SOME BACKGROUND SEMINAR MATERIAL

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Contents

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1. Generalized convergence and nets

Abstract. We discuss generalized convergence and weak topologies.

1.1. Some topological concepts.

Definition 1.1. A sequence (a_n) in a topological space (X, τ) *converges* to a point $a \in X$ (notation: $\lim a_n = a$) if every nbd of a contains almost all points of a_n.

Theorem 1.2. *Let (X, d) be a metric space with the standard topology*

$$\tau = top(d) := \{O \subseteq X : \quad x \in O \Rightarrow B_\varepsilon(x) \subseteq O\}$$

$$= \{\cup_{i \in I} B_{\varepsilon_i}(x_i) : \quad \{x_i : \ i \in I\} \subseteq X\}$$

(1) *$x \in cl(A) = \overline{A}$ if and only if $x = \lim a_n$ for some $a_n \in X$.*
(2) *A function $f : X \to Y$ to any topological space Y is continuous if and only if $\lim f(x_n) = f(x)$ for every $\lim x_n = x$.*

Define the *sequential closure* of $A \subset X$ as

$$scl(A) := \{\lim_n a_n : \quad a_n \in A\}.$$

A topological space X is said to be a *Frechet-Urysohn* space if

$$scl(A) = cl(A)$$

for every $A \subset X$. Notation: $X \in FU$.

Exercise:

(1) $A \subset scl(A) \subset cl(A)$.
(2) Every *first countable space* X is Frechet-Urysohn. In particular, every metrizable space.

$$Metr \subset B_1 \subset FU.$$

(3) For every Frechet-Urysohn space X a function $f : X \to Y$ to any topological space Y is continuous if and only if $\lim f(x_n) = f(x)$ for every $\lim x_n = x$.

For example, Sorgenfrey line $(\mathbb{R}, \tau_s)$ is first countable (hence, FU) but not metrizable.

Examples 1.3. Here we give some natural examples of spaces which are not FU.

(1) $(\mathbb{R}, \tau_{cocount})$. This example is not Hausdorff.

Proof. $2 \in cl([0,1])$ but there is no sequence in $[0,1]$ which converges to 2. □

(2) Let A be an uncountable set and $b \notin A$. Define a topology τ on $X := A \cup \{b\}$ as follows. Call a set $O \subset X$ open ($O \in \tau$) if $b \notin U$ or if $A \setminus O$ is countable. Then (X, τ) is a Hausdorff (even, normal) topological space which is not Frechet-Urysohn. In fact, every convergent sequence in this space is eventually constant.

$scl(A) = A \subsetneq cl(A) = X$

The map $1_X : (X, \tau) \to (X, \tau_{discr})$ preserves the convergent sequences but it is not continuous.

(3) $X := \{0,1\}^{[0,1]} = \{f : [0,1] \to \{0,1\}\}$

(generalized Cantor cube. [1] It is a compact Hausdorff space.

Proof. $Y := \{f \in X | \; supp(f) \text{ is finite}\}$. Here $supp(f) := \{x \in X : \; f(x) \neq 0\}$.

is dense in X (check the definition of the product topology 1.17). There is no sequence $\{f_n\}$ in Y which converges to the constant function $\mathbf{1}(x) = 1$. Indeed, use the fact that a countable union of finite sets (look at $supp(f_n)$!) is countable and we have "uncountably many coordinates"

[1] recall that $\{0,1\}^{\mathbb{N}})$ is homeomorphic to the Cantor set

indexed by $[0,1]$, where $f_n(x) = 0 \ \forall n$. (Exercise 1.19 can help here). $\square$

(4) Hilbert space l_2 in its weak topology (see Definition 1.16). [2]

Hint: Consider

$$A := \{\sqrt{n}e_n : \quad n \in \mathbb{N}\} \subset l_2$$

Then

- no subsequence of A converges weakly to $\mathbf{0}$.
  Hint: By Banach-Steinhaus theorem every weakly bounded is norm bounded.
- $\mathbf{0} \in cl_w(A)$.

Proof. (see [3]). Consider a basic nbd of the weak topology

$$O(\varepsilon, u_1, \cdots, u_n) := \{x \in l_2 : \quad |(x, u_i)| < \varepsilon \quad i = 1, \cdots, k\}$$

where

$$\{u_i : \ i = 1, 2, \cdots, k\}$$

is a finite subset in l_2.

We have for every i

$$||u_i||^2 = \sum_{n=1}^{\infty} |(u_i, e_n)|^2 < \infty$$

So

$$\sum_{n=1}^{\infty} (\sum_{i=1}^{k} |(u_i, e_n)|)^2 < \infty$$

There exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{k} |(u_i, e_{n_0})| < \frac{\varepsilon}{\sqrt{n_0}}$$

[2] Hence, (l_2, w) is not metrizable (or, even first countable)

(otherwise, we have the harmonic series)
In particular

$$|(u_i, e_{n_0})| < \frac{\varepsilon}{\sqrt{n_0}}$$

for every i. Therefore,

$$|(\sqrt{n_0}e_{n_0}, u_i)| < \varepsilon$$

for every i.

$\square$

(5) Let $\beta : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ be the *Stone-Chech compactification* of $\mathbb{N}$. In a natural sense it is the greatest compactification of $\mathbb{N}$. Precisely, for every compactification $c : \mathbb{N} \to K$ there exists a continuous onto map $\alpha : \beta\mathbb{N} \to K$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\beta"] \arrow[dr, "c"'] & \beta\mathbb{N} \arrow[d, "\alpha"] \\ & K \end{tikzcd}$$

The space $\beta\mathbb{N}$ is certainly a compact Hausdorff space. Identify $\mathbb{N}$ with its image in $\beta\mathbb{N}$. Then

$$scl(\mathbb{N}) = \mathbb{N} \neq cl(\mathbb{N}) = \beta\mathbb{N}$$

So $\beta\mathbb{N} \notin FU$. Moreover, for every $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ the **countable** subspace

$$Y_\omega := \mathbb{N} \cup \{\omega\}$$

of $\beta\mathbb{N}$ with the unique non-isolated point ω is not FU. [3]

Note that besides (1) all other spaces in Examples 1.3 are Tychonoff spaces. Recall that a space X is *Tychonoff* iff it is a topological subspace of a compact Hausdorff space iff X is T_1

[3] See for example book of R. Engelking, *General Topology*

and continuous real-valued functions on X separate points and closed subsets.

1.2. **Nets.** Let $(D, \leq)$ be a *partially ordered set.* That is,

(1) $a \leq a$.
(2) $a \leq b$ and $b \leq c$ then $a \leq c$.

We say that $(D, \leq)$ is *directed* if in addition we have

(3) For every pair $a,\ b \in D$ there exists $c \in D$ such that $a \leq c$ and $b \leq c$.

Definition 1.4. (See for example, [2], [1], [6])

A *net* (= generalized sequence) in a space X (indexed by a partially ordered set D) is a function $i : D \to X$. Possible notation:

$$(x_\lambda)_{\lambda \in D}$$

where $i(\lambda) = x_\lambda$.

Example 1.5. Usual sequence in X is just a function

$$f : \mathbb{N} \to X, \quad f(n) = x_n$$

Example 1.6. Given a point x in a topological space, let N_x denote the set of all nbd's containing x. Then N_x is a directed set, where the direction is given by reverse inclusion, so that $T \leq S$ if and only if $S \subseteq T$. For every $U \in N_x$ choose a point $x_U \in U$. Then

$$i : N_x \to X, \quad U \mapsto x_U$$

or simpler

$$(x_U)_{U \in N_x}$$

is a net in X.

Example 1.7. Let $D = (0, 1)$ with its usual order. For $t \in D$ define $f_t(x) = sin(tx)$. Then (f_t, D) is a net of real functions. It is not a sequence.

