Group representations and construction of minimal topological groups

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Abstract. For every continuous biadditive mapping ω we construct a topological group $M(\omega)$ and establish its minimality under natural restrictions. Using the evaluation mapping $G \times G^* \to \mathbb{T}$ of Pontryagin-van Kampen duality and the canonical duality $E \times E^* \to \mathbb{R}$ for a normed space E, we obtain some new results in the theory of minimal groups. In particular, it is shown that every locally compact Abelian group is a group retract of a minimal locally compact group. Every Abelian topological group is a quotient of a perfectly minimal group.

Introduction.

A Hausdorff topological group G is said to be *minimal* (Stephenson [26]) if it does not admit a strictly coarser Hausdorff group topology. If G/P is minimal for every closed normal subgroup P of G then G is called *totally minimal.* A group G is *perfectly minimal* [12, §6.1] if the product $G \times X$ is minimal whenever X is minimal. Clearly, every compact group is totally minimal. At the same time, compact groups are perfectly minimal because every sup-complete (complete with respect to its two-sided uniformity) minimal group is perfectly minimal [14, Th. 3].

Recall some interesting examples of minimal groups:

a) The symmetric topological group $S(X)$ of all bijections $X \to X$ of an infinite set X endowed with the topology τ_X of pointwise convergence is totally minimal (Dierolf and Schwanengel [10]). Moreover, every Hausdorff group topology on $S(X)$ contains τ_X (Gaughan [17]).

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b) Discrete minimal groups (i.e., groups which do not admit non-discrete Hausdorff group topologies). Such groups were constructed by Shelah [25] (assuming CH) and also by Hesse [18] and Ol'shanskii [22].

c) $\mathbb{Z}$ with the p-adic topology τ_p is a totally minimal non-perfectly minimal group (Doitchinov [13]).

d) The semidirect procuct $\mathbb{R} \lambda_\pi \mathbb{R}_+$ of the multiplicative group $\mathbb{R}_+$ of all positive numbers with $\mathbb{R}$ under the natural action $\pi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ (Dierolf and Schwanengel [11]). This was generalized in Remus and Stoyanov [23]; in particular, it is proved that arbitrary powers of $\mathbb{R} \lambda_\pi \mathbb{R}_+$ are minimal.

e) Every connected semi-simple Lie group is totally minimal iff the center is finite[23].

f) For every (real or complex) Hilbert space $\mathbb{H}$ the unitary group $U(\mathbb{H})$ (which in this paper is always endowed with the strong operator topology) is totally minimal (Stoyanov [27]).

Prodanov and Stoyanov established that every Abelian minimal group is precompact (see [12,§2.7]). A general categorical approach to minimal topologies on algebraic structures was developed by Banaschewski [5]. For comprehensive information and terminology on minimal groups, we refer to [12].

In section 1 of this paper we introduce the notion of minimality for a biadditive mapping $\omega : E \times F \to A$ and prove that the canonical mapping $\Delta : G \times G^* \to \mathbb{T}$ is minimal for every Hausdorff locally compact Abelian (LCA) group G. In this section we also establish that a minimal group G is perfectly minimal iff the center $Z(G)$ is perfectly minimal.

In section 2 we define the induced group $M(\omega)$ and establish its minimality for minimal ω and A. As a consequence, we get that every LCA group is a group retract of a minimal locally compact (LC) group.

Section 3 contains a modification of our construction for bilinear forms. By the help of the canonical bilinear form $\langle\ ,\ \rangle : c_0 \times \ell_1 \to \mathbb{R}$ and results from functional analysis a perfectly minimal group of countable weight is constructed which is not topologically isomorphic to a subgroup of the unitary group $U(\mathbb{H})$ for some Hilbert space $\mathbb{H}$ (compare (f)). This solves a problem of Stoyanov [27].

In section 4, introducing the class BR of birepresentable groups (Definition 4.7), we show that BR contains: LC groups, subgroups of $GL(E)$ (where E is a normed space and $GL(E)$ is the group of all linear topological automorphisms endowed with the uniform operator topology), additive subgroups of locally convex vector spaces and free Abelian topological groups. The last result enables us to represent every Abelian topological group as a quotient of a perfectly minimal group. This gives a positive answer (in the realm of Abelian groups) to the following question of Arkhangel'skii ([4, Problem VI.6]): Is every Hausdorff group a quotient of a minimal group?

Using the same technique, we prove that every closed subgroup G of $GL(n,\mathbb{R})$ is a group retract of a minimal Lie group of dimension $\dim G + 2n + 1$.

Some of our results were announced in [20, 21].

§1. Preliminaries.

