maximal G-compactification*

$$\beta_G \colon X \to \beta_G X.$$

By a *compactification* of X we mean a continuous dense map $c \colon X \to Y$ into a Hausdorff compact space Y. If c is a topological embedding then we say *proper compactification*. For locally compact groups G, all Tychonoff G-spaces admit proper compactifications, as was established by de Vries [55]. So, in this case, the map β_G is a topological embedding. However, in general it is not true. Resolving a question of de Vries [52], we proved in [29] that there exist noncompactifiable G-spaces (even for Polish group actions on Polish spaces).

Moreover, answering an old problem due to Smirnov, an extreme example was found by V. Pestov [40] by constructing a countable metrizable group G and a countable metrizable non-trivial G-space X for which every equivariant compactification is a singleton.

One of the most general (and widely open) attractive problems is

Problem 1.1. Clarify the structure of maximal G-compactifications $\beta_G X$ of remarkable naturally defined G-spaces X.

First of all, note that $\beta_G X$ (for nondiscrete G) usually is essentially "smaller" than βX. For instance, let G be a metrizable topological group which is not precompact. Then the canonical action $G \times \beta G \to \beta G$ is continuous iff G is discrete (Proposition 5.1).

Problem 1.2. [17, Question 1.3(b)] Study the greatest G-compactification $\beta_G\colon X \to \beta_G X$ of (natural) Polish G-spaces X. In particular: when is $\beta_G X$ metrizable?

Recall that the Ĉech–Stone compactification βX of any metrizable non-compact space X cannot be metrizable. In contrast, for several naturally defined "massive actions", $\beta_G X$ might be metrizable; sometimes even having a nice transparent geometric presentation. Perhaps the first example of this kind was a beautiful result of L. Stoyanov [49,50]. He established that the greatest $U(H)$-compactification of the unit sphere S_H in every infinite dimensional Hilbert space H is the weakly compact unit ball, where $G = U(H)$ is the unitary group of H in its standard strong operator topology.

One of the important sufficient conditions when a G-space X is G-compactifiable is the existence of a G-invariant metric on X. This was proved first by Ludescher–de Vries [25]. Another possibility to establish that such (X, d) is G-Tychonoff is to observe that in this case *Gromov compactification* $\gamma\colon (X, d) \to \gamma(X)$ is a G-compactification which is a *d-uniform* topological embedding; see the explanation in [33] using the RUC property of the distance functions $x \mapsto d(x, \cdot)$ (one may assume that d is bounded).

Pestov raised several questions in [40] about a possible coincidence between the maximal G-compactification and the Gromov compactification for some natural geometrically defined isometric actions (Urysohn sphere and Gurarij sphere, among others). These problems were studied recently in [17] (with a positive answer in the case of the Urysohn sphere and a negative answer for the Gurarij sphere).

Remark 1.3. We collect here some old and new nontrivial concrete examples when $\beta_G X$ is metrizable, usually admitting also a geometric realization.

(1) (L. Stoyanov [49,50,10]) Let $X := S_H$ be the unit sphere of the infinite-dimensional separable Hilbert space H with the unitary group $G := U(H)$. Then $\beta_G X$ is the weak compact unit ball (B_H, w) of H.
(2) [17] *Urysohn sphere* $(\mathbb{U}_1, d)$ with its isometry group $G = \mathrm{Iso}\,(\mathbb{U}_1)$. Then $\beta_G \mathbb{U}_1$ can be identified with its Gromov compactification. Moreover, $\beta_G \mathbb{U}_1$ can be identified with the compact space $K(\mathbb{U}_1)$ of all Katetov functions on X.[1]
(3) [17, Theorem 4.11] The maximal G-compactification of the unit sphere $S_{\mathfrak{G}}$ in the Gurarij Banach space $\mathfrak{G}$ (where G is the linear isometry group) is metrizable and does not coincide with its Gromov compactification. $\beta_G(S_{\mathfrak{G}})$ can be identified with the compact space $K^1_C(\mathfrak{G})$ of all normalized Katetov convex functions on $\mathfrak{G}$. These results are strongly related to some properties of the Gurarij space studied by I. Ben Yaacov [5] and Ben Yaacov–Henson [7].
(4) (Proved in [17, Theorem 4.14] thanks to an observation of Ben Yaacov) Let B_p be the unit ball of the classical Banach space $V_p := L_p[0, 1]$ for $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. Then for the linear isometry group $\mathrm{Iso}_l(V_p)$, the maximal G-compactification $\beta_G B_p$ is the Gromov compactification of the metric space B_p.
(5) [17, Theorem 4.4] For every $\aleph_0$-categorical metric structure (M, d), the maximal G-compactification of (M, d), with $G := \mathrm{A}ut\,(M)$, can be identified with the space $S_1(M)$ of all 1-types over M (and, in particular, is metrizable).
(6) (see Examples 5.4 below) Let $X = (\mathbb{Q}, \leq)$ be the rationals with the usual order but equipped with the discrete topology. Consider any dense subgroup G of the automorphism group $\mathrm{A}ut\,(\mathbb{Q}, \leq)$ with the pointwise topology (for instance, Thompson's group F). In this case $\beta_G X$ is a metrizable linearly ordered compact G-space, the actions and $\beta_G\colon X \to \beta_G$ are order preserving, where $\beta_G X$ is an inverse

[1] K. Kozlov proved in [22] that $\beta_G \mathbb{U}_1$ is homeomorphic to the Hilbert cube.

limit of finite linearly ordered spaces $\mathbb{Q}/St_F$, where $F \subset \mathbb{Q}$ is finite, St_F is the stabilizer subgroup and $\mathbb{Q}/St_F$ is the orbit space.

Whenever $\alpha\colon X \to Y$ is a compactification, one of the natural questions is which subsets A, B of X are "far" with respect to α. This means that the closures of their images do not meet in Y. This is the most basic idea of classical *proximity spaces*. See Section 2 for a description of the role of proximities and Smirnov's Theorem. This theorem shows that for every G-compactification $\alpha\colon X \to Y$ and α-far subsets A, B of X, there exists a sufficiently small neighbourhood $V \in N_e$ of the identity in G such that VA, VB are also α-far.

A natural question arises about the converse direction: when does this condition guarantee that we have a G-compactification? We show that this holds for proximities induced by a certain rich class of uniform structures on G-spaces (see Theorem 4.1). This leads to one of the main results of this paper which is to describe maximal equivariant compactification of locally compact group actions (Theorem 4.8). The local compactness of G is necessary. Indeed, there exist a Polish G-compactifiable G-space X with a Polish acting group G and G-invariant closed G-subsets A, B in X such that $A\beta_G B$ (see Example 4.10 and Remark 4.9).

A more special general problem is

Problem 1.4. For which metric G-spaces (X, d) is the following condition satisfied for every subsets A, B in X

$$A \ \overline{\beta_G} \ B \Leftrightarrow \exists V \in N_e \quad d(VA, VB) > 0.$$

Using a result from [17], we positively answer Problem 1.4 for an important class of metric G-spaces. Namely, for $\aleph_0$-*categorical metric structures* (M, d), where $G := \mathrm{A}ut\,(M, d)$ is its automorphism group. In particular, this is true for the Urysohn sphere $\mathbb{U}_1$ (Theorem 4.6).

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2. Proximities and equivariant Smirnov's theorem

2.1. Proximities and proximity spaces

In 1908, F. Riesz first formulated a set of axioms to describe the notion of closeness of pair of sets. The most useful version of proximity was introduced and studied by V.A. Efremovich [11]. We follow the setting of [38].

Definition 2.1. Let X be a nonempty set and δ be a relation in the set of all its subsets. We write $A\delta B$ if A and B are δ-related and $A\overline{\delta}B$ if not. The relation δ will be called a *proximity* on X provided that the following conditions are satisfied:

(P1) $A \cap B \neq \emptyset$ implies $A\delta B$.
(P2) $A\delta B$ implies $B\delta A$;
(P3) $A\delta B$ implies $A \neq \emptyset$;
(P4) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;
(P5) If $A\overline{\delta}B$ then there exist $C \subset X$ such that $A\overline{\delta}C$ and $(X\backslash C)\overline{\delta}B$.