Definition 1.8. We say that a net $(x_\lambda)_{\lambda\in D}$ in a topological space X *converges* to a point x if for every nbd $U \in N_x$ there exists $\lambda_0 \in D$ such that

$$x_\lambda \in U \quad \forall \lambda \geq \lambda_0$$

Definition 1.9. We say that x is a *cluster point* of the net $(x_\lambda)_{\lambda\in D}$ in a topological space X if for every nbd $U \in N_x$ and $\lambda_0 \in D$ there is a $\lambda \in D$ such that

$$\lambda \geq \lambda_0 \text{ and } x_\lambda \in U$$

Definition 1.10. A *subnet* of

$$(D, i) = (x_\lambda)_{\lambda\in D}$$

is a net (M, j) together with a function $h : M \to D$ s.t. $j = i \circ h$

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[dr, "j"'] & D \arrow[d, "i"] \\ & X \end{tikzcd}$$

and

$$\forall \lambda_0 \in D \; \exists \mu_0 \in M : \quad \lambda_0 \leq h(\mu) \;\; \forall \;\; \mu \geq \mu_0.$$

[4]

Remark 1.11. In most cases we may choose h to be *monotone* and then, in order to have a subnet, it suffices to check that

$$\forall \lambda_0 \in D \; \exists \mu_0 \in M : \quad \lambda_0 \leq h(\mu_0)$$

[4] another example producing a diagram in latex is

$$\begin{tikzcd} (X, \tau) \arrow[r, "\phi"] \arrow[d, "\alpha"'] & (Y, \mu) \arrow[d, "\nu"] \\ (X', \tau') \arrow[r, "\phi'"] & (Y', \mu') \end{tikzcd}$$

Lemma 1.12. *For every cluster point x of a net $(x_\lambda)_{\lambda \in D}$ in a topological space X there is a subnet that converges to x.*

Proof. Consider the index set

$$M := \{(\lambda, U) \in D \times N_x : \quad x_\lambda \in U\}$$

with the product order and the monotone map

$$h : M \to D, \quad h(\lambda, U) = \lambda.$$

For each $\lambda \in D$ there is $V \in N_x$ and $\nu \geq \lambda$ with $x_\nu \in V$, whence $h(\nu, V) \geq \lambda$. Thus $(x_{h(\mu)})_{\mu \in M}$ is a subnet of $(x_\lambda)_{\lambda \in D}$ converging to x (indeed, for every $V \in N_x$ the subnet is eventually in V, namely when $\mu \geq (\lambda, C)$ for some $(\lambda, C) \in M$ with $C \subseteq V$).

□

If X is metrizable (or first countable) then ”sequences are enough” for describing the topology. Nets are enough for every topological space !

Theorem 1.13. $x \in cl(A)$ *iff there is a net in A converging to x in (X, τ).*

Proof. Exercise (hint: Let $x \in cl(A)$. Then $\forall U \in N(x)$ one may choose $a_U \in U \cap A \neq \emptyset$. Use Example 1.6). □

Theorem 1.14. (X, τ) *is Hausdorff iff every converging net has a unique limit.*

Proof. Exercise. □

Theorem 1.15. *TFAE:*

(1) *$f : X \to Y$ is continuous.*
(2) $\lim x_\lambda = x \Rightarrow \lim f(x_\lambda) = f(x)$.

Proof. Exercise (hint: use the characterization of continuity in terms of the closure and apply Theorem 1.13). □

Definition 1.16. Let $\{f_i : X \to Y_i\}_{i \in I}$ be a family of maps from a set X into topological spaces (Y_i, τ_i). There is a weakest topology τ_w on X that makes all functions f_i continuous. Terminology: *weak* topology, *initial* topology.

A *subbasis* for τ_w is

$$\gamma := \{f_i^{-1}(O) : \quad O \in \tau_i, \ i \in I\}$$

So a basis is $\gamma^{\cap_{fin}}$ (= all possible finite intersections using subfamilies of γ). That is,

$$\gamma^{\cap_{fin}} := \{\cap_{j \in J} f_j^{-1}(O_j) : \quad O_j \in \tau_j, \ j \in J, \quad J \subseteq I \text{ is finite}\}$$

Hence, the topology $\tau_w = (\gamma^{\cap_{fin}})^{\cup}$.

Examples 1.17. (1) PRODUCTS.

Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. Consider the product

$$X := \prod_{i \in I} X_i = \{x : I \to \cup X_i : \quad x(i) := x_i \in X_i\}$$

The weak topology on X induced by the family of all projections

$$\pi_i : X \to X_i, \quad \pi_i(x) := x_i$$

is called the *product topology* (or, Tychonoff topology). So the basic nbd's are all possible

$$\cap_{j \in J}^{n} \pi_j^{-1}(O_j))$$

for **finite** $J \subseteq I$ and $O_j \in \tau_j$.

(2) Weak topology τ_w on l_2.

Generated by the family of all functionals

$$\{f_a : l_2 \to \mathbb{R} : \quad f_a(x) := (x, a)\}$$

basic nbd's are

$$O(\varepsilon, u_1, \cdots, u_n) := \{x \in l_2 : \ |(x, u_i)| < \varepsilon \ \ i = 1, \cdots, k\}$$

where

$$\{u_i : \ i = 1, 2, \cdots, k\}$$

is a finite subset in l_2 and $\varepsilon > 0$.

Lemma 1.18. *A net x_λ is convergent to x in (X, τ_w) iff $f_i(x_\lambda)$ is convergent to $f_i(x)$ in (Y_i, τ_i) for every $i \in I$.*

Exercise 1.19. Characterize the net converging in the product topology. In particular, observe that we have the "coordinate-wise convergence".

Exercise 1.20. A net u_λ in l_2 converges in the weak topology to u iff $(u_\lambda, a) \to (u, a)$ in $\mathbb{R}$ for every $a \in l_2$.

Exercise 1.21. Show that the product space $\prod_{i \in I} X_i$ is Hausdorff iff every X_i is Hausdorff.

Theorem 1.22. *$g : Z \to (X, \tau_w)$ is continuous iff $f_i \circ g : Z \to Y_i$ is continuous for every $i \in I$.*

Proof. (Straightforward) Exercise. □

Example 1.23. Many natural constructions in analysis are in fact limits of <u>*nets*</u>. For example, Riemann integral.

- Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Consider the net D of finite partitions. That is, subsets $\lambda := \{x_0, x_1, \cdots, x_n\}$ of $\mathbb{R}$ such that $a = x_0 < x_1 < \cdots < x_n = b\}$ *ordered by inclusion.* Construct two converging real valued nets on D. For each $\lambda \in D$ define

$$R_\lambda := \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} f(x) \ (x_k - x_{k-1})$$

$$r_\lambda := \sum_{k=1}^{n} \inf_{x\in[x_{k-1},x_k]} f(x)\ (x_k - x_{k-1})$$

Two nets R_λ and r_λ both converges in $\mathbb{R}$. If they have the same limit then it is just the Riemann integral $\int_a^b f(x)dx$.

- Another (more direct approach via *Riemann sums*) is by *tagged partitions* of an interval. It is a pair: (λ, T) a partition $\lambda := \{x_0, x_1, \cdots, x_n\}$ of an interval together with a finite sequence of numbers $T := \{t_0, \ldots, t_{n-1}\}$ such that for each i, $x_i \leq t_i \leq x_{i+1}$. In other words, it is a partition together with a distinguished point of every subinterval. In this case we have a net on D where D is a directed set of all tagged partitions (λ, T) and its value is the corresponding Riemann sum

$$s_{(\lambda,T)} := \sum_{k=1}^{n} f(t_{k-1})(x_k - x_{k-1})$$

$$s : D \to \mathbb{R}, \quad (\lambda, T) \mapsto s_{(\lambda,T)}$$

is a net and if it converges we get

$$\int_a^b f(x)dx := \lim_{(\lambda,T)\in D} s_{(\lambda,T)}$$

2. Compactness

Definition 2.1. A topological space X is *compact* if each open cover of X contains a finite part that also covers X.

Introduced by P. S. Alexandrov (1896 – 1982) and P. S. Urysohn (1898 – 1924).

Lemma 2.2. *A space is compact iff for every collection of its closed sets having the finite intersection property its intersection is nonempty.*

Proof. Use De Morgan's law. □

Proposition 2.3. *Some elementary properties:*

(1) *Any finite space is compact.*
(2) $[a, b]$ *is compact but* $\mathbb{R}$ *and* (a, b) *are not compact.*
(3) *Every closed subspace of a compact space is compact.*
(4) *Every compact subset of a metric space is bounded.*
(5) *Let* A *be a compact subset of a Hausdorff space* X *and* b *a point of* X *that does not belong to* A*. Then there exist open sets* $U, V \subset X$ *such that* $b \in V$ *,* $A \subset U$*, and* $U \cap V = \emptyset$*.*
(6) *Any compact subset of a Hausdorff space is closed.*
(7) *A compact Hausdorff space is normal.*
(8) *Every compact metric space is separable (and hence has a countable basis).*

Theorem 2.4. *(Heine-Borel) A subset of an Euclidean space* $\mathbb{R}^n$ *is compact iff it is closed and bounded (e.g., every* n*-dimensional cube* $[a, b]^n$*,* n*-dimensional torus* S^n*, etc.).*

Remark 2.5. Not true in infinite-dimensional Banach spaces. For example in the Hilbert space l_2 find a bounded closed subset which is not compact.

