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Eberlein Groups

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Abstract. We show that the algebra $W(G)$ of all weakly almost periodic functions generates the given topology on a Hausdorff topological group G (say, *Eberlein group*) if and only if there exists a reflexive Banach space X such that G is a topological subgroup of the group $Is(X)$ of all linear isometries of X endowed with the strong operator topology. For this purpose we establish a general result on coincidence of the strong and weak operator topologies on $Is(X)$ for arbitrary reflexive X. The proof is based on the Ellis-Lawson joint continuity theorem and our recent result about continuity of dual group actions on Asplund spaces. In the last part we discuss two possible ways finding non-Eberlein groups using uniformly universal Banach spaces or minimal topological groups.

§1. Introduction.

Let S be a semitopological semigroup, that is, a topologized semigroup with a separately continuous multiplication. We will denote by $C(S)$ the commutative C^*-algebra of all bounded continuous complex valued functions on S. For each $s \in S$, the right translation maps R_s of $C(S)$ into itself are defined by

$$R_s f(x) = f(xs) \quad \text{for all} \quad x \in S.$$

Recall some basic facts about weak almost periodicity (see [Eb], [LG], [BJM], [R1]). A function $f \in C(S)$ is *weakly almost periodic* if the *orbit* of f, that is, the set

$$Sf := \{R_s f \mid s \in S\}$$

is relatively weakly compact in $C(S)$. The set $W(S)$ of all such functions is a closed S-invariant subalgebra of $C(S)$. If S is compact, then $W(S) = C(S)$. Moreover, the compactification $u\colon S \to S^w$, induced by the algebra $W(S)$, is the *universal semitopological semigroup compactification* of S.

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If $(X, \| \|)$ is a Banach space, we denote by $(L(X), \| \|)$ the algebra of all bounded linear operators $X \to X$. The dual Banach space of X will be denoted by X^*. We use the following notation:

$$B(X) := \{x \in X \mid \|x\| \leq 1\}$$
$$\mathrm{Cont}(X) := \{s \in L(X) \mid \|s\| \leq 1\}.$$

The group of all linear isometries of X will be denoted by $Is(X)$. The *strong, strong*^* and *weak operator topology* (which we denote respectively by T_s, T_{s^*} and T_w) on $L(X)$ is the weakest topology generated respectively by the system of maps:

(1) $\{\tilde{x}\colon L(X) \to X,\ \tilde{x}(s) = sx \mid x \in X\}$,
(2) $\{\tilde{f}\colon L(X) \to X^*,\ \tilde{f}(s)(x) = f(sx) \mid f \in X^*\}$,
(3) $\{\psi_{x,f}\colon L(X) \to \mathbb{R},\ \psi_{x,f}(s) = f(sx) \mid x \in X, f \in X^*\}$.

If a subset P of $L(X)$ is endowed with one of the following subspace topologies $T_s \restriction_P$, $T_{s^*} \restriction_P$, $T_w \restriction_P$, then often we indicate this by writing P_s, P_{s^*} and P_w, respectively. Analogously, a subset A of X endowed with its usual weak topology is denoted by A_w. $L(X)_s, L(X)_{s^*}, L(X)_w$ are always Hausdorff semitopological semigroups.

Fact 1.1. *(Banach-Bourbaki Theorem) A Banach space X is reflexive iff $B(X)$ is weakly compact.*

Fact 1.2. *The semitopological semigroup $\mathrm{Cont}(X)_w$ is compact iff X is reflexive.*

Proof. The compactness of $\mathrm{Cont}(X)_w$ for a reflexive X is well-known (see, for instance [LG, Th. 3.1]). The converse implication follows from Fact 1.1, taking into account that for any fixed vector x_0 with $\|x_0\| = 1$, the map

$$\mathrm{Cont}(X)_w \to B(X)_w, \quad s \to sx_0$$

is continuous and *onto*. Indeed, take a continuous functional f on X such that $f(x_0) = 1$ and $\|f\| = 1$. For every $z \in B(X)$ assign to the pair (f, z) the following linear operator

$$A_{f,z}\colon X \to X, \quad A_{f,z}(x) = f(x)z$$

Clearly, $A_{f,z}$ is a contraction of X moving x_0 into z. $\square$

The following result is well known for Hilbert spaces.

