Constructing Tychonoff G-spaces which are not G-Tychonoff

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Abstract. Jan de Vries' compactification problem is whether every Tychonoff G-space can be equivariantly embedded in a compact G-space. In such a case, we say that G is a *V-group*. De Vries showed that every locally compact group G is a V-group. The first example of a non-V-group was constructed in 1988 by the first author. Until now, this was the only known counterexample. In this paper, we give a systematic method of constructing non-compactificable G-spaces. We show that the class of non-V-groups is large and contains all $\aleph_0$-bounded non-locally precompact groups. This establishes the existence of monothetic (even cyclic) non-V-groups, answering a question of the first author. As a related result, we obtain a characterization of locally compact groups in terms of "G-normality."

1. Introduction.

A *topological transformation group*, or a *G-space*, is a triple $\langle G, X, \alpha\rangle$, where G is a topological group, X is a topological space, and $\alpha\colon G\times X \to X$ is a continuous action. For basic information on G-spaces, see de Vries [16,21]. If X is Tychonoff (respectively, normal, compact, etc.), then $\langle G, X, \alpha\rangle$ (or just X, for short) is called a Tychonoff (respectively, normal, compact, etc.) G-space. A G-space is *G-Tychonoff* if it can be equivariantly embedded into a compact Hausdorff G-space. We call a group G a *V-group* if every Tychonoff G-space is G-Tychonoff.

In [17], Jan de Vries posed the "compactification problem" in its full generality, which in our terms becomes: is every topological group G a V-group? See Carlson [4] for the case $G = \mathbb{R}$. In [3], R. B. Brook investigated a more general class of groups and showed that for *any* topological group G, the G-space $\langle G, G, \alpha_L\rangle$ $(\alpha_L(g_1, g_2) = g_1g_2)$ is G-Tychonoff.

Recall de Vries' well-known result [20] which states that every locally compact group is a V-group. (For the case where G is a compact Lie group, see [12]. If G is compact, see [1,18]). Recall some G-spaces which are G-Tychonoff:

(a) [17] Every coset G-space $\langle G, G/H, \alpha_L\rangle$. (If $H = \{e\}$, this gives Brook's result mentioned above).
(b) [7] Every G-space X under an equicontinuous action.
(c) [10] Every G-group X (and, hence, every linear G-space X).
(d) [10] Every metric G-space (X, d), where G is second category and $\alpha^g : X \to X$ is d-uniformly continuous for every $g \in G$.
(e) [15] Every Baire G-space X, where G is $\aleph_0$-bounded and acts transitively on X.

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In 1988, Megrelishvili [9] answered de Vries' question negatively. He found a continuous action α of a separable, complete metrizable group G on $J(\aleph_0)$, the so-called metrizable hedgehog of spininess $\aleph_0$, such that $\langle G, J(\aleph_0), \alpha \rangle$ is not G-Tychonoff. Therefore, the group G of this example is not a V-group. By [11], no dense subgroup of G is a V-group.

It is an open question whether there are non-locally compact V-groups. In this paper, we prove the following

Main Theorem. *If G is an $\aleph_0$-bounded topological group which is not locally precompact, then G is not a V-group.*

In particular, this provides an example of a cyclic group G which is not a V-group, answering a question of Megrelishvili about monothetic groups [11]. As a related result, we characterize locally compact groups in terms of G-normality.

The compactification problem is still open in the case of a (monothetic) *precompact* group, even for a dense cyclic subgroup of the circle group $\mathbb{T}$. It is also an intriguing question whether the additive group $\mathbb{Q}$ of rational numbers is a V-group.

Our approach is to adapt techniques for constructing regular spaces which are not completely regular. For example, I. Tree [14] has constructed a machine which turns any regular topological space which is not normal into a regular space which is not completely regular. We have adapted his technique to the context of G-spaces, and we have constructed a machine which turns a normal G-space which is not G-normal in some sense into a Tychonoff G-space which is not G-Tychonoff.

Some of the results of this paper were presented by the second author at the Eighth Prague Topological Symposium in August, 1996.

