G-Minimal Topological Groups

Michael Megrelishvili
Bar Ilan University

To appear in the Proc. of Orsatty Conference, Padova, Italy, 1997, *Abelian groups, Modules and Topology, Lecture Notes in Pure and Appl. Algebra, Marcel Dekker.*

G-Minimal Topological Groups

Michael Megrelishvili
Department of Mathematics and Computer Science
Bar-Ilan University, 52 900 Ramat-Gan, Israel

Abstract. We study sufficient conditions for G-minimality of topological G-groups X and minimality of semidirect products $X \rightthreetimes G$ arising under some natural constructions.

Introduction.

A Hausdorff topological group X is called *minimal* [S] if it does not admit a strictly coarser Hausdorff group topology. The development of the theory of minimal topological groups is described in the monograph [DPS] and in several survey articles [CHR], [D2]. Here we discuss only facts that directly concern the present paper. It was shown by Stephenson [S] that every locally compact abelian minimal group must be compact. Nevertheless, every locally compact abelian Hausdorff group is a group retract of a locally compact minimal group (see [M]). The first example of a locally compact (noncompact) minimal group was found by Dierolf and Schwanengel [DS]. They proved that the semidirect product $\mathbb{R} \rightthreetimes \mathbb{R}_+$ of the multiplicative group $\mathbb{R}_+$ of all positive real numbers with $\mathbb{R}$ is minimal. Later Remus and Stoyanov [RS] established the minimality of $\mathbb{R}^n \rightthreetimes H$ for every closed subgroup H of $GL(n, \mathbb{R})$ which contains all diagonal matrices with positive entries.

In the present article we generalize these results in several directions. In section 1 we show that for every normed space X there exists an abelian subgroup H of $GL(X)$ such that $X \rightthreetimes H$ is minimal. This result helps us to establish that there exists one and only one nontrivial group topology on X which preserves the continuity of the action of $GL(X)$ on X. At the same time, the question about minimality of $X \rightthreetimes GL(X)$ is still open even for Hilbert space $X = \ell_2$.

In section 2 we prove that if a locally compact abelian Hausdorff group X contains a copy of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then $X \rightthreetimes \mathrm{Aut}(X)$ is minimal.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Section 3 is devoted to the following general question:

Question. *For which topological fields K and subgroups H of $GL(n,K)$ are the groups $K^n \leftthreetimes H$ minimal? In particular, when is $K \leftthreetimes K^\times$ minimal (here $K^\times = K \setminus \{0\}$)?*

This question leads us to the important class of *locally retrobounded division rings* introduced by Nachbin [N].

In the last section, resolving a question of Dikranjan [D2], we construct a totally disconnected minimal group which is not zero-dimensional. This answers negatively a question posed earlier by Archangel'skij: does a totally disconnected group always admit a coarser zero-dimensional Hausdorff group topology (see [D1], [D2])? Our construction uses Erdös' classical example of all rational points in ℓ_2.

The author thanks D. Dikranjan and V. Pestov for stimulating discussions.

§1. Preliminaries.

Let (X,τ) and (G,σ) be topological groups and

$$\alpha : G \times X \to X, \quad \alpha(g,x) = gx$$

be a fixed action. We say that X is a *G-group* if α is continuous and every $g-$ transition $\alpha^g : X \to X$, $\alpha^g(x) = gx$ is a group automorphism of X. For every G-group X denote by $X \leftthreetimes_\alpha G$ the standard semidirect product [Bo]. As usual, its normal subgorup $X \times \{e_G\}$ will be identified with X and, analogously, the subgroup $\{e_X\} \times G$ will be identified with G.

Definition 1.1. Let (G,σ) be a Hausdorff topological group and let (X,τ) be a Hausdorff (G,σ)-group with respect to an action α.

(a) [RS] X is a *G-minimal group* if there is no strictly coarser Hausdorff group topology $\tau' \subsetneq \tau$ on X which preserves the continuity (actually, the (σ,τ',τ')-continuity) of α.

(b) X is *strongly G-minimal* if $X \leftthreetimes_\alpha G$ is a minimal group.

(c) (see [M, Def. 1.8]) The action α is *topologically exact* (*t-exact*, in short) if there is no strictly coarser group (not necessarily Hausdorff) topology $\sigma' \subsetneq \sigma$ such that α is (σ',τ,τ)-continuous. α is *hereditarily t-exact (ht -exact)* if for every topological subgroup P of G, the corresponding action of P on X is t-exact.

Note that, since in the definition (G,σ) is Hausdorff, every t-exact action α is necessarily *algebraically exact*, that is, the normal subgroup $\mathrm{Ker}(\alpha) = \{g \in G | gx = x \quad \forall x \in X\}$ is trivial.