A pair (X, δ) is called a *proximity space*. Two sets $A, B \subset X$ are *near* (or *proximal*) in (X, δ) if $A\delta B$ and *far* (or, *remote*) if $A\overline{\delta}B$. We say a subset $A \subset X$ is *strongly contained* in $B \subset X$ with respect to δ (or, B is a δ-*neighbourhood* of A) if $A\overline{\delta}(X \setminus B)$ and write: $A \Subset B$. In Definition 2.1 one can replace (P5) by the following axiom:

(P5′) If $A\overline{\delta}B$ then there exist subsets A_1 and B_1 of X such that $A \Subset A_1,\ B \Subset B_1$ and $A_1 \cap B_1 = \emptyset$.

Every proximity space (X, δ) induces a topology $\tau := top(\delta)$ on X by the closure operator:

$$\mathrm{cl}_\delta(A) := \{x \in X :\ x\delta A\}.$$

The topology $top(\delta)$ is Hausdorff iff the following condition satisfied:

(P6) If $x, y \in X$ and $x\delta y$ then $x = y$.

Every (separated) proximity space (X, δ) is completely regular (resp., Tychonoff) with respect to the topology $top(\delta)$. A proximity δ of X is called *continuous* (or, more precisely, a τ-*continuous proximity*) if $top(\delta) \subset \tau$. In the case of $top(\delta) = \tau$, we say that δ is a *compatible* proximity on the topological space (X, τ).

Like compactifications, the family of all proximities on X admits a natural partial order. A proximity δ_1 *dominates* δ_2 (and write $\delta_2 \preceq \delta_1$) iff for every $A\delta_1 B$ we have $A\delta_2 B$.

Example 2.2.

(1) Let Y be a compact Hausdorff space. Then there exists a unique compatible proximity on the space Y defined by

$$A\delta B \Leftrightarrow cl(A) \cap cl(B) \neq \emptyset.$$

(2) Let X be a Tychonoff space. The relation β defined by

$$A\beta B \Leftrightarrow \nexists f \in C(X) \text{ such that } f(A) = 0 \text{ and } f(B) = 1$$

is a proximity which corresponds to the greatest compatible uniformity on X. The proximity β comes from the Ĉech-Stone compactification $\beta \colon X \to \beta X$.

(3) A Hausdorff topological space X is normal iff the relation

$$A\delta_n B \quad \text{iff} \quad cl(A) \cap cl(B) \neq \emptyset$$

defines a proximity relation on the set X. Then $A\delta_n B \Leftrightarrow A\beta B$.

2.2. Smirnov's theorems

Let $c \colon X \to Y$ be a compactification. Denote by δ_c the corresponding initial proximity on X defined via the canonical proximity δ_Y of Y. More precisely, for subsets A, B of X we define $A\overline{\delta_c}B$ if $c(A)\ \overline{\delta}_Y\ c(B)$, i.e., if $cl(c(A)) \cap cl(c(B)) = \emptyset$.

Conversely every continuous proximity δ on a topological space induces a totally bounded uniformity $\mathcal{U}_\delta$. Now the completion gives *Smirnov's compactification* $s_\delta \colon (X, \delta) \to s_\delta X$. It is equivalent to the Samuel compactification with respect to the uniformity $\mathcal{U}_\delta$. This leads to a description of compactifications in terms of proximities (see, for example, [42,38,12]).

Fact 2.3 *(*`Smirnov's classical theorem`*). Let X be a topological space. Assigning to any compactification $c\colon X \to Y$ the proximity δ_c on X gives rise to a natural one-to-one order preserving correspondence between all compactifications of X and all continuous proximities on the space X.*

In the case of G-spaces, it was initiated by Smirnov himself, extending in [4] his old classical purely topological results from Tychonoff spaces to the case of group actions.

For proximities of G-compactifications we simply say *G-proximity.*

Fact 2.4 *(*`Smirnov's theorem for group actions`*). In Smirnov's bijection (Fact 2.3), G-proximities are exactly proximities δ which satisfy the following two conditions:*

(1) *(G-*invariant*) $gA\ \delta\ gB$ for every $A\ \delta\ B$ and $g \in G$;*
(2) *(compatible with the action) if $A\overline{\delta}B$ then there exists $U \in N_e$ such that $UA \cap UB = \emptyset$.*

Remark 2.5. The compatibility condition (2) can be replaced by the following (formally stronger) assumption:

(2^{str}) if $A\overline{\delta}B$ then there exists $V \in N_e$ such that $VA\overline{\delta}VB$.

In order to see that (2) implies (2^{str}), apply the axiom (P5′) (from Definition 2.1). Then for $A\overline{\delta}B$ there exist subsets A_1 and B_1 of X such that $A \Subset A_1,\ B \Subset B_1$ and $A_1\overline{\delta}B_1$. By (2) there exists $V \in N_e$ such that $VA \subset A_1, VB \subset B_2$. Therefore, $VA\overline{\delta}VB$.

The same can be derived also by results of [16, Section 5.2], where a natural generalization of Smirnov's theorem (Fact 2.4) for *semigroup actions* was obtained.

Note that the G-invariantness of δ_c guarantees that the G-action on X can be extended to a G-action on the compactification Y such that all g-translations are continuous. That is, we have a continuous action $G_{discr} \times Y \to Y$, where G_{discr} is the group G with the discrete topology (however, see Fact 3.15.6 and Theorem 3.18 below).

Example 2.6. Let X be a locally compact Hausdorff space. Then the following relation

$$A\overline{\delta}_1 B \Leftrightarrow cl(A) \cap cl(B) = \emptyset \text{ where either } cl(A) \text{ or } cl(B) \text{ is compact}$$

defines a compatible proximity on X which suits the (1-point) Alexandrov compactification. If X is a G-space then this is a G-proximity. This explains Fact 3.15.2 below.

Example 2.7. Let G/H be a coset G-space with respect to the left action $\pi\colon G \times G/H \to G/H$ and a closed subgroup H. The relation δ_R defined by

$$A\delta_R B \Leftrightarrow \exists U \in N_e(G):\ UA \cap B \neq \emptyset$$

is a compatible proximity on G/H. In fact, it is a G-proximity corresponding to the maximal G-compactification (see Fact 3.15.1 below).

Note that a natural generalization of Example 2.7 is a proximity corresponding to the maximal G-compactification in case of d-open actions; see Kozlov and Chatyrko [23].

2.3. Uniform spaces and the corresponding proximity

Recall the following standard lemma about the basis of a uniform structure (defined by the entourages – reflexive binary relations) in the sense of A. Weil.

Lemma 2.8. (see, for example, [41, Prop. 0.8]) *An abstract set $\mathcal{B}$ of entourages on X is a basis of some uniformity $\mathcal{U}$ iff the following conditions are satisfied:*

(1) $\forall \varepsilon \in \mathcal{B}\ \Delta_X \subset \varepsilon$;
(2) $\forall \varepsilon \in \mathcal{B}\ \exists \delta \in \mathcal{B}\ \delta \subset \varepsilon^{-1}$;
(3) $\forall \varepsilon, \delta \in \mathcal{B}\ \exists \gamma \in \mathcal{B}\ \gamma \subset \varepsilon \cap \delta$ *($\mathcal{B}$ is a filterbase);*
(4) $\forall \varepsilon \in \mathcal{B}\ \exists \delta \in \mathcal{B}\ \delta \circ \delta \subset \varepsilon$.

The corresponding induced uniformity is just the filter generated by $\mathcal{B}$. Each uniformity $\mathcal{U}$ on X defines a topology $top(\mathcal{U})$ on X as follows: a subset $A \subset X$ is open iff for each $a \in A$ there exists $\varepsilon \in \mathcal{U}$ such that $\varepsilon(a) \subset A$, where $\varepsilon(x) := \{y \in X : (x, y) \in \varepsilon\}$.

$top(\mathcal{U})$ is Hausdorff iff $\cap\{\varepsilon : \varepsilon \in \mathcal{B}\} = \Delta$. If otherwise not stated, in the sequel we consider only Hausdorff completely regular (i.e., Tychonoff) topological spaces, Hausdorff uniformities and proper compactifications $c\colon X \to Y$ (i.e., c is an embedding).