2. Preliminaries and Conventions.

All spaces are Tychonoff, and all cardinals are assumed to be infinite. The filter of all neighborhoods of an element x of a space X is denoted by $N_x(X)$. The neutral element of a group is denoted by e. A group G is *locally precompact* if it is a subgroup of a locally compact group, or, equivalently, if its sup-completion (the completion with respect to its two-sided uniformity) is locally compact. Recall [19,20,1] that a continuous function $f : X \to \mathbb{R}$ defined on a G-space X is called *α-uniform* if for every $\varepsilon > 0$ there exists $U \in N_e(G)$ such that for all $g \in U$, $x \in X$, we have $|f(x) - f(gx)| < \varepsilon$. Denote by $C_\alpha(X)$ (respectively, $C(X)$) the set of all bounded α-uniform (respectively, continuous bounded) functions on X.

Lemma 2.1.

(i) *$C_\alpha(X)$ is a closed sub-algebra of $C(X)$.*

(ii) *If f is in $C_\alpha(X)$ and $r \in \mathbb{R}$, then the functions*

$$\mu(x) := \min\{f(x), r\} \quad \text{and} \quad h(x) := |f(x)|$$

are also in $C_\alpha(X)$. □

The compactification of X which corresponds to the algebra $C_\alpha(X)$ is the *maximal G-compactification* [19] and is denoted by $\beta_G X$.

Fact 2.2 [19,20]. *A Tychonoff G-space $\langle G, X, \alpha\rangle$ is G-Tychonoff iff $\beta_G X$ is a proper G-compactification of X iff $C_\alpha(X)$ separates the points and the closed subsets of X.* $\square$

As mentioned above, the G-space of all left translations $\langle G, G, \alpha_L\rangle$ is G-Tychonoff. In this case, the maximal (proper) G-compactification $\beta_G G$ is called the *greatest ambit.* (For further information, see [21]).

We say that subsets A and B of a G-space $\langle G, X, \alpha\rangle$ are *α-disjoint* if there exists $U \in N_e(G)$ such that $UA \cap UB = \emptyset$. Two disjoint invariant subsets are obviously α-disjoint. Moreover, if A and B are separated by an α-uniform function, then A and B are α-disjoint. In fact, we have the following stronger

Lemma 2.3. *Let $\langle G, X, \alpha\rangle$ be a G-space. Let C and D be subsets of X which are separated by an α-uniform function. Then there are sequences $\{U_n\}$, $\{O_n\}$ and $\{O'_n\}$ of nbds of e, C and D, respectively, such that for all $n \in \mathbb{N}$,*

$$cl(U_n O_n) \subseteq O_{n+1}, \quad cl(U_n O'_n) \subseteq O'_{n+1}, \quad \textit{and} \quad O_n \cap O'_n = \emptyset.$$

$\square$

We say that a G-space X is *G-normal* (*equivariantly normal* in the terminology of [8]) if every pair of α-disjoint closed subsets of X has α-disjoint nbds. Equivalently, X is G-normal iff $C_\alpha(X)$ separates the closed α-disjoint subsets of X. The continuity of the action α guarantees that for every closed subset F of X and every point $x \in X \setminus F$, the subsets F and $\{x\}$ are α-disjoint. Therefore, every G-normal G-space is G-Tychonoff. Every coset G-space $\langle G, G/H, \alpha_L\rangle$ is G-normal for arbitrary G. If G is locally compact, then every normal G-space is G-normal [8]. For the converse, see Theorem 5.2 below.

Definition 2.4. A G-space $\langle G, X, \alpha\rangle$ is called *weakly G-normal* if $C_\alpha(X)$ separates the closed invariant subsets of X.

Proposition 2.5. *Let G be an arbitrary topological group which is not sup-complete. Then there exists a normal G-space X of weight $w(X) = w(G)$ which is not G-normal.*

Proof. Denote by $\hat{G}$ the sup-completion of G. Let $\mathbb{B}$ be a base of nbds of e in G. We may suppose that the cardinality of $\mathbb{B}$ is not greater than $w(G)$. By Brook's theorem [3] the $\hat{G}$-space $\langle \hat{G}, \hat{G}, \alpha_L\rangle$ is $\hat{G}$-Tychonoff. By [11], there is a compact $\hat{G}$-extension Y of $\langle \hat{G}, \hat{G}, \alpha_L\rangle$ such that $w(Y) \le w(\hat{G}) \cdot w(\hat{G}) = w(\hat{G}) = w(G)$. We now form the topological $\hat{G}$-sum $X = \oplus\{Y_U : U \in \mathbb{B}\}$, where each $Y_U := Y \times \{U\}$ is a copy of Y. Let α denote the action of $\hat{G}$ on X restricted to G. Clearly, $\langle G, X, \alpha\rangle$ is a normal G-space and $w(X) = w(G)$. In order to show that it is not G-normal, we will construct two closed α-disjoint subsets C and D of X which are not separated by $C_\alpha(X)$.

