CHAOS THEORY AND DYNAMICAL SYSTEMS

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Abstract. This paper gives an introduction to dynamical systems and chaos. Among other things it proves a remarkable theorem called the period 3 theorem and states the more general Sarkovskii theorem.

1. Iterates and Orbits

Definition 1.1. Let S be a set and f a function mapping the set S into itself. i.e. $f : S \to S$. The functions $f^1, f^2, ..., f^n, ...$ are defined inductively as follows:
$f^1 = f$
if f^{n-1} is known than $f^n = f^{n-1} \circ f$.
Each of these functions is said to be an iterate of f.

Definition 1.2. Let $f : S \to S$. If $x_0 \in S$ then $x_0, f^1(x_0), ..., f^n(x_0), ...$ is called the **orbit** of x_0. x_0 is called the **seed** of the orbit.

Definition 1.3. Let $f : S \to S$. A point $a \in S$ is said to be a **fixed point** of f if $f(a) = a$.

2. Fixed Points and Periodic Points

Definition 2.1. Let $f : S \to S$. A point $a \in S$ is said to be **eventually fixed** if a is not a fixed point, but some point on the orbit of a is a fixed point.

Example 2.2. Let $f = x^2, S = \mathbb{R}, x_0 = -1$. We can see that $f(x_0) = 1 \neq x_0$ but $f^2(x_0) = 1 = f(x_0)$ hence the point 1 which is on the orbit of x_0 is fixed and therefore x_0 is eventually fixed.

Definition 2.3. Let $f : S \to S$. A point $x \in S$ is said to be periodic if there exists a positive integer p such that $f^p(x) = x$. If m is the least $n \in \mathbb{N}$ such that $f^n(x) = x$ then m is called the **period** of x.

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Example 2.4. Let $f = x^2 - 1$. If $x_0 = 0$ than $f(x_0) = -1$ and $f^2(x_0) = f(-1) = 0$. Therefore x_0 is periodic with period 2.

Definition 2.5. Let $f : S \to S$. The point $x_0 \in S$ is said to be **eventually periodic** if x_0 is not periodic, but some point in the orbit of x_0 is periodic.

Example 2.6. Let $f = x^2 - 1$ again. If we look at $x_0 = 1$ than we have $f(x_0) = 0$ and we have already seen than $x = 0$ is periodic with period 2 for this function. See 2.4.

3. Attracting and Repelling Fixed Points

Definition 3.1. Let a be a fixed point of $f : \mathbb{R} \to \mathbb{R}$. The point a is said to be an **attracting fixed point** if there is an open interval I containing a such that if $x \in I$, then $f^n(x) \to a$ as $n \to \infty$.

Definition 3.2. Let a be a fixed point of $f : \mathbb{R} \to \mathbb{R}$. The point a is said to be a **repelling fixed point** if there is an open interval I containing a such that if $x \in I$ and $x \neq a$, then there exists an integer n such that $f^n(x) \notin I$.

Example 3.3. Let $f(x) = x^3$. $f(x)$ has three fixed points: $-1, 1, 0$. 0 is an attracting fixed point and $-1, 1$ are repelling fixed points.

Definition 3.4. A fixed point that is neither attracting nor repelling is called a **neutral fixed point**.

4. Bifurcation

Remark 4.1. One could ask if any continuous function $f : S \to \mathbb{R}$ has a fixed point where $S \subseteq \mathbb{R}$. The answer to that is obviously no. An easy example to this is the function $f(x) = x + 1$ that obviously has no fixed points. However for $S = [0, 1]$ we can state the following theorem.

Theorem 4.2. *Let f be a continuous mapping from $[0, 1]$ into $[0, 1]$. Then there exists a point $z \in [0, 1]$ such that $f(z) = z$.*

Proof. The proof uses The Weierstrass Intermediate Value Theorem. If $f(0) = 0$ or $f(1) = 1$ then the theorem is obvious. We will assume that $f(0) > 0$ and $f(1) < 1$.
Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = x - f(x)$. g is continuous, $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. Therefore by a corollary from Weierstrass Intermediate Value Theorem there exists $z \in [0, 1]$ such that $g(z) = 0$ i.e. $f(z) = z$. $\square$

The following theorems will give a simple way of establishing whether a fixed point is attracting repelling or neutral for "well behaved functions".