Definition 2.9. Let $\mathcal{U}$ be a uniformity on X. Then the relation $\delta_{\mathcal{U}}$ defined by

$$A \delta_{\mathcal{U}} B \Longleftrightarrow \varepsilon \cap (A \times B) \neq \emptyset \quad \forall \varepsilon \in \mathcal{U}$$

is a proximity on X which is called the *proximity induced by the uniformity* $\mathcal{U}$.

Always, $top(\mathcal{U}) = top(\delta_{\mathcal{U}})$. Conversely, every proximity δ on a topological space defines canonically a totally bounded compatible uniformity $\mathcal{U}_\delta$.

We say that a proximity ν on X is $\mathcal{U}$-*uniform* if $\nu \preceq \delta_{\mathcal{U}}$.

3. Uniform *G*-spaces

Definition 3.1. Let $\pi\colon G \times X \to X$ be a group action. A uniformity $\mathcal{U}$ on X is:

(1) *equicontinuous* if (the set of all translations is equicontinuous)

$$\forall x_0 \in X\ \forall \varepsilon \in \mathcal{U}\ \exists O \in N_{x_0} \quad (gx_0, gx) \in \varepsilon \qquad \forall x \in O\ \forall g \in G;$$

(2) *uniformly equicontinuous* if (the set of all translations is uniformly equicontinuous)

$$\forall \varepsilon \in \mathcal{U}\ \exists \delta \in \mathcal{U} \quad (gx, gy) \in \varepsilon \qquad \forall (x, y) \in \delta\ \forall g \in G;$$

(3) if the conditions (1) or (2) are true for a subset $P \subseteq G$ then we say that P acts equicontinuously or uniformly equicontinuously, respectively.

If d is a G-invariant metric on X, then the corresponding uniform structure $\mathcal{U}(d)$ is a very natural case of a uniformly equicontinuous uniformity.

Definition 3.2. Let $\pi\colon G \times X \to X$ be an action of a topological group G on a set X and $\mathcal{U}$ is a uniform structure on X.

(1) We say that $\mathcal{U}$ is *saturated* if every translation $\pi^g\colon X \to X$ is uniformly continuous (equivalently, if $g\varepsilon \in \mathcal{U}$ for every $\varepsilon \in \mathcal{U}$ and $g \in G$).
(2) [8] $\mathcal{U}$ is *bounded* (or, *motion equicontinuous*) if

$$\forall \varepsilon \in \mathcal{U}\ \exists V \in N_e \quad (vx, x) \in \varepsilon \quad \forall v \in V.$$

(3) $\mathcal{U}$ is *equiuniform* if it is bounded and saturated. Notation: $(X, \mathcal{U}) \in \mathrm{EUnif}^{\mathrm{G}}$.
(4) (introduced in [28,27] $\mathcal{U}$ is *quasibounded* (or, *π-uniform at e*) if

$$\forall \varepsilon \in \mathcal{U}\ \exists V \in N_e\ \exists \delta \in \mathcal{U}\ \ (vx, vy) \in \varepsilon \quad \forall (x, y) \in \delta\ \forall v \in V.$$

π-uniform will mean quasibounded and saturated (or, π-uniform at every $g_0 \in G$); meaning that

$$\forall \varepsilon \in \mathcal{U}\ \exists V \in N_{g_0}\ \exists \delta \in \mathcal{U}\ \ (vx, vy) \in \varepsilon \quad \forall (x, y) \in \delta\ \forall v \in V.$$

Notation: $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$.

Every compact G-space (with its unique uniform structure) is equiuniform. $\mathrm{EUnif}^{\mathrm{G}} \subset \mathrm{Unif}^{\mathrm{G}}$ (by the "3ε-argument") and both are closed under G-subspaces, the supremum of uniform structures, uniform products and completions (Fact 3.5). Quasibounded uniformities give simultaneous generalization of uniformly equicontinuous and bounded uniformities on a G-space. The class $\mathrm{Unif}^{\mathrm{G}}$ is characterized by Kozlov [19] in terms of *semi-uniform maps* (in the sense of J. Isbell) on products. Bounded uniformities and G-compactifications play a major role in the book of V. Pestov [39].

Remarks 3.3.

(1) [27] There exists a natural 1–1 correspondence between proper G-compactifications of X and totally bounded equiuniformities on X.
(2) If the action on $(X, \mathcal{U})$ is uniformly equicontinuous (e.g., every isometric action) then $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$.
(3) More generally: assume that there exists a neighbourhood $V \in N_e$ such that V acts uniformly equicontinuously on $(X, \mathcal{U})$ and the action of G on X is $\mathcal{U}$-saturated. Then $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$.
(4) Let G and X both are topological groups and $\alpha\colon G \times X \to X$ be a continuous action by group automorphisms. Then $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$, where $\mathcal{U}$ is right, left, two-sided or Roelcke uniformity on X.
(5) Not every quasibounded action is bounded. For example, the natural linear action of the circle group $\mathbb{T}$ on the euclidean space $\mathbb{R}^2$ is quasibounded (even, uniformly equicontinuous) because it preserves the metric but not bounded.

Proposition 3.4. *Let $\mathcal{U}$ be an equiuniformity on a G-space X. Then $\delta_{\mathcal{U}}$ is a G-proximity (hence, the corresponding Smirnov compactification and equivalently the Samuel compactification $s\colon (X, \mathcal{U}) \to sX$ are proper G-compactifications).*

Proof. Let $A \overline{\delta}_{\mathcal{U}} B$. By Definition 2.9 there exists an entourage $\varepsilon \in \mathcal{U}$, such that $(A \times B) \cap \varepsilon = \emptyset$. Fix $g_0 \in G$. Then we claim that there exist $\varepsilon' \in \mathcal{U}$ and a neighbourhood V of g_0 in G such that $V^{-1}A$ and $V^{-1}B$ are ε'-far (this means that $V^{-1}A\ \overline{\delta}_{\mathcal{U}}\ V^{-1}B$).

Since $\mathcal{U}$ is an equiuniformity, it follows that for $\varepsilon \in \mathcal{U}$ we can choose $\varepsilon' \in \mathcal{U}$ and $V \in N_{g_0}$ such that

$$(x, y) \in \varepsilon' \Longrightarrow (g_1 x, g_2 y) \in \varepsilon, \quad \forall\ g_1, g_2 \in V. \tag{3.1}$$

Now we claim that $(V^{-1}A \times V^{-1}B) \cap \varepsilon' = \emptyset$. Assuming the contrary, we get

$$\varepsilon' \cap (V^{-1}A \times V^{-1}B) \neq \emptyset \Longrightarrow \exists\, (x, y) \in (V^{-1}A \times V^{-1}B) : (x, y) \in \varepsilon'.$$

Therefore by definition of $V^{-1}A$ and $V^{-1}B$, we conclude

$$\exists\, g', g'' \in V : \quad (g'x, g''y) \in A \times B.$$

On the other hand by Formula (3.1) for $g', g'' \in V$, we have

$$(x, y) \in \varepsilon' \Longrightarrow (g'x, g''y) \in \varepsilon.$$

This means

$$\forall\, \varepsilon \in \mathcal{U} : \ (g'x, g''y) \in (A \times B) \cap \varepsilon.$$

Hence $(A \times B) \cap \varepsilon \neq \emptyset$, a contradiction. □

Fact 3.5. (Completion theorem [32]) *Let $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$. Then the action $G \times X \to X$ continuously can be extended to the action on the completion $\widehat{G} \times \widehat{X} \to \widehat{X}$, where $(\widehat{X}, \widehat{\mathcal{U}}) \in \mathrm{Unif}^{\widehat{\mathrm{G}}}$ and $\widehat{G}$ is the Raikov completion of G.*

Corollary 3.6. *Let $G_1 \subset G$ be a dense subgroup of G. Then for every Tychonoff G-space X the maximal equivariant compactifications $\beta_G X$ and $\beta_{G_1} X$ are the same.*

Every compact G-space (with its unique uniform structure) is equiuniform.

Corollary 3.7. *For totally bounded uniformities we have the coincidence $\mathrm{EUnif}^{\mathrm{G}} = \mathrm{Unif}^{\mathrm{G}}$.*

The following lemma with full proofs can be found only in my dissertation [28].