Exercise 2.6. [8] Which of the following subsets of $Mat_n(\mathbb{R})$ are compact:
(1) $GL_n = \{A : \ detA \neq 0\}$
(2) $SL_n = \{A : \ detA = 1\}$
(3) $O_n = \{\text{orthogonal matrices}\} = \{A : \ AA^t = I\}$

2.1. **Compactness and Continuous Maps.**

Theorem 2.7. *(Generalized Weierstrass thm)*

(1) *A continuous image of a compact space is compact. (In other words, if X is a compact space and $f : X \to Y$ is a continuous map, then $f(X)$ is compact.)*
(2) *A continuous numerical function on a compact space is bounded and attains its maximal and minimal values. (In other words, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.)*

Exercise 2.8. (some standard exercises)

(1) Prove that if $f : [a, b] \to R$ is a continuous function, then $f([a, b])$ is a segment.
(2) Let A be a subset of $\mathbb{R}^n$. Prove that A is compact iff each continuous numerical function on A is bounded.
(3) Prove that if F and G are disjoint subsets of a metric space (X, d), F is closed, and G is compact, then

$$d(G, F) := \inf\{d(x, y) : \ x \in F, y \in G\} > 0$$

(4) Prove that any open set U containing a compact set A of a metric space X contains an ε-neighborhood of A (i.e., the set $\{x \in X : \ d(x, A) < \varepsilon\}$ for some $\varepsilon > 0$).
(5) Prove that if A is a compact subset in a metric space (X, d) then there exist $x, y \in X$ such that $diam(A) = d(x, y)$.

2.2. **Lebesgue number and uniform continuity.**

Lemma 2.9. *Let Y be a compact subset in a metric space (X, d). Prove that for any open family Γ which covers Y there exists a (sufficiently small) number $r > 0$ (Lebesgue number) such that each open ball of radius r is contained in an element of the cover.*

Proof. Let $\{U_1, U_2, \cdots, U_n\}$ is a finite subcover of Γ. Consider

$$f(x) := max_{i=1}^{n} f_i(x)$$

where $f_i(x) := d(x, U_i^c)$. Then $f(x) > 0$ for every $x \in X$. There exists $r > 0$ such that $f(x) \geq r$ for every $x \in X$. □

Theorem 2.10. *Let $f : X \to Y$ be a continuous map from a compact metric space X to a topological space Y, and let Γ be an open cover of Y. Then there exists a number $\delta > 0$ such that for any set $A \subset X$ with diameter $diam(A) < \delta$ the image $f(A)$ is contained in an element of Γ.*

Proof. Use Lemma 2.9. □

Theorem 2.11. *Let $f : X \to Y$ be a continuous map from a compact metric space (X, d) to a metric space (Y, ρ). Then f is uniformly continuous (that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$.*

Proof. Apply Theorem 2.10 for $\Gamma := \{B_\varepsilon(y)\}_{y \in Y}$. □

2.3. **Closed Maps.** A continuous map is *closed* if the image of each closed set under this map is closed.

Proposition 2.12. *A continuous map of a compact space to a Hausdorff space is closed.*

Theorem 2.13. *A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism.*

Proof. A continuous bijection is a homeomorphism iff it is closed. □

Corollary 2.14. *A continuous injection of a compact space into a Hausdorff space is a topological embedding.*

3. Compactness and filters

3.1. Generalized Bolzano theorem.

Theorem 3.1. *(Bolzano thm) For a metric space X TFAE:*

1. *X is compact.*
2. *X is* sequentially compact *(= every sequence in X has a convergent subsequence).*
3. *X has the Bolzano property (=any infinite subset has an* accumulation point *(=each nbd of b contains infinitely many points of A).*

Proof. (1) $\Rightarrow$ (2): Let (x_n) be a sequence in X having no convergent subsequence. Then every subset of $A := \{x_n\}_{n\in\mathbb{N}}$ is closed in X. Let $A_k := \{x_n\}_{n\geq k}$. Then $\{A_k\}_{k\in\mathbb{N}}$ has FIP but $\cap A_k = \emptyset$.

(2) $\Leftrightarrow$ (3): Hint: if A is an infinite subset in X then we can choose a sequence (a_n) in A with different members.

(2) $\Rightarrow$ (1):

First note that every sequentially compact metric space (X, d) is bounded (why ?). Now we show that X has a countable basis. It is equivalent for metric spaces to show the separability. Choose arbitrary $p_0 \in X$ and consider

$$r_0 := \sup_{x\in X} d(p_0, x)$$

Then r_0 is finite by the boundedness of X. Now inductively, let p_{i+1} be chosen so that

$$\min_{0\leq n\leq i} d(p_n, p_{i+1}) \geq \frac{r_i}{2}$$

where

$$r_i := \sup_{x\in X} \min_{0\leq n\leq i} d(p_n, x)$$

Then clearly, $r_0 \geq r_1 \geq \cdots$. We claim that $r_n \to 0$. Indeed, otherwise the sequence (p_n) has no convergent subsequence (because it has no Cauchy subsequence). So, $r_n \to 0$. This means that for every $x \in X$ and every $\varepsilon > 0$ there is a p_n such that $d(p_n, x) < \varepsilon$. That is, $\{p_n\}_{n\in\mathbb{N}}$ is dense in X. So we prove that X is separable. Therefore, X has a countable basis. It follows that for every open cover there exists a countable subcover. So it is enough to prove now that the compactness property for countable covers.

Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable open cover of X with no finite subcover. Then the *decreasing family*

$$\{M_n := \cap_{i=1}^{n} U_i^c\}_{n\in\mathbb{N}}$$

has FIP and

$$\cap_{n\in\mathbb{N}} M_n = \cap_{n\in\mathbb{N}} U_n^c = \emptyset.$$

In every $M_n := \cap_{i=1}^{n} U_i^c$ choose a point x_n. We can assume that all the members are different (otherwise, $\cap_{n\in\mathbb{N}} U_n^c \neq \emptyset$). By (3) there exists an accumulation point $x \in X$ for $A := \{x_n\}_{n\in\mathbb{N}}$. Then (since the family is decreasing and every M_n is closed) we get in fact

$$x \in \cap_{n\in\mathbb{N}} U_n^c \neq \emptyset$$

$\square$

What for general (not necessarily metrizable) compact spaces ? A generalized approach can be developed by nets and *filters*.

3.2. **Filters, compactness and Tychonoff theorem.**

Definition 3.2. (Filters)

- A *filter* on X is a system α of *nonempty* subsets s.t.:
  (1) $A \cap B \in \alpha$ (equivalently, α has FIP).
  (2) If $A \in \alpha$ and $A \subset B$ then $B \in \alpha$.
- A filter on X is called **ultrafilter** if it is not properly contained (=maximality !) in any other filter on X.

- A filter α *converges* to a point $x \in X$ if $N_x \subset \alpha$.

Lemma 3.3. *Every filter (even any system with FIP) on X is contained in an ultrafilter on X.*

Proof. "Zornification argument": Let α_0 be a filter on X. Denote by B the collection of all filters on X which contains α_0. B is partially ordered by $\subseteq$. If T is a some totally ordered subset of B then

$$\alpha_T := \{A : \ A \in \alpha \in T\}$$

is a filter on X and $\alpha \subseteq \alpha_T$ for every $\alpha \in B$. By Zorn's lemma there is a maximal element which is obviously on X (containing α_0). $\square$

Lemma 3.4. *If α is an ultrafilter in X and $Y \subset X$ then precisely one of the sets Y or Y^{co} is contained in α.*

Proof. If $Y \notin \alpha$ then there exists $A \in \alpha$ s.t. $A \cap Y = \emptyset$ (otherwise, $\alpha \cup \{Y\}$, having FIP, is contained in an ultrafilter, a contradiction). So $A \subset Y^c$. Therefore, $Y^c \in \alpha$. $\square$

Corollary 3.5. *Let α be an ultrafilter on X. If $A_1 \cup A_2 \cup \cdots A_n \in \alpha$ then at least one $A_i \in \alpha$.*

Proof. Exercise. $\square$

Theorem 3.6. *A topological space X is compact iff every ultrafilter on X converges to a point in X.*

Proof. Let α be a ultrafilter. If it does not converge to any $p \in X$ then some open nbd $U_p \notin \alpha$ and thus its complement U_p^c is in α (Lemma 3.4). So X is covered by $\{U_p : \ p \in X\}$. By compactness there is a finite subcover

$$X = \cup_{i=1}^n U_{p_i}.$$

We get that

$$\emptyset = \cap_{i=1}^n U_{p_i}^c.$$

Means that α is not a filter.

