FIXED POINT THEOREMS
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Contents

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Abstract. We shall present Banach and Brouwer fixed point theorems and some of their applications in analysis and differential equations. We shall also discuss some of the generalizations of Brouwer fixed point theorem for general topological spaces.

1. The Banach fixed point theorem

Definition 1.1. Let (X, d) be a metric space, let $f : X \to X$ be a function, then we say that f is a contraction mapping on X if f satisfies the following condition:
There is a real number $0 \leq q < 1$ such that, for every $x, y \in X$ $d(f(x), f(y)) \leq qd(x, y)$.

Examples 1.2. 1. Every differentiable function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $|f'(x)| \leq q, \forall x \in \mathbb{R}$ for some real q, $(0 \leq q < 1)$ is a contraction.
(This follow immediately by Lagrange mean value theorem).
2. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a $n \times n$ matrix, suppose that A satisfies $\sum_{i=1}^{n} ||(R_i)||^2 \leq q < 1$ where R_i is the i row of A, then A is a contraction with respect to the usual norm on $\mathbb{R}^n$.
(This follows from the Cauchy-Schwarz inequality).
3. Define $F : C[0, 1] \to C[0, 1]$ (where $C[0, 1]$ is equipped with the sup norm) by the formula $F(f)(x) = \int_0^x f(t)dt$.
define $G = F \circ F$, then G is a contraction, one can see that $G(f)(x) = \int_0^x \int_0^t f(s)dsdt$.
we shall see that $||G(f)|| \leq \frac{1}{2}||f||$,$\forall f \in C[0, 1]$.
(This is sufficient by the linearity of G), indeed, we have:
$||G(f)|| = \sup_{x\in[0,1]} | \int_0^x \int_0^t f(s)dsdt| \leq \sup_{x\in[0,1]} \int_0^x \int_0^t |f(s)|dsdt$
$\leq \sup_{x\in[0,1]} \int_0^x \int_0^t ||f||dsdt = ||f|| \sup_{x\in[0,1]} \int_0^x \int_0^t dsdt =$
$||f|| \int_0^1 \int_0^t dsdt = \frac{1}{2}||f||$.

Theorem 1.3. *Let (X, d) be a complete metric space, if $f : X \to X$ is a contraction, then f has a fixed point.*

(i.e. a point $x \in X$ which satisfies $f(x) = x$) and this point is unique.

Proof. If f is a contraction, then there is a real number $0 \leq q < 1$ which satisfies $d(f(x), f(y)) \leq qd(x, y)$ for every $x, y \in X$. Choose a point $x_0 \in X$ and define a sequence $(x_n)_{n=1}^{\infty}$ by the recursion $x_{n+1} = f(x_n)$, for every n we have $d(x_n, x_{n+1}) \leq q^n d(x_0, x_1)$, hence, for $k > n \geq 0$ we get:
$0 \leq d(x_n, x_k) \leq d(x_n, x_{n+1}) + .. + d(x_{k-1}, x_k) \leq$
$q^n d(x_0, x_1) + .. + q^{k-1} d(x_0, x_1) = q^n(1 + q + .. + q^{k-n-1}) d(x_0, x_1) \leq$
$q^n(1 + q + q^2 + ..) d(x_0, x_1) = q^n d(x_0, x_1) \frac{1}{1-q} = q^n \cdot \alpha$
Where α is independent of n, thus $q^n \cdot \alpha \to 0 \ as \ n \to \infty$.
Thus $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, but we have assumed that X is complete, hence the sequence converge.
Assume that $x_n \to x$, then $f(x) = f(lim\ x_n) = lim f(x_n) = lim\ x_{n+1} = x$. (Every contraction is in particular continuous.) Thus x is a fixed point of f. To prove uniqueness let's assume that $f(x) = x, f(y) = y$, where $x, y \in X$, then we have $d(x, y) = d(f(x), f(y)) \leq qd(x, y)$, so we must have $d(x, y) = 0$ which means that $x = y$. □

We shall give now an important application of the previous theorem.