Let $U \in \mathbb{B}$, and let $cl_{\hat{G}}(U)$ be the closure of U in $\hat{G}$. Since G is not sup-complete, we can pick $g_U \in cl_{\hat{G}}(U) \backslash G$. Set $C := \{(g_U, U) : U \in \mathbb{B}\}$ and $D := \{(e, U) : U \in \mathbb{B}\}$. Clearly, C and D are closed α-disjoint subsets of X. It is easy to show, however, that for all $V \in \mathbb{B}$

$$(g_V, V) \in cl_X(VD) \cap C.$$

By lemma 2.3, C and D are not separated by $C_\alpha(X)$. □

For a compact space X, $H(X)$ denotes the (topological) group of all homeomorphisms of X, endowed with the compact-open topology. Let $\langle G, (X, \mu), \alpha \rangle$ be an arbitrary G-space, where μ is a compatible uniformity for X, and let $A \subseteq G$. We say that A acts *μ-uniformly equicontinuously* if for every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that for all $g \in A$ and for all $x, y \in X$, $(x, y) \in \delta$ implies $(gx, gy) \in \varepsilon$. We say that G acts *locally uniformly equicontinuously* if there exists $U \in N_e(G)$ such that U acts μ-uniformly equicontinuously.

Lemma 2.6. *Let (G, τ) be a topological group. For $g \in G$, let $\tilde{g} : \beta_G G \to \beta_G G$ be the extension of the transition map $\alpha^g : G \to G$. Then the map $\varphi : G \to H(\beta_G G)$ defined by $\varphi(g) = \tilde{g}$ is a topological group embedding.*

Proof. Straightforward. □

Lemma 2.7. *The action of G on the greatest ambit $\beta_G G$ is locally uniformly equicontinuous iff G is locally precompact.*

Proof. Suppose the action $\tilde{\alpha} : G \times \beta_G G \to \beta_G G$ is locally uniformly equicontinuous. That is, there is a neighborhood U of e which acts μ-uniformly equicontinuously, where μ is the unique uniformity for $\beta_G G$. Then, the Ascoli-Arzela theorem implies that the closure of $\varphi(U)$ in $H(\beta_G G)$ is compact. Therefore, the closure of $\varphi(G)$ in $H(\beta_G G)$ is a locally compact group containing the group $G = \varphi(G)$ (Lemma 2.6).

Conversely, suppose G is locally precompact. By the equivariant completion theorem [11], there is a continuous extending action of the sup-completion $\hat{G}$ of G on $\beta_G G$. By our assumption, there exists a compact neighborhood V of e in $\hat{G}$. Easy compactness arguments now imply that V acts μ-uniformly equicontinuously on $\beta_G G$. Hence the neighborhood $V \cap G$ of e in G also acts μ-uniformly equicontinuously on $\beta_G G$. □

Recall that a topological group G is called *$\aleph_0$-bounded* [2,6] if for every $V \in N_e(G)$ there exists a countable subset S of G such that $SV = G$. Guran [6] proved that G is $\aleph_0$-bounded iff G is a topological subgroup of a product of second countable topological groups. If G is separable, Lindelöf, or ccc, then G is $\aleph_0$-bounded.

Fact 2.8 (Equivariant Approximation Theorem) [10]. *Let G be an $\aleph_0$-bounded group, and let X be a compact G-space. Then X can be represented as a G-limit of an inverse G-system $\{\langle G, X_i, \alpha_i \rangle : i \in I\}$ of compact metrizable G-spaces X_i such that dim $X_i \leq$ dim X.*

3. From non-G-normality to non-G-Tychonoff-ness.

We now adapt the original construction of Tree [14] to the context of G-spaces. Let $\langle G, X, \alpha\rangle$ be a regular G-space. We describe the construction of a related G-space $\langle G, X^+, \alpha^+\rangle$. We assume that there are disjoint closed G-subspaces $C, D \subseteq X$.

Let $\omega = \mathbb{N} \cup \{0\}$ in the discrete topology. Let Y be the quotient space formed from $X \times \omega$ by identifying the pairs $(c, 2i+1)$ and $(c, 2i+2)$ for $c \in C, i \in \omega$, and the pairs $(d, 2i)$ and $(d, 2i+1)$ for $d \in D, i \in \omega$. Let $p\colon X \times \omega \to Y$ be the quotient map. For $n \in \omega$, let $i_n\colon X \to X \times \omega$ be the canonical injection $x \mapsto (x, n)$.