Conversely, suppose that every ultrafilter converges in X. Let F be a family of closed subsets in X with FIP. There exists an ultrafilter α which contains F. There exists a point p s.t. α converges to p. Then $p \in cl(A)$ for every $A \in \alpha$ (why ?). In particular, $p \in cl(A) = A$ for every $A \in F$. $\square$

Lemma 3.7. *Every net in X has a* universal subnet *(=for every subset $A \subseteq X$ the net is either eventually in A or eventually in A^c).*

Theorem 3.8. [2, 5] *For a topological space (X, τ) TFAE:*

1. *X is compact.*
2. *Every system of closed subsets with FIP property has the nonempty intersection.*
3. *Every net in X has a cluster point.*
4. *Every ultrafilter in X is convergent.*
5. *Every net in X has a convergent subnet.*

Theorem 3.9. *A product $X := \prod_{i \in I} X_i$ is compact if (and only if) each X_i is compact.*

Proof. "Only if" is trivial (why ?).

"If": Let α be an ultrafilter in X then the set

$$\alpha_i := \{\pi_i(A) : \ A \in \alpha\}$$

is ultrafilter (why ?) for every $i \in I$. Since every X_i is compact each α_i converges (Theorem 3.6) in X_i to some p_i. Then α converges to $p \in X$ where $p(i) := p_i$. $\square$

Corollary 3.10. *The following spaces are compact:*

(1) $[0,1]^I$ *(Tychonoff cube).*

$\Downarrow$

(2) *In particular, the Hilbert cube* $[0,1]^{\mathbb{N}}$.

(3) *Generalized Cantor cube* $\{0,1\}^I$.

$\Downarrow$

(4) *In particular, Cantor cube* $\{0,1\}^{\mathbb{N}}$ *(which is homeomorphic to usual Cantor set).*

Exercise 3.11. Show that $\{0,1\}^{\mathbb{N}}$ is homeomorphic with the Cantor set $C \subset [0,1]$.

Proof. Hint: First recall that $C := \cap_{n\in\mathbb{N}} C_n$, where $C_1 := [0,1]$ and C_{n+1} is obtained from C_n by deleting the "open middle third". Furthermore, C can be identified with

$$\{x \in [0,1] : \quad x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad a_n \in \{0,2\}\}$$

Now by Theorem 2.13 show that C is homeomorphic with $\{0,2\}^{\mathbb{N}}$. $\square$

Exercise 3.12. Show that $[0,1]$ is a continuous image of the Cantor set.

Proof. Hint: Consider the function

$$f : C \to [0,1], \; f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}} \quad a_n \in \{0,2\}\}$$

for every $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in C$. $\square$

Exercise 3.13. Is it true that $C^2 \cong C$, $C^n \cong C$ for every $n \in \mathbb{N}$, $C^{\mathbb{N}} \cong C$?

Exercise 3.14. (see for example [7]) Prove that the Hilbert cube is a continuous image of the Cantor set.

Remark 3.15. ** (Alexandrov & Urysohn, [7]) Every compact metric space X is a continuous image of C.

Corollary 3.16. *(Peano, "space-filling curves") There exists a continuous onto map* $p : [0,1] \to [0,1] \times [0,1]$.

Hint: Use the Tietze extension theorem.

3.3. **Universality of the Hilbert cube.**

Theorem 3.17. *Every compact metrizable space X can be embedded topologically into the Hilbert cube $[0,1]^{\mathbb{N}}$.*

Proof. Hint: There exists a countable family of functions $f_n : X \to [0,1]$ separating points of X. Then the diagonal product

$$F : X \to \prod_{n \in \mathbb{N}} [0,1]^{\mathbb{N}}, \;\; F(x) = (f_n(x))_{n \in \mathbb{N}}$$

is a topological embedding (by Corollary 2.14). □

In fact we have a more general result

Theorem 3.18. *(Urysohn's theorem) Every separable metrizable space X can be embedded topologically into the Hilbert cube $[0,1]^{\mathbb{N}}$.*

Proof. Hint: There exists a countable family of functions $f_n : X \to [0,1]$ separating points and closed subsets of X (Urysohn functions for every pair U, V of a countable basis where $cl(U) \subset V$). Then the diagonal product

$$F : X \to \prod_{n \in \mathbb{N}} [0,1]^{\mathbb{N}}, \;\; F(x) = (f_n(x))_{n \in \mathbb{N}}$$

is a topological embedding. □

Exercise 3.19. Is it true that every separable metrizable space X can be embedded topologically into $(0,1)^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$?

Exercise 3.20. Prove that a topological space X is compact and metrizable if and only if X is a closed subspace of the Hilbert cube $[0,1]^{\mathbb{N}}$.

Definition 3.21. A (proper) compactification of X is a pair (K, j) where: K is a compact Hausdorff space, $j : X \to Y$ is a continuous map (resp., an embedding) and $cl(j(X)) = K$.

Theorem 3.22. *The following conditions are equivalent:*

(1) *X is a Tychonoff space (= Hausdorff and for every closed subset A and $b \notin A$ there exists a separating function $f : X \to \mathbb{R}$ with $f(b) = 0, f(A) = 1$).*
(2) *X has a proper compactification.*
(3) *X is a subspace of a compact space.*

Exercise 3.23. (1) Prove that every metrizable space X is Tychonoff.

Hint (even normal !): For every pair of closed disjoint subsets $A,\ B$ consider the function

$$f : X \to \mathbb{R},\ f(x) := \frac{d(x, A)}{d(xA) + d(x, B)}$$

(2) Is it true that every separable metrizable space X has a separable metrizable compactification ?
(3) Prove or disprove: Any metrizable space has a proper compactification.
(4) Prove that every locally compact Hausdorff space admits the *one-point compactification* (=Alexandrov compactification).

4. Completeness and topological groups

4.1. Completeness type conditions.

Definition 4.1. (1) A sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ s.t.

$$d(a_i, a_j) < \varepsilon \quad \forall\, i, j \geq n_0.$$

Equivalently,

$$diam\{x_{n_0}, x_{n_0+1}, \cdots\} \leq \varepsilon.$$

(2) A filter α in (X, d) is Cauchy if for every $\varepsilon > 0$ there exists $A \in \alpha$ s.t. $diamA \leq \varepsilon$.

(3) A **net** $(x_i)_{i\in I}$ in (X, d) is Cauchy if for every $\varepsilon > 0$ there exists $i_0 \in I$ s.t.

$$d(a_i, a_j) < \varepsilon \quad \forall\, i, j \geq i_0.$$

(equivalently, the associated filter is Cauchy).

(4) (X, d) is *complete* if every Cauchy sequence is convergent (the converse is always true !). Equivalently, if every Cauchy net is convergent.

(5) A *Banach space* is a normed space $(X, \|\|)$ which is complete wrt the metric induced by given norm.

Proposition 4.2. (1) *Every convergent sequence is Cauchy.*

(2) *A Cauchy sequence is convergent iff it has a convergent subsequence.*

(3) *A closed subspace of a complete metric space is complete.*

(4) *A complete subspace of a metric space is closed.*

(5) *If $f : X \to Y$ is uniformly continuous and (x_n) is Cauchy then $(f(x_n))$ is Cauchy in Y.*

(6) ** Every complete metric space X is a **Baire** space (every sequence of dense open subsets has a nonempty*

*interior). Therefore X is of **second category in itself** (if $X = \cup_{n\in\mathbb{N}} F_n$ of closed subsets F_n then at least one F_n contains a nonempty open set).*

Theorem 4.3. *TFAE:*
1. *(X, d) is compact.*
2. *(X, d) is totally bounded and complete.*

Proof. (1) $\Rightarrow$ (2):
(total boundedness) For every $\varepsilon > 0$ the ε-cover has a finite subcover...
(completeness) Let (x_n) be a Cauchy sequence. Since (X, d) is a compact metric space it is sequentially compact. So there exists a convergent subsequence. Now apply Proposition 4.2.2.
(2) $\Rightarrow$ (1):
Let α be an ultrafilter in X. It is enough (by Theorem 3.6) to show that α is convergent. By completeness it is enough to show that α is Cauchy. Let $\varepsilon > 0$. By total boundedness, X is a union of finitely many subsets of diameter $< \varepsilon$. Since $X \in \alpha$, by Corollary 3.5, we get that α contains a set of diameter $< \varepsilon$. Thus α is Cauchy. $\square$

Examples 4.4. (1) The Euclidean space $\mathbb{R}^n$ is a Banach space.
(2) The Hilbert space l_2 is a Banach space.
(3) $(F^b(X), \|\|_{sup})$ of all **bounded** functions is a Banach space for every set X.
(4) $(C(X), \|\|_{sup})$ of all **bounded continuous** functions is a Banach space for every topological space X.
(5) $l_\infty := C(\mathbb{N}) = \{\text{bounded sequences}\}$

Theorem 4.5. *(Frechet) Every metric space (X, d) is isometric to a metric subspace of the Banach space $C(X)$.*

Proof. Sketch: fix $z \in X$ and define

$$j : X \to C(X), \;\; j(x)(t) := d(x, t) - d(t, z) \;\; \forall t \in X$$

Then j is a distance preserving map.