Lemma 3.8. *[27,28,30] Let X be a G-space with a topologically compatible uniformity $\mathcal{U}$. Assume that $\mathcal{U}$ is quasibounded. Then there exists a topologically compatible uniformity $\mathcal{U}^G \subseteq \mathcal{U}$ on X such that $\mathcal{U}^G$ is bounded. Furthermore,*

(1) *if $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$ then $(X, \mathcal{U}^G) \in \mathrm{EUnif}^{\mathrm{G}}$;*
(2) *if $(X, \mu) \in \mathrm{EUnif}^{\mathrm{G}}$ and $\mu \subset \mathcal{U}$ then $\mu \subset \mathcal{U}^G$.*
(3) *if $\mathcal{U}$ is totally bounded and $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$, then $\mathcal{U} = \mathcal{U}^G$ (is an equiuniformity);*
(4) *if $\mathcal{U}$ and G are metrizable, then $\mathcal{U}^G$ is also metrizable;*

Proof. For every $U \in N_e$ and $\varepsilon \in \mathcal{U}$, consider

$$[U, \varepsilon] := \{(x, y) \in X \times X : \quad \exists u_1, u_2 \in U \quad (u_1 x, u_2 y) \in \varepsilon\}. \tag{3.2}$$

Then $\Delta_X \subset \varepsilon \subset [U, \varepsilon]$ and $[U_1, \varepsilon_1] \subset [U_2, \varepsilon_2]$ for every $U_1 \subset U_2, \varepsilon_1 \subset \varepsilon_2$.

It follows that $[U_1 \cap U_2, \varepsilon_1 \cap \varepsilon_2] \subset [U_1, \varepsilon_1] \cap [U_2, \varepsilon_2]$. It is also easy to see that if $U = U^{-1}$ and $\varepsilon^{-1} = \varepsilon$ are symmetric, then also $[U, \varepsilon]^{-1} = [U, \varepsilon]$ is symmetric.

The system $\alpha := \{[U, \varepsilon]\}_{U \in N_e, \varepsilon \in \mathcal{U}}$ is a filter base on the set $X \times X$.

Define by $\mathcal{U}^G$ the corresponding filter generated by α. We show that $\mathcal{U}^G$ is a uniformity on the set X. The conditions (1), (2), (3) of Lemma 2.8 are satisfied. We have to show only condition (4) for the members of the base α.

Let $[U,\varepsilon] \in \alpha$. We have to show that there exists $[V,\delta] \in \alpha$ such that

$$[V,\delta] \circ [V,\delta] \subseteq [U,\varepsilon].$$

Choose $\varepsilon_1 \in \mathcal{U}$ such that $\varepsilon_1^2 \subseteq \varepsilon$. By the quasiboundedness condition for ε_1 there exist $\delta \in \mathcal{U}$ and $V \in N_e$ such that

$$(x,y) \in \delta, v \in V \Rightarrow (vx,vy) \in \varepsilon_1. \tag{3.3}$$

Without loss of generality (by properties of topological groups), we can assume in addition that

$$V = V^{-1}, \quad V^2 \subset U.$$

We check now that $[V,\delta]^2 \subseteq [U,\varepsilon]$. Let $(x,y),(y,z) \in [V,\delta]$. Then there exist $v_1,v_2,v_3,v_4 \in V$ such that

$$(v_1x, v_2y), (v_3x, v_4y) \in \delta.$$

Then by (3.3) (taking into account that $V = V^{-1}$), we get

$$(v_2^{-1}v_1x, y) \in \varepsilon_1, \quad (y, v_3^{-1}v_4z) \in \varepsilon_1.$$

Therefore,

$$(v_2^{-1}v_1x, v_3^{-1}v_4z) \in \varepsilon_1 \circ \varepsilon_1 \subseteq \varepsilon.$$

Since $v_2^{-1}v_1$ and $v_3^{-1}v_4$ both are in $V^{-1}V \subset U$, we conclude that $(x,z) \in [U,\varepsilon]$.
It is easy to see other axioms. So $\mathcal{U}^G$ is a uniformity.

The uniformity $\mathcal{U}^G$ is topologically compatible with X. That is, $top(\mathcal{U}^G) = top(\mathcal{U})$. Clearly, $\varepsilon \subseteq [U,\varepsilon]$ for all $U \in N_e, \varepsilon \in \mathcal{U}$. Hence, $\mathcal{U}^G \subseteq \mathcal{U}$. As to the inverse direction $\mathcal{U}^G \supseteq \mathcal{U}$, one may show that for every $\varepsilon \in \mathcal{U}$ and $x_0 \in X$ there exist $[V,\delta] \in \alpha$ such that $[V,\delta](x_0) \subseteq \varepsilon(x_0)$. Indeed, using the continuity of the action, one may choose $\delta, \gamma \in \mathcal{U}$ and $V \in N_e$ such that:

a) $g\,\gamma(x_0) \subset \varepsilon(x_0) \quad \forall g \in V$

b) $(gx_0, x_0) \in \delta \quad \forall g \in V$

c) $\delta^2 \subset \gamma, V = V^{-1}, \gamma = \gamma^{-1}$.

Now, if $y \in [V,\delta](x_0)$ then $(v_1y, v_2x_0) \in \delta$ for some $v_1, v_2 \in V$. We obtain that $(v_1y, x_0) \in \delta \circ \delta \subset \gamma$. Hence, $v_1y \in \gamma(x_0)$. So, $y \in v_1^{-1}\gamma(x_0) \subseteq \varepsilon(x_0)$.

$\mathcal{U}^G$ is bounded. Indeed, $(x, ux) \in [U,\varepsilon]$ for every $x \in X$ and every $u \in U$ (because if we choose $u_1 := u, u_2 := e \in U$ then $(u_1x, u_2ux) = (ux, ux) \in \Delta_X \subseteq \varepsilon$).

$\mathcal{U}^G$ is saturated if $\mathcal{U}$ is saturated. Let $g_0 \in G$. Then $g_0^{-1}Ug_0 \in N_e$ for every $U \in N_e$ and $g_0^{-1}\varepsilon \in \mathcal{U}$ for every $\varepsilon \in \mathcal{U}$ because $\mathcal{U}$ is saturated. Now observe that $g_0[U,\varepsilon] = [g_0^{-1}Ug_0, g_0^{-1}\varepsilon]$.

The assertions (1), (2) and (4) easily follow now from the construction. In order to check (3) consider the completion $(X,\mathcal{U}) \hookrightarrow (\widehat{X}, \widehat{\mathcal{U}})$, which, in fact is a compactification because $\mathcal{U}$ is totally bounded. According to the completion theorem [32] we have $(\widehat{X}, \widehat{\mathcal{U}}) \in \mathrm{Unif}^G$ and the action $G \times \widehat{X} \to \widehat{X}$ is continuous. Now, by Remark 3.3.1, we obtain that $\mathcal{U}$ is an equiuniformity. Hence, by (2) we conclude that $\mathcal{U} = \mathcal{U}^G$. $\square$

Recall that X is G-Tychonoff iff X admits a compatible bounded uniformity (according to an old result which goes back at least to R. Brook [8] and J. de Vries [52]).

Theorem 3.9. *[28] Let X be a Tychonoff G-space. The following are equivalent:*

(1) *X is G-Tychonoff.*
(2) *$(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$ for some compatible uniform structure $\mathcal{U}$ on X.*
(3) *There exists a compatible uniform structure $\mathcal{U}$ on X which is quasibounded.*

Proof. (1) $\Rightarrow$ (2) Let X be G-Tychonoff. Consider a proper G-compactification $c\colon X \hookrightarrow Y$. Since the natural uniformity on Y is an equiuniformity, then it induces on X a (precompact) equiuniformity $\mathcal{U}$. So, $(X, \mathcal{U}) \in \mathrm{EUnif}^{\mathrm{S}}$ with respect to some compatible uniformity $\mathcal{U}$. Now recall that $\mathrm{EUnif}^{\mathrm{G}} \subseteq \mathrm{Unif}^{\mathrm{G}}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let ξ be a quasibounded uniformity on X. Then one may easily find a finer quasibounded uniformity $\mathcal{U}$ which, in addition, is G-saturated. Indeed, the system $\Sigma := \{g\varepsilon : g \in G, \varepsilon \in \mathcal{U}\}$, where $g\varepsilon := \{(gx, gy) \in X \times X : \ (x, y) \in \varepsilon\}$ is a subbase of a filter of subsets in $X \times X$. Denote by $\mathcal{U}$ the corresponding filter generated by Σ. Then $\mathcal{U}$ is a saturated uniformity on X, $\xi \subseteq \mathcal{U}$ and $top(\xi) = top(\mathcal{U})$. If Σ is quasibounded or bounded, it is straightforward to show (use that the conjugations are continuous in any topological group), then $\mathcal{U}$ respectively is quasibounded or bounded.