Fix a point $a \notin Y$, and let $X^+ = Y \cup \{a\}$. Topologize X^+ by setting Y to be an open subset with its quotient topology and the nth basic nbd of a to be

$$N_n(a) = \{a\} \cup p(i_{2n}(X \setminus C)) \cup \bigcup\{p(i_m(X))\colon m > 2n\}.$$

It is trivial to check that this generates a topology τ on X^+. We remark only that if O is an open subset of Y, then $O \cap N_n(a)$ is also open in Y. To see this, it is enough, by definition of the quotient topology, to show that $p^{-1}(O \cap N_n(a))$ is open. Observe that $p^{-1}(O)$ is open by continuity of p, and $p^{-1}(N_n(a)) = N_n(a) \setminus \{a\}$ is open by definition of $N_n(a)$.

We now define a function $\alpha^+ : G \times X^+ \to X^+$. For any $g \in G$, set $\alpha^+(g, a) = a$. For $p((x, n)) \in Y$, set $\alpha^+(g, p((x, n))) = p(i_n(\alpha(g, x)))$. Note that α^+ is well-defined because C and D are invariant.

Claim 3.1. *$\langle G, X^+, \alpha^+\rangle$ is a G-space.*

Proof. It is easy to see that α^+ is an action on X^+. It remains to show that α^+ is continuous. Continuity of α^+ at points (g, y) for $y \in Y$ is easily proved using the continuity of α. For continuity at the point (g, a), note that for any $n \in \mathbb{N}$, $GN_n(a) = N_n(a)$.

Theorem 3.2. *Let $\langle G, X, \alpha\rangle$ be a normal G-space with closed disjoint invariant subsets $C, D \subseteq X$ which are not separated by $C_\alpha(X)$. (Hence X is a non-weakly G-normal G-space). Then $\langle G, X^+, \alpha^+\rangle$ is a normal (and, hence, Tychonoff) G-space which is not G-Tychonoff.*

Proof. First we show that X^+ is normal.

Claim 1. *If $A \subseteq Y$, U is an open subset of $X \times \omega$, and $p^{-1}(A) \subseteq U$, then there is an open subset U' of $X \times \omega$ such that $p^{-1}(A) \subseteq U' \subseteq U$, and $p(U')$ is open in the space Y.*

Proof. For each $(x, n) \in p^{-1}(A)$, we will define an open set $U_{x,n} \subseteq X \times \{n\}$ such that $(x, n) \in U_{x,n} \subseteq U$. If $x \notin C \cup D$ (or if $n = 0$ and $x \notin D$), then there is an open set $U' \subseteq X$ such that $x \in U'$ and $U' \cap (C \cup D) = \emptyset$. Since $(x, n) \in U$, there is an open set $U'' \subseteq X \times \{n\}$ such that $(x, n) \in U'' \subseteq U$. Let $U_{x,n} := (U' \times \{n\}) \cap U''$. Then $U_{x,n}$ is open, $(x, n) \in U_{x,n}$, $U_{x,n} \subseteq U$, and $p(U_{x,n})$ is open in Y, since $p^{-1}(p(U_{x,n})) = U_{x,n}$. If $x \in C$, then assume without loss of generality that n is odd. So both (x,n) and $(x,n+1)$ belong to $p^{-1}(A)$. Since $(x,n) \in U$, there is an open set $U_n \subseteq X \times \{n\}$ such that $(x,n) \in U_n \subseteq U$. Similarly, there is an open set $U_{n+1} \subseteq X \times \{n+1\}$ such that $(x,n+1) \in U_{n+1} \subseteq U$. Let

$$\begin{aligned} U_{x,n} &:= \{(y,n) \in U_n \colon (y,n+1) \in U_{n+1}\} \\ U_{x,n+1} &:= \{(y,n+1) \in U_{n+1} \colon (y,n) \in U_n\}. \end{aligned}$$

The case $x \in D$ is handled similarly.

Finally, define $U' = \bigcup_{(x,n)\in p^{-1}(A)} U_{x,n}$. This proves claim 1.

Claim 2. *If $C \subseteq X^+$ is closed, and $a \notin C$, then there exists an $n \in \omega$ such that $C \subseteq \bigcup_{k<n} p(X \times \{k\})$.*

Proof. If not, then for every $n \in \omega$, there are $m \geq n$ and $x_n \in p(X \times \{m\})$ such that $x_n \in C$. Then $x_n \to a$, so $a \in C$.

