Linear Group Actions On Asplund Banach Spaces

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Contents

Abstract

In this lecture we will give a brief sketch about linear group actions on Asplund spaces, focusing on the fact that in this case, the continuity of the dual action can be derived from the continuity of the original action.

1 Predefinitions

Let $(X, \|\ \|)$ be a mormed space. The strong topology on X is the the topology induced by the norm. The weak topology on X is the minimal topology on X for which all the linear functionals in the dual space X^* remain continuous. Clearly it is generated by the sets:

$$\varphi^{-1}O \qquad \varphi \in X^*, O \text{ is open in } \mathbb{R}$$

On X^* one can define another weaker topology named the weak* topology. This is the minimal topology on X^* for which the functionals:

$$F_x \in X^{**} \qquad F_x\varphi = \varphi x \qquad (x \in X, \varphi \in X^*)$$

are continuous.

A set of continuous functions F from a metric space X to a metric space Y is said to be equicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$ and $f \in F$, $d(x, y) < \delta$ derives that $d(fx, fy) < \varepsilon$.

2 Dual Actions

Let $\pi : G \times X \longrightarrow X$ a linear continuous action of a topological group G over a normed linear space $(X, \| \ \|)$. One can define the dual action of G on X^* by:

$$\pi^* : X^* \times G \longrightarrow X^* \qquad (\varphi g)(x) = \varphi(gx)$$

Unfortunately, it turns out that the dual action is usually not continuous even for "simple cases" such as $\mathbb{R}$.

Example 2.1 Let G be an infinite compact not-discrete metric group (i.e. the topology on G is metrizable. We do not demand the left and right translations to be isometries. For example, take $S = \{z \in \mathbb{C} : |z| = 1\}$). Consider the Banach space $C(G)$ of all continuous functions from G to $\mathbb{R}$ assigned with the supremum norm.

Define the "natural" action of G on $C(G)$ by:

$$g \bullet f(x) = f(gx)$$

In order to prove $\bullet$ is continuous recall that every $f \in C(G)$ is continuous on a compact set and, hence, equicontinuous. Therefore, given $\varepsilon > 0$ and $g \in G$ we can choose $\delta > 0$ such that:

$$d(x,y) < \delta \Longrightarrow d(f(x), f(y)) < \frac{\varepsilon}{2}$$

Since the left translation $x \longmapsto gx$ is continuous, there is $U \in N_g$ such that $h \in U \Longrightarrow d(gx, hx) < \delta$. Now, for every $f' \in B(f, \frac{\varepsilon}{2})$ and $h \in U$ holds:

$$\begin{aligned} d(g \bullet f(x), h \bullet f'(x)) &= d(f(gx), f'(hx)) \leq \\ &\leq d(f(gx), f(hx)) + d(f(hx), f'(hx)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and therefore, $\bullet$ is continuous.

Let us now observe in the (right) dual action of G on $C(G)^*$. For every $g \in G$ define the functional φ_g by:

$$\varphi_g(f) = f(g)$$

It is easy to verify that $\varphi_g \in X^*$.

Notice that for $h \in G$ holds:

$$(\varphi_g \bullet h)(f(x)) = \varphi_g(h \bullet f(x)) = f(hg) = \varphi_{hg}(f(x))$$

Now, consider a series $\{h_n\}$ of different elements converging to h (such exist since G is infinite and not discrete). Let us prove that not only $\varphi_g \bullet h_n \nrightarrow \varphi_{hg}$, but $\varphi_g \bullet h_n$ is not even a Couchy series.

For $a, b \in G$, $a \neq b$ define the "cone" function $c_a^b : G \longrightarrow \mathbb{R}$ by:

$$c_a^b(x) = \begin{cases} 1 - \frac{d(x,b)}{d(a,b)} & d(x,b) \leq d(a,b) \\ 0 & d(x,b) > d(a,b) \end{cases}$$

Note that:

(1) $c_a^b \in C(G)$

(2) $\|c_a^b\| = 1$

(3) $c_a^b(a) = 0$ and $c_a^b(b) = 1$

Now, for $n \neq m$:

$$[\varphi_g \bullet h_n - \varphi_g \bullet h_m] c_{h_m g}^{h_n g} = (\varphi_{h_n g} - \varphi_{h_m g}) c_{h_m g}^{h_n g} = 1 - 0 = 1$$

This implies that $\|\varphi_g \bullet h_n - \varphi_g \bullet h_m\| \geq 1$ and hence, $\varphi_g \bullet h_n$ cannot converge and the dual action is not continuous.