Hence, we obtain $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$. Then $(X, \mathcal{U}^G) \in \mathrm{EUnif}^{\mathrm{G}}$ according to Lemma 3.8. By Proposition 3.4, $\delta_{\mathcal{U}^G}$ is a G-proximity (hence, the corresponding Smirnov's compactification is a G-compactification). $\square$

Corollary 3.10. (Ludescher–de Vries [25]) *Every continuous uniformly equicontinuous action of a topological group G on $(X, \mathcal{U})$ is G-Tychonoff. In particular, it is true if X admits a G-invariant metric.*

Lemma 3.11. *Let $\pi\colon G \times X \to X$ be a continuous action and $\Sigma := \{d_i\}_{i \in I}$ be a bounded system of pseudometrics on X such that the induced uniform structure $\mathcal{U}$ on X is topologically compatible. Assume that M is a family of nonempty subsets in G such that:*

$$\forall A \in M \ \forall d_i \in \Sigma \ \forall x, y \in X \quad d_{A,i}(x, y) := sup_{g \in A} d_i(gx, gy) < \infty.$$

Define by Σ_M the system of pseudometrics $\{d_{A,i} : \ A \in M, i \in I\}$ on X. Let $\xi = \xi(\Sigma, M)$ be the corresponding uniform structure on X generated by the system Σ_M.

(1) *If every $A \in M$ acts equicontinuously on $(X, \mathcal{U})$, then $top(\xi) = top(\mathcal{U})$.*
(2) *If for every $A \in M$ there exist $V \in N_e$ and $B \in M$ such that $AV \subseteq B$, then the action is ξ-quasibounded.*
(3) *If there exists $A \in M$ such that $e \in A$, then $\mathcal{U} \subseteq \xi$.*

Proof. (1) and (3) are straightforward.

(2) Observe that if $A \subset B$ then $d_{A,i}(x, y) \leq d_{B,i}(x, y)$. Therefore, if $VA \subseteq B$ then

$$d_{A,i}(vx, vy) = sup_{g \in A} d(gvx, gvy) \leq sup_{t \in AV} d(tx, ty) \leq sup_{t \in B} d(tx, ty) = d_{B,i}(x, y).$$

For every triple $A \in M, i \in I, \varepsilon > 0$ (such triples control the natural uniform subbase of ξ), we have

$$d_{B,i}(x, y) < \varepsilon \Rightarrow d_{A,i}(vx, vy) < \varepsilon \ \ \forall v \in V. \quad \square$$

Theorem 3.12. (de Vries [56,57] and also [28]) *Let $G \times X \to X$ be a continuous action. Suppose that a neighbourhood U of e acts equicontinuously on X with respect to some compatible uniformity $\mathcal{U}$. Then X is G-Tychonoff.*

Proof. By Theorem 3.9, it is enough to show that there exists a compatible finer uniformity $\xi \supseteq \mathcal{U}$ on the topological space X which is quasibounded.

By our assumption there exists a neighbourhood U of e in G and a compatible uniformity ξ on the topological space X such that U acts equicontinuously on $(X, \mathcal{U})$. Choose a sequence $U_n \in N_e$ such that $U_n^{-1} = U_n$, $U_{n+1}^2 \subset U_n \subset U$ for every $n \in \mathbb{N}$. Now define inductively the sequence $M := \{V_n\}_{n \in \mathbb{N}}$ of subsets in G where

$$V_n := U_1 U_2 \cdots U_n.$$

Choose also a family of pseudometrics $\Sigma := \{d_i\}_{i \in I}$ on X such that Σ generates the uniformity $\mathcal{U}$. One may assume that $d_i \leq 1$ for every i. Now we define the uniformity ξ as in Lemma 3.11 generated by the system of pseudometrics Σ_M. □

For every Tychonoff space X there exists the greatest compatible uniformity on X. We denote it by $\mathcal{U}_{max}$.

Theorem 3.13. *[28] Let G be a locally compact group. Then for every G-space X we have $(X, \mathcal{U}_{max}) \in \mathrm{Unif}^{\mathrm{G}}$.*

Proof. Any compact neighbourhood of a locally compact group acts equicontinuously. According to the proof of Theorem 3.12 there exists a compatible finer uniformity $\xi \supseteq \mathcal{U}_{max}$ on the topological space X which is quasibounded. Then by the maximality of $\mathcal{U}_{max}$ we have $\xi = \mathcal{U}_{max}$. Therefore $\mathcal{U}_{max}$ is quasibounded. In fact, again by the maximality property we obtain that $\mathcal{U}_{max}$ is also saturated. Hence, $(X, \mathcal{U}_{max}) \in \mathrm{Unif}^{\mathrm{G}}$. □

Combining Theorems 3.9 and 3.13, one directly gets the following well-known important result of de Vries:

Fact 3.14. (J. de Vries [55]) *Let G be a locally compact group. Then every Tychonoff G-space is G-compactifiable.*

Fact 3.15. *Here we list several sufficient conditions of G-compactifiability. Some of these results were already mentioned above.*

(1) *[52] Every coset G-space G/H (with natural action).*
(2) *[52] Every locally compact G-space X.*
(3) *[55] Every G-space X, where G is locally compact.*
(4) *[30,31] Let G and X both be topological groups and $\alpha\colon G \times X \to X$ is a continuous action by group automorphisms. Then X is G-Tychonoff (see Remark 3.3.4 and Corollary 4.2).*
(5) *[30] For every metric G-space (X, d), where G is a Baire space and every g-translation is d-uniform we have $(X, \mathcal{U}(d)) \in \mathrm{Unif}^{\mathrm{G}}$ (and X is G-compactifiable).*
(6) *[30] If G is Baire then every* **metrizable** *G_{discr}-compactification of a G-space X is a G-compactification.*
(7) *[51] Every G-space X, where the action is algebraically transitive, X is Baire and G is $\aleph_0$-bounded. More generally, every d-open action.*
(8) *[25] Every metric space X with a G-invariant metric. More generally, every continuous uniformly equicontinuous action of a topological group G on $(X, \mathcal{U})$.*
(9) ([56,57] and also [28]) *Let $G \times X \to X$ be a continuous action. Suppose that a neighbourhood U of e acts equicontinuously on X with respect to some compatible uniformity $\mathcal{U}$. Then X is G-Tychonoff.*
(10) (Theorem 3.18 below) *Every ordered G_{discr}-compactification of a G-space X is a G-compactification.*

A good example illustrating connection between quasibounded uniformity $\mathcal{U}$ and $\mathcal{U}^G$ is a natural action of G on a coset G/H with respect to a neutral subgroup H and quasibounded uniformity $\mathcal{L} \vee \mathcal{U}$ (where $\mathcal{L}$ and $\mathcal{U}$ are quotient uniformities of left and right uniformities on G respectively). Then $(\mathcal{L} \vee \mathcal{U})^G = \mathcal{U}$ which is the maximal equiuniformity on G/H, [22, Theorem 3.18].

3.1. Linearly ordered G-compactifications

A linearly ordered topological space (LOTS) X will mean that X is a topological space which topology is the usual interval topology for some linear order on X. In this subsection, compactifications are proper. We show that every linearly ordered G_{discr}-compactification of a G-space X with the interval topology is necessarily a G-compactification. Recall the following result of V. Fedorchuk which gives an analog of Smirnov's theorem for linearly ordered compactifications.

