Equivariant Completions

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ABSTRACT. An important consequence of a result of Katetov and Morita states that every metrizable space is contained in a complete metrizable space of the same dimension. We give an equivariant version of this fact in the case of a locally compact σ acting group.

Introduction.

Let $\alpha : G \times X \to X$ be a continuous action of a topological group G on a uniform space (X, μ). We give a sufficient condition for the existence of a continuous extension $\hat{\alpha} : \hat{G} \times \hat{X} \to \hat{X}$ where $\hat{G}$ is the sup-completion (i.e., the completion with respect to its two-sided uniformity) and $(\hat{X}, \hat{\mu})$ is the completion of (X, μ). Our sufficient condition is necessary in the following important situation: $\hat{G}$ is Baire, μ is metrizable and for every $g \in \hat{G}$ the g-transition $\hat{X} \to \hat{X}$ is $\hat{\mu}$-uniformly continuous. As an application of a general equivariant completion theorem we unify the verification of the sup-completeness property for some natural groups.

An important consequence of a result of Katetov [10] and Morita [14] states that every metrizable space is contained in a complete metrizable space of the same dimension (see Engelking [6, 7.4.17]). Using the G-factorization theorem [13] we obtain an equivariant generalization of the last fact in the case of locally compact σ-compact acting group G. This generalization, at the same time, improves some "equivariant results" of de Groot [7] and de Vries [16].

A sufficient condition for the existence of G-completions in the case of locally compact G was obtained by Bronstein [3, 20.3].

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1. Conventions and known results.

All spaces are assumed to be Tychonoff. The filter of all neighborhoods (nbd's) of an element x of a space X is denoted by $N_x(X)$. If μ is a compatible entourage uniformity on a topological space X then for every $\varepsilon \in \mu$ and $x \in X$ denote by $\varepsilon(x)$ the nbd $\{y \in X : (x,y) \in \varepsilon\}$. The greatest compatible uniformity is denoted by $\mu_{\max}$. If G is a topological group then G_d denotes the topological group with the same underlying group as in G, but provided with the discrete topology. The left, right and two-sided uniformity is denoted by U_L, U_R and U_{LR} respectively. The neutral element is always denoted by e.

If $\alpha : G \times X \to X$ is an action then the *g-transition* of X is the mapping $\alpha^g : X \to X$, $\alpha^g(x) = \alpha(g,x)$. As usual, instead of $\alpha(g,x)$, we will write gx. For $x \in X$ the *x-orbit mapping* is $\alpha_x : G \to X$, $\alpha_x(g) = gx$. If α is continuous then X is called a G-space. Let μ be a uniformity on X. Then the system $\langle G, (X,\mu), a\rangle$ (or simpler: α) is called *saturated* if each g-transition is μ-uniformly continuous. For any such system there exists a canonical action $\hat{\alpha}_0$ of G on the completion $\hat{X}$. We will say that $(\hat{X}, \hat{\mu})$ is a *G-completion* if $\hat{\alpha}_0$ is continuous. The following natural questions are central in the paper:

(a) When does the continuity of $\alpha : G \times X \to X$ imply the continuity of the canonical action $\hat{\alpha}_0 : G \times \hat{X} \to \hat{X}$?

(b) Let (X,μ) be complete. Under what conditions does there exist a continuous action $\hat{\alpha} : \hat{G} \times X \to X$ which extends α?

(c) When does a metrizable G-space admit metric G-completions (of the same dimension)?

Examples 1.2 and 3.5 for (a), 3.2 for (b), 3.11 and 3.12 for (c) will show that these questions are non-trivial.

If a G-completion $\hat{X}$ is compact, then we get a compact G-extension of X. Due to J. de Vries, a G-space X is said to by *G-Tychonoff* if X admits compact G-

extensions. Denote by Tych^G the class of all G-Tychonoff triples $\langle G, X, \alpha\rangle$. Recall [18] that if G is locally compact then Tych^G coincides with the class of all Tychonoff G-spaces.