In this lecture we will present the class of Asplund spaces and prove that when X is an Asplund Banach space, the dual action π^* is continuous at any point.

Note that G might not be injected in $GL(X)$ for it is assigned with its own topology (which might not be Hausdorff).

3 Actions of Topological Spaces

We start by defining some properties of abstract actions:

Let $\pi : P \times X \longrightarrow X$ be an action (not necessarily continuous) of a topological space P on a metric space X. The functions $\tilde{x} : P \longrightarrow X$ and $\tilde{p} : X \longrightarrow X$ are defined in the usual manner to be $\tilde{x}p = px = \tilde{p}x$. We also define:

$$\mathrm{Con}_p^\ell(\pi) = \{x \in X : \ \tilde{x} \text{ is continuous at } p\} \qquad \mathrm{Con}^\ell(\pi) = \cap\{\mathrm{Con}_p^\ell(\pi) \ : \ p \in P\}$$

$$\mathrm{Con}_x^r(\pi) = \{p \in P : \ \tilde{p} \text{ is continuous at } x\} \qquad \mathrm{Con}^r(\pi) = \cap\{\mathrm{Con}_x^r(\pi) \ : \ x \in X\}$$

$$\mathrm{Con}_p(\pi) = \{x \in X : \ \pi \text{ is continuous at } (p, x)\}$$

$$\mathrm{Con}_x(\pi) = \{p \in P : \ \pi \text{ is continuous at } (p, x)\}$$

$$\mathrm{Con}(\pi) = \{(p, x) \in X \times P : \ \pi \text{ is continuous at } (p, x)\}$$

($\mathrm{Con}^r(\pi)$ is the set of $p \in P$ such that $\tilde{p}$ is continuous for every $x \in X$).

Note that all the claims presented in this lecture holds for both left and right actions. The proofs are similar.

Definition 3.1 *Let $\pi : P \times X \longrightarrow X$. We say that X is* π-uniform at $p_0 \in P$ *if **for every** $\varepsilon > 0$ **there are** $\delta > 0$ and $U \in N_{p_0}$ **such that for every** $p \in U$ and $a, b \in X$ for which $d(a, b) < \delta$ **holds** $d(pa, pb) < \varepsilon$.*

X is called π-uniform *if it is π-uniform at any point of P.*

Lemma 3.2 *Let X be π-uniform at $p_0 \in P$ such that $\mathrm{Con}^r(\pi) = P$ (that is, each $\tilde{p}$ is continuous), then:*

$$\mathrm{Con}_{p_0}^\ell(\pi) = \mathrm{Con}_{p_0}(\pi)$$

Lemma 3.3 *Let $(X, \|\ \|)$ be a normed space and let $\pi : P \times X \longrightarrow X$ be a linear action on X, then X is π-uniform at p_0 if and only if $(p_0, 0) \in \mathrm{Con}(\pi)$.*

Proof: Assume that X is π-uniform at p_0. Let $\varepsilon > 0$. By our assumption there is $U \in N_{p_0}$ and $\delta > 0$ such that $x, y \in X$, $d(x, y) < \delta$ and $p \in P$ derives that $d(px, py) < \varepsilon$. If we choose $y = 0$, we get that for $(p, x) \in U \times B(0, \delta)$ holds:

$$d(px, p_0 0) = d(px, 0) = d(px, p0) < \varepsilon$$

and therefore, π is continuous at $(p_0, 0)$.