Lemma 1.2. *Let* $(X \leftthreetimes_\alpha G, \gamma)$ *be a semidirect product of Hausdorff topological groups* (X, τ) *and* (G, σ)*. Suppose that a minimal group* Y *is a topological subgroup of* $X \leftthreetimes_\alpha G$*.*

(i) *If* $X \subseteq Y$ *then* X *is* G*-minimal.*

(ii) *If* X *is strongly* G*-minimal then* X *is* G*-minimal.*

(iii) *If* $G \subseteq Y$ *and* α *is algebraically exact then* α *is* t*-exact.*

Proof. (i) Let $\tau' \subseteq \tau$ be a coarser Hausdorff group topology on X such that the action α is (σ, τ', τ')-continuous. Then the semidirect product $(X, \tau') \leftthreetimes (G, \sigma)$ is well-defined. Denote the corresponding product Hausdorff group topology by γ'. Since $\gamma' \subseteq \gamma$ and Y is minimal then $\gamma'|_Y = \gamma|_Y$. Therefore, $\gamma'|_X = \gamma|_X$. On the other hand, $\gamma'|_X = \tau'$ and $\tau = \gamma|_X$. Thus, we obtain $\tau' = \tau$.

(ii) Directly follows from (i).

(iii) Let $\sigma' \subseteq \sigma$ be a strictly coarser group topology on G such that α is (σ', τ, τ)-continuous. Since α is algebraically exact then necessarily σ' is Hausdorff. The rest of the proof, via the semidirect product $(G, \alpha') \leftthreetimes_\alpha (X, \tau)$, is similar to the case of (i). □

Fact 1.3. [M, Corollary 2.8] *Let* $(X \leftthreetimes_\alpha G, \gamma)$ *be a topological semidirect product and let* α *be* t*-exact. Suppose that* X *is abelian and* $\gamma' \subseteq \gamma$ *is a group topology which agrees with* γ *on* X*. Then* $\gamma' = \gamma$*.*

Theorem 1.4. *Let* (G, σ) *be a Hausdorff group and let* (X, τ) *be a Hausdorff abelian* G*-group. If* X *is* G*-minimal and the action* α *is* t*-exact then* $X \leftthreetimes G$ *is minimal (i.e.,* X *is strongly* G*-minimal).*

Proof. Denote by γ the original product topology of $(X, \tau) \leftthreetimes_\alpha (G, \sigma)$. Assume that $\gamma' \subseteq \gamma$ is a coarser Hausdorff group topology on $X \leftthreetimes_\alpha G$. Then the map

$$\alpha : (G, \gamma'|_G) \times (X, \gamma'|_X) \to (X, \gamma'|_X)$$

is continuous (see [Bo, Ch.III, §2]). Clearly, $\gamma'|_G \subseteq \gamma|_G = \sigma$ and $\gamma'|_X = \gamma|_X$. By Fact 1.3 we get $\gamma' = \gamma$. Therefore $X \leftthreetimes_\alpha G$ is minimal, as required. □

The last result is not true in general for non-abelian X. It was shown in [EDS, Example 10] that there exists a totally minimal precompact torsion group X such that a certain semidirect product $X \leftthreetimes G$ is not minimal, where $G = \mathbb{Z}_2$ is the discrete cyclic group of order 2.

A topological group G is called *perfectly minimal* [St, p.107] if $G \times H$ is minimal for every minimal group H.

Lemma 1.5. *Let G be a minimal group. Then G will be perfectly minimal in each of the following cases:*

(a) [EDS] *G is complete in its two-sided uniformity (e.g., compact);*
(b) [M, Th.1.14] *The center $Z(G)$ is perfectly minimal (e.g., compact);*
(c) *Algebraically, G is a semidirect product $X \leftthreetimes_\alpha H$ such that the subgroups* $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Coker}(\alpha) = \{x \in X \mid gx = x \quad \forall g \in H\}$ *are compact (e.g., trivial).*

Proof. $Z(X \leftthreetimes_\alpha H) = \mathrm{Coker}(\alpha) \times \mathrm{Ker}(\alpha)$ generally holds. Therefore, (c) directly follows from (b). □

Fact 1.6. [M, Th.1.15] *The arbitrary product $\prod X_i$ of minimal groups X_i with trivial center is perfectly minimal.*

§2. GL-minimality of normed spaces.