Example 1.1. [12] There exists a continuous action α of a separable complete metrizable group G on $J(\aleph_0)$ (the so-called *hedgehog space of spininess* $\aleph_0$ [6]) such that $\langle G, J(\aleph_0), \alpha\rangle$ has no compact G-extensions.

For every topological group G the left translations define a triple $\langle G, G, \alpha_L\rangle$ which always is G-Tychonoff (see Brook [4]).

Example 1.2. [4],[17, p.147] Let G be a locally compact group. Consider the triple $\langle G, G, \alpha_L\rangle$. If G is non-compact and non-discrete then the canonical action $G \times \beta G \to \beta G$ is not continuous.

A system $\langle G, (X, \mu), \alpha\rangle$ is called

(a) *quasibounded* [13] if for every $\varepsilon \in \mu$ there exists a pair $(\delta, U) \in \mu \times N_e(G)$ such that $(gx, gy) \in \varepsilon$ whenever $(x, y) \in \delta$ and $g \in U$;

(b) *bounded* [18] if for every $\varepsilon \in \mu$ there exists $U \in N_e(G)$ satisfying $(x, gx) \in \varepsilon$ for every $(g, x) \in U \times X$.

Denote by Unif^G the class of all quasibounded saturated systems $\langle G, (X, \mu), a\rangle$ with continuous α and by Comp^G the class of all compact G-spaces. Since every compact G-space is bounded with respect to its unique uniformity (see [4]) then we obtain

Lemma 1.3. $\mathrm{Comp}^G \subset \mathrm{Unif}^G$.

The following result directly follows from the definitions.

Lemma 1.4. *If $\langle G, (X, \mu), \alpha\rangle$ is uniformly equicontinuous and α is continuous, then $\langle G, (X, \mu), \alpha\rangle \in \mathrm{Unif}^G$.*

Theorem 1.5. *Let $\alpha : G \times X \to X$ be a continous action. The statements are equivalent:*

(i) $\langle G, X, \alpha \rangle \in \mathrm{Tych}^G$;

(ii) $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$ *for a certain compatible* μ;

(iii) $\langle G, (X, \xi), \alpha \rangle$ *is quasibounded for a certain compatible* ξ.

Proof. Since the quasiboundedness is hereditary by 1.3 we obtain (i) $\Rightarrow$ (ii). Trivially (ii) $\Rightarrow$ (iii). An argument for (iii) $\Rightarrow$ (i), see [13],[11]. $\square$

We will use the following known results.

Theorem 1.6. [13] *Let X be a topogical group and $\alpha : G \times X \to X$ be a continous action of a topological group G on X by automorphisms. Then $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$ whenever $\mu \in \{U_L, U_R, U_{LR}\}$.*

Theorem 1.7. [13] *Suppose that G is Baire and $\alpha : G \times X \to X$ is a d-saturated action on a metrizable uniform space (X, d) such that for a certain dense subset $Y \subset X$ orbit mappings $\alpha_y : G \to X$, where $y \in Y$ are continuous. Then*

(a) α *is continuous;*

(b) $\langle G, (X, d), \alpha \rangle \in \mathrm{Unif}^G$;

(c) X *is G-Tychonoff.*

2. Inherited actions of dense subgroups.

Lemma 2.1. *Let $\alpha : G \times X \to X$ be an action with continuous g-transitions, H be a dense subgroup of G and Y be a dense H-subspace of X such that the orbit mapping $\alpha_y : G \to X$ is continuous for every $y \in Y$. If μ is a compatible uniformity on X and $\langle H, (Y, \mu|_Y), \alpha|_{H \times Y} \rangle$ is quasibounded, then*

(a) $\langle G, (X, \mu), \alpha \rangle$ *is quasibounded;*

(b) α *is continuous;*

(c) *If $\langle H, (Y, \mu|_Y), \alpha|_{H \times Y} \rangle$ is saturated, then $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$.*

