COMPACTIFICATIONS
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Contents

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1. Basic Definitions

Abstract. In this chapter we define compactifications, and add a sort theorem, In chapters 2 and 3 we show two specific constructions with interesting properties, In chapter 4 we study compactifications who are T_2, and show that the constructions form 2 and 3 are, in a way, minimal and maximal among them.

Definition 1.1. A Compactification (Y, h) of a topological space X is a topological space Y and an embedding $h : X \to Y$ such that $h[X]$ is dense and Y is compact.

Examples 1.2. Basic examples:

(1) $([0,1], i)$ with i the identity is a compactification of (0,1).
(2) $(\hat{\mathbb{C}}, h)$ the riemann sphere is a compactification of $\mathbb{C}$, where h is the stereographic projection.

Remark 1.3. After we define (Y, h) and prove h is an embedding we usually identify X with $h[X]$ and say $X \subset Y$, like commonly done with $\hat{\mathbb{C}}$.

As always, it is preferred to work with Hausdorff spaces, so we would like to create a compactification that is a T_2 space, however, this is not always possible:

Theorem 1.4. *A space X has a Hausdorff compactification only if X is Tychonoff.*

Proof. If Y is a T_2 compactification of X then Y is compact and T_2 and there for T_4 [lecture 2, prop 2.3.7] so Y is $T_{3\frac{1}{2}}$ and $X \subset Y$ so X is $T_{3\frac{1}{2}}$. □

Remark 1.5. This is in fact an "if and only if", we will show it in chapter 3, theorem 3.1.

Corollary 1.6. *If $f : X \to Y$ is an embedding and Y is compact, then $(\overline{f[X]}, f)$ is a compactification of X.*

Proof. $\overline{f[X]}$ is closed in the compact space Y and so $\overline{f[X]}$ is compact, $f[X]$ is dense in $\overline{f[X]}$, and f is still an embedding. □

2. The one-point compactification

$\mathbb{C}$ can be extended to $\hat{\mathbb{C}}$ by adding the point at ∞ We would like to create an analog construction for every non-compact topological space X, which we'll call X^+.

First we add a point $\infty \notin X$ to X^+, for formality we can take $\infty = \{X\}$, we would like X to be a subspace of X^+.

We will like X to be open in X^+, (like $\mathbb{C}$ is in $\hat{\mathbb{C}}$) an so $A \subseteq X$ is open in X iff it's open in X^+.

If $\infty \in A \subset X^+$ is open, then $A^c \subseteq X$ is closed in X, and in X^+ which we want to be compact, so A^c is compact, in $\hat{\mathbb{C}}$ all such sets are compact, we will do the same in here.

Definition 2.1. Let (X, T) be a topological space, its One-Point Compactification is (X^+, T') where $X^+ = X \cup \{\infty\}$ and $T' = T \cup \{A \cup \{\infty\} | A \in T \text{ and } A^c \text{ is compact } \}$

Remark 2.2. $B \subseteq X^+$ is closed iff $B \subseteq X$ and is closed and compact, or $B = A \cup \{\infty\}$ were $A \subseteq X$ is closed.

Theorem 2.3. *X^+ is a compactification of X, meaning:*

(1) *T' is a topology on X^+.*
(2) *X is a subspace of X^+.*
(3) *X^+ is compact.*
(4) *X is densee in $X^+ \Leftrightarrow X$ isn't compact.*

Proof. (1) Exercise! (work with closed sets, use remark 2.2).

(2) If $A \subseteq X$ is open, then $A \subseteq X^+$ is open.
And if $A \subseteq X^+$ is open, then $(A \cap X) \subseteq X^+$ is open, and there for $(A \cap X) \subseteq X$ is open.

(3) Let $\{U_a\}$ be an open cover of X^+, so there is a $U \in \{U_a\}$ such that $\infty \in U$, by definition, U^c is compact, so there is a finite subcover of U^c, $\{U_{a_1}...U_{a_n}\} \subseteq \{U_a\}$. and so:

$$X^+ = U \cup U^c \subseteq U \cup \bigcup_{i=1}^{n} U_{a_i}$$

(4) If X is compact then by definition $X \subseteq X^+$ is closed, and so of $cl_{X^+}(X) = X \neq X^+$.
If X isn't compact then $X \subseteq X^+$ isn't closed, and so $X \subsetneqq cl_{X^+}(X) \subseteq X^+$, and the only set like that is X^+.