Theorem 1.4. *Let $f : [a, b] \times (-\infty, \infty) \to \mathbb{R}$ be a continuous function, suppose that f also satisfies Lipschitz condition in the y variable, i.e, there exist $M > 0$ such that*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \ \forall x \in [a, b].$$

If $(b - a)M < 1$, then for every $x_0 \in [a, b], y_0 \in (-\infty, \infty)$ there exist a unique solution to the following differential equation:

$$y' = f(x, y) \ , y(x_0) = y_0 \ (x, y) \in [a, b] \times (-\infty, \infty)$$

Proof. The differential equation $y' = f(x,y)$, $y(x_0) = y_0$ is equivalent to the following integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Define the map $A : C[a,b] \to C[a,b]$ by the formula

$$(A(y))(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \ , y \in C[a,b].$$

We will show that A is a contraction (with $C[a,b]$ equipped with the sup norm), Let $y_1, y_2 \in C[a,b]$, then:
$||A(y_1) - A(y_2)|| = \sup_{x\in[a,b]} |\int_{x_0}^{x}[f(t,y_1(t)) - f(t,y_2(t))]dt| \leq \sup_{x\in[a,b]} \int_{x_0}^{x} |f(t,y_1(t)) - f(t,y_2(t))|dt$
$\leq \sup_{x\in[a,b]} \int_{x_0}^{x} M|y_1(t) - y_2(t)|dt$
$\leq \ M \sup_{x\in[a,b]} \int_{x_0}^{x} ||y_1 - y_2||dt \leq M(b-a)||y_1 - y_2||.$
But $(b-a)M < 1$, thus A is a contraction.
Since $C[a,b]$ is complete (with the sup norm), by the previous theorem we get that A has a unique fixed point y. This y will satisfies the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt$.

□

2. The Brouwer fixed point theorem

Before introducing the Brouwer fixed point theorem we shall give a few definitions and lemmas.

Definition 2.1. A function $f : A \to \mathbb{R}^m (A \subset \mathbb{R}^n$ an open set) is differentiable at $a \in A$ if we can write

$$f(a+x) = f(a) + (D_f{}^a)(x) + r_a(x)$$

where ${D_f}^a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map and $r_a(x) = \circ(||x||)$. In general, a function $f : A \longrightarrow \mathbb{R}^m$ is of class C^1 if f is differentiable at every point of A and the function
$D_f : A \longrightarrow Hom(\mathbb{R}^n, \mathbb{R}^m)$ $a \mapsto ({D_f}^a)$ is continues.

Remark 2.2. From advanced calculus it is known that
$({D_f}^a) = (\frac{\partial f_i}{\partial x_j}(a)), j = 1...n, i = 1...m.$
Hence, f is of class C^1 in A if and only if f have continues partial derivatives in A.

Lemma 2.3. *If A is an open convex set of $\mathbb{R}^n$,$f : A \longrightarrow \mathbb{R}^m$ differentiable in A and $\sup\{||{D_f}^a|| : a \in A\} = M < \infty$, then we have the following inequality:*

$$||f(y) - f(x)|| \leq M||y - x|| \ \forall x, y \in A$$

Lemma 2.4. *(The open mapping theorem) If $A \subset \mathbb{R}^n$ is an open set and $f : A \longrightarrow \mathbb{R}^m$ is of class C^1 and satisfies $rank({D_f}^x) = m$ for all $x \in A$, then f is an open mapping. (i.e for every open set $G \subset A$, $f(G)$ is an open set in $\mathbb{R}^m$).*

Lemma 2.5. *If A is a $n \times n$ matrix and $||A|| < 1$ then $Det(I + A) > 0$.*

Proof. Define the polynomial $g(x) = Det(I + xA)$, if
$Det(I + A) \leq 0$,then we have $g(0) = Det(I) = 1$ and
$g(1) \leq 0$. Hence we have $x_0 \in (0, 1]$ which satisfies $g(x_0) = 0$, or $Det(I + x_0A) = 0$. Hence we have $v \in \mathbb{R}^n$, $v \neq 0$ which satisfies $(I + x_0A)v = 0$, or $-v = x_0Av$, thus we have
$||v|| = ||x_0Av|| \leq x_0||A||||v|| < x_0||v||$ which is a contradiction.