Let (G, σ) and (X, τ) be topological groups. As usual, a representation $\alpha : G \to \operatorname{Aut} X$ is called continuous if the corresponding action $\tilde{\alpha} : G \times X \to X, \tilde{\alpha}(g, x) = \alpha(g)(x)$ is continuous. In this situation, we say that X is a topological G-group. It is well-known that the semidirect product $X \leftthreetimes_\alpha G$ is a topological group with respect to the product topology $\tau \times \sigma$ iff α is continuous. In the sequel we will often write e (the neutral element) instead of e_G, e_X or (e_X, e_G). We identify G with the subgroup $\{e\} \times G$, as well as X with the normal subgroup $X \times \{e\}$. Clearly, the canonical projection $pr : X \leftthreetimes_\alpha G \to G$ is a group retraction and $\ker pr = X$. For further details on semidirect products see [24, Ch. 6].

The filter of all neighborhoods of an element x of a topological group (G, σ) will be denoted by $N_x(G, \sigma)$, the zero element of an Abelian group X by o_X. Sometimes we write $\prod X_i$ instead of $\prod_{i \in I} X_i$. If P is a subgroup of G then $\sigma|_P$ will denote the relative topology on P induced by σ and σ/P will be the quotient topology on the left coset space G/P. The following result is standard.

Lemma 1.1. *([11], [12, Lemma 7.2.3]). Let P be a subgroup of a group G and let σ_1 and σ_2 be (not necessarily Hausdorff) group topologies on G with $\sigma_1 \subset \sigma_2$, $\sigma_1|_X = \sigma_2|_X$ and $\sigma_1/X = \sigma_2/X$. Then $\sigma_1 = \sigma_2$.*

Let E, F, A be Abelian groups. As usual, a mapping $\omega : E \times F \to A$ is called *biadditive* if the induced mappings $\omega_x : F \to A$, $\omega_f : E \to A$ for every $x \in E$ and $f \in F$ are homomorphisms. Recall that ω is called *separated* if for every pair (x_0, f_0) of non-zero elements there exists a pair (x, f) such that $f(x_0) \neq o_A$ and $f_0(x) \neq o_A$, where $f(x) = \omega(x, f)$. Note that in [16, Ch. X] a biadditive mapping is called "bilinear function."

Definition 1.2. Let $(E, \sigma), (F, \tau), A$ be Abelian Hausdorff groups. A continuous separated biadditive mapping $\omega : (E, \sigma) \times (F, \tau) \to A$ will be called *minimal* if for every coarser pair (σ_1, τ_1) of Hausdorff group topologies $\sigma_1 \subset \sigma$, $\tau_1 \subset \tau$ such that $\omega : (E, \sigma_1) \times (F, \tau_1) \to A$ is continuous (in such cases, we will say that (σ_1, τ_1) is ω-compatible), it follows that $\sigma_1 = \sigma$ and $\tau_1 = \tau$.

Remark 1.3. Since ω is separated and A is Hausdorff, every ω-compatible pair (σ_1, τ_1) automatically is a pair of Hausdorff topologies. The contents of Definition 1.2 will not be changed if we omit the word "separated" provided that σ_1, τ_1, are not necessarily Hausdorff.

Example 1.4. Let A be a Hausdorff topological ring with a unit. Then the multiplication defines a minimal biadditive mapping $\omega : (A, +) \times (A, +) \to (A, +)$, where $(A, +)$ is the additive group of A.

The following Lemma is elementary.

Lemma 1.5. *Let $\omega : E \times F \to A$ be a continuous biadditive mapping, and P be a (not necessarily closed) subgroup of E. Then the natural mapping $\omega/P : E/P \times P^{\perp} \to A$ is continuous provided that $P^{\perp} = \{f \in F : f(x) = o_A \quad \forall x \in P\}$.*

Lemma 1.6. *Let $\omega_i : E_i \times F_i \to A_i$ be a minimal biadditive mapping for every $i \in I$. Then the natural biadditive mapping $\prod \omega_i : \prod E_i \times \prod F_i \to \prod A_i$, defined coordinatewise, is minimal.*

Proof. Denote by σ_i and τ_i the given topologies of E_i and F_i respectively. Let $\sigma' \subset \sigma = \prod \sigma_i$ and $\tau' \subset \tau = \prod \tau_i$ be new coarser group topologies on $\prod E_i$ and $\prod F_i$ such that (σ', τ') is $\prod \omega_i$-compatible. In order to check the equalities $\sigma' = \sigma, \tau' = \tau$, it suffices to show that for every $k \in I$ the projections

$$p_k : \left(\prod E_i, \sigma'\right) \to (E_k, \sigma_k), \quad q_k : \left(\prod F_i, \tau'\right) \to (F_k, \tau_k)$$

are continuous. Clearly, $\ker p_k = \prod\{E_i : i \neq k\}$. Set $P_k = \ker p_k$. Since every ω_i is separated (Definition 1.2), we have $P_k^{\perp} = F_k$. By Lemma 1.5 (with $\omega := \prod \omega_i$) we obtain that $\omega_k : (E_k, \sigma'/P_k) \times (F_k, \tau_k) \to A_k$ is continuous. The minimality of ω_k implies $\sigma'/P_k = \sigma_k$. So every projection p_k is σ'-continuous. This means that $\sigma' = \sigma$.