$\square$

Remark 2.4. We just proved every topological space X has a compactification, if X is compact then X is it's own compactification, otherwise X^+ will do.

Remark 2.5. $X \subseteq Y$ it doesn't mean $X^+ \subseteq Y^+$.

Example 2.6. $(0,1) \subseteq (0,1]$, it is easily shown that $(0,1)^+ \cong S^1$ and $(0,1]^+ \cong [0,1]$, so $(0,1]^+$ is not a subspace of $(0,1)^+$.

We prefer to work with Hausdorff spaces, and will like to know for which X's X^+ is T_2.

Theorem 2.7. *X^+ is $T_2 \Leftrightarrow X$ is locally compact and T_2.*

Proof. If X is locally compact and T_2, let $a, b \in X^+, a \neq b$:

If $a, b \in X$ then there exist $A, B \subseteq X$ two disjoint and open sets such that $a \in A$ and $b \in B$, and by definition A and B are also open in X^+.
Else $b = \infty$, X is locally compact so there is a compact nbd K of a, $a \in U \subseteq K \Rightarrow U \cap K^c = \emptyset$, U is open in X and X^+, and $K \subseteq X$ is compact and closed since X is T_2 [lecture 2, prop 2.3.6] and so K^c is open and $b = \infty \in K^c$.

If X^+ is T_2 then the subspace X is T_2, and if $x \in U \subseteq X$ is open, then $x \in U \subseteq X^+$ is open, and X^+ is compact and $T_2 \Rightarrow T_4 \Rightarrow$ regular, so there is a closed subnbd $a \in K \subseteq U$ in X^+, and by definition,K is a compact subnbd in X. $\square$

Result 2.8. *A locally compact Hausdorff space is Tychonoff.*

Proof. If X is compact and T_2 then it's $T_4 \Rightarrow T_{3\frac{1}{2}}$,
Else X^+ is a T_2 compactification, so we can use theorem 1.4. $\square$

Example 2.9. $\mathbb{Q}$ is $T_{3\frac{1}{2}}$ but not locally compact, and so $\mathbb{Q}^+$ is not T_2.

Theorem 2.10. *If Y is compact and T_2, and $Y = X \cup \{a\}$ with $(a \notin X)$ then the function $f : Y \to X^+$, defied as: $f(a) = \infty$ and $\forall x \in X : f(x) = x$ is a homeomorphism.*

Proof. By definition, f is one to one and onto,
We will show $U \subseteq Y$ is open $\Leftrightarrow f[U] \subseteq X^+$ is open:

(1) If $a \notin U$ meaning $U \subseteq X$:
Y is T_2 so $\{a\} \subseteq Y$ is closed, so $\{a\}^c = X \subseteq Y$ is open, and so U is open in $Y \Leftrightarrow U$ is open in X (X is open) $\Leftrightarrow f[U] = U$ is open in X^+ (by definition of (X^+, T')).
(2) If $a \in U$:
$U \subseteq Y$ is open $\Rightarrow U^c \subseteq X$ is closed (in $Y \Rightarrow$ in X) and because Y is compact U^c is compact $\Rightarrow f[U^c] = U^c$ is closed in $x^+ \Rightarrow f[U] = f[U^c]^c$ is open in X^+.
And if $f[U] = f[U^c]^c$ is open in X^+ then $f[U^c] = U^c$ is closed in X^+ and is there for compact, $U^c \subseteq Y$ is compact and Y is T_2, so U^c is closed and $U \subseteq Y$ is open.

□

Remark 2.11. We just proved that if X has a T_2 compactification Y, such that $Y \setminus X$ has one point, then X is locally compact. (because $Y \cong X^+$, so X^+ is T_2).

Let's show that theorem 2.10 doesn't work without the requirement that Y is T_2.

Exercise 2.12. For every topological space (X, T) you can define $X^- = X \cup \infty$ with the topology $T" = T \cup \{X^-\})$.
Prove this is a compactification of X, and that $Id : X^+ \to X^-$ is not a homeomorphism, even if X is locally compact and T_2.