Theorem 4.3. *Let S be an interval in $\mathbb{R}$ and a be a point in the interior of S. Furthermore, let a be a fixed point of a function $f : S \to \mathbb{R}$. If f is differentiable at the point a and $|f'(a)| < 1$, then a is an attracting fixed point of f.*

Proof. As $|f'(a)| < 1$, we have $|f'(a)| < k < 1$ where $k = \frac{|f'(a)|+1}{2}$.
By definition, $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$. There exists $\delta > 0$ and interval $I = [a - \delta, a + \delta]$ such that $|\frac{f(x)-f(a)}{x-a}| \leq k$ for all $x \in I$ with $x \neq a$.
Since a is a fixed point $f(a) = a$. So we get $|f(x) - a| \leq k|x - a|$, $\forall x \in I$.

noting that $k < 1$ the last result implies that $f(x)$ is closer to a than x. Therefore, we get that $f(x) \in I$. So we can repeat the argument using $f(x)$ instead of x. We get:

$$|f^2(x) - a| \leq k|f(x) - a| \quad \forall x \in I$$

Combining this two results we get:

$$|f^2(x) - a| \leq k^2|x - a| \quad \forall x \in I$$

Since $|k| < 1$ $k^2 < 1$. We can repeat the argument again and we can obtain by induction

$$|f^n(x) - a| \leq k^n|x - a| \quad \forall x \in I \quad and \quad \forall n \in \mathbb{N}$$

As $|k| < 1$, $\lim_{n\to\infty} k^n = 0$. By the result this implies that $f^n(x) \to a$ as $n \to \infty$. And we have proved that a is an attracting point. □

Theorem 4.4. *Let S be an interval in $\mathbb{R}$ and a be a point in the interior of S. Furthermore, let a be a fixed point of a function $f : S \to \mathbb{R}$. IF f is differentiable at a and $|f'(a)| > 1$, than a is a repelling fixed point of f.*

Proof. Same as the proof of the previous theorem. □

Definition 4.5. A **Quadratic Map** is a function $Q_c : \mathbb{R} \to \mathbb{R}$ where $c \in \mathbb{R}$ and $Q_c(x) = x^2 + c$.

Remark 4.6. Quadratic maps are a family of functions with the surprising feature that the dynamics of Q_c changes as c changes.

Theorem 4.7. ***The first Bifurcation Theorem***. *Let Q_c be the quadratic function for $c \in \mathbb{R}$.*

(1) *If $c > \frac{1}{4}$, then all orbits tend to infinity. That is, $\forall x \in \mathbb{R}$, $(Q_c)^n(x) \to \infty$ as $n \to \infty$.*
(2) *If $c = \frac{1}{4}$, then Q_c has exactly one fixed point at $x = \frac{1}{2}$ and this is a neutral fixed point.*
(3) *If $c < \frac{1}{4}$, then Q_c has two fixed points $a_+ = \frac{1}{2}(1+\sqrt{1-4c})$ and $a_- = \frac{1}{2}(1-\sqrt{1-4c})$.*
   - *The point a_+ is always repelling.*
   - *If $-\frac{3}{4} < c < \frac{1}{4}$, than a_- is attracting.*
   - *If $c < -\frac{3}{4}$, then a_- is repelling.*

Definition 4.8. Let f be a function mapping the set S into itself. If the point $x \in S$ has period m, then the orbit of x is $\{x, f(x), ..., f^{m-1}(x)\}$ and the orbit is called an **m-cycle**.

Definition 4.9. Let a be a periodic point of a function $f : S \to S$ of period m, for some $m \in \mathbb{N}$. a is clearly a fixed point of $f^m : S \to S$. a is said to be an **attracting period point** of f if it is an attracting fixed point of f^m. Similarly a is said to be a **repelling period point** if f if it is a repelling fixed point of f^m.

Theorem 4.10. ***The Second Bifurcation Theorem*** *Let Q_c be the quadratic function for $c \in \mathbb{R}$.*

(1) *If $-\frac{3}{4} \leq c < \frac{1}{4}$, then Q_c has no 2-cycles.*
(2) *If $-\frac{5}{4} < c < -\frac{3}{4}$, then Q_c has an attracting 2-cycle, $\{q_-, q_+\}$, where $q_+ = \frac{1}{2}(-1 + \sqrt{-4c-3})$ and $q_- = \frac{1}{2}(-1 - \sqrt{-4c-3})$.*
(3) *If $c < -\frac{5}{4}$, then Q_c has a repelling 2-cycle $\{q_-, q_+\}$.*

5. The Magic of Period 3: Period 3 Implies Chaos

In this section we wish to prove the period 3 theorem. The period 3 theorem was proved in 1975 by Yorke and Li (see [4]). We will first prove some helpful lemmas.