Let E be a real normed space and let $\alpha : GL(E) \times E \to E$ be the usual continuous action of $GL(E)$ on E, where $GL(E)$ denotes the topological group of all topological automorphisms of E endowed with the uniform operator topology. The standard dual normed space of E will be denoted by E^*. We need a method from [M] which enables us to construct a minimal group which is denoted by $M_+(E)$ and called *induced group* of the canonical bilinear form $E \times E^* \to \mathbb{R}$, $(x, f) \mapsto f(x)$. Summarizing for convenience some definitions of [M], recall that $M_+(E)$ is the group

$$M_+(R) = ((\mathbb{R} \times E) \leftthreetimes_\nu E^*) \leftthreetimes_\pi \mathbb{R}_+$$

where the actions ν and π are defined as follows:

$$\nu : E^* \times (\mathbb{R} \times E) \to \mathbb{R} \times E, \quad \nu(f, (a, x)) = (a + f(x), x)$$

$$\pi : \mathbb{R}_+ \times ((\mathbb{R} \times E) \leftthreetimes_\nu E^*) \to (\mathbb{R} \times E) \leftthreetimes_\nu E^*, \quad \pi(t, (a, x, f) = (ta, tx, f).$$

Fact 2.1. [M, Theorem 3.10] *For every normed space E the induced group $M_+(E)$ is perfectly minimal.*

Theorem 2.2. *For every normed space E there exists an abelian subgroup H of $GL(E)$ such that the corresponding semidirect product $E \leftthreetimes H$ is perfectly minimal. Hence E is $GL(E)$-minimal.*

Proof. The conclusion part follows from Lemma 1.2(i). The proof of the main part of our assertion is divided into two subcases.

Case I. $\dim E = 1$. In this case use minimality of $\mathbb{R} \leftthreetimes \mathbb{R}_+$ [DS] and Lemma 1.5.

Case II. $\dim E > 1$. Represent E as a product $\mathbb{R} \times X$, where X is a closed linear subspace of codimension 1. Consider the topological group product $X^* \times \mathbb{R}_+$ and the following action

$$\alpha : (X^* \times \mathbb{R}_+) \times E \to E, \quad \alpha((f,t),(a,x)) = (ta + tf(x), tx).$$

It is straightforward to show that:

(a) α is a continuous linear action;
(b) The corresponding homomorphism $i_\alpha : X^* \times \mathbb{R}_+ \to GL(E)$ is a topological group embedding;
(c) The semidirect product $E \leftthreetimes_\alpha (X^* \times \mathbb{R}_+)$ is naturally topologically isomorphic to the induced group $M_+(X)$.

By Fact 2.1, the group $M_+(X)$ is minimal. Therefore we can complete the proof by defining $H := X^* \times \mathbb{R}_+$. $\square$

Note that, as it was pointed out us by the referee, the abelian group H, as well as the resulting semidirect product $E \leftthreetimes H$, are both connected Banach-Lie groups rather than merely topological groups.

Proposition 2.3. *Let z be a fixed non-zero element of a normed space $(E, \| \ \|)$. Then the orbit map*

$$\tilde{z} : GL(E) \to E \setminus \{0\}, \quad \tilde{z}(g) = gz$$

is surjective and open.

Proof. Assign to every pair $(f, y) \in E^* \times E$ a continuous linear operator

$$A_{f,y} : E \to E, \quad A_{f,y}(x) = x + f(x)y.$$

It is easy to show that:

(a) $\|A_{f,y} - I\| = \|f\| \cdot \|y\|$ (where I denotes the neutral element of $GL(E)$);
(b) $A_{f,y}$ is invertible if and only if $f(y) \neq -1$. The inverse is $A^{-1}_{f,y} = A_{tf,y}$, where $t = -(1 + f(y))^{-1}$.

In order to prove that $\tilde{z}$ is surjective, consider an arbitrary non-zero element $x \in E \setminus \{0\}$. There exists a continuous functional f such that $f(z) = 1$ and $f(x) \neq 0$. Define $y = x - z$ and observe that $f(y) \neq -1$. Therefore, $A_{f,y} \in GL(E)$. At the same time, $A_{f,y}$ moves z into x.

Now we prove that $\tilde{z}$ is open. It suffices to show (see, for example [RD, Proposition 5.7]) that for every neighborhood U of I in $GL(E)$ the image $\tilde{z}(U) = Uz$ is a neighborhood of z in $X \setminus \{0\}$. There exists an $\varepsilon > 0$ such that $\varepsilon < 1$ and

$$V_\varepsilon := \{g \in GL(E) \mid \|g - I\| < \varepsilon\} \subseteq U.$$

The properties (a), (b) imply that the set

$$P_\varepsilon := \{A_{f,y} \mid \|f\| \leq 1, \ \|y\| < \varepsilon\}$$

is contained in V_ε. Thus, $P_\varepsilon z \subseteq V_\varepsilon z \subseteq Uz$. On the other hand, $P_\varepsilon z$ contains the Ball $B(z, r)$ with center z and radius $r := \|z\|\varepsilon$. Therefore, Uz is a neighborhood of z, as required. $\square$

Note that the use of the automorphisms $A_{f,y}$ in the proof of Proposition 2.3 was inspired by [P].