3. The Stone-Cech compactification

As promised in remark 1.5 we will now prove every Tychonoff space has a Hausdorff compactification:

Theorem 3.1. *Let X be a Tychonoff space, and let C_x be the set of all continuous functions from X to C then X can be embedded in I^{C_x}.*

Proof. For every $x \in X$ we define a map $\hat{x} : C_x \to I$ as

$$\forall \varphi \in C_x : \hat{x}(\varphi) = \varphi(x).$$

and we define $e : X \to I^{C_x}$ by $e(x) = \hat{x}$, we will prove this is a topological embedding:

(1) e is 1 to 1:
Let $x, y \in X$ and $x \neq y$, X is $T_{3\frac{1}{2}}$ so there is $\phi \in C_x$ such that $\phi(x) = 0$ and $\phi(y) = 1$, and so $\hat{x}(\phi) = 0 \neq 1 = \hat{y}(\phi)$ and $e(x) \neq e(y)$.

(2) e is continuous :
By the universal property of a product space, the function $e : X \to I^{C_x}$ is continuous iff for every $\varphi \in C_x$,
$P_\varphi \circ e : X \to I$ is continuous (P_φ is a projection to I)
but by definition , for every $x \in X$ and $\varphi \in C_x$:

$$P_\varphi \circ e(x) = e(x)(\varphi) = \hat{x}(\varphi) = \varphi(x)$$

so $P_\varphi \circ e = \varphi$, and we know φ is continuous .

(3) $e^{-1} : e(X) \to X$ is continuous :
Let $x_\delta, x \in X$ with $e(x_\delta) \to e(x)$, we will prove $x_\delta \to x$:

For every $\varphi \in C_x, P_\varphi$ is continuous , and so $P_\varphi \circ e(x_\delta) \to P_\varphi \circ e(x)$, meaning $\varphi(x_\delta) \to \varphi(x)$. (*)

If we assume $x_\delta \nrightarrow x$ than there is an open $x \in U \subseteq X$ such that for every δ_o there is $\delta_o \leq \delta$ with $x_\delta \in U^c$.

but U^c is closed and X is $T_{3\frac{1}{2}}$, so there is a $\phi \in C_x$ such that $\phi[U] = \{0\}$ and $\phi(x) = 1$!

And so for every δ_o there is $\delta_o \leq \delta$ with $\phi(x_\delta) = 0$, and so $\phi(x_\delta) \nrightarrow 1 = \phi(x)$, in contradiction to (*).

□

I^{C_x} is compact and T_2, and by corollary 1.6 $(\overline{e[X]}, e)$ is a compactification of X, that is also Hausdorff, we will define $\beta X = \overline{e[X]}$ to be the Stone-Cech compactification of X.
we should also add that:

Theorem 3.2. *If X is compact then $e : X \to \beta X$ is a homeomorphism.*

Proof. Theorem 3.1 proves everything except that $e[X] = \beta X$, $\beta X = \overline{e[X]}$ so it's enough to show $e[X] \subseteq I^{C_x}$ is closed, which is true since it is compact and I^{C_x} is T_2. □

Like before, we identify X with e[X], $(x \leftrightarrow \hat{x})$ and find that βX has some interesting properties:

Lemma 3.3. *Let X be $T_{3\frac{1}{2}}$, and let $f : X \to I$ be continuous, then f has a continuous extension $F : I^{C_x} \to I$.*

Proof. For every $a \in I^{C_x}$, $(a : c_x \to I)$ we define $F(a) = a(f)$, and so $F(\hat{x}) = \hat{x}(f) = f(x)$ like we need.
F is continuous because it is the F_f projection. □

This can be generalized:

Lemma 3.4. *Let X be $T_{3\frac{1}{2}}$, S be a set, and $f : X \to I^S$ be continuous, then f has a continuous extension*
$F : I^{C_x} \to I^S$.