Lemma 5.1. *If $f : \mathbb{R} \to \mathbb{R}$ then for every interval $I \subset \mathbb{R}$ $f(I)$ is also an interval*

Proof. This is a result of the fact that a continuous mapping maps connected sets to connected sets, and that all connected sets in $\mathbb{R}$ are intervals. □

Lemma 5.2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : I = [a, b] \to \mathbb{R}$ a continuous function. If $f(I) \supseteq I$ f has a fixed point in I.*

Proof. We define $f[I] = [c, d]$ where $c < a$ and $d > b$. There exists two points $s, t \in [a, b]$ such that $f(s) = c \leq a \leq s$ and $f(t) = d \geq b \geq t$ otherwise $f(I)$ will not cover I.
We define $g(x) : I \to \mathbb{R}$ as follows: $g(x) = f(x) - x$. $g(x)$ is continuous and $g(s) \leq 0$, $g(t) \geq 0$. By Weirstrass Intermediate value theorem we get that there exists a point $a \in I$ such that $g(a) = 0 \to f(a) = a$ and a is a fixed point of f in I. □

Remark 5.3. Note that this is a generalization of the theorem about the interval $[0, 1]$ proved in 4.2.

Lemma 5.4. *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $f([a, b]) \supseteq J = [c, d]$, for $c, d \in \mathbb{R}$ with $c < d$. Then there exists a subinterval $I' = [s, t]$ of $I = [a, b]$ such that $f(I') = J$.*

Proof. We first observe that c and d are closed sets and since $f([a, b]) \supseteq J = [c, d]$ and f is continuous we get that $f^{-1}\{c\}$ and $f^{-1}\{d\}$ are non empty closed sets. Therefore there is a largest number s such that $f(s) = c$.
We first assume that there is $x > s$ such that $f(x) = d$. Therefore there is a smallest number t such that $t > s$ and $f(t) = d$. Now we suppose in contradiction that there is a $y \in [s, t]$ such that $f(y) < c$. If so we can look at the interval $[y, t]$ and by Weirstrass intermediate value theorem we get that there is $z > y > s$ such that $f(z) = c$ in contradiction with s maximality. In a similar fashion we can deduce that there is no $y \in [s, t]$ such that $f(y) > d$. We get that $f([s, t])$ contains no points outside the interval $[c, d]$ and by Weirstrass intermediate value theorem we get that $f[s, t]$ covers $[c, d]$, therefore $f[s, t] = [c, d] = J$ as required.
We now assume that there is no $x > s$ such that $f(x) = d$. Let s' be the largest number that $f(s') = d$. Clearly $s' < s$. Let t' be the smallest number such that $t' > s'$ and $f(t') = c$. Just like the argument before we can deduce that $f([s', t']) = [c, d]$ J as required.

□

We can now turn to state and prove the period 3 theorem

Theorem 5.5. **{*The Period 3 Theorem*}** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a point of period 3, then for each $n \in \mathbb{N}$ it has a periodic point of period n.*

Proof. There exists a point $a \in \mathbb{R}$ such that $f(a) = b$, $f(b) = c$ and $f(c) = a$ where $a \neq b \neq c$. We shall consider the case where $a < b < c$, the other cases are dealt similarly.
We define $I_0 = [a, b]$ and $I_1 = [b, c]$. Using 5.1 we get that $f(I_0) \supseteq I_1$. Using 5.1 again we can get that $f(I_1) \supseteq I_0 \cup I_1$. If we apply 5.4 on this result we get that there exists an interval $A_1 \subseteq I_1$ such that $f(A_1) = I_1$. Note that $f(A_1) = I_1 \supseteq A_1$, using 5.4 again will give us $A_2 \subseteq A_1$ such that $f(A_2) = A_1$.