Theorem 1.3. *Let X be a reflexive Banach space. Then the strong and weak operator topologies coincide on $Is(X)$.*

Proof. In [Me1, Corollary 6.9] we have already proved that the reflexivity of X guarantees the continuity of the dual action of $Is(X)$ on X^*. This leads (see [Me1, Corollary 6.11] to the equality $T_s \restriction_{Is(X)} = T_{s^*} \restriction_{Is(X)}$. Therefore, for our purposes it suffices to show that $T_{s^*} \restriction_{Is(X)} \subseteq T_w \restriction_{Is(X)}$.

Consider the canonical separately continuous semigroup action

$$\alpha\colon \operatorname{Cont}(X)_w \times B(X)_w \to B(X)_w.$$

The reflexivity of X guarantees that $B(X)_w$ and $\operatorname{Cont}(X)_w$ are compact. Therefore we can apply the *Ellis-Lawson Theorem* [La, Corollary 5.2] which implies that α is jointly continuous at each point (g,x), where g is an arbitrary unit of $\operatorname{Cont}(X)$ and x is an arbitrary point of $B(X)$. Thus, the restricted group action

$$\alpha'\colon Is(X)_w \times B(X)_w \to B(X)_w$$

is jointly continuous. Let $f \in X^*$. Clearly, the restricted map $f\mid_{B(X)}\colon B(X)_w \to \mathbb{C}$ is continuous. Using the joint continuity of α' and the compactness of $B(X)_w$, we obtain that for a given $\varepsilon > 0$ and $g_0 \in Is(X)$ there exists a neighborhood $O(g_0)$ of g_0 in $Is(E)_w$ such that

$$|f(gx) - f(g_0x)| < \varepsilon$$

for every $x \in B(X)$ and $g \in O(g_0)$. Or, equivalently,

$$\|fg - fg_0\|^* < \varepsilon$$

for every $g \in O(g_0)$. This means that the orbit map

$$\tilde{f}\colon Is(X)_w \to X^*$$

is continuous for every $f \in X^*$. Therefore, $T_{s^*}\mid_{Is(X)} \subseteq T_w\mid_{Is(X)}$, as required. $\square$

Remark 1.4. In general, $T_s\mid_{Is(X)} \neq T_w\mid_{Is(X)}$. The following general construction, providing many counterexamples, is based on an idea of Helmer [H, Ex. 13]. Let Y be a compact Hausdorff space and let G be a subgroup of the group $H(Y)$ of all autohomeomorphisms of Y. Denote by G_p the group G endowed with the topology of pointwise convergence. Then the evaluation map $\alpha\colon G_p \times Y \to Y$ is separately continuous. Consider the induced action

$$\pi\colon G \times C(Y) \to C(Y), \qquad (g \circ f)(y) = f(g^{-1}y)$$

and the induced injective group homomorphism $j\colon G \to Is(X)$, where $X = C(Y)$. Now, suppose that α is *not* jointly continuous and G is a k-space (for example, in [H, Ex. 13], G is the topological group of all rationals and Y is the square $[-1,1]^2$). Then, by Grothendieck's classical result [G, Th. 5], the map $j\colon G_p \to Is(X)_w$ is compact-preserving and, hence, continuous by our assumption on G. It is easy to show that $T_w\mid_{j(G)} \neq T_s\mid_{j(G)}$. Indeed, assuming the contrary, we will obtain that the restricted dual action

$$\pi_B^*\colon G_p \times B(X^*)_{w^*} \to B(X^*)_{w^*}\ , \qquad (g \circ \psi)(f) = \psi(g^{-1} \circ f)$$

is continuous, where $B(X^*)_{w^*}$ is the unit ball of X^* endowed with the weak* topology. Then the original action α (being canonically equivalent to a subaction of π_B^*) is jointly continuous. This contradicts our assumption.