Analogously, using the dual version of Lemma 1.5, we obtain $\tau' = \tau$. $\square$

Lemma 1.7. *Let $(E, \sigma), (F, \tau), A$ be Abelian Hausdorff groups and let $\omega : (E, \sigma) \times (F, \tau) \to A$ be a continuous biadditive mapping. Denote by $\overline{\omega} : (\overline{E}, \overline{\sigma}) \times (\overline{F}, \overline{\tau}) \to \overline{A}$ the corresponding completion* [8, III, §6.5]*. Then $\overline{\omega}$ is minimal if and only if ω is minimal.*

Proof. Sufficiency: Let ω be minimal and (ν, μ) be a $\overline{\omega}$-compatible pair. Since (ν, μ) is coarser than $(\overline{\sigma}, \overline{\tau})$, the minimality of ω implies $\nu|_E = \overline{\sigma}|_E = \sigma$, $\mu|_F = \overline{\tau}|_F = \tau$. Clearly, E is ν-dense and $\overline{\sigma}$-dense in $\overline{E}$. Therefore, ν/E and $\overline{\sigma}/E$ are trivial topologies; in particular $\nu/E = \overline{\sigma}/E$. Using Lemma 1.1 we get $\nu = \overline{\sigma}$. Similarly, it can be proved that $\mu = \overline{\tau}$.

Necessity: If $\overline{\omega}$ is minimal, then according to Definition 1.2, $\overline{\omega}$ is separated. Since E and F are dense subgroups, then ω is separated too. Let (σ_1, τ_1) be a ω-compatible pair of Hausdorff group topologies. Consider the corresponding completion $\hat{\omega} : (\hat{E}, \hat{\sigma}_1) \times (\hat{F}, \hat{\tau}_1) \to \overline{A}$ and the natural mappings $p : (\overline{E}, \overline{\sigma}) \to (\hat{E}, \hat{\sigma}_1), q : (\overline{F}, \overline{\tau}) \to (\hat{F}, \hat{\tau}_1)$, where $p|_E = \mathrm{Id}_E$ and $q|_F = \mathrm{Id}_F$. Since E and F are dense and all groups are Hausdorff then the equality $\hat{\omega}(p(x), q(f)) = \overline{\omega}(x, f)$ holds for every $(x, f) \in (\overline{E}, \overline{F})$. Since $\overline{\omega}$ is separated, p and q are injective. Then $(p^{-1}(\hat{\sigma}_1), q^{-1}(\hat{\tau}_1))$ is a $\overline{\omega}$-compatible pair of Hausdorff group topologies. The minimality of $\overline{\omega}$ implies $p^{-1}(\hat{\sigma}_1) = \overline{\sigma}$, $q^{-1}(\hat{\tau}_1) = \overline{\tau}$, which yields $\sigma_1 = \sigma, \tau_1 = \tau$. $\square$

Definition 1.8. Let (G, τ) be a topological group and $\psi = \{\psi_i : G \to \operatorname{Aut} X_i\}_{i \in I}$ be a system of representations in Hausdorff groups X_i. We say that ψ is *topologically exact* (or shortly: *t-exact*) if ψ is algebraically exact (i.e., $\cap \ker \psi_i = \{e\}$), τ is ψ-compatible (i.e., each $\tilde{\psi}_i : (G, \tau) \times X_i \to X_i$ is continuous) and every strictly coarser group topology τ' is not ψ-compatible. ψ will be called *hereditarily t-exact (ht-exact)* if for every topological subgroup P of G the system $\{\psi_i|_P\}_{i \in I}$ is t-exact.

Remark 1.9.

(i) For a normed space $(E, \|\ \|)$ denote by $Is(E)$ the group of all linear isometries of E onto itself endowed with the strong operator topology. Then the natural action $Is(E) \times E \to E$ defines an ht-exact representation.

(ii) Let (G, τ) be an LC group and let $\operatorname{Aut} G$ be the group of all topological automorphisms of G endowed with the Birkhoff topology [19, §26]. Denote this topology by τ_B and recall that τ_B has a local base at the identity formed by the sets:

$$\mathcal{B}(C, U) = \{\varphi \in \operatorname{Aut} G : \varphi(c) \in Uc \quad \text{and} \quad \varphi^{-1}(c) \in Uc \quad \forall c \in C\},$$

where C runs over the compact subsets of G and U runs over the neighborhoods of e in G. Then the natural action $\alpha : \operatorname{Aut} G \times G \to G$ defines an ht-exact representation. Moreover, every $\alpha|_{P \times G}$-compatible group topology σ on a subgroup P of $\operatorname{Aut} G$ is finer than $\tau_B|_P$.