Theorem 2.4. *Let $\tau_{\| \|}$ be the norm topology of a normed space $(E, \| \|)$ and let $\sigma_{\| \|}$ be the uniform operator topology on $GL(E)$. Suppose that τ' is a non-trivial group topology on E such that the action*

$$\alpha : (GL(E), \sigma_{\| \|}) \times (E, \tau') \to (E, \tau')$$

remains continuous. Then $\tau' = \tau_{\| \|}$.

Proof. Case I. τ' is not Hausdorff. Then the τ'-closure $cl_{\tau'}\{0\}$ of $\{0\}$ contains a non-zero element. Moreover, since α is $(\sigma_{\| \|}, \tau', \tau')$-continuous and $\{0\}$ is $GL(E)$-invariant then the subgroup $cl_{\tau'}\{0\}$ is $GL(E)$-invariant too. By the transitivity of the action on $E \setminus \{0\}$ we obtain that $cl_{\tau'}\{0\} = E$. Therefore τ' is a trivial topology contradicting our assumption.

Case II. τ is Hausdorff. First denote $\tau_0 := \tau_{\| \|}\big|_{E \setminus \{0\}}$, $\tau_0' := \tau'\big|_{E \setminus \{0\}}$ and fix a non-zero element z of E. Then by Proposition 2.3, the $GL(E)$-space $(E \setminus \{0\}, \tau_0)$ can be identified with the topological coset $GL(E)$-space $GL(E)/St_z$, where St_z is the stabilizer $\{g \in GL(E) \mid gz = z\}$. On the other hand, the $(\sigma_{\| \|}, \tau', \tau')$-continuity of α implies the continuity of the orbit map:

$$\tilde{z} : (GL(E), \sigma_{\| \|}) \to (X \setminus \{0\}, \tau_0').$$

By Proposition 2.3 and the characterization property of the quotient topology τ_0 on $GL(X)/St_z = X \setminus \{0\}$ we obtain that $\tau_0' \subseteq \tau_0$. Since $X \setminus \{0\}$ is open under *group* topologies τ' and $\tau_{\| \|}$, then $\tau' \subseteq \tau_{\| \|}$. Now, by Theorem 2.2, we can conclude $\tau' = \tau_{\| \|}$. $\square$

Question 2.5. *For what infinite-dimensional normed spaces E is the group $E \leftthreetimes GL(E)$ minimal?*

By Theorem 2.2, Theorem 1.4 and Lemma 1.2, it is equivalent to ask:

Question 2.6. *Is the action of $GL(E)$ on a normed space E t-exact?*

This question seems to be unclear even for $E = \ell_2$.

§3. Aut-minimality of locally compact groups.

Let (X, τ) be a locally compact group and let $\mathrm{Aut}(X)$ be the group of all topological automorphisms of X endowed with the *Birkhoff topology* τ_B (see for example [HR, §26]). Recall that τ_B has a local base at the identity formed by the sets:

$$\mathcal{B}(C, U) := \{\varphi \in \mathrm{Aut}(X) \bigm| \varphi(c) \in Uc \quad \text{and} \quad \varphi^{-1}(c) \in Uc \ \forall c \in C\},$$

where C runs over the compact subsets of X and U runs over the neighborhoods of e in X.

Lemma 3.1. *For every locally compact Hausdorff group X the action*

$$\alpha : (\mathrm{Aut}(X), \tau_B) \times (X, \tau) \to (X, \tau)$$

is hereditarily t-exact.

Proof. See, for example [M, Remark 1.9]. □

Lemma 3.2. *For every locally compact abelian Hausdorff group X TFAE:*

(a) *X is $\mathrm{Aut}(X)$-minimal;*
(b) *$X \leftthreetimes \mathrm{Aut}(X)$ is minimal;*
(c) *$X \leftthreetimes H$ is minimal for a certain topological subgroup H of $\mathrm{Aut}(X)$.*

Proof. The first implication (a) ⇒ (b) follows from Theorem 1.4 and Lemma 3.1. The implication (b) ⇒ (c) is trivial. If $X \leftthreetimes H$ is minimal then by Lemma 1.2(ii), X is H-minimal. Then, clearly, X is $\mathrm{Aut}(X)$-minimal too. Hence, (c) ⇒ (a). □

In the sequel we say that a locally compact group X is *Aut-minimal* if X is $\mathrm{Aut}(X)$-minimal.