So far we have $A_2 \subseteq A_1 \subseteq I_1$ and $f^2(A_2) = I_1$.
We now use induction to extend this result. Lets assume that for $n-3$ there exists a series $A_{n-3} \subseteq A_{n-4} ... \subseteq A_2 \subseteq A_1 \subseteq I_1$ and $f(A_i) = A_{i-1}$ $\forall 2 \leq i \leq n-2$ and $f(A_1) = I_1$. Now using 5.4 again we can get $A_{n-2} \subseteq A_{n-3}$ such that $f(A_{n-2}) = A_{n-3}$. Using this result we get that $f^{n-2}(A_{n-2}) = I_1$ and $A_{n-2} \subseteq I_1$.
Now noting that $f(I_0) \supseteq I_1 \supseteq A_{n-2}$ we can have a closed interval $A_{n-1} \subseteq I_0$ such that $f(A_{n-1}) = A_{n-2}$. Finally as $f(I_1) \supset I_0 \supseteq A_{n-1}$ there is a closed interval $A_n \subseteq I_1$ such that $f(A_n) = A_{n-1}$.
Putting the above parts together we get that $\forall 2 \leq i \leq n$ $f(A_i) = A_{i-1}$ and $f^n(A_n) = I_1$. Now using the fact that $A_n \subset I_1$ and using 5.2 we get that there is a point $x_0 \in A_n$ such that $f^n(x_0) = x_0$, i.e. x_0 is a periodic point of f of period n. We now have to show that x_0 period is n.
Lets assume in contradiction that x_0 is a periodic point of period k where $k < n$. We notice that $f(x_0) \in A_{n-1} \subseteq I_0$ and $\forall 2 < i < n$ $f^i(x_0) \in I_1$. Now $I_0 \cap I_1 = \{b\}$. We first assume that $f(x_0) \neq b$. If so we get that $f^k(x_0) = x_0$ by assumption and therefore $f^{k+1}(x_0) = f(x_0) \in I_0$ in contradiction to the fact that $f^i(x_0) \in I_1 \forall 2 < i < n$. Now we show that $f(x_0) \neq b$. We assume in contradiction that $f(x_0) = b$. We see that $f^2(x_0) = f(b) = a \notin I_1$. Therefore we get that for any $n \geq 3$ there exists a point x_0 such that x_0 has a period n of f.
We now deal with the case $n = 1$. We use the fact that $f(I_1) \supseteq I_1$ and by 5.2 we get that there is a fixed point of f in I_1.
Now we deal with the case $n = 2$. We have that $f(I_0) \supseteq I_1$ and $f(I_1) \supseteq I_0$. Using 5.4 there is a closed interval $B \subseteq I_0$ such that $f(B) = I_1$. Now $f^2(B) \supseteq I_0$, using 5.2 we get there exists a point $x_1 \in B$ such that $f^2(x_1) = x_1$. Now $x_1 \in B \subseteq I_0 = [a, b]$ and $f(x_1) \in f(B) \subseteq I_1$ $[b, c]$. Moreover $x_1 \neq b$ as $f^2(b) = f(c) = a \neq b$. Therefore x_1 is of period 2.
This completes the proof of the period 3 theorem. $\square$

We now state a more general theorem called the Sarkovskii theorem (see [5]). This result is a generalization of the period 3 theorem. It was proved earlier in 1964 though remained unnoticed.

Definition 5.6. Sarkovskii's ordering of the natural numbers. We introduce the following ordering of the natural numbers by Sarkovskii.

$$\begin{gathered} 3, 5, 7, 9, ... \\ 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, ... \\ 2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, ... \\ 2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, ... \\ \vdots \\ ..., 2^n, 2^{n-1}, ..., 2^2, 2^1, 1 \end{gathered}$$

We now state the Sarkovskii theorem

Theorem 5.7. ***Sarkovskii's Theorem*** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a point of period n and n precedes k in Sarkovskii's ordering of the natural numbers, then f has a periodic point of period k.*

Remark 5.8. Note that as the number 3 is the first number if Sarkovskii's ordering this theorem gives the Period 3 Theorem and is a generalization.

The amazing thing is that the converse of Sarkovskii's theorem is also true.

Theorem 5.9. ***Converse of Sarkovskii's Theorem.*** *let $n \in \mathbb{N}$ and l precedes n in Sarkovskii's ordering of the natural numbers. Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of period n, but no periodic point of period l.*

6. Chaotic Dynamical Systems

This section will give an introduction to Chaotic Dynamical Systems.