□

Theorem 2.6. *(The Brouwer fixed point theorem)*
Let $B(0,1) \subset \mathbb{R}^n$ be the closed unit ball. Then if $f : B(0,1) \to B(0,1)$ is a continuous function, then f has a fixed point.

Proof. It turns out that it suffice to prove the theorem only for C^1 functions which are defined on a neighborhood of $B(0,1)$ (i.e, function which are defined on $B(0,r), r > 1$, where r is dependent on the function).
Indeed, if the theorem true for C^1 functions and false for some continuous function $g : B(0,1) \to B(0,1)$, then we denote $\varepsilon = \frac{1}{3} min_{x \in B(0,1)} ||g(x) - x||$, from our assumption $\varepsilon > 0$.
From Weierstrass approximation theorem we can find a polynomial $h : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$||g(x) - h(x)|| < \varepsilon \quad \forall x \in B(0,1)$$

.
Define $f(x) = \frac{h(x)}{1+\varepsilon}$, then f map $B(0,1)$ to itself (because g map $B(0,1)$ to itself). Now f is defined on all of $\mathbb{R}^n$ and is of class C^1, and for $x \in B(0,1)$ we have:
$||f(x)-x|| = ||(g(x)-x)-(g(x)-h(x))-(h(x)-f(x))|| \geq ||g(x) - x|| - ||g(x) - h(x)|| - ||h(x) - f(x)|| = ||g(x)-x||-||g(x)-h(x)||-\varepsilon||f(x)|| \geq ||g(x)-x||-2\varepsilon > 0.$
So f also does not have a fixed point. Contradiction.
Now we have arrived to the main step of the theorem, we will prove that there is no f of class C^1, defined on a neighborhood of $B(0,1)$ and satisfying

$$f(B(0,1)) = \partial B(0,1), \ f(x) = x \ , \forall x \in \partial B(0,1)$$

.
For if such f exist, we define (on the domain of f) the functions
$h(x) = f(x) - x$
$f^t(x) = x + th(x) = (1-t)x + tf(x), \ t \in [0,1].$

Now h is zero on $\partial B(0,1)$ and hence $f^t(x) = x$ on $\partial B(0,1)$. Since h is of class C^1 we have $M = max_{x\in B(0,1)}||D_h{}^x|| < \infty$. From Lemma 2.3 $||h(x)-h(y)|| \leq M||x-y||$ $(x, y \in B(0,1))$. Hence, for $t \in [0, \frac{1}{M})$, f^t is one to one on $B(0,1)$, because if $f^t(x) = f^t(y)$ then:
$||x-y|| = t||h(x)-h(y)|| \leq tM||x-y||$, but $tM < 1$ and hence we must have $x = y$.
From the definition of f^t we have for all $t \in [0,1]$, $x \in B(0,1)$

$$D_{(f^t)}{}^x = I + tD_h{}^x.$$

If $t \in [0, \frac{1}{M})$, $||tD_h{}^x|| < 1$, hence from Lemma 2.5, $J_{(f^t)}{}^x > 0$, which means that $Rank(D_{(f^t)}{}^x) = n$, hence from Lemma 2.4, for all $t \in [0, \frac{1}{M})$ $f^t(B_0(0,1))$ is an open set which contained in $B(0,1)$ and hence $f^t(B_0(0,1)) \subseteq B_0(0,1)$.
We will in fact prove that $f^t(B_0(0,1)) = B_0(0,1)$, otherwise, there is some point $e \in B_0(0,1) \backslash f^t(B_0(0,1))$, suppose J is the segment which connect e with some point $g \in f^t(B_0(0,1))$. There is some point b in this segment such that

$$b \in \partial f^t(B_0(0,1)) = Clf^t(B_0(0,1)) \backslash f^t(B_0(0,1)).$$

from the continuity of f we have $Clf^t(B_0(0,1)) \subseteq f^t(B(0,1))$. Hence

$$b \in f^t(B(0,1)) \backslash f^t(B_0(0,1)) = f^t((B(0,1)) \backslash B_0(0,1))$$

$$= f^t(\partial B(0,1)) = \partial B(0,1)$$

.