Definition 3.16. [13] Let $\leq$ be a linear order on X. A proximity δ on X is said to be an *ordered proximity* (with respect to $\leq$) if δ induces the interval topology $\tau_{\leq}$ on X and the following two properties are satisfied:

(a) for every $x < y$ we have $(-\infty, x] \ \overline{\delta} \ [y, +\infty)$;
(b) for every $A \ \overline{\delta} \ B$ there exists a finite number O_i, $i \in \{1, 2, \cdots, n\}$ of open $\leq_X$-convex subsets[2] such that

$$A \subset \cup_{i=1}^n O_i \subset X \setminus B.$$

Fact 3.17. (V. Fedorchuk [13]) *Let $c\colon X \to Y$ be a compactification of a LOTS X and δ_c be the corresponding proximity on X. The following conditions are equivalent:*

(1) *There exists a linear order $\leq_Y$ on Y such that Y is LOTS.*
(2) *The proximity δ_c is an ordered proximity with respect to the linear order $\leq_X$ on X inherited from $\leq_Y$.*

Note that if $c\colon X \to Y$ is a compactification, where (Y, τ) is compact with respect to some linear order $\leq_Y$ on Y (i.e., $\tau = \tau_{\leq_Y}$), then the subspace topology on X, in general, is stronger than the interval topology of the inherited order $\leq_X$ on X. The coincidence $\tau_{\leq_X} = \tau|_X$ we have iff the proximity δ_c is an ordered proximity.

Theorem 3.18. *Let X be a linearly ordered space with the interval topology of a linear order $\leq$. Let $G \times X \to X$ be a continuous action which preserves the order $\leq$. Assume that δ_c is an ordered proximity of a linearly ordered compactification $c\colon (X, \leq) \to (Y, \leq_Y)$ such that c is a G_{discr}-compactification (i.e., δ_c is G-invariant). Then c is a G-compactification.*

Proof. Since δ_c is already G-invariant, it is enough to show (by Smirnov's theorem, Fact 2.4) that the proximity is compatible with the action. That is, if $A\overline{\delta}B$ then there exists $U \in N_e$ such that $UA \cap B = \emptyset$. One may assume that A is closed. Also, in condition (b) of Fact 3.17.2, we may assume that the open convex subsets O_i are disjoint. Let $A_i := A \cap O_i$ for every $i \in \{1, 2, \cdots, n\}$. Then

$$A_{i_0} = A \setminus \cup_{i \neq i_0} O_i$$

is closed in X. Since we have finitely many i, it is enough to prove the following (this will cover also condition (a) in the definition of ordered proximity).

[2] as usual, C is said to be convex if $a, b \in C$ implies that the interval (a, b) is a subset of C.

Claim. *Let A be a closed subset of X and O is an open $\leq$-convex subset of X which contains A. Then there exists $U \in N_e$ such that $UA \subset O$.*

Proof. We may assume that $O \neq \emptyset, O \neq X$. Consider other cases for convex open subsets.

(a) $O = (-\infty, b)$.

There exists a convex neighbourhood P of b such that $A \cap P = \emptyset$. One may assume that $P = (c, \infty)$. Then $a \leq c$ for every $a \in A$. By the continuity of the action, there exists $U \in N_e$ such that $U^{-1} = U$ and $Ub \subset (c, \infty)$. For every $b \leq x$ and every $u \in U$ we have $a \leq c < ub \leq ux$ (action preserves the order). Hence, $A \cap U[b, \infty) = \emptyset$. This implies that $U^{-1}A = UA \subset O = (-\infty, b)$.

(b) $O = (b, \infty)$. This case is completely similar to (a).

(c) $O = (-b_1, b_2)$. Combine (a), (b) (taking the intersection of two neighbourhoods of e).

(d) $O = (-\infty, b]$.

Then, since O is open, b is an internal point of $(-\infty, b]$. There exists $U \in N_e$ such that $Ub \subset (-\infty, b]$. Then $UA \subset U(-\infty, b] \subset (-\infty, b]$.

(e) $O = [b, \infty)$. Similar to (d).

(f) $O = [b_1, b_2]$. Combine (d) and (e).

(g) $O = (b_1, b_2]$ or $O = [b_1, b_2)$. Combine (a), (b), (d) and (e). $\square$

4. *G*-compactifications and proximities

Theorem 4.1. *Let $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$, where G is an arbitrary topological group. Then the following rule*

$$A \ \nu \ B \Leftrightarrow \forall V \in N_e \quad VA \ \delta_{\mathcal{U}} \ VB \tag{4.1}$$

*defines a G-proximity on X such that $\delta_{\mathcal{U}^G} = \nu$ and it corresponds to the greatest $\mathcal{U}$-**uniform** G-compactification of X (that is, $\rho \preceq \nu$ for any G-proximity $\rho \preceq \delta_{\mathcal{U}}$).*

Proof. We have to show that $\delta_{\mathcal{U}^G} = \nu$. That is, ν coincides with the canonical proximity $\delta_{\mathcal{U}^G}$ of the uniformity $\mathcal{U}^G$ (where $\mathcal{U}^G$ is defined in Lemma 3.8).

Let A and B be $\mathcal{U}^G$-far subsets in X. That is, $A \ \overline{\delta_{\mathcal{U}^G}} \ B$. Lemma 3.8 guarantees that $(X, \mathcal{U}^G) \in \mathrm{EUnif}^{\mathrm{G}}$. Since $\mathcal{U}^G$ is an equiuniformity, its proximity $\delta_{\mathcal{U}^G}$ is a G-proximity by Proposition 3.4. Hence,

$$\exists \ V \in N_e \quad VA \ \overline{\delta_{\mathcal{U}^G}} \ VB.$$

Since $\mathcal{U}^G \subseteq \mathcal{U}$, we have $\delta_{\mathcal{U}^G} \preceq \delta_{\mathcal{U}}$. Therefore, $VA \ \overline{\delta_{\mathcal{U}}} \ VB$. By definition of ν this means that $A \ \overline{\nu} \ B$.

In the converse direction, we assume now that $A \ \overline{\nu} \ B$. That is, $VA \ \overline{\delta_{\mathcal{U}}} \ VB$ for some $V \in N_e$. There exists $\varepsilon \in \mathcal{U}$ such that

$$\varepsilon \cap (VA \times VB) = \emptyset.$$

This implies that

$$[V, \varepsilon] \cap (A \times B) = \emptyset,$$

where, as in Lemma 3.8, $[V, \varepsilon] = \{(x, y) \in X \times X : \ \exists v_1, v_2 \in U \quad (v_1 x, v_2 y) \in \varepsilon\}$. By definition of the uniformity $\mathcal{U}^G$, this means that $A \ \overline{\delta_{\mathcal{U}^G}} \ B$. So, we can conclude that $\nu = \delta_{\mathcal{U}^G}$ and ν is a G-proximity (because, $\delta_{\mathcal{U}^G}$ is).

Since, $\mathcal{U}^G \subseteq \mathcal{U}$, we have $\nu = \delta_{\mathcal{U}^G} \preceq \delta_{\mathcal{U}}$. This shows that the G-compactification ν of X is $\mathcal{U}$-uniform.

Finally we show the *maximality property* of ν. Let ρ be any $\mathcal{U}$-uniform G-compactification and $A\overline{\rho}B$. Then by Smirnov's Theorem 2.4 (and Remark 2.5), there exists $V \in N_e$ such that $VA\ \overline{\rho}\ VB$. Since ρ is $\mathcal{U}$-uniform, we have $\rho \preceq \delta_{\mathcal{U}}$. Therefore, $VA\ \overline{\delta_{\mathcal{U}}}\ VB$ holds. So, by Equation (4.1) we can conclude that $VA\ \overline{\nu}\ VB$. This means that $\rho \preceq \delta_{\mathcal{U}^G} = \nu$. $\square$

Corollary 4.2. *Let G and X both be topological groups and $\alpha\colon G \times X \to X$ is a continuous action by group automorphisms. Then the following condition*

$$A\ \overline{\delta}\ B \Leftrightarrow \exists V \in N_e(X)\ \exists U \in N_e(G) \quad V\{g(A)\}_{g\in U}\ \cap V\{g(B)\}_{g\in U} = \emptyset$$

defines a G-proximity which corresponds to the greatest $\mathcal{R}(X)$-uniform G-compactification, where $\mathcal{R}(X)$ is the right uniformity of X.

Proof. Observe that $\langle (X, \mathcal{R}(X)), \alpha\rangle \in \mathrm{Unif}^{\mathrm{G}}$. Now apply Theorem 4.1 to $(X, \mathcal{R}(X))$. $\square$

Definition 4.3. Let (X, d) be a metric space and $\pi\colon G \times X \to X$ is any action with uniform translations. We say that this action is *d-majored* if the greatest G-compactification of X is d-uniform.