Theorem 3.3. *Let a locally compact abelian Hausdorff group X contain as a subgroup the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then X is* Aut*-minimal.*

Proof. The group $\mathbb{T}$ *splits* in X by [HR, §25.31]. Thus, X can be represented as a topological group product $\mathbb{T} \times G$. Denote by G^* the *dual group* of all continuous characters of G endowed with the compact open topology. Following [M] consider the continuous action:

$$\alpha : G^* \times (\mathbb{T} \times G) \to \mathbb{T} \times G, \quad \alpha(\chi, (t, g)) = (t + \chi(g), g).$$

The dual form of [M, Theorem 2.11] shows that the semidirect product $M(G) := (\mathbb{T} \times G) \leftthreetimes_\alpha G^*$ is a minimal group. Moreover, it is straightforward to show that the homomorphism $\pi_\alpha : G^* \to \mathrm{Aut}(\mathbb{T} \times G)$ associated to the action α is a topological group embedding. Therefore, there exists a topological subgroup $H := \pi_\alpha(G^*)$ of $\mathrm{Aut}(\mathbb{T} \times G)$ such that the group $X \leftthreetimes H$ (being topologically isomorphic to $M(G)$) is minimal. Now, we can apply the implication (c) $\Rightarrow$ (a) from Lemma 3.2. $\square$

§4. Minimal groups generated by division rings.

Let (K, τ) be a Hausdorff topological division ring. Denote by $(K^\times, \tau^\times)$ the topological multiplicative group consisting of all non-zero elements endowed with the subspace topology $\tau^\times := \tau|_{K^\times}$. The natural group action

$$\alpha : (K^\times, \tau^\times) \times (K, \tau) \to (K, \tau), \quad \alpha(a, b) = ab$$

is t-exact. Indeed, if $\eta \subseteq \tau^\times$ is a coarser group topology such that the action is (η, τ, τ)-continuous, then the map

$$(K^\times, \eta) \to (K^\times, \tau^\times), \quad g \mapsto g \cdot 1$$

is continuous. Therefore, $\eta \supseteq \tau^\times$. Thus, $\eta = \tau^\times$, as required.

Lemma 4.1. *For every Hausdorff topological division ring K the group $K \leftthreetimes_\alpha K^\times$ is minimal if and only if K is a $K^\times$-minimal group.*

Proof. As we have seen the action α is t-exact. Hence we may apply Lemma 1.2 and Theorem 1.4. $\square$

Let (E, σ) be a Hausdorff topological K-vector space. A Hausdorff topology σ' is called *admissible* (with respect to σ) if $\sigma' \subseteq \sigma$ and (E, σ') is also a K-vector space. As usual, (E, σ) is called a *minimal K-vector space* if σ is the unique admissible topology on E. In the particular case, when $E = K$ is a one-dimensional K-vector space, we obtain the definition of *strictly minimal division ring* in the sense of Nachbin [N] (see also *straight division ring* [W]).

Fact 4.2. ([N, Theorem 2], [W, Theorem 24.2]) *Let K be a non-discrete locally retrobounded division ring. Then K is strictly minimal.*

Recall some important resources of locally retrobounded division rings.

Fact 4.3. ([W, §19], [Wi, §5]) *A Hausdorff topological division ring K is locally retrobounded in each of the following cases:*

(a) *K is locally compact;*

(b) *K is topologized by an absolute value or a valuation;*

(c) *K is a linearly ordered field.*

Lemma 4.4. *Let (K, σ) be a Hausdorff topological division ring and let (E, τ) be a K-vector space with respect to the action $\alpha : K \times E \to E$.*

(i) *α is continuous if and only if the restriction $\alpha^{\times} : K^{\times} \times E \to E$ is continuous.*

(ii) *If K is a linearly ordered field then α is continuous if and only if the restriction $\alpha_{+} : K_{+} \times E \to E$ is continuous, where K_{+} is the multiplicative group of all positive elements in K.*