Proof. (a) Given any $\varepsilon_0 \in \mu$, choose a symmetric entourage $\varepsilon_1 \in \mu$ such that $\varepsilon_1^5 \subset \varepsilon_0$. Since $\alpha\,|_{H\times Y}$ is quasibounded with respect to $\mu\,|_Y$, there exists a pair $(\delta, U) \in \mu \times N_e(H)$ such that

$$(1) \qquad (gp, gq) \in \varepsilon_1 \text{ for every } g \in U \text{ and every } (p,q) \in \delta \cap (Y \times Y).$$

Take $\delta_1 \in \mu$ with the property $\delta_1^2 \subset \delta$. Since H is a dense subgroup of G then the closure $cl_G(U)$ belongs to $N_e(G)$. Therefore, it suffices to show that $(gp, gq) \in \varepsilon_0$ whenever $g \in cl_G(U)$ and $(p,q) \in \delta_1$. Assuming the contrary take $g_0 \in cl_G(U) \setminus U$ and $(x_1, x_2) \in \delta_1$ such that

$$(2) \qquad (g_0x_1, g_0x_2) \notin \varepsilon_0.$$

Since $\alpha^{g_0} : X \to X$ is continuous and Y is dense then a certain pair $(y_1, y_2) \in \delta \cap (Y \times Y)$ satisfies

$$(g_0x_1, g_0y_1) \in \varepsilon_1, \ (g_0x_2, g_0y_2) \in \varepsilon_1.$$

Using the continuity of the orbit mappings $\alpha_{g_0y_1}, \alpha_{g_0y_2}$ we pick $V \in N_e(G)$ such that

$$(3) \qquad (g_0x_1, gg_0y_1) \in \varepsilon_1^2, \ (g_0x_2, gg_0y_2) \in \varepsilon_1^2 \text{ for every } g \in V.$$

Since $\varepsilon_1^5 \subset \varepsilon_0$ it follows from (2) and (3) that $(gg_0y_1, gg_0y_2) \notin \varepsilon_1$ for every $g \in V$. By our assumption, $g_0 \in cl_G(U)$. Therefore, $Vg_0 \cap U$ is not empty. Hence, $hg_0 \in U$ for a certain $h \in V$. Then we get $(hg_0y_1, hg_0y_2) \notin \varepsilon_1$ which contradicts (1).

(b) Since all g-transitions are continuous, it suffices to check the continuity of α at (e, x_0) for an arbitrary $x_0 \in X$. For a given $\varepsilon_0 \in \mu$ take a symmetric $\varepsilon_1 \in \mu$ such that $\varepsilon_1^4 \subset \varepsilon_0$. According to (a) choose a symmetric $\delta \in \mu$ and $U_1 \in N_e(G)$ satisfying $(gp, gq) \in \varepsilon_1$ for every $g \in U_1$ and $(p,q) \in \delta$. Fix an element $y_0 \in Y \cap \delta(x_0)$. Since $\alpha_{y_0} : G \to X$ is continuous there exists $U_2 \in N_e(G)$ such that $(y_0, gy_0) \in \varepsilon_1$ for every $g \in U_2$. Now, if $x \in \delta(x_0)$ and $g \in U_1 \cap U_2$ then $gx \in \varepsilon_0(x_0)$. This proves the continuity of α.