Now assume that π is continuous at $(p_0, 0)$. Then for any $\varepsilon > 0$ there is $U \in N_{p_0}$ and $\delta > 0$ such that for every $(p, x) \in U \times B(0, \delta)$: $\|px\| = d(px, p0) < \varepsilon$. Thus, if $d(x, y) < \delta$ and $p \in P$:

$$d(px, py) = \|px - py\| = \|p(x - y)\| < \varepsilon$$

Thus X is π-uniform at p_0. **q.e.d.**

Definition 3.4 *Let $\pi : P \times X \longrightarrow X$ be a linear action on a normed space $(X, \| \, \|)$. We say that π is* locally bounded *at p_0 if for every bounded subset $B \subseteq X$ there is $U \in N_{p_0}$ such that $UB = \{px : p \in U, x \in B\}$ is bounded.*

π is called locally equicontinuous *at p_0 if there is $U \in N_{p_0}$ such that the family $\tilde{U}$ is equicontinuous.*

Recall that a set of linear transformations in $L(X, Y)$ is equicontinuous if and only if their norm is uniformly bounded.

Lemma 3.5 *Let $\pi : P \times X \longrightarrow X$ be a linear action on a normed space $(X, \| \, \|)$, then the following holds:*

(i) If π is continuous at $(p_0, 0)$ then π is locally equicontinuous at p_0.

(ii) If π is locally equicontinuous at p_0 then π is locally bounded at p_0.

(iii) If π is locally bounded at p_0 then $\pi^* : X^* \times P \longrightarrow X^*$ is continuous at $(0, p_0)$.

(iv) If $(p_0, 0) \in \mathrm{Con}(\pi)$ and $P = \mathrm{Con}^r(\pi)$ then $\mathrm{Con}^{\ell}_{p_0}(\pi) = \mathrm{Con}_{p_0}(\pi)$.

Proof: (i) π is continuous at $(p_0, 0)$. Therefore, there exists $U \in N_{p_0}$ and $\delta > 0$ such that for every $x \in \overline{B}(0, \delta)$ and $p \in U$ holds $\|px\| \leq 1$. Now, for arbitrary $x \in X$ and $p \in U$:

$$\|px\| = \frac{1}{\delta} \|x\| \|p(\frac{\delta}{\|x\|} x)\| \leq \frac{1}{\delta} \|x\|$$

Thus, U is norm bounded and, hence, equicontinuous.

(ii) Since π is locally equicontinuous at p_0, there is $U \in N_{p_0}$ such that $\tilde{U}$ is equicontinuous and, hence, norm bounded. Now, if $B \subseteq X$ is bounded then clearly UB is bounded and π is locally bounded at p_0.

(iii) Let $\varepsilon > 0$. We need to find $\delta > 0$ and $U \in N_{p_0}$ such that for $\varphi \in B^*(0, \delta) \subseteq X^*$ and $p \in U$ holds: $\|\varphi p\| < \varepsilon$. That is, $\|\varphi(px)\| < \varepsilon$ for $\|x\| \leq 1$. Since the set $B = B(0, 1) \subseteq X$ is bounded and X is locally bounded at p_0, we may choose U such that UB is norm-bounded, say by M. Now choose $\delta < \frac{\varepsilon}{M}$ and we are done since:

$$\|\varphi(px)\| \leq \|\varphi\| \|px\| < \frac{\varepsilon}{M} M = \varepsilon$$

(iv) Simply follows form lemmas 3.2 and 3.3. **q.e.d.**

Remark 3.6 What lemma 3.5 means is that given a continuous linear action $\pi : P \times X \longrightarrow X$, it is enough to check that the map $\tilde{\varphi} : P \longrightarrow X^*$ is continuous for all $\varphi \in X^*$ in order to prove that π^* is continuous.

4 Fragmentability

In order to continue we need to present the fragmentability term.

Definition 4.1 *Let (X, τ) be a topological space, (Y, d) be a metric space, $f : X \longrightarrow Y$ (not necessarily continuous) and Γ a system of subsets of X. We say that Γ is* fragmented *by f if for every $\phi \neq A \in \Gamma$ and $\varepsilon > 0$ there is $O \in \tau$ such that $A \cap O \neq \phi$ and $\operatorname{diam}(A \cap O) < \varepsilon$.*

In case $\Gamma = N_x$ we say that X is locally fragmented *at x (by f) and in case $\Gamma = \tau$ we say that X is* fragmented *(by f). If $\Gamma = P(A)$ (the power set), we say that A is* fragmented.