□

Exercise 4.6. (1) Prove all details of Theorem 4.5.

(2) Show that every metric space (X, d) is isometric to a metric subspace of the Banach space $C(D)$, where D is a dense subset of X.
(3) Show that every separable metric space (X, d) is isometric to a metric subspace of the Banach space l_∞.
(4) Show that l_∞ is not separable.

Theorem 4.7. ** (Banach) Every separable metric space is isometric to a subset of the Banach space $C[0, 1]$ (which is separable).*

4.2. **Completion.** A *completion* [5] of a metric space (X, d) is a pair (Y, j), where Y is a complete metric space and $j : X \to Y$ is an isometric dense embedding.

Theorem 4.8. *Every metric space has a completion.*

Proof. (I: not constructive) Use Thm 4.5.

(II: constructive) Sketch:

[similar to the construction of $\mathbb{R}$ from $\mathbb{Q}$]

The needed metric space Y will be the natural quotient of a semimetric space Y_0. The latter is defined as the set of all Cauchy sequences in (X, d). The semimetric is

$$\rho_0((a_n), (b_n)) := \lim \ d(a_n, b_n)$$

□

Theorem 4.9. *(Principle of extension of continuity) Let X and Y be metric spaces and let Y be complete. If $f : A \to Y$ is uniformly continuous on the dense subset $A \subset X$ then f has a unique continuous extension which is uniformly continuous.*

[5] uniquely defined up to isometries

Proof. For $x \in X$ take a sequence (a_n) in A s.t. $\lim(a_n) = x$ in X. Since (a_n) is a Cauchy s. and $f : A \to Y$ is **uniformly** continuous, the sequence $(f(a_n))$ is Cauchy in Y. Since Y is complete, there is a point $y \in Y$ with $\lim f(a_n) = y$. Define

$$g : X \to Y,\ g(x) = y$$

This map is well defined because $g(x)$ depends only on x. Indeed, if (b_n) is another sequence in A with $\lim b_n = x$ then $\lim d(a_n, b_n) = 0$. By uniform continuity of f we get

$$\lim d(f(a_n), f(b_n)) = 0.$$

Therefore, $\lim f(b_n) = g(x)$.

It is easy to see the uniform continuity of $g : X \to Y$. The uniqueness of g is obvious. $\square$

5. (Semi)groups with a topology

5.1. Topological groups.

Definition 5.1. Let $(G, \cdot)$ be a group and τ be a topology on G. $(G, \cdot, \tau)$ (or, simply, G) is a topological group if the group operations

$$G \times G \to G, \quad (x, y) \mapsto xy$$

$$G \to G, \quad x \mapsto x^{-1}$$

are continuous.

As usual a function $\|\cdot\| : X \to [0, \infty)$ on a vector space X is said to be a *norm* on X if:

(1) $\|cx\| = |c|\|x\|$ for every $(c, x) \in \mathbb{R} \times X$.
(2) $\|x\| = 0 \Rightarrow x = \mathbf{0}$.
(3) $\|x + y\| \leq \|x\| + \|y\|$.

Example 5.2. Every normed space $(X, \|\cdot\|)$ is a topological group wrt induced topology. (For example, $\mathbb{R}^n, l_2, l_p, l_\infty, c_0, C(K)$)

More generally, we have

Lemma 5.3. *Let $(G, \cdot)$ be a group and $p : G \to \mathbb{R}$ be a* generalized absolute value function, *i.e.,*

(1) $p(x) \geq 0, \quad \forall x \in G.$
(2) $p(x^{-1}) = p(x) \quad \forall x \in G.$
(3) $p(xy) \leq p(x) + p(y) \quad \forall x, y \in G.$
(4) $\lim p(x_i) = 0 \Rightarrow \lim p(a x_i a^{-1}) = 0 \quad \forall a \in G.$ [6]

Then $d(x, y) := p(x^{-1}y)$ defines a (left invariant) semimetric on G which induces a group topology.

[6] one may drop it if G is abelian

Example 5.4. Let

$H_+[0,1] := \{\text{homeom. } f : [0,1] \to [0,1] : \ f(0) = 0, f(1) = 1\}$.

Define

$$p(f) := \max_{x\in[0,1]} |f(x) - x|.$$

Then p is an absolute value function (Lemma 5.3).

Proposition 5.5. *(weak topologies on groups)*

(1) *Any subgroup of a topological group is a topological group.*
(2) *Any product of topological groups is a topological group.*
(3) *The* ***weak topology*** τ_w *on a group* G *wrt a system* $h_i : G \to (G_i, \tau_i)$ *of group homomorphisms into topological groups is a group topology.*

Proposition 5.6. *Let* X *be a group with the identity* e *and a topology* τ. *Then* τ *is a group topology if and only if:*

(1) *Every left translate of an open set is open*

$$(aO \in \tau, \quad \forall (a, O) \in X \times \tau)$$

(2) $U^{-1} \in N_e$ *for every nbd* $U \in N_e$.
(3) $\forall\ U \in N_e \quad \exists V \in N_e : \quad VV \subset U$.
(4) $\forall\ (a, U) \in X \times N_e \quad \exists V \in N_e : \quad aVa^{-1} \subset U$.

Example 5.7. (1) Every group with its discrete topology is a topological group.
(2) Every linear topological space (in particular, normed space) is a t.gr.
(3) $\mathbb{Q}$.
(4) Every (multiplicative) subgroup of $\mathbb{C}\backslash\{0\}$.
(5) $GL_n(\mathbb{R})$ is a t. gr. wrt pointwise topology.
(6) $\mathbb{Z}_2^{\mathbb{N}}$ (where $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$). One may call it, "Cantor group" !

(7) * $(\mathbb{Z}, \tau_p)$. $\mathbb{Z}$ with the p-adic topology is a non-discrete metrizable t.gr. Its completion is a group of p-adic integers.

Exercise 5.8. * For every metric space (X, d) the group $Is(X)$ of all onto isometries (=*symmetries*) $X \to X$ is a topological group wrt the pointwise topology inherited from X^X.

Exercise 5.9. (1) Show that $\overline{A} \cdot \overline{B} \subset \overline{A \cdot B}$ in every t.gr. $(G, \cdot)$.
(2) Find two closed subsets A, B in $\mathbb{R}$ s.t. $A+B$ is not closed.
(3) Let $(G, \cdot)$ be a Hausdorff t. gr. Show that if K is a **compact** subset then KF is closed for every closed subset $F \subset G$.
Hint: Use nets, let $(x_i) \in KF$ and $x_i \to x \in G$. We have to show that $x \in KF$...
(4) Show that the inversion map $x \mapsto x^{-1}$ in a t. gr. is a homeomorphism.
(5) Show that every left (right) $x \mapsto ax$ translation in a t. gr. is a homeomorphism. Derive that every topological group is *homogeneous*.
(6) Show that every conjugation $x \mapsto axa^{-1}$ in a t. gr. is a homeomorphism.
(7) Show that the closure of a (normal) subgroup is a (normal) subgroup in every t.gr.
(8) show that every open subgroup is clopen in every t.gr.

Proposition 5.10. *Let G be a t. gr.*

(1) *If G is commutative then its completion naturally becomes a topological group (with the dense subgroup G, like $\mathbb{Q}$ and its completion $\mathbb{R}$).*
(2) *A group homomorphism $h : G \to X$ is continuous iff h is continuous at some point (at e for example).*

(3) *Let H be a subgroup of G. On the coset set $G/H := \{gH : g \in G\}$ define the quotient topology $\tau_{G/H}$. That is, $O \subset G/H$ is open iff $q^{-1}(O)$ is open in G. Then:*
- *$q : G \to G/H$ is an open continuous map.*
- *The action $G \times G/H \to G/H$ is continuous.*
- *G/H is Hausdorff iff H is closed in G.*
- *If H is a normal subgroup in G then G/H is a t.gr. wrt the natural group operation.*

Proof. For every open $O \subset G$ the subset

$$q^{-1}q(O) = HO \subset G$$

is also open. Hence $q(O)$ is open in G/H (by definition of quotient topology). $\square$

Proposition 5.11. *Let V be a normed space. Then its completion can be treated as a Banach space.*

As an example: $(C[0,1], \|\cdot\|)$ wrt $\|f\| := \int |f(t)|dt$ is a normed (but noncomplete) space Its completion is a Banach space denoted as usual by $L_1[0,1]$.