(c) According to the definition of Unif^G, we have only to show that $g\varepsilon \in \mu$ for any $(g, \varepsilon) \in G \times \mu$. Using (a) we pick a pair $(\delta, U) \in \mu \times N_e(G)$ such that $\delta \subset t\varepsilon$ for every $t \in U$. Clearly, $g = ht$ for a certain pair $(h, t) \in H \times U$. Therefore, $h\delta \subset ht\varepsilon = g\varepsilon$. By our hypothesis, $\alpha^h|_Y$ is $\mu|_Y$-uniformly continuous. Since $\alpha^h : X \to X$ is a homeomorphism and Y is dense, then α^h is μ-uniformly continuous. Therefore, $h\delta \in \mu$, which yields $g\varepsilon \in \mu$. $\square$

Proposition 2.2. *Let H be a dense subgroup of G. Then*

$$\langle G, X, \alpha \rangle \in \mathrm{Tych}^G \quad \textit{iff} \quad \langle H, X, \alpha|_{H \times X} \rangle \in \mathrm{Tych}^G .$$

Proof. Necessity is trivial. The converse follows from Theorem 1.5 and Lemma 2.1(a). $\square$

Combining 1.1 and 2.2 we get

Example 2.3. There exists a continuous action $\alpha : H \times J(\aleph_0) \to J(\aleph_0)$ of a metrizable *countable* group H such that $\langle H, J(\aleph_0),\ \alpha \rangle$ is not H-Tychonoff.

Question 2.4. Let G be a monothetic group. Is it true that every Tychonoff G-space is G-Tychonoff?

3. Main results.

Theorem 3.1. *Let $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$. Then there exists a continuous action $\hat{\alpha} : \hat{G} \times \hat{X} \to \hat{X}$ which extends α and satisfies $\langle \hat{G}, (\hat{X}, \hat{\mu}), \hat{\alpha} \rangle \in \mathrm{Unif}^G$.*

Proof. Let φ and γ be Cauchy filters in (G, U_{LR}) and (X, μ), respectively. Denote by $\varphi\gamma$ the system $\{AB : A \in \varphi, B \in \gamma\}$. An essential step in our proof is the following

Claim. *$\varphi\gamma$ is a μ-Cauchy filter basis.*

Proof of Claim. For a given $\varepsilon \in \mu$ choose a pair $(\delta_1, U) \in \mu \times N_e(G)$ such that

$$(gx, gy) \in \varepsilon \text{ for every } (x, y) \in \delta_1 \text{ and } g \in U. \tag{1}$$

The inclusion $U_R \subset U_{LR}$ implies that φ is U_R-Cauchy. Therefore, there exists $g_0 \in G$ such that $Ug_0 \in \varphi$. Since α^{g_0} is μ-uniformly continuous one can choose $\delta_2 \in \mu$ with the property

$$(g_0 x, g_0 y) \in \delta_1 \text{ for every } (x, y) \in \delta_2. \tag{2}$$

There exists a symmetric entourage $\delta_3 \in \mu$ such that $\delta_3^3 \subset \delta_2$. Using the quasiboundedness we pick a symmetric $\delta_4 \in \mu$ and $V_1 \in N_e(G)$ satisfying

$$(vx, vy) \in \delta_3 \text{ whenever } (x, y) \in \delta_4 \text{ and } v \in V_1. \tag{3}$$

Since γ is μ-Cauchy, then $\delta_4(x_0) \in \gamma$ for a certain $x_0 \in X$. The continuity of $\alpha_{x_0} : G \to X$ implies the existence of $V_2 \in N_e(G)$ such that

$$(x_0, vx_0) \in \delta_4 \text{ for } v \in V_2. \tag{4}$$

It follows from (3),(4) and the inclusions $\delta_4 \subset \delta_3$, $\delta_3^3 \subset \delta_2$ that

$$(x_1, vx_2) \in \delta_2 \text{ for } x_1, x_2 \in \delta_4(x_0) \text{ and } v \in V_1 \cap V_2 = V. \tag{5}$$