Indeed, fix a pair (C, U). Choose a neighborhood $U_1 \in N_e(G, \tau)$ such that $U_1 U_1^{-1} \subset U$. Since $\alpha|_{P \times G} : (P, \sigma) \times (G, \tau) \to (G, \tau)$ is continuous, for every $c \in C$ there exist $O(c) \in N_c(G, \tau), V_c \in N_e(P, \sigma)$ such that $V_c^{-1} = V_c$ and $\varphi(x) \in U_1 c$ for every $x \in O(c)$ and $\varphi \in V_c$; in particular, $O(c) \subset U_1 c$. Since C is compact there exists a finite subset $\{c_1, c_2, \ldots, c_n\}$ of C such that $C \subset \bigcup\limits_{i=1}^{n} O(c_i)$. Now, if we put $V = \bigcap\limits_{i=1}^{n} V_{c_i}$, then $\varphi(c) \in$

$U_1U_1^{-1}c \subset Uc$ for every $c \in C$ and $\varphi \in V = V^{-1}$. Therefore, $V \subset \mathcal{B}(C,U) \cap P$. This means that $\tau_B|_P \subset \sigma$.

Proposition 1.10. *For every LCA group G the canonical biadditive mapping $\Delta : G^* \times G \to \mathbb{T}$ is minimal.*

Proof. Let τ^* and τ be the given (compact-open) topologies of G^* and G and let (τ_1^*, τ_1) be any Δ-compatible pair. For a τ-compact subset C of G and a number $\varepsilon > 0$ denote by $[C, \varepsilon]$ the set $\{\chi \in G^* : \chi(C) \subset O_\varepsilon\}$, where O_ε is the ε-neighborhood of the zero in $\mathbb{T}$. Since C is τ_1-compact there exists $U \in \tau_1^*$ such that $e \in U$ and $\chi(C) \subset O_\varepsilon$ for every $\chi \in U$. Therefore, $e \in U \subset [C, \varepsilon]$, which yields $\tau_1^* = \tau^*$.

Using duality, we get $\tau_1 = \tau$. $\square$

Recall that a subset S of a group G is called unconditionally closed (u.c.) if S is closed for every Hausdorff group topology on G. The centralizer $\mathrm{cen}(g)$ of any element $g \in G$ and the center $Z(G)$ of G are u.c. in G (see [24, Lemma 3.11]).

Lemma 1.11. *Let X and Y be groups. Then $X \times Z(Y)$ and $Z(X) \times Y$ are u.c. in $X \times Y$.*

Proof. Observe that $X \times Z(Y) = \cap\{\mathrm{cen}(e, y) : y \in Y\}$ and $Z(X) \times Y = \cap\{\mathrm{cen}(x, e) : x \in X\}$. $\square$

Now we need the following known result.

Proposition 1.12. [12, Proposition 7.2.5] *Let G be a minimal group, and let H be a closed subgroup of the center $Z(G)$ of G. Then H is minimal.*

Proposition 1.13. *Let X, Y be minimal groups. Then $X \times Y$ is minimal if and only if $Z(X) \times Z(Y)$ is minimal.*

Proof. Necessity: If $X \times Y$ is minimal then, by Proposition 1.12, $Z(X \times Y)$ (which coincides with $Z(X) \times Z(Y)$) is minimal.

Sufficiency: Suppose that $Z(X) \times Z(Y)$ is minimal and $\gamma_1 \subset \gamma$ be a coarser Hausdorff group topology on $X \times Y$. By Lemma 1.11 the subgroup $Z(X) \times Z(Y)$ is u.c. in $X \times Z(Y)$. The minimality of $Z(X) \times Z(Y)$ implies that $Z(Y)$ is γ_1-closed in $Z(X) \times Z(Y)$. Therefore, $Z(Y)$ is γ_1-closed in $X \times Z(Y)$. Since X is minimal, using Lemma 1.1 for the groups $X \times Z(Y) \supset Z(Y)$, we get $\gamma_1|_{X \times Z(Y)} = \gamma|_{X \times Z(Y)}$. In particular, X is γ_1-closed in $X \times Z(Y)$. On the other hand, $X \times Z(Y)$ is u.c. in $X \times Y$ (Lemma 1.11). Hence, X is γ_1-closed in $X \times Y$. Using the minimality of Y and Lemma 1.1, we conclude $\gamma_1 = \gamma$. □

Theorem 1.14. *A minimal topological group X is perfectly minimal iff $Z(X)$ is perfectly minimal.*

Proof. Suppose that $Z(X)$ is perfectly minimal. For every minimal Y the center $Z(Y)$ is minimal (Proposition 1.12). Therefore, $Z(X) \times Z(Y)$ is minimal. From Proposition 1.13 it follows that $X \times Y$ is minimal. So, X is perfectly minimal.