Remark 1.5. In general, $T_s \big|_{Is(X)} \neq T_{s^*} \big|_{Is(X)}$. Indeed, it is well known that there are many continuous norm invariant continuous linear group actions on Banach spaces X such that the corresponding dual actions on X^* are not continuous. Consider, for example, $X = \ell_1$ and define a subgroup $S(\mathbb{N})$ of $Is(X)_s$ consisting of all permutations of "coordinates." Then the dual action of $S(\mathbb{N})$ on $\ell_1^* = m$ is not continuous. Hence, it is not even true that $T_{s^*} \big|_{Is(X)} \subseteq T_s \big|_{Is(X)}$. For arguments in the case of $X = C[0,1]$, see [Me1]. Note also that if X is *Asplund* (by Stegall's result [St], it is equivalent to saying that the dual X^* has the Radon-Nikodym property), then, necessarily, $T_{s^*} \big|_{Is(X)} \subseteq T_s \big|_{Is(X)}$ (cf. [Me1, Corollary 6.9]).

§2. Compact semitopological semigroups "live" in reflexive spaces.

For every reflexive Banach space X, the semitopological semigroup $\mathrm{Cont}(X)_w$ is compact. The aim of this section is to show

Theorem 2.1. *Let (S, τ) be a compact Hausdorff semitopological semigroup. Then there exists a reflexive Banach space X such that (S, τ) is a subsemigroup of the compact semitopological semigroup* $\mathrm{Cont}(X)_w$.

Proof. Without loss of generality, we can assume that S has the identity e. Consider the natural monoid action

$$\alpha \colon S \times C(S) \to C(S) \qquad \alpha(s, f) = sf = R_s(f).$$

Clearly, each s-translation $\alpha^s = R_s$ is continuous. Moreover, by [G, Th. 5], each orbit map

$$\alpha_f \colon S \to C(S)\ , \qquad s \mapsto sf$$

is weakly continuous (see [LG]). Therefore, for every fixed $f \in C(S)$, the orbit Sf is weakly compact. Denote by E_f the Banach subspace of $C(S)$ linearly and topologically generated by Sf. Since a^s is continuous for every $s \in S$, E_f is S-invariant. By the Hahn-Banach Theorem , the weak topology of E_f is the same as its relative weak topology as a subset of $C(S)$. In particular, Sf is weakly compact in E_f. By the Krein-Smulian Theorem, the convex hull $co(-Sf \cup Sf) = W$ of the weakly compact symmetric subset $-Sf \cup Sf$ is relatively weakly compact. That is, the (weak) closure of W in E_f is weakly compact. Since W is a convex, bounded and symmetric subset of E_f, we can apply the factorization procedure discovered by Davis, Figiel, Johnson and Pelczynski [DFJP]. For each natural n, set

$$U_n = 2^n W + 2^{-n} B(E_f).$$

Let $\| \ \|_n$ be the gauge of the set U_n. That is,

$$\|x\|_n = \inf \ \{\lambda > 0 \mid x \in \lambda U_n\}.$$

Then, using [DFJP, §2] we obtain:
(1) $\| \, \|_n$ is a norm on E_f equivalent to the given norm $\| \, \|$ of E_f;
(2) For $x \in E_f$, let

$$N(x) = \left(\sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2},$$

and let

$$X_f = \{x \in E_f \mid N(x) < \infty\}.$$

Denote by $j\colon X_f \to E_f$ the inclusion map;
(3) $f \in Sf \subseteq W \subseteq B(X_f)$;
(4) (X_f, N) is a Banach space and $j\colon X_f \to E_f$ is a continuous linear injection;
(5) X_f is reflexive;
(6) The restriction of $j\colon X_f \to E_f$ on each bounded subset A of X_f induces a homeomorphism of A and $j(A)$ in the weak topologies.

Proof. Consider the weak closure $cl_w(A)$ of A in X_f. By the reflexivity of X_f, the set $cl_w(A)$ is weakly compact. Hence, j, being weakly continuous and injective, induces a homeomorphism of $cl_w(A)$ and $j(cl_w(A))$ with respect to the weak topologies. This proves assertion (6).
(7) $N(sx) \leq N(x)$ for every $x \in X_f$ and every $s \in S$.

Proof. It suffices to show that $\|sx\|_n \leq \|x\|_n$ for every $n \in \mathbb{N}$. By our construction $sW \subseteq W$ and $sB(E_f) \subseteq B(E_f)$ (R_s is a contraction of E_f). Then, from $x \in \lambda(2^n W + 2^{-n} B(E_f))$ we obtain that $sx \in \lambda(2^n(sW) + 2^{-n}s(B(E_f))) \subseteq \lambda(2^n W + 2^{-n}B(E_f))$. Hence, $\|sx\|_n \leq \|x\|_n$, as required. This proves assertion (7).