Definition 6.1. Let (X, d) be a metric space and $f : X \to X$ a continuous mapping of the set X to itself. Then (X, f) is said to be a **dynamical system.**

Definition 6.2. Let (X, d) be a metric space and $f : X \to X$ a mapping of X into itself. Then the dynamical system (X, f) is said to be **transitive** if there exists $x_0 \int X$ such that the orbit of x_0 i.e. $\{x_0, f(x_0), ..., f^n(x_0), ...\}$ is dense in X.

Definition 6.3. The dynamical system (X, f) is said to be **chaotic** if

(1) the set of all periodic points of f is dense in the set X, and

(2) (X,f) is transitive.

Definition 6.4. The dynamical system (X, f) is said to **depend sensitively on initial conditions** if there exists $\beta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$ there is $n \in \mathbb{N}$ and $y \in X$ with $d(x, y) < \varepsilon$ such that $d(f^n(x), f^n(y)) > \beta$.

Theorem 6.5. *When $X = \mathbb{R}$ every chaotic dynamical system depends sensitively on initial conditions.*

Remark 6.6. In fact until 1992 depending sensitively on initial conditions was a part of a chaotic dynamical system definition given by Devaney (see [1]). However in 1992 it was proven by Banks Davis Stacey Brooks and Cairns that this property follows our definition (see [2]).

Remark 6.7. A paper that generalizes this result to more general case is the "Note on Sensitivity of Semigroup Actions" by E. Kontorovich and M. Megrelishvili (see [6]). This paper gives a sufficient condition for a dynamical system defined using a topological semigroup actions to be sensitive.

Definition 6.8. Let (X_1, d_1) and (X_2, d_2) be metric spaces and (X_1, f_1) and (X_2, f_2) dynamical systems. Then (X_1, f_1) and (X_2, f_2) are said to be **conjugate dynamical systems** if there is a homeomorphism $h : (X_1, d_1) \to (X_2, d_2)$ such that $f_2 \circ h = h \circ f_1$ i.e. $f_2(h(x)) = h(f_1(x)) \quad \forall x \in X_1$. The map h is called the **conjugate map**.

Remark 6.9. The concept of conjugate map and conjugate dynamical systems gives an equivalence between dynamical systems. Many times analysis of complex dynamical system is possible by showing that it is conjugate to a different dynamical system which properties are already studied.

Remark 6.10. The following commutative diagram illustrates the concepts of conjugate maps.

$$\begin{tikzcd} a \arrow[r, "f_1"] \arrow[d, "h"] & b \arrow[d, "h"] \\ c \arrow[r, "f_2"] & d \end{tikzcd}$$

Where $a, b \in X_1$ and $c, d \in X_2$.

Theorem 6.11. *Let (X_1, f_1) and (X_2, f_2) be conjugate dynamical systems, where h is the conjugate map.*

(1) *A point $x \in X_1$ is a fixed point of f_1 in $X_1 \iff h(x)$ is a fixed point of f_2 in X_2.*
(2) *A point $x \in X_1$ is a periodic point of period $n \in \mathbb{N}$ of f_1 in $X_1 \iff h(x)$ is a periodic point of period n of f_2 in X_2.*
(3) *The dynamical system (X_1, f_1) is chaotic $\iff$ the dynamical system (X_2, f_2) is chaotic.*

Proof. Parts (1) and (2) are obvious. We will prove part (3). Lets assume that (X_1, f_1) is chaotic. Let P be the set of periodic points of f_1. As (X_1, f_1) is chaotic, P is dense in X_1. As h is continuous, it is $h(P)$ is dense in $h(X_1) = X_2$. As $h(P)$ is the set of all periodic points if (X_2, f_2), it follows that (X_2, f_2) satisfies condition 1 in definition 6.3.
To complete the proof we need to show that (X_2, f_2) is transitive. We know that there exists $x_0 \in X_1$ such that its orbit $x_0, f(x_0), ..., f^n(x_0), ...$ is dense in X_1. We know that the orbit of $h(x_0)$ will be dense in X_2 as the map h is a homeomorphism and h maps an orbit of the point x_0 to the orbit of the point $h(x_0)$. Hence the point $h(x_0) \in X_2$ has an orbit dense in X_2. $\square$

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