Conversely, let X be perfectly minimal. Then $X \times (\mathbb{Z}, \tau_p)$ is minimal for every prime p (see Introduction (c)). By Proposition 1.12 the groups $Z(X) \times (\mathbb{Z}, \tau_p)$ and $Z(X)$ are minimal. Hence, $Z(X)$ is precompact by the Prodanov-Stoyanov result (cf. Introduction). Now, Theorem 6.1.8 from [12] shows that $Z(X)$ is perfectly minimal. □

Theorem 1.15. *The arbitrary product $(\prod X_i, \sigma = \prod \sigma_i)$ of minimal groups (X_i, σ_i) with trivial center is perfectly minimal.*

Proof. By Theorem 1.14 it is sufficient to show that $X = \prod X_i$ is minimal. Since $Z(X_k) = \{e\}$ then the kernel $\ker q_k = \prod\{X_i : i \neq k\}$ of any projection q_k is u.c. in X (Lemma 1.11). Now, using the minimality of X_k we obtain that q_k is (ν, σ_k)- continuous for every Hausdorff coarser group topology $\nu \subset \sigma$. Thus, $\nu = \sigma$. □

The last result gives an analog of Theorem 7.3.9(c) from [12] for minimal groups and, consequently, partially solves a problem posed in [12, p. 235].

§2. Induced groups of biadditive mappings.

Let E, F and A be Abelian topological groups and $\omega : E \times F \to A$ be a biadditive mapping. For all $(f, a, x) \in F \times A \times E$ let $\omega^{\nabla}(f, (a, x)) = (a + f(x), x)$, where

$f(x) = \omega(x, f)$. Since ω is biadditive, ω^∇ defines an action of F on the product $A \times E$. Moreover, every transition under this action is an automorphism. Denote the corresponding semidirect product $A \times E \ \lambda_{\omega^\nabla} F$ by $M(\omega)$. We call this group the *induced group* of ω.

Lemma 2.1.

(i) If $u = (a, x, f),\ v = (b, y, \varphi) \in M(\omega)$, *then:*

(a) $uv = (a + b + f(y), x + y, f + \varphi)$;

(b) $u^{-1} = (f(x) - a, -x, -f)$;

(c) $uvu^{-1} = (f(y) - \varphi(x) + b, y, \varphi)$;

(ii) If ω *is separated then* $Z(M(\omega)) = A$.

Proof (ii). Let $u = (a, x, f) \in Z(M(\omega))$ and $v = (o_A, o_E, \varphi)$. Then the equality $uvu^{-1} = v$ implies $\varphi(x) = o_A$ for every $\varphi \in F$. Since ω is separated, we conclude that $x = o_E$. Similarly, considering $v = (o_A, y, o_F)$, we get $f = o_F$. On the other hand, clearly $(a, o_E, o_F) \in Z(M(\omega))$ for every $a \in A$. $\square$

Lemma 2.2.

(i) The action $\omega^\nabla : F \times (A \times E) \to A \times E$ *is continuous iff* ω *is continuous.*

(ii) If ω *is minimal then the induced representation defined by* ω^∇ *is* t*-exact.*

Proof.

(i) It easily follows from the identity $\omega^\nabla(f, (a, x)) - (a, x) = (f(x), o_E)$.

(ii) Let $\tau_1 \subset \tau$ be a ω^∇-compatible group topology on F. By (i) the mapping $\omega : E \times (F, \tau_1) \to A$ is continuous. The minimality of ω implies $\tau_1 = \tau$. $\square$

Proposition 2.3. *For every LCA group* G *the mappings*

$$i_1 : \ G \to \mathrm{Aut}(\mathbb{T} \times G^*) \quad , \quad i_1(g)(t, \chi) = (t + \chi(g), \chi),$$

$$i_2 : \ G^* \to \mathrm{Aut}(\mathbb{T} \times G) \quad , \quad i_2(\chi)(t, g) = (t + \chi(g), g)$$

are topological group embeddings.

Proof. By Proposition 1.10, $\Delta : G^* \times G \to \mathbb{T}$ is minimal. Combining Lemma 2.2(ii) and Remark 1.9(ii) we obtain that i_1 is a topological embedding. Using duality we get our assertion for i_2. $\square$

Definition 2.4. Let $q : X \to Y$ be a (not necessarily group) retraction of a group X on a subgroup Y. We say that q is *central* if $q(xyx^{-1}) = y$ for each $(x, y) \in X \times Y$.