5.2. **Semigroups.** Let $(S, \cdot)$ is a semigroup with a topology. We say that S is a right topological if all right translates are continuous. Instead of "right and left topological" we say "semitopological".

Example 5.12. ** Let V be a normed space.

(1) the semigroup $\Theta(V)$ of all contractive linear selfmaps is a semitopological monoid wrt weak operator topology.
(2) $\Theta(V)$ is a semitopological monoid wrt strong operator topology.
(3) the group $Is(V)$ of all linear self-isometries is a topological group wrt strong operator topology.

A *dynamical system* is a semigroup action $S \times X \to X$ on X (mostly with compact X). *Cascade* is a dynamical system generated by iterations of a continuous selfmap $X \to X$.

Example 5.13. (1) For every topological space X the semigroup X^X is right topological wrt product topology.

(2) ** enveloping semigroups of an action (the pointwise closure of the (semi)group of all translates $s : X \to X$ in X^X) of compact dynamical systems are right topological.

** Example: Consider the cascade on $X := [0,1]$ generated by

$$\sigma : [0,1] \to [0,1], \quad \sigma(t) := t^2.$$

What is his enveloping semigroup ?

Theorem 5.14. ***(Ellis Theorem)*** *Let E be a right (or, left) topological semigroup. Then E contains an idempotent (i.e., an element $u \in E$ s.t. $u^2 = u$).*

Proof. Zorn's Lemma (and compactness of E) imply that there exists a minimal element Y in the set of all closed non-empty subsemigroups of E. Fix $u \in Y$. We claim that $u^2 = u$ (and, hence $Y = \{u\}$). The set Yu being a closed subsemigroup of Y, is equal to Y. It follows that the closed subsemigroup

$$Z := \{y \in Y : \quad yu = u\}$$

is non-empty. Hence $Z = Y$ and $yu = u$ for every $y \in Y$. In particular, $u^2 = u$. □

6. Banach spaces

As usual Banach spaces are normed spaces which are complete wrt its norm.

One of the most important examples is the Banach space $C(X)$ of all *bounded* continuous real valued functions on a topological space X wrt sup-norm $||f|| := sup\{|f(x)| : x \in X\}$.

Every finite dimensional normed space (say, n-dimensional) is topologically isomorphic to the Euclidean space $\mathbb{R}^n$.

A Banach space with an inner (scalar) product is a *Hilbert space*. For example, $\mathbb{R}^n$, l_2 and $L_2[0,1]$ are (separable) Hilbert spaces. A Hilbert space is *separable* if and only if it admits a countable orthonormal basis. By Riesz-Fisher theorem all infinite-dimensional separable Hilbert spaces are topologically isomorphic to l_2.

Suppose X is a normed space. We denote by X^* its dual, i.e. the space of all continuous linear maps (functionals) from X to the base field $\mathbb{R}$. Then $X^* := L(X, \mathbb{R})$ is a Banach space wrt the natural norm

$$||f|| := sup\{|f(x)| : \ ||x|| \leq 1\} = sup\{|f(x)| : \ ||x|| = 1\}.$$

The weak-star topology w^* on the dual V^* is the (pointwise) topology inherited from the product $\mathbb{R}^V$.

Fact 6.1. *(Banach-Alaouglu thm) Let V be a normed space. Then the unit closed ball $B^* := B_{V^*}$ of the dual V^* is compact wrt weak-star topology.*

Proof. Define the natural identification map

$$\alpha : B^* \to K := \prod_{v \in V} [-||v||, ||v||], \quad f \mapsto < f(v) >_{v \in V}$$

This map is a topological embedding by the definition of the weak-star topology w^*. The product space $\prod_{v \in V}[-||v||, ||v||]$ is compact by Tychonov thm. Now, observe that the $\alpha(B^*)$ is closed (hence, compact). Indeed, if a net $< f_i(v) >_{v \in V}$ converges to $< s_v >_{v \in V}$ in K then the mapping

$$s : V \to \mathbb{R}, \quad v \mapsto s_v$$

defines a functional from B^*. Indeed, the linearity of s is easy. Now observe that $s(v) \in [-||v||, ||v||]$. Therefore, $|s(v)| \leq ||v||$ for every $v \in V$. Hence, $||s|| \leq 1$, that is, $s \in B^*$. $\square$

The *weak topology* on V is the weakest topology generated by the family V^*. The net v_i converges to v in V wrt weak topology iff $\psi(v_i)$ converges to $\psi(v)$ for every $\psi \in V^*$.

We can form the double dual X^{**}, the dual of X^* for every normed space X. There is a natural continuous linear transformation

$$j : X \to X^{**}$$

defined by $j(x)(\phi) = \phi(x)$ for every $x \in X$ and $\phi \in X^*$. That is, j maps x to the functional on X^* given by evaluation at x. As a consequence of the Hahn - Banach theorem, j is norm-preserving (i.e., $||j(x)|| = ||x||$) and hence injective. The space X is called *reflexive* if j is bijective. In this case X must be isomorphic to X^{**}. Every Hilbert space is reflexive.

Some examples, of the dual spaces:

- $C(K)^* \cong M(K)$ (By Riesz thm for every compact space K the dual of $C(K)$ is isomorphic to the space of all regular measures).
- $c_0^* \cong l_1$
- $l_1^* \cong l_\infty = C(\mathbb{N}, \mathbb{R})$.
- $l_2^* \cong l_2$.
- $l_p^* \cong l_q$ with $1/p + 1/q = 1$ for every $p > 1$.

Hence, l_p is reflexive if and only if $p > 1$. For instance, l_3 is reflexive but not Hilbert.

A separable Banach space is said to be *Asplund* if its dual is also separable. It is well known (but not trivial) that every separable reflexive space is Asplund. An easy example of Asplund which is not reflexive is c_0. At the same time l_1 is not Asplund (because its dual l_∞ is not separable !).

6.1. **Operator topologies.** Let X and Y be Banach spaces. Then the set $L(X,Y)$ (in particular, the dual $X^* := L(X,\mathbb{R})$ of all continuous linear operators from X to Y is a Banach space wrt the usual operator norm

$$||f|| := sup\{|f(x)| : \ x \in B_X\} = sup\{|f(x)| : \ x \in S_X\}$$

where B_X and S_X are the (closed) unit ball and the unit sphere of X. Then $L(X,Y)$ becomes a Banach space wrt that norm and usual operations. There are two additional important locally convex topologies on $L(X,Y)$.

The *strong operator topology* is the weakest topology of $L(X,Y)$ relative to which the mapping

$$L(X,Y) \to Y, \quad f \mapsto f(x)$$

is continuous for each $x \in X$.

The *weak operator topology* is the weakest topology of $L(X,Y)$ relative to which the mapping

$$L(X,Y) \to \mathbb{R}, \quad f \mapsto \psi(f(x))$$

is continuous for each $x \in X$ and $\psi \in Y^*$.

Consider the particular case of $X = Y = V$. Then the strong operator topology on $L(V,V)$ is just the pointwise (product) topology which comes from V^V. A net f_i in $L(V,V)$ converges to f iff $f_i(v)$ converges wrt the norm to $f(v)$ in V for each $v \in V$.

Similarly, the weak operator topology on $L(V,V)$ is the pointwise (product) topology which comes from V_w^V, where V_w means the space V endowed with the weak topology. A net f_i in $L(V,V)$ converges to f iff $f_i(v)$ converges wrt the weak topology to $f(v)$ in V for each $v \in V$. Precisely this means that $\psi(f_i(v))$ converges to $\psi(f(v))$ for every $v \in V$ and every $\psi \in V^*$.

Denote by $\Theta(V)$ the set of all contractive linear operators of V into itself. That is,

$$\Theta(V) = \{\sigma \in L(V, V) : ||\sigma|| \leq 1\}.$$

Of course $\Theta(V) = B_{L(V,V)}$.