Claim 3. *X^+ is normal (and, hence, Tychonoff).*

Proof. Let $A, B \subseteq X^+$ be closed and disjoint.

Case 1: $a \notin A \cup B$.

In this case, A and B are closed subsets of Y. So $p^{-1}(A)$ and $p^{-1}(B)$ are closed disjoint subsets of $X \times w$. Since $X \times \omega$ is normal, there are disjoint open sets $U, V \subseteq X \times \omega$ such that $p^{-1}(A) \subseteq U$ and $p^{-1}(B) \subseteq V$. By Claim 1, let $U', V' \subseteq X \times \omega$ be open sets such that $p^{-1}(A) \subseteq U' \subseteq U$, $p^{-1}(B) \subseteq V' \subseteq V$, and $p(U'), p(V')$ are open in Y. Note that U' and V' are disjoint. Now

$$A = p(p^{-1}(A)) \subseteq p(U') \quad \text{and} \quad B = p(p^{-1}(B)) \subseteq p(V').$$

Subclaim. $p(U') \cap p(V') = \emptyset$.

Proof. Suppose $z \in p(U') \cap p(V')$. If $z = p((x,n))$, where $x \notin C \cup D$, then $(x,n) \in U' \cap V' = \emptyset$. Contradiction. Hence we must have $x \in C \cup D$. Without loss of generality, assume that $x \in C$, n is odd, and $(x,n) \in U', (x,n+1) \in V'$. From the construction of U' in Claim 1, we see that there must $c \in C$ such that $(x,n) \in U_{c,n}$. So $(x,n+1) \in U_{c,n+1}$. Hence $(x,n+1) \in U' \cap V'$. Contradiction.

Case 2: $a \in A \cup B$.

Without loss of generality, assume that $a \in A \setminus B$. So $A \cap Y = A \setminus \{a\}$ is closed in Y. So we apply Case 1 to separate $A \cap Y$ and B by disjoint open sets U and V, respectively. By Claim 2 there is $n \in \omega$ such that

$$B \subseteq \bigcup_{k<n} p(X \times \{k\}).$$

Assume without loss of generality that n is odd. Define $V' := V \cap [(\bigcup_{k<n} p(X \times \{k\})) \bigcup p((X \setminus C) \times \{n\})]$. Let $U' = U \cup N_n(a)$. Then $A \subseteq U'$, $B \subseteq V'$, $U' \cap V' = \emptyset$, and U' and V' are open.

This completes the proof that X^+ is normal.

Next, we will show that $\langle G, X^+, \alpha^+\rangle$ is *not* G-Tychonoff. Our proof follows closely the proof in [14]. For the sake of completeness, we have included many of the details.

Notation. For a subset $A \subseteq X$ and $n \in \omega$, let $A^+_n = p(i_n(A))$.

We claim that the point a and the set C^+_0 are not separated by a bounded α^+-uniform function. The proof is by contradiction. Suppose $f : X^+ \to \mathbb{R}$ is a bounded α^+-uniform function such that $f(C^+_0) = 0$ and $f(a) = 1$. As in Tree [14], there is an $n \in \omega$ such that for all $m \geq n$ and for all $x \in C^+_m$, $|f(x)| \geq \dfrac{1}{n}$. Fix such an n. Note that $f(C^+_n) \cap \left(-\dfrac{1}{n}, \dfrac{1}{n}\right) = \emptyset$. Assume without loss of generality that n is odd. Let $n = 2r + 1$. Define a function $\mu : X^+ \to \mathbb{R}$ by $\mu(x) = \min\{n|f(x)|, 1\}$. By lemma 2.1, μ is α^+-uniform. Note that $\mu(C^+_0) = 0$ and $\mu(C^+_n) = 1$. We now define $n+1$ functions $h_0, h_1, \ldots, h_n : X \to \mathbb{R}$ by restricting the domain of μ to X^+_k for $k = 0, 1, \ldots, n$. More precisely,

$$h_k(x) = \mu(p(i_k(x))).$$

Claim 4. *Each h_k is α-uniform.*

Proof. Fix $\varepsilon > 0$. Let U be a nbd of e in G such that if $x \in X^+$ and $g \in U$, then $|\mu(x) - \mu(gx)| < \varepsilon$. Fix $x \in X, g \in U$ and $k \in \{0, 1, \ldots, n\}$. Then $|h_k(x) - h_k(gx)| = |\mu(p(i_k(x))) - \mu(p(i_k(gx)))| = |\mu(p(x,k)) - \mu(p(gx,k))| = |\mu(p(x,k)) - \mu(gp(x,k))| < \varepsilon$. This proves claim 4.