Lemma 2.5. *For every biadditive mapping $\omega : E \times F \to A$ the natural projections $q_E : M(\omega) \to E, q_F : M(\omega) \to F$ are central.*

Proof. If $u = (a, x, f) \in M(\omega), y \in E$ and $\varphi \in F$ then by Lemma 2.1(c) holds $uyu^{-1} = (f(y), y, o_F), u\varphi u^{-1} = (-\varphi(x), o_E, \varphi)$. $\square$

Proposition 2.6. *Let (M, γ) be a topological group such that M is algebraically a semidirect product $M = X\lambda_\alpha G$. If $q : X \to Y$ is a continuous central retraction of the topological subgroup X on a topological G-subgroup Y of X, then the action*

$$\alpha|_{G\times Y} : (G, \gamma/X) \times (Y, \gamma|_Y) \to (Y, \gamma|_Y) \tag{1}$$

is continuous.

Proof. Clearly, each g-transition $(Y, \gamma|_Y) \to (Y, \gamma|_Y)$ is continuous. We have to show that $\alpha|_{G\times Y}$ in (1) is continuous at (e, y) for every $y \in Y$. Fix an arbitrary $y \in Y$ and a neighborhood $O(y) \in N_y(Y, \gamma|_Y)$. Since $q : (X, \gamma|_X) \to (Y, \gamma|_Y)$ is continuous, there exists $U_1 \in N_e(M, \gamma)$ such that

$$q(U_1 \cap X) \subset O. \tag{2}$$

Choose $V, U_2 \in N_e(M, \gamma)$ with the property

$$vU_2v^{-1} \subset U_1 \quad \forall v \in V. \tag{3}$$

Now, if $pr : M \to G$ denotes the canonical projection, then

$$\alpha(g, z) \in O \quad \forall g \in pr(V), \quad \forall z \in U_2 \cap Y,$$

Indeed, if $v=(x,g)\in V$ and $z\in U_2\cap Y$ then by (3), $vzv^{-1}\in U_1$. ¿From the normality of X we have $vzv^{-1}\in X$. Thus, $vzv^{-1}\in U_1\cap X$. Using (2) and the identity $vzv^{-1}=x\alpha(g,z)x^{-1}$, we get $q(x\alpha(g,z)x^{-1})\in O$. Since q is central and $\alpha(g,z)\in Y$, we obtain $\alpha(g,z)\in O$. Finally, observe that $pr(V)\in N_e(G,\gamma/X)$ and $U_2\cap Y\in N_y(Y,\gamma\,|_Y)$. $\square$

Proposition 2.7. *Let $(X\lambda_\alpha G,\gamma)=M$ be a topological semidirect product and $\{Y_i\}_{i\in I}$ be a system of G-subgroups in X such that the system $\{\alpha|_{G\times Y_i}: G\times Y_i\to Y_i\}_{i\in I}$ is t-exact and for each $i\in I$ there exists a continuous central retraction $q_i: X\to Y_i$. Suppose that $\gamma_1\subset\gamma$ is a coarser group topology on M such that $\gamma_1\,|_X=\gamma\,|_X$. Then $\gamma_1=\gamma$.*

Proof. Proposition 2.6 shows that each action $\alpha|_{G\times Y_i}:(G,\gamma_1/X)\times Y_i\to Y_i \qquad (i\in I)$ is continuous. By Definition 1.8, γ_1/X coincides with the given topology of G. Therefore, $\gamma_1/X=\gamma/X$. Now, Lemma 1.1 completes the proof. $\square$

Corollary 2.8. *Let $(X\lambda_\alpha G,\gamma)$ be a topological semidirect product and let α be t-exact. Suppose that X is Abelian and $\gamma_1\subset\gamma$ is a group topology which agrees with γ on X. Then $\gamma_1=\gamma$.*

Proof. Since X is Abelian, the identity mapping $\mathrm{Id}_X: X\to X$ is central. $\square$

The commutativity of X is essential. It was shown in ([14, Example 10]) that there exists a totally minimal precompact torsion group X such that a certain semidirect product $X\lambda G$ is not minimal, where G is the discrete cyclic group of order 2.

Proposition 2.9. *Let $\omega:(E,\sigma)\times(F,\tau)\to A$ be a minimal biadditive mapping and $(M(\omega),\gamma)=(A\times E\ \lambda_{\omega^\triangledown}F,\gamma)$ be the induced group. If $\gamma_1\subset\gamma$ is a group topology such that $\gamma_1\,|_A=\gamma\,|_A$, then $\gamma_1=\gamma$.*