As a corollary, we get that X_f is an S-invariant subset of E_f. Therefore, the restricted action

$$\alpha_f\colon S \times X_f \to X_f$$

is well-defined.

(8) For every $z \in X_f$, the orbit map

$$\tilde{z}\colon S \to X_f \quad , \qquad \tilde{z}(s) = sz$$

is weakly continuous.

Proof. Indeed, by assertion (7), the orbit $\tilde{z}(S) = Sz$ is an N-normed bounded subset in X_f. Our assertion follows from (6) (for $A = Sz$), taking into account that $\tilde{z}\colon S \to E_f$ is weakly continuous.

By (7), for every $s \in S$, the translation map $\alpha_f^s\colon X_f \to X_f$ is a linear contraction of (X_f, N). Therefore, we get the map

$$\gamma_f\colon S \to \mathrm{Cont}(X_f) \quad , \quad \gamma_f(s) = \alpha_f^s.$$

Now, directly from (8) we obtain

(9) $\gamma_f\colon S \to \mathrm{Cont}(X_f)_w$ is a continuous monoid homomorphism.

Now we are ready to construct the desired reflexive Banach space X. Consider the family $F = \{X_f \mid f \in C(S)\}$ of reflexive Banach spaces and the family

$$\{\gamma_f\colon S \to \mathrm{Cont}(X_f) \mid f \in C(S)\}$$

of monoid homomorphisms . Define X as the ℓ_2-product (cf. [Da, p. 35] or [NP, p. 743]), $X = \prod_2 X_f$ of the family F. Recall that it is the space of all functions $x = (x_f)$ such that $x_f \in X_f$ for each $f \in C(S)$, and the norm on X is defined by

$$\|x\| = \left(\sum_{f \in C(S)} \|x_f\|^2\right)^{1/2} < \infty.$$

Then $(X, \|\ \|)$ is reflexive. Moreover, $(\prod_2 X_f)^* = \prod_2 X_f^*$ and the corresponding pairing for $x = (x_f) \in \prod_2 X_f$, $h = (h_f) \in \prod_2 X_f^*$ is defined by

$$h(x) = \sum_{f \in C(S)} h_f(x_f).$$

Now we define a linear representation of S in X as the ℓ_2-product of old representations. Precisely, we define

$$\gamma\colon S \to \mathrm{Cont}(X) \quad , \quad \gamma(s)(x_f) = (sx_f).$$

First observe that by assertion (7), X is well-defined. Clearly, γ is a monoid homomorphism. By assertion (9) and the above-mentioned description of X^*, it is easy to show that γ is weakly continuous. In order to establish that γ is the desired embedding, by the compactness of S, we have only to show that γ is injective. Equivalently, it suffices to check that $\{\gamma_f \mid f \in C(S)\}$ separates the points of S. Let s_1, s_2 be distinct points of S. Choose a continuous function $f \in C(S)$ with $f(s_1) \neq f(s_2)$. Since $(s_1 f)(e) = f(s_1)$ and $(s_2 f)(e) = f(s_2)$, it follows that $s_1 f$ and $s_2 f$ are distinct elements of $C(S)$ and of E_f. Moreover, by our construction, $X_f \subseteq E_f$ and $s_1 f, s_2 f$ both belong to X_f (see assertion (3)). Therefore, $\gamma_f(s_1) \neq \gamma_f(s_2)$. This implies that $\gamma(s_1) \neq \gamma(s_2)$, as required. $\square$

Remark 2.2. In Theorem 2.1 we may choose X as having the same topological weight as S. That is, $w(X) = w(S)$. Indeed, we can easily modify the second part of the proof, taking the family $\{X_f \mid f \in P\}$, where P separates the points of S.

§3. Eberlein groups and Ruppert's problem.

Recall that for every semitopological semigroup S, the compactification $u\colon S \to S^w$ induced by the algebra $W(S)$ is just the universal semitopological compactification of S.

Definition 3.1. We say that G is an *Eberlein group* if one of the following equivalent conditions holds:

(i) $W(G)$ separates the points from the closed sets;
(ii) $u\colon G \to G^w$ is a topological embedding;
(iii) G is a topological subgroup of a compact semitopological semigroup.