Observation.

(1) For $i = 0, 1, \ldots, r-1$,

$$h_{2i+1} \restriction C = h_{2i+2} \restriction C.$$

(2) For $i = 0, 1, \ldots, r$,

$$h_{2_i} \restriction D = h_{2i+1} \restriction D.$$

(3) $h_0 \restriction C = 0$.

(4) $h_n \restriction C = 1$.

Finally, define $h : X \to \mathbb{R}$ by

$$h(x) = \sum_{k=0}^{n} (-1)^{k+1} h_k(x).$$

By lemma 2.1, h is α-uniform. Now $c \in C \Rightarrow h(c) = -h_0(c) + h_n(c) = 0 + 1 = 1$, by (1), (3) and (4) above, while $d \in D \Rightarrow h(d) = 0$ by (2).

This contradicts the assumption that C and D cannot be separated by an α-uniform function. This completes the proof of theorem 3.2. $\square$

4. Proof of the Main Theorem.

Definition 4.1. Let $\langle G, X, \alpha\rangle$ be a G-space. Let $S \subseteq G$. Subsets A and B of X are said to be *S-near* if for every pair O_1, O_2 of neighborhoods of A and B, respectively, there exist $g_1, g_2 \in S$ such that $g_1O_1 \cap g_2O_2 \neq \emptyset$, *i.e.*, $SO_1 \cap SO_2 \neq \emptyset$.

Lemma 4.2. *Let $\langle G, X, \alpha\rangle$ be a G-space with a fixed point z. Let $S \subseteq G$. Let $x \in X \setminus \{z\}$ be such that x and z are S-near. Set $Y = X^2 \setminus \{(z,z)\}$, $C = (\{z\} \times X) \setminus \{(z,z)\}$, and $D = (X \times \{z\}) \setminus \{(z,z)\}$. Define $\alpha' : G \times Y \to Y$ by the "one-coordinate" action $\alpha'(g,(x,y)) = (\alpha(g,x), y)$. Then $\langle G, Y, \alpha'\rangle$ is a G-space, and C and D are closed disjoint invariant S-near subsets of Y.*

Proof. Only the statement about the S-nearness of C and D is non-trivial. Fix nbds O_1 and O_2 of C and D, respectively. Since $(x,z) \in D$, we have $(x,z) \in O_2$, and hence there are open nbds $O_x(x), O_z(z)$ of x and z, respectively, such that

$$(x,z) \in (O_x(x) \times O_z(z)) \setminus \{(z,z)\} \subseteq O_2.$$

It is easy to see that z is not isolated. Hence there exists $y \in O_z(z)$ such that $y \neq z$. Then $(z,y) \in O_1$, so there are open nbds $O'_z(z), O_y(y)$ of z and y, respectively, such that

$$(z,y) \in (O'_z(z) \times O_y(y)) \setminus \{(z,z)\} \subseteq O_1.$$

By hypothesis, x and z are S-near, so there are $g_1, g_2 \in S$ such that $g_1O_x(x) \cap g_2O'_z(z) \neq \emptyset$. Therefore,

$$g_1(O_x \times O_z) \cap g_2(O'_z \times O_y) \neq \emptyset.$$

This easily implies that $g_1O_1 \cap g_2O_2 \neq \emptyset$. □

Main Theorem 4.3. *Let G be a non-locally precompact $\aleph_0$-bounded topological group. Then G is* **not** *a V-group,* i.e., *there is a Tychonoff G-space which is* **not** *G-Tychonoff.*

Proof. Since G is not locally precompact, Lemma 2.7 implies that G does not act locally uniformly equicontinuously on $X = \beta_G G$.

We now build a G-space X_U for every U in a collection $\mathbb{B}$ (of cardinality $\chi(G)$) of basic nbds of e in G. By the approximation theorem 2.8, X is the G-limit of an inverse G-system of compact metrizable G-spaces X_i $(i \in I)$. Let μ and μ_i denote the unique uniformity on X and X_i, respectively.