Proof. By Lemma 1.1 it suffices to show that $\gamma_1/A=\gamma/A$. Therefore, we have to check the continuity of the projection

$$q_{E\times F}:(M(\omega),\gamma_1)\to(E\times F,\sigma\times\tau). \tag{4}$$

By Proposition 2.6 (with $X:=A\times E,\ q:=\mathrm{Id}_X,\ G:=F$) the action

$$\omega^\triangledown:(F,\gamma_1/A\times E)\times(A\times E,\gamma_1\,|_{A\times E})\to(A\times E,\gamma_1\,|_{A\times E})$$

is continuous. Therefore, by Lemma 2.2 (i) the biadditive mapping

$$\omega : (E, \gamma_1 |_E) \times (F, \gamma_1 / A \times E) \to (A, \gamma_1 |_A)$$

is continuous. From the equality $\gamma_1 |_A = \gamma |_A$ and the minimality of ω, it follows that the topology $\gamma_1 / A \times E$ on F coincides with the given topology τ. Thus, the projection $q_F : (M(\omega), \gamma_1) \to (F, \tau)$ is continuous. Now, observe that $A \times F$ (which is identified with $A \times \{e\} \times F$) is a normal subgroup of M and $(A \times F) \cap E = \{e\}$. This means (see [24, Definition 6.10]) that $M(\omega)$ is algebraically a semidirect product $A \times F \ \lambda \ E$. Moreover, the corresponding action (by means of the inner automorphisms) of E on $A \times F$ is

$$\omega_*^{\nabla} : E \times A \times F \to A \times F, \quad \omega_*^{\nabla}(x, (a, f)) = (a + \omega_*(f, x), f),$$

where $\omega_* : F \times E \to A$ is a biadditive mapping defined by $w_*(f, x) = -\omega(x, f)$. Clearly, ω_* is minimal iff ω is minimal. Therefore, as in the case of q_F, using Proposition 2.6 (with $X := A \times F,\ q := \mathrm{Id}_X,\ G := E$) and Lemma 2.2(i) (for ω_*), we can establish the continuity of $q_E : (M(\omega), \gamma_1) \to (E, \sigma)$. Thus, $q_{E \times F}$ from (4) is continuous. $\square$

Theorem 2.10. *Let $\omega : E \times F \to A$ be a minimal biadditive mapping. Then the induced group $M(\omega)$ is (perfectly) minimal iff A is (perfectly) minimal.*

Proof. The "minimal case" follows from Proposition 1.12, Proposition 2.9 and Lemma 2.1(ii), for the "perfectly minimal case" we use, in addition, Theorem 1.14. $\square$

In the sequel, instead of $M(\Delta)$ we will write $M(G)$.

Theorem 2.11. *Let $\Delta : G^* \times G \to \mathbb{T}$ be the canonical biadditive mapping for an LCA group G. Then the induced group $M(G) = \mathbb{T} \times G^* \lambda \ G$ is minimal. In particular, every LCA group is a group retract of a minimal LC group.*

Proof. Apply Proposition 1.10 and Theorem 2.10. $\square$

Remus and Stoyanov [23] proved that every compactly generated LCA group is a quotient of a minimal LC group.

Theorem 2.12. *Every locally precompact Abelian group G is a group retract of a locally precompact perfectly minimal group.*

Proof. Consider the natural mapping $\Delta : (\overline{G})^* \times \overline{G} \to \mathbb{T}$, where $\overline{G}$ is the completion of G. By Proposition 1.10 and Lemma 1.7 the restricted mapping $\Delta|_{(\overline{G})^* \times G} : (\overline{G})^* \times G \to \mathbb{T}$ is minimal. Now, it follows from Theorem 2.10 that the induced group $\mathbb{T} \times (\overline{G})^* \lambda\, G$ is perfectly minimal. □

Question 2.13.

(i) Let $\underline{\mathcal{A}}$ be a certain class of topological groups and $\underline{\text{min}}$ denotes the class of all minimal groups. The general questions which naturally arise here are the following: Is it true that every $G \in \underline{\mathcal{A}}$ is a group retract of a group $M \in \underline{\mathcal{A}} \cap \underline{\text{min}}$? What happens if $\underline{\mathcal{A}}$ is the class of all locally compact (or Lie) groups?

(ii) Arkhangel'skii [4, Problem VI.6] posed the question if every Hausdorff group is a quotient of a minimal group. V. Uspenskii announced a positive answer (see [29], Theorem 3.3E.2).

In the sequel we give some partial results concerning Question 2.13. The following is one of them.

Theorem 2.14. *An arbitrary product of LCA groups is a group retract of a minimal group M which may be represented as a product of LC groups.*

The proof follows from Proposition 1.10 and the following generalization of Theorem 2.10.

Theorem 2.15. *Let for every $i \in I$, $\omega_i : E_i \times F_i \to A_i$ be a minimal biadditive mapping. Then $\prod M(\omega_i)$ is (perfectly) minimal iff $\prod A_i$ is (perfectly) minimal.*

Proof. The natural mapping

$$\prod M(\omega_i) \to M\left(\prod \omega_i\right), \quad (a_i, x_i, f_i)_{i\in I} \to ((a_i)_{i\in I}, (x_i)_{i\in I}, (f_i)_{i\in I})$$

is a topological isomorphism. Now, our assertion follows from Lemma 1.6 and Theorem 2.10 (for $\prod \omega_i$). □

§3. Bilinear forms and induced groups.