Fact 6.2. (1) *$\Theta(V)_s$ is a topological semigroup.*
(2) *$\Theta(V)_w$ is a semitopological semigroup.*
(3) *The pair $(\Theta(V)_w, B(V)_w)$ is a semitopological flow.*

Recall some natural ways getting topological monoids and monoidal actions.

Let V be a normed space.

Examples 6.3. (1) For every metric space (M, d) the semigroup $\Theta(M, d)$ of all *d-contractive maps $f : X \to X$* (that is, $d(f(x), f(y)) \leq d(x, y)$) is a topological monoid with respect to the topology of pointwise convergence. Furthermore, the evaluation map $\Theta(M, d) \times M \to M$ is a jointly continuous monoidal action. The subspace $Is(X, d)$ of all linear onto isometries is a topological group.

(2) For every normed space $(V, || \cdot ||)$ the semigroup $\Theta(V)$ of all contractive linear operators $V \to V$ endowed with the *strong operator topology* (being a topological submonoid of $\Theta(V, d)$ where $d(x, y) := ||x - y||$) is a topological monoid. The subspace $Is(V)$ of all linear onto isometries is a topological group.

7. Why we study actions ?

Some examples of groups

S_n, $S_X := \{f : X \to X\}$ Symmetric group w.r.t. $\circ$

$G \leq GL(n, \mathbb{R})$ Matrix groups

$Iso(X, d)$ Group of isometries

$Homeo(X)$ Group of homeomorphisms

and semigroups $(Map(X, X), \circ)$, $C(X, X), \circ)$,

$S := (f^n\}_{n\in\mathbb{N}}$ for every $f : X \to X$, where $f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n}$.

`(Semi)Groups come as usual by transformations of some` X`.`

$S_X \times X \to X, \quad (\sigma, x) \mapsto \sigma(x)$ is the *natural action.*

Moreover for every homomorphism $h : G \to S_X$ we have the associated action:

$G \times X \to X, \quad (g, x) \mapsto h(g)(x)$

Definition 7.1. Action of a semigroup S on X is a function

$$\pi : S \times X \to X, \pi(s, x) := sx$$

s.t. $s_1(s_2x) = (s_1s_2)x$ (and $ex = x$ if e is the neutral element of S).

If S and X are topological spaces one can define **continuous** actions. (S, X) (sometimes write also as (X, S)) is a **dynamical system**.

Lemma 7.2. *Let $S = G$ be a group and $\pi : S \times X \to X$ be a (continuous) action. Then*

(1) *every s-*translation *$\hat{s} : X \to X, x \mapsto sx$ is a homeomorphism and $\phi : S \to Homeo(X) \leq S_X$ is a homomorphism of groups.*
(2) *$X := \bigsqcup_{x\in X}[x]$ where*

$$[x] = Sx := \{sx\}_{s\in S}$$

is the **orbit** *of x.*

If X is discrete and finite with m orbits then

$$|X| = \sum_{1\leq i\leq m} [G : St(x_i)]$$

"Finite combinatorics" ...

General Problem: What if X and S are not finite ? Study **topological** behavior of the action, orbits & etc.

Definition 7.3. For $f : X \to X$ define the action $\mathbb{N} \times X \to X, \pi(n, x) := f^n(x)$, where $f^n := f \circ f \circ \cdots \circ f$. Then (X, f), or $(X, \mathbb{N})$ is said to be a cascade (generated by f). If f is a homeomorphism one can define $\mathbb{Z} \times X \to X$.

(S, X) is a dynamical system. We say that (S, X) is:
a) *cascade* if $S := \mathbb{N}$ or $S := \mathbb{Z}$.
b) *flow* if $S := \mathbb{R}$.

$Y \subseteq X$ is S-*invariant* if $SY \subseteq Y$ (that is, $sy \in Y$ for all $y \in Y$). Then (Y, S) is a *subsystem* of X.

8. Representations

Let G be a (topological) group and V be a Banach space ($\mathbb{R}^n$, $H = l_2$, ... for example). An representation of G on V is a homomorphism $h : G \to GL(V)$. If $h(G) \subset Is(V)$, we say "isometric representation". h is (topologically) exact if h is a (topological) embedding.

Example 8.1. (1) Standard representation by rotations

$$\mathbb{Z}_n \hookrightarrow Is(\mathbb{R}^2) = O(2, \mathbb{R})$$

$$[k] \mapsto \begin{pmatrix} cos\frac{2\pi k}{n} & sin\frac{2\pi k}{n} \\ -sin\frac{2\pi k}{n} & cos\frac{2\pi k}{n} \end{pmatrix}$$

(2)

$$\exists \;\; D_n \hookrightarrow O(3, \mathbb{R})$$

(3)

$$h : \mathbb{R}^2 \hookrightarrow GL(3, \mathbb{R})$$

$$(x_1, x_2) \mapsto \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

is an exact representation (which is not isometric).

Exercise0: There is no topologically exact representation by isometries $\mathbb{R} \to Is(\mathbb{R}^n)$ of $\mathbb{R}$ on $V := \mathbb{R}^n$.

Question 8.2. *Whether G admits a faithful representation on a good (well behaved class) of Banach space ?*

Some "good" classes (increasing):
(1) Finite dimensional spaces $\mathbb{R}^n$.
(2) Hilbert spaces (l_2, $L_2[0, 1], \cdots$).
(3) Reflexive spaces (l_p, $1 < p < \infty, \cdots$)
(4) Asplund spaces ($c_0, \cdots$).
(5) Rosenthal spaces.

Exercise1: Every finite group G is representable on a finite dimensional space.

Hint: Use $\mathbb{R}^G$ or do it for the symmetric group S_n on $\mathbb{R}^n$.

Exercise2: Every discrete group G is representable on a Banach space V.

Hint: Take $V := C_b(G) = l_\infty(G)$. For example, $\mathbb{Z}$ is representable on l_∞.

Theorem 8.3. *(Theorem 1 of Teleman) Every topological group G is a topological subgroup of $Is(V)$ for some Banach space V.*

Hint: canonical representation on $V := RUC(G)$.

Theorem 8.4. *(Theorem 2 of Teleman) Every topological group G is a topological subgroup of $Homeo(K)$ for some compact space K.*

Hint: Take K = weak-star compact unit ball of the dual V^* of V, where $V := RUC(G)$.

Question 8.5. *(General question) Whether V can be Hilbert, reflexive, Asplund, ... ?*

Remark 8.6. Describe finite subgroups of $SO(3, \mathbb{R})$ (up to isomorphisms).

Hint: $\{C_n, D_n, Is(K)\}$, where K is: a) cube; b) tetrahedron; c) icosahedron.

Question 8.7. *Whether a given action $G \times X \to X$ admits a linearization into a good Banach space ?*

9. Orbit hierarchy

Definition 9.1. Orbit hierarchy for the cascade (X, T) and a point $x_0 \in X$:

a) *fixed* $\quad Tx_0 = x_0$.

b) *periodic* $\quad \exists p > 1 : T^p x_0 = x_0$.

c) *almost periodic* (or, *uniformly recurrent*)
$\forall$ nbd V of $x_0 \quad \exists l > 0$:

$$\forall n > 0 \quad \exists i \in \{0, \cdots, l\} : T^{n+i} x \in V$$

c*) (reformulation of (c)) : $\forall$ nbd V the set of "return times"

$$R(x, V) := \{n \in \mathbb{N} : \quad T^n x \in V\}$$

is a *syndetic set*, where A is **syndetic** means that $F^{-1}A = \mathbb{N}$ for some finite (**"compact"** in **general case**) $F \subset \mathbb{N}$.

d) *(positively) recurrent* $\quad x \in cl(Orb_+(x))$
(i.e. for any nbd V of x there exists $n \geq 1$ with $T^n x \in V$).
If X is a metric then it is equivalent to saying that

$$\lim_{k \to \infty} T^{n_k} x = x$$

for some sequence n_k of natural numbers.

"a point is periodic if it returns to itself every hour on the hour and is almost periodic if it returns to an arbitrary nbd every hour within the hour (where the length of the "hour" depends on the nbd)"

Definition 9.2. Let (S, X) be a dynamical system.

1. (1) X is *(topologically) transitive* if $X = cl(Sx_0)$ for some $x_0 \in X$.
2. (2) A closed subset A of X is called *minimal* whenever $A \neq \emptyset$, A is invariant (that is, $SA \subseteq A$) and A has no closed invariant proper subsets.

SOME EXAMPLES

Example 9.3. For every topological group G and a subgroup $H \leq G$ the natural action

$$G \times G/H \to G/H, \quad (s, tH) \mapsto stH$$

on the *coset space* G/H is the "1-orbit action" $=$ alg. transitive (hence, minimal (hence, top. transitive)).