(where in the first passage we use the fact that f^t is one to one when $t \in [0, \frac{1}{M})$ and in the third passage we use the fact that

f^t is the identity on the boundary).
Hence we have $e, g \in B_0(0,1)$ and b a point between e and g such that $b \in \partial B(0,1)$ which is a contradiction to the convexity of $B_0(0,1)$.
Now, for every $t \in [0, \frac{1}{M})$ $f^t(B_0(0,1)) = B_0(0,1)$ and $f^t(\partial B(0,1)) = \partial B(0,1)$, thus, for those t, f^t is a homeomorphism from $B(0,1)$ to itself. We have shown also that ${J_{(f^t)}}^x > 0 \ \forall x \in B(0,1)$, hence:

$P(t) = \int_{B(0,1)} {J_{(f^t)}}^x dx = V(f^t(B(0,1))) = V(B(0,1))$. Hence $P(t)$ is a constant polynomial for $t \in [0, \frac{1}{M})$, hence P is constant for every $t \in [0,1]$. In particular, for $t = 1$ we have: $V(B(0,1)) = P(1) = \int_{B(0,1)} {J_{(f^1)}}^x dx = \int_{B(0,1)} {J_f}^x dx$. But we have assumed that $f(B(0,1)) = \partial B(0,1)$, thus:

$$< f(x), f(x) >= 1, \ \ \forall x \in B(0,1),$$

hence:

$$< \frac{\partial f(x)}{\partial x_i}, f(x) >= 0, \ \ 1 \leq i \leq n$$

(here we use the fact that f is of class C^1). But $f(x) \neq 0$ and hence ${J_f}^x = 0$ and $\int_{B(0,1)} {J_f}^x dx = 0$ which is a contradiction.
Now, the proof of our theorem is immediate, for if $f : B(0,r) \to B(0,1)$ $(r > 1)$ is of class C^1 and doesn't have a fixed point in $B(0,1)$, we can define a function h from $B(0,r)$ to $B(0,1)$ such that $h(B(0,1)) = \partial B(0,1)$, $h \equiv I \ on \ \partial B(0,1)$ which correspond to each $x \in B(0,1)$ the intersection point of $\partial B(0,1)$ with the ray starting at $f(x)$ and passes through x. It is easy to see that h is of class C^1 and can be defined on a neighborhood of $B(0,1)$ which contradicts our main step. $\square$

3. Generalizations of Brouwer fixed point theorem

Definition 3.1. A topological space X is said to have the fixed point property, if for every continuous mapping $f : X \to X$, there exist a $p \in X$ with $f(p) = p$.

Examples 3.2. If we define the Hilbert cube as
$C = \{(x_n)_n | x_n \in \mathbb{R},\ x_n \leq \frac{1}{n}\}$, then C has the fixed point property. Indeed, let $T : C \to C$ be a continues map.
Define $P_n : C \to C$ be the map defined by:

$$P_n([x_1, ..., x_n, x_{n+1}, ...]) \mapsto [x_1, ..., x_n, 0, 0, ...]$$

The set $C_n = P_n(C)$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$. Since the mapping $p_n \circ T|_{C_n} : C_n \to C_n$ is continuous, the Brouwer theorem implies that it has a fixed point $y_n \in C_n \subset C$.
Hence if $T(y_n) = [T_1, ..., T_n, T_{n+1}, ...]$, then
$[T_1, ..., T_n, 0, ...] = [y_{n,1}, ..., y_{n,n}, 0...] = y_n$, so we have
$|y_n - T(y_n)| = \sqrt{\sum_{i=n+1}^{\infty} T_i^2} \leq \sqrt{\sum_{i=n+1}^{\infty} \frac{1}{i^2}}$.
Since C is compact, y_n has a convergent subsequence. The limit of this sequence is clearly a fixed point of T.

We turn now to prove the main theorem in this section, the Kakutani theorem, but first we need a definition.

Definition 3.3. A family F of linear transformations on a linear topological space X is said to be *equicontinuous* on a subset K of X if for every neighborhood V of the origin in X there is a neighborhood U of the origin such that if $k_1, k_2 \in K$ and $k_1 - k_2 \in U$ then $F(k_1 - k_2) \subseteq V$, that is:
$T(k_1 - k_2) \in V$ for all $T \in F$.