Proof. For every $s \in S$, $P_s \circ f$ can be extended to $F_s : I^{C_x} \to I$, the map $\prod F_s : I^{C_x} \to I^S$ is the extension we need. □

And from here follows the main property of βX:

Theorem 3.5. *Let X be $T_{3\frac{1}{2}}$, Y be compact and T_2, and $f : X \to Y$ then f has a continuous extension $\beta f : \beta X \to Y$!*

Proof. Without loss of generality, we can assume Y is a closed subset of I^A for some set A, otherwise Y is homeomorphic to βY [theorem 3.2] which is.
[If $e_Y \circ f : X \to \beta Y$ can be extended to $F : \beta X \to \beta Y$ then $e^{-1} \circ F : \beta X \to Y$ is an extension of f]

By lemma 3.4 $f : X \to Y \subseteq I^A$ can be extended to $F : I^{C_x} \to I^A$, were F is continuous and $X \subseteq F^{-1}[Y]$, Y is closed, so $F^{-1}[Y]$ is closed, so $\beta X = \overline{X} \subseteq F^{-1}[Y]$. and so $F|_{\beta X} : \beta X \to Y$ is an extension of $f : X \to Y$. □

Remark 3.6. Formally this means that $\beta f : \beta X \to Y$ is continuous and $\beta f \circ e = f$ [as we identified x with $e(x)$].

Remark 3.7. This property is unique to $(\beta X, e)$ among T_2 compactifications, as we will show in result 4.8.

The Stone-Cech compactification is, in a way, the biggest compactification for any given space [see Theorem 4.7]. Unfortunately, it's size give's it some pathologic properties, as seen in the following Theorem and example.

Theorem 3.8. *If X is T_4 then for every $x \in \beta X \setminus X$:*

(1) *No sequence $X_n \in X$ converges to x.*
(2) *βX is not first countable at x.*

Proof. (1) If $x_n \to x$ then $\{x\} \cup \{x_n\}_{n=1}^{\infty}$ is compact, [any open set containing x will contain almost all of $\{x_n\}_{n=1}^{\infty}$] so it's closed in βX [who's T_2] so $\{x_n\}_{n=1}^{\infty}$ is closed in X. for the same reason $\{x_{2n-1}\}_{n=1}^{\infty}$ and $\{x_{2n}\}_{n=1}^{\infty}$ are closed in X who's T_4, so there is a $f : X \to I$ such that $f(x_{2n-1}) = 0$ and $f(x_{2n}) = 1$ [Urysohn's lemma] which can be extended to βf so $0 = \lim \beta f(x_{2n-1}) = \beta f(x) = \lim \beta f(x_{2n}) = 1$ which is a contradiction.

(2) If βX had a countable local base at x, $\{B_n\}_{n=1}^{\infty}$ we could pick $x_n \in \bigcap_{i=1}^{n} B_i \cap X$ [X is dense, so it intersect every environment] so $x_n \to x$ in contradiction of (1).

□

Remark 3.9. (2) is actually true for every $T_{3\frac{1}{2}}$ space, but it's beyond the reach of this presentation.

Result 3.10. *If X is not compact $\Leftrightarrow \beta X \setminus X \neq \emptyset$ then βX isn't first countable, and so isn't metrizable.*

Example 3.11. Let's look at $\beta(0,1]$, define $f : (0,1] \to [-1,1]$ as $f(x) = sin\frac{1}{x}$, for every $t \in [-1,1]$ there is a sequence $x_n \in (0,1]$ such that $x_n \to 0$ and $f(x_n) \to t$, βX is compact, so x_n has a cluster point (see Wilansky [1, p.122]) $x_t \in \beta(0,1] \setminus (0,1]$ (we know it's not in $(0,1]$) and so $\beta f(x_t)$ is a cluster point of $\beta f(x_n) = t$, and there for $\beta f(x_t) = t$, so for every $t \in [-1,1]$ there is a different $x_t \in \beta(0,1] \setminus (0,1]$.
In a similar way $\beta(0,1)$ has two such points for every $t \in [-1,1]$, x_t and a y_t cluster point of $y_n \to 1$ such that $f(y_n) \to t$, and $\beta Id_{(0,1)}(x_t) = 0$ and $\beta Id_{(0,1)}(y_t) = 1$ so $x_t \neq y_t$.

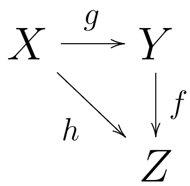
Remark 3.12. Take another look at example 3.11, every sequence x_n from there has a cluster point in $(0,1]$, but it has no partial limit in $(0,1]$ nor in $\beta(0,1] \setminus (0,1]$ because $(0,1]$ is T_4, so this is an example of sequences with cluster points but without converging subsequences, also $\beta(0,1]$ is compact but not sequentially compact.