For instance, every Hausdorff locally compact group is Eberlein.

Problem 3.2. (Ruppert [R1, p.114-115]) Find a Hausdorff topological group which is not Eberlein.

This problem is open even in the case when G is algebraically isomorphic to $\mathbb{Z}$ (cf. [R2]).

Every topological group G is a topological subgroup of $Is(X)_s$ for a certain Banach space X (take, for example, $X = C_r(G)$, the space for all right uniformly continuous functions on G (as in Teleman [T])). The natural question is: how good may X be ? When X may be Asplund or even reflexive? It turns out that the case of a reflexive X gives a characterization of Eberlein groups. Indeed, by combining Theorem 2.1 and Theorem 1.3 we obtain

Theorem 3.2. *For every topological group G TFAE:*

(i) *G is an Eberlein group;*
(ii) *G is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space X.*

Below we discuss some related results and questions (having independent interest) which may help resolve Ruppert's problem.

Proposition 3.3. *Let G be a separable metrizable group and let $U_\ell(G)$ denote its left uniform structure. If G is Eberlein, then $(G, U_\ell(G))$, as a uniform space, is embedded into a separable reflexive Banach space Y.*

Proof. By Theorem 3.2, G is a topological subgroup of $Is(X)_s$ for a certain reflexive Banach space X. Proceeding as in [Me2, Counterexample 3.13], without loss of generality we may suppose that X is separable. By the definition of the strong operator topology, the system of all orbit maps on G generates the uniformity $U_\ell(G)$. Since G is second countable, we may suppose that there exists a sequence z_n in X such that the corresponding sequence of orbit maps

$$\tilde{z}_n\colon G \to X \quad , \quad \tilde{z}_n(g) = gz_n$$

generates $U_\ell(G)$. Moreover, we may suppose that $\|z_n\| = 2^{-n}$. Consider the ℓ_2-product $\prod_2 X_n = Y$ of the family $\{(X_n, \|\ \|_n) \bigm| n \in \mathbb{N}\}$, where each $(X_n, \|\ \|_n)$ is a copy of $(X, \|\ \|)$. Clearly, Y is a reflexive Banach space. Since $\|z_n\| = 2^{-n}$, it is easy to show that the diagonal product map

$$\gamma\colon G \to \prod_2 X_n = Y, \qquad \gamma(g) = (gz_n)$$

provides the desired uniform embedding. □

We say that a separable Banach space U is *uniformly universal* if every separable Banach space, as a uniform space, can be uniformly embedded into U. Aharoni [Ah] proved that c_0 is uniformly universal. In response to a question by Yu. Smirnov, in 1969 Enflo [En] found a separable metrizable uniform space which is not uniformly embedded into a Hilbert space. That is, ℓ_2 is not uniformly universal. However, it is not clear if here "Hilbert" may be replaced by "reflexive."

Question 3.4. (a) Is it true that $C[0,1], c_0$ (or any other uniformly universal space) is a uniform subset of a certain reflexive Banach space?

(b) Equivalently, is it true that there exists a uniformly universal reflexive Banach space?

Note that if c_0 is an Eberlein group, then, by Theorem 3.2 and Proposition 3.3, there exists a uniformly universal reflexive Banach space. Note also that there is no *Lipschitz embedding* of c_0 into a reflexive Banach space [Ma]. For additional results in the theory of uniform (or Lipschitz) Banach embeddings, we refer to the survey of J. Lindenstrauss [Li].

In order to explain another possible link to Problem 3.2, let us recall that a Hausdorff topological group G is said to be *minimal* [Step] if there is no strictly coarser Hausdorff group topology on G.

Question 3.5. (Arhangel'skij (cf. [Di], [Me2]) Is it true that every Hausdorff topological group G is a quotient of a minimal topological group?

The following proposition implies a positive answer to Question 3.5 in the case of an Eberlein group.

Proposition 3.6. *Every Eberlein group is a group retract of a minimal topological group.*

Proof. In [Me1, Theorem 6.12] we proved that, if X is an Asplund space, then every topological subgroup G of $Is(X)_s$ is a group retract of a minimal topological group. Now apply Theorem 3.2. □

If the answer to Arhangel'skij's question or to Question 3.4 is negative, then this will provide the solution to Ruppert's problem.

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