For the next argument observe that φ is U_L-Cauchy. Choose $A \in \varphi$ such that $g_1^{-1} g_2 \in V$ for every $g_1, g_2 \in A$. Since $Ug_0 \in \varphi$ then without restriction of generality we may suppose that $A \subset Ug_0$. Now, if $g_1, g_2 \in A$ then $g_1 = t_1 g_0, g_2 = t_2 g_0$ for certain $t_1, t_2 \in U$. Since $g_1^{-1} g_2 \in V$ then by (5) we have $(x_1, g_0^{-1} t_1^{-1} t_2 g_0 x_2) \in \delta_2$ for every $x_1, x_2 \in \delta_4(x_0)$. Using (2) we obtain $(g_0 x_1, t_1^{-1} t_2 g_0 x_2) \in \delta_1$. By (1) we get $(t_1 g_0 x_1, t_2 g_0 x_2) \in \varepsilon$. Therefore, $(g_1 x_1, g_2 x_2) \in \varepsilon$ for every $x_1, x_2 \in \delta_4(x_0)$ and every $g_1, g_2 \in A$. Since $A \in \varphi$ and $\delta_4(x_0) \in \gamma$, this means that $\varphi\gamma$ is μ-Cauchy.

Now we are ready for the construction of $\hat{\alpha}$. Denote by $i : X \to \hat{X}$, the canonical embedding and consider the composition $f = i \circ \alpha : G \times X \to \hat{X}$. Let ξ be a Cauchy filter in the uniform product $(G, U_{LR}) \times (X, \mu)$. There exists a U_{LR}-Cauchy filter φ and a μ-Cauchy filter γ such that $\varphi \times \gamma$ is a filter basis of ξ. Since $i(\varphi\gamma)$ is a subfilter of $f(\xi)$ in $\hat{X}$, the above claim implies that $f(\xi)$ generates a convergent filter in $\hat{X}$. Therefore, we can use the well-known *extension theorem* [1, ch.I, sec. 8.5, Th.1]. Taking into account that the completion of the product $(G, U_{LR}) \times (X, \mu)$ is canonically equivalent to the uniform product $(\hat{G}, \hat{U}_{LR}) \times (\hat{X}, \hat{\mu})$, we get the *continuous* mapping $\hat{\alpha} : \hat{G} \times \hat{X} \to \hat{X}$ which extends α. By the *principle of the extension of identities* [1, Ch.I, Sec.8.1, Cor.1] $\hat{\alpha}$ is an action. Finally, from Lemma 2.1 it directly follows that $\langle \hat{G}, (\hat{X}, \hat{\mu}), \hat{\alpha} \rangle \in \mathrm{Unif}^{\hat{G}}$. $\square$

The paper [11] (which can be regarded as a preprint) contains a slightly different proof of Theorem 3.1. The weaker result (the continuity of the canonical completion $\hat{\alpha}_0 : G \times \hat{X} \to \hat{X}$) can be found in [13].

By Example 1.2 it is clear that the quasiboundedness in Theorem 3.1 is essential. The following example shows that this condition cannot be dropped even for complete uniformity μ.

Example 3.2. Let $\mathbb{Q}$ be the topological group of all rational numbers. Consider the system $\langle \mathbb{Q}, (\mathbb{Q}, \mu_{\max}), \alpha_L \rangle$. Then $\mu_{\max}$ is complete but there is no continuous non-trivial action of $\hat{\mathbb{Q}} = \mathbb{R}$ on $\mathbb{Q}$.

Results in section 1 (and Corollary 3.4 and Proposition 3.7 below) show that Theorem 3.1 is often applicable. There is an important case where our sufficient condition, at the same time, is necessary.

Proposition 3.3. *Let $\alpha : G \times X \to X$ be a continuous action on a metrizable uniform space (X, μ). If $\hat{G}$ is Baire then the following statements are equivalent:*

(a) *There exists a $\hat{\mu}$-saturated continuous extension $\hat{\alpha} : \hat{G} \times \hat{X} \to \hat{X}$;*

(b) $\langle G, (X, \mu), \alpha\rangle \in \mathrm{Unif}^G$.

Proof. (b) $\Rightarrow$ (a) is true by Theorem 3.1. For the converse use Theorem 1.7. $\square$

Corollary 3.4. [13] *If G is Baire then every metrizable G_d-completion of a G-space is a G-completion.*

Example 1.2 emphasizes the importance of the metrizability condition in Corollary 3.4 and Theorem 1.7. Now we show that the assumption concerning G also is not superfluous.