Let $U \in \mathbb{B}$. Since U does not act μ-uniformly equicontinuously on $\beta_G G$, there exists an index $i \in I$ such that U does not act μ_i-uniformly equicontinuously on (X_i, μ_i). Therefore, there is $\varepsilon \in \mu_i$ such that for every $\delta \in \mu_i$ there exist $(x_\delta, y_\delta) \in \delta$ and $g_\delta \in U$ such that

$$(g_\delta x_\delta, g_\delta y_\delta) \notin \varepsilon. \qquad (*)$$

Thus we obtain nets $\langle x_\delta\rangle, \langle y_\delta\rangle, \langle g_\delta x_\delta\rangle$, and $\langle g_\delta y_\delta\rangle$ in (X_i, μ_i), indexed by the elements δ of μ_i. Passing to subnets if necessary, we may assume that there exist $x^U, a^U, b^U \in X$ such that $x_\delta \to x^U$, $y_\delta \to x^U$, $g_\delta x_\delta \to a^U$, and $g_\delta y_\delta \to b^U$. By $(*)$ we have $a^U \neq b^U$. Hence $(x^U, x^U) \in \Delta$, and $(a^U, b^U) \notin \Delta$, where $\Delta = \{(x,x)|x \in X_i\}$. Clearly, (x^U, x^U) and (a^U, b^U) are U-near in the G-space $(X_i, \mu_i) \times (X_i, \mu_i)$, with the natural "two-coordinate" action.

We now form the quotient G-space $Y_i = (X_i \times X_i)/\Delta$. Consider the quotient G-map $p : X_i \times X_i \to Y_i$. Let $z := p(x,x)$. Then z and $p(a^U, b^U)$ are U-near in Y_i.

Since z is a fixed point of Y_i, we may apply Lemma 4.2, with $X = Y_i, S = U$, and $x = p(a^U, b^U)$. That is, let $X_U = (Y_i \times Y_i) \setminus \{(z,z)\}, C^U = (\{z\} \times Y_i) \setminus \{(z,z)\}$, and $D^U = (Y_i \times \{z\}) \setminus \{(z,z)\}$. Define the "one-coordinate" action $\alpha' : G \times X_U \to X_U$ by

$$\alpha'(g, (x,y)) = (\alpha(g,x), y),$$

where α is the action of G on Y_i. Then, by lemma 4.2, $\langle G, X_U, \alpha'\rangle$ is a G-space, and C^U and D^U are closed disjoint invariant U-near subsets of X_U. This completes the contruction of X_U from U.

Now form the topological G-sum $S = \oplus\{X_U : U \in \mathbb{B}\}$. Let $\alpha^* : G \times S \to S$ be the natural action. Define

$$C = \bigcup_{U \in \mathbb{B}} C^U, \quad D = \bigcup_{U \in \mathbb{B}} D^U.$$

Then C and D are closed disjoint invariant subsets of S.

Claim 1. *C and D are not separated by $C_{\alpha^*}(S)$.*

Proof. By lemma 2.3, it is enough to show that there do not exist α^*-disjoint nbds of C and D. Let O_C, O_D be nbds of C and D, respectively. Let $U \in N_e(G), U' \subseteq U$, with $U' \in \mathbb{B}$. Define $O_1 = O_C \cap X_{U'}, \quad O_2 = O_D \cap X_{U'}$. Then $U'O_1 \cap U'O_2 \neq \emptyset$. Hence $UO_C \cap UO_D \neq \emptyset$.

Claim 2. *S is metrizable.*

Proof. It is sufficient to show that each X_U in the topological sum is metrizable. Each X_i is a metrizable compact space and Δ is closed in $X_i \times X_i$. Clearly, $Y_i = (X_i \times X_i)/\Delta$ is metrizable and compact. Hence every X_U is metrizable, being a subspace of $Y_i \times Y_i$.

By the above claims, $\langle G, S, \alpha^*\rangle$ is a normal G-space which is not weakly G-normal. Therefore, by theorem 3.2, $\langle G, S^+, (\alpha^*)^+\rangle$ is a Tychonoff G-space which is not G-Tychonoff. This completes the proof of the Main Theorem. □

We discuss here some topological properties of S^+. Observe that by our construction, $w(S^+) = w(S) \cdot \chi(G)$. If G is second countable, then we can choose S such that $w(S) = w(G)$. In this case, we get $w(S^+) = w(G) = \aleph_0$.

Note that the space S^+ has only one point, namely a, of non-local compactness. This is interesting because any locally compact G-space is G-Tychonoff.