Proof. (i) If α is continuous then clearly $\alpha^{\times}$ is continuous too. Now suppose that $\alpha^{\times}$ is continuous. We have to show that α is continuous at each point $(t, x) \in K \times X$. If t is not the zero element 0_K of K then the proof becomes trivial because $K^{\times}$ is open in K. Thus, it is enough to establish the continuity of α at $(0_K, x)$ for arbitrarily fixed $x \in E$. Let U be a neighborhood of the zero element $0_E = 0_K \cdot x$ of E. Choose a neighborhood U_1 of 0_E such that $U_1 - U_1 \subseteq U$. Let 1 be the multiplicative unit of $K^{\times}$. Since $\alpha^{\times}$ is continuous at $(1, x)$ we can pick a $\sigma^{\times}$-neighborhood W of 1 in $K^{\times}$ and a τ-neighborhood V of x in E such that $WV \subseteq U_1$. Since $K^{\times}$ is σ-open in K then W is a σ-neighborhood of 1 in K. Whence $W_1 := W - 1$ is a σ-neighborhood of 0_K. Then $W_1 V \subseteq U$. Indeed, by our choice $WV = (W_1 + 1)V \subseteq U_1$. Thus $W_1 V + V \subseteq U_1$ and hence $W_1 V \subseteq U_1 - V \subseteq U_1 - U_1 \subseteq U$. Therefore the proof of (i) is completed.

(ii) The proof is similar to (i) taking into account that K_{+} is open in K. □

Corollary 4.5. *Let K be a Hausdorff topological division ring and let E be a topological K-vector space. TFAE:*

(i) *E is a minimal K-vector space.*

(ii) *E is a $K^{\times}$-minimal group.*

Theorem 4.6. *For every Hausdorff topological division ring K TFAE:*

(i) $K \leftthreetimes K^{\times}$ *is minimal.*

(ii) K *is strictly minimal.*

Proof. By Lemma 4.1, $K \leftthreetimes K^{\times}$ is minimal iff K is a $K^{\times}$-minimal group. On the other hand, applying Corollary 4.5 for $E := K$ we obtain that K is $K^{\times}$-minimal iff K is strictly minimal. Thus, (i) $\Leftrightarrow$ (ii) is proved. $\square$

For every integer $n \geq 1$ denote by K^n the standard n-dimensional K-vector space endowed with the product topology. Then the scalar action $K \times K^n \to K^n$ leads to the well-defined semidirect product $K^n \leftthreetimes K^{\times}$.

Theorem 4.7. *Let K be a non-discrete locally retrobounded division ring and let ν be an arbitrary cardinal.*

(a) *The group $(K \leftthreetimes K^{\times})^{\nu}$ is perfectly minimal.*

(b) *If K is complete then $(K^n \leftthreetimes K^{\times})^{\nu}$ is perfectly minimal.*

(c) *If K is an ordered field then $K^{\times}$ can be replaced by K_+ in both cases* (a) *and* (b).

Proof. (a) By Fact 4.1 and Theorem 4.6, the group $K \leftthreetimes K^{\times}$ is minimal. Since $Z(K \leftthreetimes K^{\times})$ is trivial we may apply Fact 1.6.

(b) By *Nachbin's uniqueness theorem* [N, Theorem 7] K^n has only one admissible topology. Therefore, K^n is a minimal K-vector space. By Corollary 4.5, K^n is a $K^{\times}$-minimal group. Theorem 1.4 implies that $K^n \leftthreetimes K^{\times}$ is minimal. Then Fact 1.6 is again applicable.

(c) The given proofs may be easily modified taking into account Lemma 4.4(ii). $\square$

Let K be a Hausdorff topological field. Denote by $GL(n, K)$ the topological group of all invertible $n \times n$ matrices over K endowed with the topology inherited from K^{n^2}.

Theorem 4.8. *Let K be a non-discrete locally retrobounded topological field and let H be a topological subgroup of $GL(n, K)$. Then $(K^n \leftthreetimes H)^{\nu}$ is perfectly minimal in each of the following cases:*

(a) *H contains all diagonal matrices with non-zero entries.*

(b) *K is complete and H contains all scalar matrices with non-zero entries.*

(c) *K is a (complete) ordered field and H contains all (resp., scalar) diagonal matrices with positive entries.*

Proof. In each case $Z(K^n \rightthreetimes H)$ is trivial (see the proof of Lemma 1.5(c)). Therefore by Lemma 1.5(b) and fact 1.6 it suffices to show the minimality of $K^n \rightthreetimes H$. Moreover, since H carries the topology of pointwise convergence then the action is t-exact. By Theorem 1.4 we have only to establish that K^n is H-minimal. Consider the group $(K \rightthreetimes K^\times)^n$. As we know from Theorem 4.7, this group is minimal. On the other hand, $(K \rightthreetimes K^\times)^n$ is topologically isomorphic to $Y := K^n \rightthreetimes (K^\times)^n$. If the assertion (a) is satisfied then $K^n \subseteq Y \subseteq K^n \rightthreetimes H$. Now Lemma 1.2(i) implies that K^n is H-minimal. Therefore, (a) is proved.