Theorem 3.4. *(Kakutani) Let K a be non empty compact convex subset of a locally convex linear topological space X,* *and G be a group of linear mappings which is equicontinuous on K and such that $G(K) \subseteq K$. (i.e, $T(K) \subseteq K$, $\forall T \in G$). Then there is a point $p \in K$ such that $T(p) = p$,$\forall T \in G$.*

Proof. By Zorn's lemma, K contains a minimal non-empty compact convex subset K_1 such that $G(K_1) \subseteq K_1$.(for every chain of non-empty compact convex sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ which satisfies $G(A_i) \subseteq A_i$ we can define $B = \cap_i A_i$ which is convex, compact, non-empty and satisfies $G(B) \subseteq B$ and hence B can be taken as an upper-bound).
If K_1 contains only one point, then our proof is complete. If this is not the case, the compact set $K_1 - K_1$ contains some point other than the origin, thus there is a neighborhood V of the origin such that V does not contain $K_1 - K_1$.
X is locally convex, hence there is a convex neighborhood V_1 of the origin such that $\alpha V_1 \subseteq V$ for $|\alpha| \leq 1$. By the equicontinuity of G on the set K_1, there is a neighborhood U_1 of the origin such that if $k_1, k_2 \in K$ and $k_1 - k_2 \in U_1$ then $G(k_1 - k_2) \subseteq V_1$.
Define $U_2 = convexhull(G(U_1))$, since G is a group $G(U_2) = U_2$ and by the linearity of G we have $G(Cl(U_2)) = Cl(U_2)$.
Let $\delta = inf\{a | a > 0, aU_2 \supseteq K_1 - K_1\}$, and define $U = \delta U_2$. It follows that for each $0 < \varepsilon < 1$ the set $K_1 - K_1$ is not contained in $(1 - \varepsilon)Cl(U)$, while $K_1 - K_1 \subseteq (1 + \varepsilon)U$.
The family of open sets $\{\frac{1}{2}U + k\}$, $k \in K_1$ is a covering of K_1. Let $\{\frac{1}{2}U + k_1, .., \frac{1}{2}U + k_n\}$ be a finite sub-covering and let $p = (k_1 + .. + k_n)/n$. If k is any point in K_1, then $k_i - k \in \frac{1}{2}U$ for some i between 1 and n. Since $k_i - k \in (1 + \varepsilon)U$ for $1 \leq i \leq n$ and $\varepsilon > 0$, we have $p \in n^{-1}(\frac{1}{2}U + (n-1)(1+\varepsilon)U) + k$.
Substituting $\varepsilon = \frac{1}{4(n-1)}$ we have $p \in (1 - \frac{1}{4n})U + k$ for each $k \in K_1$. Now let $K_2 = K_1 \cap (\cap_{k \in K_1}((1 - \frac{1}{4n})Cl(U) + k))$.
$K_2 \neq \emptyset$ because $p \in K_2$ and since $(1 - \frac{1}{4n})Cl(U)$ does not contain $K_1 - K_1$,$K_2 \neq K_1$.
Since K_2 is closed it is compact and it's clearly convex. Further,

since $T(aCl(U)) \subseteq aCl(U)$ for $T \in G$, we have
$T(aCl(U) + k) \subseteq aCl(U) + Tk$ for $T \in G$,$k \in K_1$,
but $T(K_1) \subseteq K_1$, hence, $Tk = k'$ for some $k \in K_1$ and thus $T(aCl(U) + k) \subseteq aCl(U) + k'$, and for every $k' \in K_1$ there is some $k \in K_1$ such that $T(k) = k'$ ($T(K_1) = K_1$, because G is a group). Hence $T(aCl(U) + k) \subseteq aCl(U) + k'$ and in the end we get that $\cap_{k \in K_1} T(aCl(U) + k) \subseteq \cap_{k \in K_1} aCl(U) + k$.
This implies that $G(K_2) \subseteq K_2$, which is a contradiction to the minimality of K_1.
Thus K_1 contains only one point and this point is a fixed point for the group G. □

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