Idea 3.13. One last thought on the size on βX:
consider $\beta\mathbb{N}$, $\mathbb{N}$ is a countable discrete space, and is homeomorphic to $\mathbb{Z}^n \subseteq \mathbb{R}^n$, every two sequences in there that you can separate with a function to I have at list two different cluster points in $\beta\mathbb{N}$, including two disjoint subsequences of the same sequence, thats far larger then any matric space.

4. T_2-COMPACTIFICATIONS

We introduce an ordering on all T_2 compactifications of X.

Definition 4.1. Let X be $T_{3\frac{1}{2}}$, we define a Partial Ordering on the set $\Psi(X)$ of T_2 compactifications of X by :
$(Y, g) \geq (Z, h)$ if there is a continuous $f : Y \twoheadrightarrow Z$ [Onto] such that $h = f \circ g$.



Theorem 4.2. *That means $f|_{g[X]}$ is a homeomorphism, and $f[Y \setminus g[X]] = Z \setminus h[X]$.*

Proof. $f|_{g[X]}^{-1} = g \circ h|_{h[x]}^{-1}$ which is continuous,
$f[Y \setminus g[X]] \supseteq Z \setminus h[X]$ is trivial,(f is onto) to prove $\subseteq$:
if $y \in Y \setminus g[X]$ and $f(y) \in h[X]$, let $y_\delta \in g[X]$, $y_\delta \to y$ ($g[X]$ is dense) than $h[X] \ni f(y_\delta) \to f(y) \in h[X]$, and so $y_\delta = f|_{g[X]}^{-1} \circ f(y_\delta) \to f|_{g[X]}^{-1} \circ f(y) \in g[X]$ and $y_\delta \to y \notin g[X]$ a contradiction, because Y is T_2 so a net's limit is unique. □

Example 4.3. Let $X = (0, 1)$, $Y = [0, 1]$, $Z = S^1$, $g(x) = x$ and $h(x) = e^{2\pi i x}$. (Y, g) and (Z, h) are compactifications of X, $(Y, g) \geq (Z, h)$ by $h_1(x) = e^{2\pi i x}$, but $(Y, g) \not\leq (Z, h)$, because if there was a $h_2 : Z \to Y$ like we need, then we will get $h_2[\{1\}] = \{0, 1\}$ (throw theorem 4.2).

'$\leq$' is obviously reflexive and transitive, though equivalence is possible:

Theorem 4.4. *If $(Y, g), (Z, h)$ are T_2 compactifications of X and $(Y, g) \leq (Z, h) \leq (Y, g)$ then there is a homeomorphism $f : Y \to Z$ such that $h = f \circ g$ (the same f form the definition of $\leq$).*

Proof. There are $f : Y \to Z$ and $k : Z \to Y$ such that $h = f \circ g$ and $g = k \circ h$, so $g = k \circ f \circ g$ so (g is 1 to 1) $Id_(g[X]) = (k \circ f)|_{g[X]}$ and $g[X]$ is dense so for every $y \in Y$ there is $g[X] \ni y_\delta \to y$ so $y \leftarrow y_\delta = k \circ f(y_\delta) \to k \circ f(y)$ and since Y is T_2, $y = k \circ f(y)$, so $k \circ f \equiv Id_Y$, for the same reason $f \circ k \equiv Id_X$, so f is a homeomorphism. □

Remark 4.5. If (Y, g) and (Z, h) are equivalent $[(Y, g) \sim (Z, h)]$ then f is a homeomorphism and $f[g[X]] = h[X]$, so we can treat them as the same compactification.

Now that we have a partial order, it is natural to ask about minimal and maximal members.

Lemma 4.6. *If $X \subseteq Y$ is dense, X locally compact and Y is T_2 than $X \subseteq Y$ is open.*

Proof. Let $x \in X$ then there are $x \in U \subseteq K \subseteq X$ such that U is open in X and K is compact. so K is closed in Y and $U = X \cap G$for some open $x \in G \subseteq Y$. $G \cap X \subseteq K \subseteq X$ so $cl_Y(G \cap X) \subseteq cl_Y(K) = K \subseteq X$, but $X \subseteq Y$ is dense and $G \subseteq Y$ is open, so $cl(G) = cl(G \cap X)$ [exercise] and so $G \subseteq cl(G) \subseteq X$ and so X is a neighborhood of x for every $x \in X$ so X is open. □