The proofs of (b) and (c) are similar to the case (a). For (b) we need only to replace Y by the group $K^n \rightthreetimes K^\times$. In the case (c) we consider $(K \rightthreetimes K_+)^n$ (or, respectively $K^n \rightthreetimes K_+$). $\square$

Using completely different methods, Remus and Stoyanov [RS] have shown that $(\mathbb{R}^n \rightthreetimes H)^\nu$ is minimal for every cardinal ν and a closed subgroup H of $GL(n, \mathbb{R})$ which contains all diagonal matrices with positive entries. Our Theorems 4.7 and 4.8 improve this result. Even in the case $K = \mathbb{R}$ we may consider any (not necesarily closed) subgroup H which contains all scalar (not necessarily all diagonal) matrices with positive entries. In particular, this implies that $\mathbb{R}^n \rightthreetimes \mathbb{R}_+$ is minimal. This seems to be new even for $n = 2$.

§5. A counterexample.

Recently Dikranjan asked[1] [D2, Question 7.9]: is a minimal totally disconnected group always zero-dimensional? This was inspired by the more general question of Arhangel'skij [D1, Question 1.2]: does a totally disconnected group always admit a coarser zero-dimensional group topology?

In this section we negatively answer these questions by constructing the following

Example 5.1. There exists a totally disconnected (separable metrizable) perfectly minimal group G which is not zero-dimensional.

Construction. Let $\omega : \ell_2 \times \ell_2 \to \mathbb{R}$ be the standard scalar product, where $\omega(a, b) = \sum_{i=1}^{\infty} a_i b_i$ for $a = \langle a_i \rangle$, $b = \langle b_i \rangle \in \ell_2$. As was mentioned above in section 2, the induced

[1] The author thanks Dikranjan for providing him the survey article [D2] before its publication.

group

$$M_+(\omega) = ((\mathbb{R} \times \ell_2) \leftthreetimes \ell_2) \leftthreetimes \mathbb{R}_+$$

is minimal. It will be convenient to present here explicitly the operations of this group.

Lemma 5.2. *Let* $u_1 = (r_1, x_1, y_1, t_1)$ *and* $u_2 = (r_2, x_2, y_2, t_2)$ *be two elements of* $M_+(w)$. *Then*

(a) $u_1 \cdot u_2 = (r_1 + t_1 r_2 + t_1 w(y_1, x_2), x_1 + t_1 x_2, y_1 + y_2, t_1 t_2)$.
(b) $u_2^{-1} = (t_2^{-1} w(x_2, y_2) - t_2^{-1} r_2, -t_2^{-1} x_2, -y_2, t_2^{-1})$.
(c) $[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1} =$
$= ((1-t_2) r_1 - (1-t_1) r_2 + t_1 w(y_1, x_2) - t_2 w(x_1, y_2), (1-t_2) x_1 - (1-t_1) x_2, \mathbb{O}, 1)$,
where $\mathbb{O}$ *denotes the zero element of* ℓ_2.

Define two subgroups of ℓ_2 by letting:

$$E := \{\langle a_i \rangle \in \ell_2 \bigm| a_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N}\},$$

$$F := \{\langle a_i \rangle \in E \bigm| a_i = 0 \text{ for all but finitely many i's}\}$$

Fact 5.3. [E] *E is a totally disconnected but not zero-dimensional group.*

Observe that if $a \in E$ and $b \in F$ then $\omega(a, b)$ is a rational number. Therefore, the restriction $\omega|_{E \times F} : E \times F \to \mathbb{Q}$ is a well-defined $\mathbb{Q}$-bilinear map. This fact together with Lemma 5.2 easily implies that the subset

$$G := \{(q, x, y, t) \in M_+(\omega) \bigm| q \in \mathbb{Q}, x \in E, y \in F, t \in \mathbb{Q}_+\}$$

forms a subgroup of $M_+(\omega)$. Moreover, G can be represented as $G = ((\mathbb{Q} \times E) \leftthreetimes F) \leftthreetimes \mathbb{Q}_+$. We show that G is the desired group.

First note that by Fact 5.3, G is a totally disconnected non-zero-dimensional group. By Lemma 1.5(c) it suffices to show that G is a minimal group. Since G is a *dense subgroup* of a minimal group $M_+(\omega)$, due to the Banaschewskii criterion [B], we have only to check that G is an *essential* subgroup of $M_+(\omega)$. That is, we need to check that $G \cap N$ is not trivial for every closed non-trivial normal subgroup N of $M_+(\omega)$. We will prove that $G \cap N$ always contains the element $z = (1, \mathbb{O}, \mathbb{O}, 1)$ which is different from the neutral element $e = (0, \mathbb{O}, \mathbb{O}, 1)$. Obviously, $z \in G$. It suffices to show that always $z \in N$. Let $u_2 = (r_2, x_2, y_2, t_2) \in N \setminus \{e\}$. Since N is normal then the commutator $[u_1, u_2]$ also belongs to N for arbitrary $u_1 \in M_+(\omega)$. Our aim is to find $u_1 \in M_+(\omega)$ such that $[u_1, u_2] = z$.