Proposition 4.4. *Let (X, d) be a metric space and a G-space such that $(X, \mathcal{U}(d)) \in \mathrm{Unif}^{\mathrm{G}}$ (e.g., d is G-invariant). Then the following condition*

$$A\ \overline{\delta}\ B \Leftrightarrow \exists V \in N_e \quad d(VA, VB) > 0 \tag{4.2}$$

*defines a G-proximity which corresponds to the greatest d-**uniform** G-compactification of X (and coincides with the proximity of $\mathcal{U}(d)^G$).*

If, in addition, the action is d-majored then (4.2) describes the proximity of $\beta_G X$.

Proof. Let $\mathcal{U}(d)$ be the uniformity of the metric d. Its proximity δ_d is defined as follows:

$$A\delta_d B \Leftrightarrow d(A, B) = 0.$$

Now apply Theorem 4.1 to $(X, \mathcal{U}(d))$. $\square$

Remark 4.5. Let $X := (\mathbb{U}_1, d)$ be the Urysohn sphere and $G := \mathrm{Iso}\,(\mathbb{U}_1)$ be the Polish isometry group (pointwise topology). In the joint work [17] with T. Ibarlucia, we prove that for the G-space X the maximal G-compactification of X is just the Gromov compactification $\gamma(\mathbb{U}_1, d)$ (in particular, $\beta_G(\mathbb{U}_1)$ is metrizable) and $\mathrm{RUC}_G(\mathbb{U}_1)$ is the unital algebra generated by the distance functions. Since $\gamma\colon (X, d) \to \gamma X$ is a d-uniformly continuous topological G-embedding, we obtain that the greatest G-compactification of X is d-uniform. So, the action is d-majored.

Theorem 4.6. *Let $X := (\mathbb{U}_1, d)$ be the Urysohn sphere and $G := \mathrm{Iso}\,(\mathbb{U}_1)$. Then for subsets A, B in $\mathbb{U}_1$ we have:*

$$A\ \overline{\beta_G}\ B \Leftrightarrow \exists V \in N_e(G) \quad d(VA, VB) > 0.$$

Proof. Combine Remark 4.5 and Proposition 4.4. $\square$

Remark 4.7. Theorem 4.6 remains true for a large class of all $\aleph_0$-*categorical metric* G-*structures* $M = (X, d)$ (including Urysohn sphere), where $G := \mathrm{A}ut\,(M, d)$ is its automorphism group (for definitions, motivation and related tools, we refer to [6] or, [17]). In this case the action is d-majored as it follows from [17, Theorem 4.4].

Another condition which guarantees that the action is d-majored, is the *uniform micro-transitivity* of the action in the sense of [17].

Theorem 4.8. (Maximal G-compactification for locally compact group G) *Let G be a locally compact group. Then for every G-space X the maximal G-proximity β_G can be characterized by the maximal topological proximity β as follows:*

$$A\ \beta_G\ B \Leftrightarrow \forall V \in N_e \quad VA\ \beta\ VB.$$

So, if X, as a topological space is normal, then we obtain

$$A\ \beta_G\ B \Leftrightarrow \forall V \in N_e \quad Vcl(A) \cap Vcl(B) \neq \emptyset.$$

Proof. By Theorem 3.13 for the maximal compatible uniformity, we have $(X, \mathcal{U}_{max}) \in \mathrm{Unif}^{\mathrm{G}}$. In fact, $\mathcal{U}_{max}$ is the greatest compatible uniformity on X. Therefore, its proximality defines just the usual maximal compactification βX. We use the notation β for this proximity. Note that $VA\ \beta\ VB$ means that VA and VB cannot be functionally separated. Now, we use Theorem 4.1 in order to complete the proof.

If X is normal then the condition $VA\ \beta\ VB$ means that $cl(VA) \cap cl(VB) \neq \emptyset$ (see Example 2.2.3). Since G is locally compact, we can suppose that V is compact. Hence VM is closed for every closed subset $M \subset X$. In particular, we get $Vcl(A) = cl(VA)$ and $Vcl(B) = cl(VB)$. $\square$

Remark 4.9. Local compactness of G is essential. Indeed, for every Polish group G which is not locally compact, there exists a second countable G-Tychonoff space X and closed disjoint (hence, far in βX) G-**invariant** subsets A, B such that $A\ \beta_G\ B$ (cannot be separated by RUC_G functions). This follows from the proof of [36, Theorem 4.3].

Since the proof of [36, Theorem 4.3] is quite complicated, we give here a concrete simpler example for the sake of completeness. The idea is similar to [29] (see also [34]).

Example 4.10. Let $I = [0, 1]$ be the unit interval and

$$G_1 = H_+[0, 1] = \{g \in \mathrm{Homeo}\,(I) : \ g(0) = 0, g(1) = 1\}.$$

Denote by $\pi_1 \colon G_1 \times I \to I$ the natural action of G_1 on I. Then $cl(G_1O) = [0, 1]$ for every neighbourhood of 0.

Let $\{(G_n, I_n, \pi_n) : n \in \mathbb{N}\}$ be a countable system of ttg's, where each (G_n, I_n, π_n) is a copy of (G_1, I, π_1). Consider the special equivariant sum (G, X, π) of the actions π_n. So, $G = \prod_{k \in \mathbb{N}} G_k$ and $X = \bigsqcup_{k \in \mathbb{N}} I_k$. Clearly, G is a Polish group and X is a separable metrizable G-space. Define two naturally defined subsets A, B of X, where $A := \{i_n(0) : n \in \mathbb{N}\}$ is the set of all left end-points and $B := \{i_n(1) : n \in \mathbb{N}\}$ is the set of all right end-points. Then X, being locally compact, clearly is G-Tychonoff. The subsets A, B are closed disjoint and G-invariant subsets in X. So, $GA \cap GB = \emptyset$. However, they cannot separate by any RUC_G function. Hence we have $A\ \beta_G\ B$.

4.1. Equivariant normality

Two subsets A, B of a G-space X are said to be π-*disjoint* if $UA \cap UB = \emptyset$ for some $U \in N_e$. It is equivalent to require that $VA \cap B = \emptyset$ for some $V \in N_e$.

We introduced the following definition in order to answer some questions of Yu.M. Smirnov. Among others, to have a generalized *Urysohn Lemma* and a *Tietze extension theorem* for G-spaces.

Definition 4.11. [26,28] Let G be a topological group G. A G-space X is *G-normal* (or, *equinormal*) if for every pair of π-disjoint closed subsets A_1, A_2 in X there exists a pair of disjoint neighbourhoods O_1, O_2 such that O_1 and O_2 are π-disjoint.

This concept is closely related to G-proximities as the following result shows.

Fact 4.12. The following are equivalent:

(1) X is G-normal;
(2) every pair of π-disjoint closed subsets A_1, A_2 in X can be separated by a function $f \in \mathrm{RUC}_G(X)$;
(3) the relation

$$A\delta_\pi B \Leftrightarrow \forall\ V \in N_e \quad V cl(A) \cap V cl(B) \neq \emptyset$$

is a proximity on X;
(4) the relation from (3) is a G-proximity on X;
(5) the relation from (3) is the maximal G-proximity β_G on X.

Every G-normal space is G-Tychonoff. The natural action of $G := \mathbb{Q}$ on $X := \mathbb{R}$ is not G-normal. By [36], G is locally compact if and only if every normal G-space is G-normal. For some additional properties of G-normality, we refer to [1] and [34].

4.2. Description of $\beta_G X$ by filters

Remark 4.13. For every proximity space (X, δ) there exists the *Smirnov's compactification*

$$s_\delta \colon X \to s_\delta X,$$

where $s_\delta X$ is the set of all *δ-ends* (*maximal centered δ-systems*) ξ. See [42] for details. Recall that *δ-system* means that every member $A \in \xi$ is a *δ-neighbourhood* of some $B \in \xi$. That is, $B \Subset A$ holds (meaning that B and $A^c := X \setminus A$ are δ-far).

Let us apply this to the case of β_G for a G-space X.

• If G is locally compact then by Corollary 4.8, $B\ \overline{\beta_G}\ A^c$ if and only if VB and VA^c are functionally separated for some $V \in N_e$.

• Similar results for an arbitrary topological group G are not true in general. However, it is true if, in addition, the action is G-normal in the sense of Definition 4.11.