Proposition 3.5. *Let G be a countable non-trivial metrizable group such that $\hat{G}$ is connected. Consider the triple $\langle G, X := G, \alpha_L\rangle$. Then there exists a compact metrizable G_d-extension of X which is not G-extension.*

Proof. Clearly, $\dim X = 0$. Since G_d is discrete and countable, it follows from a result of de Groot and Mcdowell [8, Cor.2.5] that there exists a *zero-dimensional* compact G_d-extension cX of X. Suppose that this is a G-extension. Then the corresponding action $\alpha_c : G \times cX \to cX$ is continuous. Since cX is a compact G-space, by Lemma 1.3 and Theorem 3.1 there exists a continuous extension $\hat{\alpha}_c : \hat{G} \times cX \to cX$. By our assumption $\hat{G}$ is connected. Therefore, $\hat{\alpha}_c(\hat{G}, e) = e$. This implies that G is trivial. $\square$

An application for topological groups. Theorem 3.1 makes possible an easy and unified verification of the sup-completeness property in some natural cases.

Proposition 3.6. *In each case listed below G is sup-complete:*

(a) [2, Ch.X, §3, Ex.16] $G = \mathrm{Unif}(X, \mu)$ — *the group of all unimorphisms of a complete uniform space (X, μ) endowed with the topology of uniform convergence.*

(b) [2, Ch.X,§3, Ex.19] $G = \mathrm{Is}(X, d)$ — *the group of all isometries of a complete metric space (X, d) endowed with the topology of pointwise convergence.*

(c) *([15] or [5, 7.8.9]) $G = \mathrm{Aut}\,X$ — the group of all topoligical automorphisms of a locally compact group X endowed with the Birkhoff topology.*

Proof. (a) Let $\alpha : G \times X \to X$ be a natural action. Clearly, α is μ-saturated and μ-bounded; in particular, $\langle G, (X,\mu), \alpha\rangle \in \mathrm{Unif}^G$. By Theorem 3.1, $\hat{G} \subset \mathrm{Unif}(X,\mu) = G$. Therefore, $\hat{G} = G$.

(b),(c) Combine 3.1 with 1.4 and 1.6 respectively. The continuity of $\hat{\alpha} : \hat{G} \times X \to X$ implies

(1) For every $g \in \hat{G}$ the g-transition is a homeomorphism of X;

(2) The topology of $\hat{G}$ contains the topology of pointwise convergence.

¿From (1) and (2) it easily follows that $g \in \mathrm{Is}(X,d)$ and $g \in \mathrm{Aut}\,X$, respectively. □

Proposition 3.7. [11] *Let $\alpha : G \times X \to X$ be a continuous action of a locally compact group G. Then $\langle G, (X, \mu_{\max}), \alpha\rangle \in \mathrm{Unif}^G$.*

Proof. Let B be a compact nbd of the identity and let a system $\{d_i : i \in I\}$ of pseudometrics generate $\mu_{\max}$. Denote by θ the uniformity on X induced by the system $\{d_i^{(n)} : i \in I,\ n \in \mathbb{N}\}$ where

$$d_i^{(n)}(x,y) = \sup\{d_i(gx, gy) : g \in B^n\}.$$

Since $\{\alpha^g : g \in B^n\}$ is d_i-equicontinuous, θ is compatible with the original topology. Evidently $\langle G, (X,\theta), a\rangle \in \mathrm{Unif}^G$ and $\mu_{\max} \subset \theta$. Finally, observe that the maximality of $\mu_{\max}$ implies $\mu_{\max} = \theta$. □

By Theorem 3.1 and Example 3.2 it is clear that the assumption "locally compact" in Proposition 3.7 cannot be dropped.