Proposition 4.2 *Let (X, τ), (Y, σ) be topological spaces and (Y, d) a metric space. Also let $f : X \longrightarrow Y$ be continuous and $g : Y \longrightarrow Z$ an arbitrary function. Then, if for $A \subseteq X$, $f(A)$ is fragmented by g, then A is fragmented by $g \circ f$.*

Proof: Let $\varepsilon > 0$ and $A' \subseteq A$. Since $f(A)$ is fragmented by g, there is $O \in \sigma$ such that $f(A') \cap O \neq \phi$ and $\operatorname{diam}(g(f(A') \cap O)) < \varepsilon$. Since f is continuous, $f^{-1}(O) \in \tau$. Now, $A' \cap f^{-1}(O) \neq \phi$ since there is some $a \in A'$ for which $a \in f(A') \cap O$ and:

$$\operatorname{diam}(g \circ f)(A' \cap f^{-1}(O)) \leq \operatorname{diam}(g)(f(A') \cap O) < \varepsilon$$

Thus, A is fragmented by $g \circ f$. **q.e.d.**

We now make some conventions: Let $(X, \| \ \|)$ be a normed space. We say that $A \subseteq X$ (X^*) is fragmented if it is fragmented by the inclusion map $A \hookrightarrow X$ ($A \hookrightarrow X^*$) when the topology on A is chosen to be the weak (weak*) topology. (The metric on X (X^*) is the metric induced by the norm.)

Proposition 4.3 *Let $\pi : G \times X \longrightarrow X$ be a group action on a metric space (X, d). Also assume that $x_0 \in X$, $g \in G$ and the following terms hold:*

(1) X is π-uniform at g.

(2) G is locally fragmented at e by the orbit map $\tilde{x}_0$.

Then $\tilde{x}_0$ is continuous at g.

Proof: Let $\varepsilon > 0$. Since X is π-uniform at g, there is $P \in N_g$ and $\delta > 0$ such that for all $h \in P$ and $x, y \in X$ for which $d(x, y) < \delta$ holds $d(hx, hy) < \varepsilon$.

We may choose $U \in N_e$ such that $gU \subseteq P$. Since G is locally fragmented at e by $\tilde{x}_0$, there is an open non-empty $O \subseteq U^{-1}$ such that:

$$\operatorname{diam}(\tilde{x}_0 O) = \operatorname{diam}(O x_0) < \delta$$

Now, choose some $u^{-1} \in O \subseteq U^{-1}$. Since $gU \subseteq P$, $gu \in P$. In addition, $g \in guO$ ($g = guu^{-1}$) and hence, $guO \in N_g$. But:

$$\mathrm{diam}(\tilde{x}_0 guO) = \mathrm{diam}(guOx_0) < \varepsilon$$

and this means that $\tilde{x}_0$ is continuous at g (since guO is a neighborhood of g), as wanted. **q.e.d.**

5 Asplund Spaces

Let $(X, \| \ \|)$ be a normed space. A function $f : X \longrightarrow \mathbb{R}$ defined on an open convex $U \subseteq X$ is called Fréchet differentiable at $x_0 \in U$ if there is $\varphi \in X^*$ such that for every $\varepsilon > 0$ and bounded $B \subseteq X$ there is $\delta > 0$ such that for all $x \in B$ and $0 \neq t \in (-\delta, \delta)$:

$$\left| \frac{f(x_0 + tx) - f(x_0)}{t} - \varphi x \right| < \varepsilon$$

Definition 5.1 *Let $(X, \| \ \|)$ be a normed space. A function f defined on an open convex subset $U \subseteq X$ is called* convex *if for every $x, y \in U$ and $0 \leq t \leq 1$:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Definition 5.2 *A normed space $(X, \| \ \|)$ is called* Asplund *if every continuous convex function (defined on a convex open subset) is Fréchet differentiable at a G_δ dense subset of its domain.*

Remark 5.3 The G_δ condition may be dropped if X is Banach.

Now one can ask what is the topological meaning of being Asplund?