Example 9.4. On the "circle" $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ define the **"irrational rotation"**

$$\sigma_\theta : \mathbb{T} \to \mathbb{T} \quad \sigma_\theta([x]) := [x + \theta].$$

(Equiv.: $\sigma_\theta(e^{2\pi i t}) = e^{2\pi i t + 2\pi i \theta}$)

If θ is irrational then the corresponding system is minimal (not so trivial !). Every point is almost periodic (but not periodic !).

Example 9.5. Logistic map $f : [0,1] \to [0,1], f(t) := 4t(1-t)$. Not transitive but has a subsystem isomorphic to the shift on the Cantor space $\{0,1\}^{\mathbb{N}}$.

Example 9.6. Shift systems (Symbolic cascades).

$\Lambda := \{0, 1, \cdots, r\}, \quad \Omega := \Lambda^{\mathbb{Z}} = \{\omega : Z \to \{1, \cdots, r\}\}$

$$d(x, y) := \frac{1}{1 + min\{|i| : x_i \neq y_i\}}$$

(homeomorphic to the "Cantor set" !).

Define the "shift": $\sigma : \Lambda^{\mathbb{Z}} \to \Lambda^{\mathbb{Z}}, \quad (\sigma\omega)(n) := \omega(n+1)$.

$$x := \cdots, x_{-2}, x_{-1}, \breve{x}_0, x_1, \cdots$$

$$\leftarrow$$

$$\sigma x := \cdots, x_{-1}, x_0, \breve{x}_1, x_2, \cdots$$

(Ω, σ) is a cascade.

(clopen) *cylinders*

$$x[_j; k] := \{y \in \Omega : y_i = x_i\}_{j \le i \le k}$$

local basis at x.

Exercise 1:
(1) $x[-j; j] = \{y \in \Omega : d(x, y) \le \frac{1}{1+(j+1)}\}$.
(2) For $x, y \in \Omega$ one has $\sigma^r y \in x[0; k]$ iff the block $x_0, \cdots, x_k$ occurs in y at place r that is iff $x_i = y_{r+i}$ for $i = 0, \cdots, k$.
(3) $x \neq y \Rightarrow \exists k \in \mathbb{Z} : d(\sigma^k x, \sigma^k y) = 1$.

Exercise 2: Ω is *sensitive* (i.e. there exists $\varepsilon > 0$ s.t. for every open $\emptyset \neq O \subseteq \Omega$ there exists $k \in \mathbb{Z}$ s.t. $diam(\sigma^k O) \ge \varepsilon$).

Exercise 3: Let $x \in \Omega$. Then:
(a) Fixed points = constant maps.
(b) x is periodic iff $\exists\ p \in \mathbb{N}$ s.t. $x_{i+p} = x_i$ for all $i \in \mathbb{Z}$.
(c) A point $y \in \Omega$ is in the orbit closure of x iff every finite block which occurs in y also occurs in x (then it occurs at infinitely many different places in x).
(d) x has a dense orbit in Ω iff every finite block occurs in x at some place j.
(e)* Ω is topologically transitive (has a dense orbit).
(f) x is positively recurrent iff every block which occurs in x does so at places j for arbitrarily large j.
(g) x is almost periodic iff every block which occurs in x with bounded gaps, i.e. for each block b occuring in x the set

$$\{j \in \mathbb{Z} : b \quad \text{occurs in} \quad x \quad \text{at place} \quad j\}$$

is syndetic in $\mathbb{Z}$.

Remark: if $\Omega := \{0, 1\}^{\mathbb{Z}}$. Then
(1) Ω is transitive but not minimal.
(2) The set of all periodic points is dense.
(3) is chaotic !

Definition 9.7. A closed subset A of X is called *minimal* whenever $A \neq \emptyset$, A is invariant and A has no closed invariant proper subsets.

Lemma 9.8. *An invariant subset $A \subseteq X$ is minimal iff $cl(Orb(x)) = A$ for every $x \in A$ (iff every orbit is dense in A).*

Proof. Indeed $cl(Orb(x))$ is invariant and closed. □

For example:

- Ω is topologically transitive but not minimal.
- $(\sigma_\theta, \mathbb{T})$ is minimal for every irrational θ.

Theorem 9.9. *Every* ***compact*** *dynamical system (S, X) contains a minimal subset.*

Proof. Apply Zorn's Lemma to the partially ordered (with respect to inclusion) family of all non-empty closed invariant subsets of X. This family is non-empty (X belongs to it !) and the intersection of a chain in it is a non-empty because X is compact. □

Theorem 9.10. [Birkhoff Recurrence Theorem 1927] *Every compact cascade (X, T) contains a point which is recurrent under T.*

Proof. By Theorem 9.9 (with $S := \mathbb{N}$) there exists a minimal subset, say, $Y \subseteq X$. Every point $y \in Y$ of the minimal set Y is recurrent. Indeed, by Lemma 9.8 we have $cl(Orb_+(y)) = Y$. In particular, $y \in cl(Orb_+(y))$. □

Theorem 9.11. [Gottschalk 1944] *For every compact system (S, X) and a point $x_0 \in X$ TFAE:*

1. *The subsystem $Y := cl(Sx_0)$ is minimal.*
2. *For every open nbd V of x_0 in X there exists a finite set $F := \{s_1, \cdots, s_n\} \subseteq S$ s.t. $F^{-1}V \supseteq Y$ (i.e. $\bigcup_{i=1}^{n} s_i^{-1}V \supseteq Y$).*
3. *x_0 is almost periodic in X.*

Proof. (1) $\Rightarrow$ (2): Suppose (Y, S) is minimal. Then for every $y \in Y$ there exists $s \in S$ s.t. $sy \in V$. Therefore,

$$\bigcup_{s \in S} s^{-1}V \supseteq Y.$$

By compactness of Y we can choose a finite subset $F \subseteq S$ s.t. $F^{-1}V \supseteq Y$.

(1) $\Leftarrow$ (2): If (Y, S) is not minimal and M is a closed invariant nonempty subset of Y then define $V := X \backslash M$. We have

$$\bigcup_{s \in S} s^{-1}V \not\supseteq Y.$$

(2) $\Leftrightarrow$ (3): Clear by the reformulation of Definition 9.1(c).

(3) $\Rightarrow$ (1): $Y = cl(Sx_0)$ is nonempty, closed and invariant. It remains to show that if $y \in Y$ then $x_0 \in cl(Sy)$. Assume otherwise, so that $x_0 \notin cl(Sy)$. Choose an open nbd V of x_0 s.t. $cl(V) \cap cl(Sy) = \emptyset$. Since x_0 is almost periodic (by (2)) there is a **finite** set $F := \{s_1, \cdots, s_n\}$ so that for each $s \in S$ some $s_i s x_0 \in V$. In other words, each

$$sx_0 \in F^{-1}V = \bigcup_{i=1}^{n} s_i^{-1}V.$$

Hence

$$Sx_0 \subseteq \bigcup_{i=1}^{n} s_i^{-1}V.$$

Then

$$y \in cl(SX) \in cl(\bigcup_{i=1}^{n} s_i^{-1}V) = \bigcup_{i=1}^{n} s_i^{-1}cl(V).$$

But then $Sy \cap cl(V) \neq \emptyset$ contrary to our assumption. $\square$

Theorem 9.12. *Every compact dynamical system contains an almost periodic point.*

Proof. Let Y be a minimal subsystem (exists by Theorem 9.9) of X. Then every $y \in Y$ is almost periodic (by Theorem 9.11) in Y (and hence also in X). $\square$

Theorem 9.13. [Pouncare Recurrence Theorem 1899] *Let $T : X \to X$ be a measure preserving transformation of a measure space $(X, \mathcal{B}, \mu)$, with $\mu(X) < \infty$. If B is an arbitrary measurable set in X with positive measure $\mu(B) > 0$ then there is some point $x \in B$ and $n \in \mathbb{N}$ with $T^n x \in B$.*

Proof. $\mu(T^{-n}B) = \mu(B) > 0$ for every $n \in \mathbb{N}$. Then $T^{-i}B \cap T^{-j}B \neq \emptyset$ for some $i < j \in \mathbb{N}$ (otherwise $\mu(X) = \infty$!). Take $y \in T^{-i}B \cap T^{-j}B$. Then $T^{j-i}x \in B$ for $x := T^i y \in B$. $\square$

Since every arbitrarily small set of positive measure returns to itself it follows that almost all point are recurrent.

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