Since $u_2 \neq e$ then one of the following possibilities hold:

(1) $t_2 \neq 1$;
(2) $t_2 = 1,\ \ y_2 \neq \mathbb{O}$;
(3) $t_2 = 1,\ \ y_2 = \mathbb{O},\ \ x_2 \neq \mathbb{O}$;
(4) $t_2 = 1,\ y_2 = \mathbb{O},\ x_2 = \mathbb{O},\ \ r_2 \neq 0$.

We indicate below how we will choose the appropriate $u_1 = (r_1, x_1, y_1, t_1)$ according to each case:

(1) $u_1 = ((1 - t_2)^{-1}, \mathbb{O}, \mathbb{O}, 1)$;
(2) $u_1 = (0, x_1, \mathbb{O}, 1)$, where $x_1 \in E$ and $\omega(x_1, y_2) = -1$;
(3) $u_1 = (0, \mathbb{O}, y_1, 1)$, where $y_1 \in F$ and $\omega(x_2, y_1) = 1$;
(4) $u_1 = (0, \mathbb{O}, \mathbb{O}, 1 + r_2^{-1})$.

It should be noted that in the case (4) we may suppose that $r_2 > 0$ (otherwise, take $u_2^{-1} = (-r_2, \mathbb{O}, \mathbb{O}, 1)$ instead of u_2). Then $1 + r_2^{-1} \in Q_+$ and hence u_1 is well-defined. By Lemma 5.2(c) in each case we obtain $[u_1, u_2] = z$. $\square$

References

[B] B. Banaschewskii, *Minimal topological algebras*, Math. Ann. **211** (1974), 107-114.

[Bo] N. Bourbaki, *Topologie Générale, Ch.III-IV*, Hermann, Paris, 1960.

[CHR] W.W. Comfort, K.H. Hofmann and D. Remus, *Topological groups and semigroups*, in: Recent Progress in General Topology (M. Hušek and J. van Mill, ed.), North-Holland, Amsterdam, 1992, pp. 57-144.

[DS] S. Dierolf and U. Schwanengel, *Examples of locally compact non-compact minimal topological groups*, Pacific J. Math. **82** (1979), 349-355.

[D1] D. Dikranjan, *Proc. of Matsuyama Conference (1994), Japan* (to appear).

[D2] ———, *Recent advances in minimal topological groups*, Topology Appl. (to appear).

[DPS] D. Dikranjan, Iv. Prodanov, L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, vol. 130, Pure Appl. Math., Marcel Dekker, New York-Basel, 1989.

[E] P. Erdös, *The dimension of rational points in Hilbert space*, Ann. of Math. (2) **41** (1940), 734-736.

[EDS] V. Eberhardt, S. Dierolf, U. Schwanengel, *On products of two (totally) minimal topological groups and the three-space-problem*, Math. Ann. **251** (1980), 123-128.

[HR] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

[M] M. Megrelishvili, *Group representations and construction of minimal groups*, Topology Appl. **62** (1995), 1-19.

[N] L. Nachbin, *On strictly minimal topological division rings*, Bull. AMS **55** (1949), 1128-1136.

[P] V. Pestov, *An example of a non-metrizable minimal topological group which unit has type G_δ*, Ukr. Math. J. **37** (1986), 655.

[RS] D. Remus, L. Stoyanov, *Complete minimal topological groups*, Topology Appl. **42** (1991), 57-69.

[RD] W. Roelcke, S. Dierolf, *Uniform Structures on Topological Groups and their Quotients*, McGraw-Hill, 1981.

[S] R. Stephenson, *Minimal topological groups*, Math. Ann. **192** (1971), 193-195.

[St] L. Stoyanov, *On products of minimal and totally minimal groups*, Proc. 11th Conf. of the Union of Bulg. Math., Sunny Beach (1981), 203-208.

[W] S. Warner, *Topological Fields*, North Holland Mathematics Studies, vol. 154, North-Holland-Amsterdam, New York, Oxford, Tokyo, 1989.

[Wi] W. Wieslaw, *Topological Fields*, vol. 119, Pure Appl. Math., Marcel Dekker, New York-Basel, 1988.

E-mail: megereli@macs.biu.ac.il