Theorem 4.7. *Let X be a $T_{3\frac{1}{2}}$ space and (Y, g) be a T_2 compactification of X, then $(\beta X, e) \geq (Y, g)$ and if X is locally compact then $(Y, g) \geq (X^+, i)$.*

Proof. (1) $g : X \to Y$ is continuous, and Y is compact T_2, so by theorem 3.5 [and remark 3.6] there is a continuous $\beta g : \beta X \to Y$ such that $\beta g \circ e = g$, to prove βg is onto: $\beta g[\beta X] \subseteq Y$ is compact, so it's closed [Y is T_2] and $g[X] = \beta g \circ e[X] \subseteq \beta g[\beta X]$ so
$Y = Cl_Y(g[X]) \subseteq \beta g[\beta X]$ because $g[X]$ is dense in Y.

(2) We define $f : Y \to X^+$ as: for $x \in g[X]$, $f(x) = g^{-1}(x)$ and for $x \in Y \setminus g[X]$, $f(x) = \infty$ it's clear that f is onto and $f \circ g = i$, as for continuity:
Let $U \subseteq X^+$ be open:
If $U \subseteq X$ then U is open in X so $f^{-1}[U] = g[U]$ is open in $g[X]$ so $f^{-1}[U] = V \cap g[X]$ for an open $V \subseteq Y$, but $g[X]$ is open in Y [lemma 4.6] so $f^{-1}[U]$ is open in Y.
If $\infty \in U$ then $U^c \subseteq X$ is compact, and
$(f^{-1}[U])^c = f^{-1}[U^c] = g[U^c] \subseteq Y$ is compact, so it's closed. [Y is T_2] so $f^{-1}[U] \subseteq Y$ is open.

□

Result 4.8. *If (cX, f) is a compactification of X such that for every pair (Y, g) were Y is a compact T_2 space and $g : X \to Y$ is continuous there is a continuous $cg : cX \to Y$ such that $cg \circ f = g$ than $(cX, f) \sim (\beta X, e)$.*

Proof. By theorem 4.7 $(cX, f) \leq (\beta X, e)$, but (cX, f) has all the properties of $(\beta X, e)$ we used to prove $(\beta X, e)$'s minimality. so $(cX, f) \geq (\beta X, e)$ and from theorem 4.4 $(cX, f) \sim (\beta X, e)$.

□

Remark 4.9. Since βX is compact and Y is a Hausdorff compactification, then $\beta f : \beta X \to Y$ is a quotient map, which means:
"Every T_2 compactification of X is a quotient space of βX"

Exercise 4.10. We have seen $(0, 1)$ has a one-point T_2 compactification S^1 and a two-point T_2 compactification $[0, 1]$, show it doesn't have a three-point T_2 compactification:

If $Y = \{p_1, p_2, p_3\} \cup (0,1)$ is T_2 than p_1, p_2, p_3 have pair-wise disjoint open neighborhoods G_1, G_2, G_3 so $G = G_1 \cup G_2 \cup G_3$ is open so $Y \setminus G \subseteq Y$ is closed and compact (Y is compact).

$Y \setminus G \subseteq (0,1)$ so $Y \setminus G \subseteq [a,b]$ for $0 < a \leq b < 1$ and $(0,a) \cup (b,1) \subseteq G$, now $\{(0,a) \cap G_1, (0,a) \cap G_2, (0,a) \cap G_3\}$ is a partition of $(0,a)$ so it must be trivial [$(0,a)$ is connected] so only one G_i may intersect $(0,a)$ [we can say it's G_1 WLOG], for the same reason, only G_2 may intersect $(b,1)$, so
$G_3 \cap ((0,a) \cup (b,1)) = \emptyset$.

Since $G_3 \setminus \{p_3\} \subseteq (0,1)$, we get $G_3 \setminus \{p_3\} \subseteq [a,b]$,
$[a,b] \subseteq Y$ is compact so it's closed so, $G_3 \cap (Y \setminus [a,b]) = \{p_3\}$ so $p_3 \subseteq Y$ is open! $\Rightarrow p_3 \notin Cl_Y(0,1)$ so $(0,1)$ is not dense in Y, in contradiction to Y being a compactification of X.

References

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