Theorem 3.8. *Let $\alpha : G \times X \to X$ be a continuous action of a locally compact σ-compact group G on a metrizable space X. Then there exists a metric uniformity d on X such that the following conditions hold:*

(a) *Each $\alpha^g : X \to X$ $(g \in G)$ is d-uniformly continuous;*

(b) *The canonical action* $\hat{\alpha} : G \times (\hat{X}, \hat{d}) \to (\hat{X}, \hat{d})$ *is continuous and* $\langle G, (\hat{X}, \hat{d}), \hat{\alpha} \rangle \in \mathrm{Unif}^G$;

(c) $\dim \hat{X} = \dim X$.

Proof. Let ρ be any compatible metric uniformity on X and consider the identity $\mathrm{Id}_X : (X, \mu_{\max}) \to (X, \rho)$. By Proposition 3.7, $\langle G, (X, \mu_{\max}), \alpha \rangle \in \mathrm{Unif}^G$. Therefore, we can apply the G-factorization theorem [13, Th.2.6]. Taking into account the σ-compactness of G we can choose a *metric* uniformity μ on X such that $\langle G, (X, \mu), \alpha \rangle \in \mathrm{Unif}^G$ and the uniform dimension [9] $\Delta d \mu X$ is not greater than $\Delta d \mu_{\max} X$. By Theorem 3.1, $\langle G, (\hat{X}, \hat{\mu}), \hat{\alpha} \rangle \in \mathrm{Unif}^G$. Since (X, μ) is uniformly dense in $(\hat{X}, \hat{\mu})$, then $\Delta d \mu X = \Delta d \hat{\mu} \hat{X}$ [9, p.78]. On the other hand, $\Delta d \mu_{\max} X = \dim X$ [9, p.147]. Therefore, $\Delta d \hat{\mu} \hat{X} \leq \dim X$. For every metrizable space Y, the number $\dim Y$ is the minimum of $\Delta d \theta$ for all compatible metric uniformities θ on Y [9, p.153]. Thus, $\dim \hat{X} \leq \Delta d \hat{\mu} \hat{X}$. This establishes $\dim \hat{X} \leq \dim X$. Since $\hat{X}$ is perfectly normal, the inequality $\dim X \leq \dim \hat{X}$ follows from Čech's monotonicity theorem. □

Remark 3.9. Part (a) of Theorem 3.8 is contained in de Groot [7]. Part (b) improves a result of de Vries [16, Th.4.7].

Definition 3.10. Let $\alpha : G \times X \to X$ be a continuous action. We say that a G-space X is (*weakly*) *G-metrizable* if there exists a metric uniformity d on X such that ($\langle G, (X, d), \alpha \rangle$ is saturated) $\langle G, (X, d), \alpha \rangle \in \mathrm{Unif}^G$.

If $\langle H, J(\aleph_0), \alpha \rangle$ is as in Example 2.3, then by Theorem 1.5, $J(\aleph_0)$ is not H-metrizable. On the other hand, since H is countable, $J(\aleph_0)$ is weakly H-metrizable. Indeed, for every metric d on $J(\aleph_0)$ with the bounded diameter we may consider a new metric $d^*(x, y) = \sup\{2^{-n} d(g_n x, g_n y) : n \in \mathbb{N}, g_n \in H\}$ (see [8, Lemma 2.2]). In contrast to this example, Theorem 1.7 implies that if G is Baire then a G-space X is weakly G-metrizable iff X is G-metrizable. The last fact explains the following

Example 3.11. The polish transformation group $\langle G, J(\aleph_0), \alpha\rangle$ from Example 1.1 is not weakly G-metrizable.

Example 3.12. [8, Ex.2.10] Let $X = \mathbb{Q}$ be the space of rational numbers and G the group of all autohomeomorphisms of $\mathbb{Q}$ endowed with the discrete topology. Then X is not weakly G-metrizable.

As Examples 3.11 and 3.12 show, local compactness and σ-compactness are essential in Theorem 3.8(a).

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