For answering this question, consider the unit ball B in $(X, \| \ \|)$. We may assign two topologies to B namely the strong topology (denoted as B_s) and the weak topology (denoted B_w). We can now observe in the "identity" map:

$$i : B_s \longrightarrow B_w$$

Clearly this map is continuous. But what about its inverse i^{-1}? One can prove that i^{-1} is continuous if and only if X is finite dimensional and i^{-1} is continuous in a G_δ dense subset of B if and only if X is Asplund.

The following criteria also holds:

Theorem 5.4 *Let $(X, \| \ \|)$ be separable, then X is Asplund if and only if X^* is separable.*

This results in the following example:

Example 5.5 c_0 is an Asplund Banach space while $\ell_1 \cong c_0^*$ is not Asplund (though being the dual of an Asplund space).

In general we have:

Theorem 5.6 *A Banach space $(X, \| \ \|)$ is Asplund if and only if the dual of every separable subspace of X is separable.*

The class of Asplund spaces generalizes the class of reflexive spaces as demonstrated below:

Theorem 5.7 *Every reflexive space is Asplund.*

We now turn to present the connection between Asplund spaces and fragmentability.

Proposition 5.8 *Let $(X, \| \|)$ be a normed space and let $F \subseteq X^*$ be an equicontinuous subset of X^*. Suppose that for every non-empty relatively weak* closed subspace A of F the (not necessarily linear) functional:*

$$\sigma_A(x) = \sup\{\varphi x : \varphi \in A\}$$

is Fréchet differentiable at some point of X, then F is fragmented.

Corollary 5.9 *If F is a bounded subset of X^* where is X is Asplund, then F is fragmented.*

Proof: By the last proposition it is enough to check that σ_A is convex for every bounded $A \subseteq X^*$. Indeed, when $x, y \in X$ and $0 \leq t \leq 1$ we have:

$$\begin{aligned}
\sigma_A(tx + (1-t)y) &= \sup\{t\varphi x + (1-t)\varphi y : \varphi \in A\} \leq \\
&\leq \sup\{t\varphi x : \varphi \in A\} + \sup\{(1-t)\varphi y : \varphi \in A\} = \\
&= t\sigma_A x + (1-t)\sigma_A y
\end{aligned}$$

Thus, we are done. **q.e.d.**

The following theorem generalizes our result when X is Banach and demonstrates that one can check if a Banach space is Asplund by inspecting its dual:

Theorem 5.10 *(Namioka-Phelps) A Banach space is Asplund if and only if every bounded subset of X^* is fragmented.*

6 Linear Group Actions on Asplund Spaces

Lemma 6.1 *Let $\pi : P \times X \longrightarrow X$ be a linear action of a topological space P on a normed space $(X, \| \|)$ such that $P = \mathrm{Con}^r(\pi)$ and $(p_0, 0) \in \mathrm{Con}(\pi)$. Consider the dual action:*

$$\pi^* : X^* \times P \longrightarrow X^*$$

Then for every equicontinuous subset $F \subseteq X^*$ there is $U \in N_{p_0}$ such that FU is equicontinuous in X^*.

Theorem 6.2 *Let $(X, \| \|)$ be an Asplund Banach space and let $\pi : G \times X \longrightarrow X$ be a continuous linear action of a topological group G on X, then the dual action π^* is continuous.*

Proof: Due to lemma 3.5, it is enough to prove that the orbit map:

$$\tilde{\varphi} : G \longrightarrow X^*$$

is continuous for all $\varphi \in X^*$.

Let $g \in G$. By lemma 3.5(iii), π^* is continuous at $(0, g)$ and therefore, by lemma 3.3 it is π-uniform at g.

Now, due to lemma 6.1, there is $U \in N_e$ such that φU is equicontinuous in X^* and by theorem 5.10, φU is fragmented. Since π is continuous, the map $\tilde{\varphi}$ is weak*-continuous. Thus, by proposition 4.2, U is fragmented by $\tilde{\varphi}$. That derives that $\tilde{\varphi}$ is locally fragmented at e and, therefore, by proposition 4.3, $\tilde{\varphi}$ is continuous at g. **q.e.d.**

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