5. Further Results.

The following corollary to the Main Theorem provides a negative answer to Megrelishvili's question [11] whether every monothetic group is a V-group.

Corollary 5.1. *There exists a cyclic (metrizable) topological group G which is not a V-group.*

Proof. Take, for example, a dense cyclic subgroup G of the monothetic non-locally precompact group from [13]. □

It is interesting to note that local compactness of the acting group G can be characterized in terms of G-normality.

Theorem 5.2. *The following are equivalent for a topological group G:*

(i) *G is locally compact.*
(ii) *Every normal G-space is G-normal.*

Proof. $(i) \Rightarrow (ii)$ see [8].

$(ii) \Rightarrow (i)$ First observe that by Proposition 2.5, G is sup-complete. Therefore, we may assume that G is not locally precompact. Then, by lemma 2.7, the action of G on $X = \beta_G G$ is not locally uniformly equicontinuous. Let $U \in N_e(G)$, and let μ be the unique uniformity for X. Then U does not act μ-uniformly equicontinuously. As in the proof of the Main Theorem, we obtain nets $\langle x_\delta \rangle, \langle y_\delta \rangle$ in X and a net $\langle g_\delta \rangle$ in U, each indexed by the elements δ of μ. Without loss of generality, we may assume that there exist $x^U, a^U, b^U \in X$ such that $x_\delta \to x^U, y_\delta \to x^U, g_\delta x_\delta \to a^U$, $g_\delta y_\delta \to b^U$, and $a^U \neq b^U$. Form the topological G-sum $S = \oplus\{Y_U : U \in N_e(G)\}$, where each Y_U is a copy of $X \times X$, with the "two-coordinate" action. Clearly, S is normal. Define $C = \{(x^U, x^U)_U : U \in N_e(G)\}$ and $D = \{(a^U, b^U)_U : U \in N_e(G)\}$. Then C and D are closed and α-disjoint subsets of S. However, for every $U \in N_e(G)$ and for every pair of nbds O_1, O_2 of C and D, respectively, we have

$$cl(UO_1) \cap cl(UO_2) \neq \emptyset.$$

Now lemma 2.3 implies that C and D are not separated by $C_\alpha(X)$. This proves that S is not G-normal. □

In a topological space X, points a and b are called *twins* if $f(a) = f(b)$ for every continuous function $f : X \to \mathbb{R}$. In a G-space $\langle G, X, \alpha \rangle$, points a and b are called *α-twins* if $f(a) = f(b)$ for every α-uniform function $f : X \to \mathbb{R}$.

In [14], Tree shows how to modify any regular space which is not completely regular in order to obtain a regular space with twins. We have adapted his technique to the context of G-spaces.

Let $\langle G, X, \alpha \rangle$ be a regular G-space. We construct a related G-space $\langle G, X^*, \alpha^* \rangle$. We assume that there is a closed invariant subset C of X. Let X^* be the quotient space formed from $X \times \{0, 1\}$ by identifying the pairs $(c, 0)$ and $(c, 1)$ for $c \in C$.

Let $p : X \times \{0,1\} \to X^*$ be the quotient map. Let $i_0, i_1 : X \to X \times \{0,1\}$ be the canonical injections. Define $\alpha^* : G \times X^* \to X^*$ by $\alpha^*(g, p((x,n))) = p(i_n(\alpha(g,x)))$.

The action α^* is well-defined because C is invariant. It is easy to see that $\langle G, X^*, \alpha^* \rangle$ is a G-space. The corresponding proof in [14] works here since $C_\alpha(X)$ is an algebra. Thus we obtain

Theorem 5.3. *Let $\langle G, X, \alpha \rangle$ be a normal G-space with a fixed point z and a closed invariant subset C which are not separated by $C_\alpha(X)$. Then $\langle G, X^*, \alpha^* \rangle$ is a Tychonoff G-space with α^*-twins. (In fact, X^* is even normal.)*

Combining the Main Theorem 4.3 and Theorem 5.3, we see that for every non-locally precompact, $\aleph_0$-bounded group G, there is a Tychonoff G-space with α-twins.

Questions.

(1) *What can be said in the Main Theorem if G is locally precompact (non-locally compact)? For example, is the additive group $\mathbb{Q}$ of rational numbers a V-group?*

(2) *In Theorem 5.2, can we replace "G-normal" by "weakly G-normal"?*

(3) *(Yu.Smirnov) Is there a Tychonoff G-space X on which every α-uniform function is constant?*

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