• If (M, d) is an $\aleph_0$-categorical metric structure, then by Remark 4.7 $B\ \overline{\beta_G}\ A^c$ if and only if $d(VA^c, VB) > 0$ for some $V \in N_e(\mathrm{A}ut\,(M))$.

5. Additional notes about maximal G-compactifications

5.1. Coset spaces and the greatest ambit

For every coset G-space $X := G/H$, the standard *right uniformity* $\mathcal{U}_r$ (see, for example, [41]) is the largest possible topologically compatible equiuniformity on the G-space G/H. So, G/H is a G-Tychonoff space (de Vries [52]). Moreover, the Samuel compactification of the right uniform space $(G/H, \mathcal{U}_r)$ is the greatest (proper) G-compactification. In particular, every topological group G is G-compactifiable with respect to the standard left action (Brook [8]). This G-space $\beta_G G$ is just the *greatest ambit* of G which is widely used in topological dynamics.

Let us compare this G-compactification with the usual topological greatest compactification $G \to \beta G$. The translations in this case are continuous. So, it is a G_{discr}-compactification.

Proposition 5.1. *Let G be a metrizable topological group which is not precompact. Then the canonical action $G \times \beta G \to \beta G$ is continuous if and only if G is discrete.*

Proof. Let G be not discrete. We have to show that there exists a pair of closed disjoint subsets A, B which are near with respect to right uniformity $\mathcal{U}_R$. Since G is not precompact, there exists an infinite uniformly $\mathcal{U}_R$-discrete sequence $\{a_n\}_{n=1}^{\infty}$. This means that

$$\exists U_0 \in N_e \ U_0 x_n \cap U_0 x_m = \emptyset \ \ \forall m \neq n.$$

Choose a symmetric neighbourhood $V \in N_e$ such that $V^2 \subset U_0$. Since G is metrizable and not discrete, one may choose a sequence $v_n \in V$ such that $\lim v_n = e$ and all members of this sequence are distinct. Define $A := \{a_n\}_{n=1}^{\infty}$, $B := \{v_n a_n\}_{n=1}^{\infty}$. Then $UA \cap B \neq \emptyset$ for all $U \in N_e$. Therefore, $A \beta_G B$. On the other hand, A and B are closed disjoint subsets in the normal space G. Hence, $A \overline{\beta} B$. □

Note that if G is a pseudocompact group then βG is a topological group naturally containing G (see [9]). So, in this case, $G \times \beta G \to \beta G$ is continuous and $\beta_G G = \beta G$.

5.2. Massive actions

Definition 5.2. Let $\pi \colon G \times X \to X$ be an action of a topological group G on a uniform space $(X, \mathcal{U})$. We say that the action is $\mathcal{U}$-*massive* if the uniform structure $\mathcal{U}^G$ (from Lemma 3.8) is totally bounded.

Proposition 5.3. *Let $(X, \mathcal{U}) \in \mathrm{Unif}^{\mathrm{G}}$. Consider the following conditions:*

(1) *the greatest $\mathcal{U}$-uniform G-compactification (induced by the proximity $\delta_{\mathcal{U}^G}$) of X is metrizable;*
(2) *the action is $\mathcal{U}$-massive.*

Then always (1) ⇒ (2). If, in addition, the uniformity $\mathcal{U}$ is metrizable and G is a metrizable topological group, then (2) ⇒ (1).

Proof. (1) ⇒ (2): By Theorem 4.1, the greatest $\mathcal{U}$-uniform G-compactification of X is the Smirnov compactification of the proximity $\delta_{\mathcal{U}^G}$ (which is the same as the Samuel compactification of $\mathcal{U}^G$).

Assume the contrary that $\mathcal{U}$ is not G-massive. Then by Definition 5.2, $\mathcal{U}^G$ is not totally bounded. Equivalently, X contains an infinite sequence which is $\mathcal{U}^G$-uniformly discrete. This implies that the corresponding Samuel compactification of $\mathcal{U}^G$ is not metrizable.

(2) ⇒ (1): Since $\mathcal{U}$ and G are metrizable, then $\mathcal{U}^G$ is also metrizable. By (2), $\mathcal{U}^G$ is totally precompact. Then its completion is metrizable. On the other hand, this completion is the greatest $\mathcal{U}$-uniform G-compactification by Theorem 4.1. □

Many naturally defined uniform structures are G-massive as it follows from the examples of Remark 1.3 making use of Proposition 5.3.

An extreme (but useful) sufficient condition is the case of the discrete uniform space $(X, \mathcal{U}_{discr})$. Let us say that the action is *strongly G-massive* if for every finite subset $F \subset X$ the stabilizer subgroup action $St_F \times X \to X$ has finitely many orbits.

Examples 5.4. Here we give some examples of strongly G-massive actions.

(a) $S_\infty \times \mathbb{N} \to \mathbb{N}$. In this case $\beta_G X$ (of the discrete space $X := \mathbb{N}$ with the action of the Polish symmetric group $G := S_\infty$) is the Alexandroff compactification $\mathbb{N} \cup \{\infty\}$.

(b) $X = (\mathbb{Q}, \leq)$ the rationals with the usual order but equipped with the discrete topology. Consider the automorphism group $G = \mathrm{A}ut\,(\mathbb{Q}, \leq)$ with the pointwise topology. In this case the action of G on the discrete uniform space $(X, \mathcal{U}_{discr})$ is G-massive. Hence, $\beta_G X$ is metrizable by Proposition 5.3. In fact, one may show that $X \to \beta_G X$ is a proper G-compactification such that $\beta_G X$ is a linearly ordered G-space. By Corollary 3.6, the same is true for every dense subgroup G of $\mathrm{A}ut\,(\mathbb{Q}, \leq)$ (for instance, Thompson's group F).

Sketch: We use an idea and results of [14] and [35]. Let $F := \{t_1 < t_2 < \cdots < t_m\}$ be a finite chain in $\mathbb{Q}$. Using the ultrahomogeneity of the action of $\mathrm{A}ut\,(\mathbb{Q}, \leq)$ on $(\mathbb{Q}, \leq)$, the corresponding (finite) orbit of the stabilizer subgroup St_F is

$$X_F := \{(-\infty, t_1), t_1, (t_1, t_2), t_2, (t_2, t_3), \cdots, (t_{m-1}, t_m), t_m, (t_m, \infty)\}.$$

Therefore, the present action is strongly G-massive and in particular $\mathcal{U}_{discr}$-massive ($\mathcal{U}^G_{discr}$ is totally bounded). The proximity of the uniformity $\mathcal{U}^G_{discr}$ corresponds to $\beta_G X$. On the other hand, the completion of $\mathcal{U}^G_{discr}$ can be realized as a certain inverse limit $X_\infty = \varprojlim (X_F, I)$ of finite linearly ordered sets X_F, where $F \in I$ and the finite orbit space $X_F := X/St_F$ carries the natural linear order. The G-space X_∞ is the maximal G-compactification of X. This can be done using linear modification of [14, Lemma 4.5] (which originally was related to the more complex case of circularly ordered set X).

(c) A similar result is valid for the circular version of (b). Namely, for the rationals on the circle with its circular order $X = (\mathbb{Q}/\mathbb{Z}, \circ)$, the automorphism group $G = \mathrm{A}ut\,(\mathbb{Q}/\mathbb{Z}, \circ)$ and its dense subgroups G (for instance, Thompson's circular group T). In this case $\beta_G X \setminus X$ is the universal minimal G-space $M(G)$.

6. Some open questions

The following questions are still open.

Question 6.1. (S. Antonyan–M. Megrelishvili) *Is it true that* $\dim \beta_G G = \dim G$ *for every locally compact group* G*? What if* G *is a Lie group?*

Question 6.2. (Yu.M. Smirnov (see [33] and [3])) *Let* $G = \mathbb{Q}$ *be the topological group of all rationals. Is it true that there exists a Tychonoff* G*-space which is not* G*-Tychonoff?*

Question 6.3. (H. Furstenberg and T. Scarr (see [33,40])) *Let* $G \times X \to X$ *be a continuous action with one orbit (that is, transitive) of a (metrizable) topological group* G *on a (metrizable) space* X*. Is it true that* X *is* G*-Tychonoff?*

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