As usual, the dual space $(E^*, \| \ \|^*)$ of a real normed space $(E, \| \ \|)$ is the Banach space of all continuous linear functionals $f : E \to \mathbb{R}$ equipped with the norm $\| f \|^* = \sup\{|f(x)| : \| x \| \leq 1\}$.

Definition 3.1. Let E, F be normed spaces. We say that a continuous bilinear form $\omega : E \times F \to \mathbb{R}$ is a *strong duality* if for every norm-unbounded sequences $(x_n) \subset E$ and $(f_n) \subset F$ the subsets $S_1 = \{f(x_n) : \| f \| \leq 1, n \in \mathbb{N}\}$, $S_2 = \{f_n(x) : n \in \mathbb{N}, \| x \| \leq 1\}$ are unbounded in $\mathbb{R}$.

Remark 3.2. Considering in Definition 3.1 the sequences $x_n = nx_0$ and $f_n = nf_0$, where $(x_0, f_0) \in E \times F$, we obtain that every strong duality is a separated biadditive mapping.

Lemma 3.3. *For every normed space E the canonical bilinear form $\langle \ , \ \rangle : E \times E^* \to \mathbb{R}$ defined by $\langle x, f \rangle = f(x)$, is a strong duality.*

Proof. Suppose $(x_n) \subset E$ and $(f_n) \subset E^*$ are norm-unbounded. For every x_n there exists $\varphi_n \in E^*$ such that $\| \varphi_n \|^* = 1$ and $\varphi_n(x_n) = \| x_n \|$ (cf. [6, Theorem (40.10)]. Hence, S_1 (from Definition 3.1) is unbounded. The unboundedness of S_2 directly follows from the definition of $\| \ \|^*$. $\square$

Proposition 3.4. *Let G be an LC group and μ be the standard Haar measure on G. Let $(L^1(G), \| \ \|_1)$ be the Banach space of all equivalence classes of 1-integrable real functions on G, and denote by $(\mathcal{K}(G), \| \ \|)$ the normed space of all continuous real functions with compact supports endowed with the* sup*-norm. Then the natural bilinear form $\omega : L^1(G) \times \mathcal{K}(G) \to \mathbb{R}, \omega(f, \varphi) = \int_G f\varphi d\mu$ is a strong duality.*

Proof. Considering for every $\varphi_0 \in \mathcal{K}(G)$ the corresponding functional $\hat{\varphi}_0 : L^1(G) \to \mathbb{R}$, $\hat{\varphi}_o(f) = \int_G f\varphi_0 d\mu$ we obtain an embedding of $\mathcal{K}(G)$ into the normed space $(L^1(G))^* = L^\infty(G)$ (see the remark in [8, IV, §6, 3]). Therefore, the norm $\| \varphi_o \|$ is equal to the number

$$\| \varphi_0 \| = \sup\{\omega(f, \varphi_0) : f \in L^1(G), \| f \|_1 \leq 1\}. \tag{1}$$

On the other hand, by [19, Theorem (12.13)], we have

$$\| f_o \|_1 = \sup\{\omega(f_o, \varphi) : \varphi \in \mathcal{K}(G), \| \varphi \| \leq 1\}. \tag{2}$$

Clearly, (1) and (2) imply that ω is a strong duality. $\square$

Lemma 3.5. *Let $(E, \| \ \|)$ be a normed space and let $X \subset E$ be an additive topological subgroup. Suppose that τ is a strictly coarser group topology on X. Then every τ-neighborhood $U \in N_0(X, \tau)$ of the zero in X is norm-unbounded.*

Proof. Since τ is strictly coarser, there exists $\varepsilon_0 > 0$ such that every $O \in N_0(X, \tau)$ contains an element $x \in O$ with $\| x \| \geq \varepsilon_0$. Fix $U \in N_0(X, \tau)$. Since τ is a group topology, for each natural n there exists $V_n \in N_0(X, \tau)$ such that $nV_n \subset U$. Choose $x_n \in V_n$ with the property $\| x_n \| \geq \varepsilon_0$. Then $\| nx_n \| \geq n\varepsilon_0$. Since $nx_n \in U$, this means that U is norm-unbounded. $\square$

Proposition 3.6. *Every strong duality is a minimal biadditive mapping.*

Proof. Apply Lemma 3.5. $\square$

Let E, F be normed spaces and $\omega : E \times F \to \mathbb{R}$ be a continuous bilinear form. Consider the induced group $M(\omega) = A \times E \leftthreetimes_{\omega^\triangledown} F$ of the biadditive mapping ω and define the following action:

$$\pi : \mathbb{R}_+ \times M(\omega) \to M(\omega), \qquad \pi(t, (a, x, f)) = (ta, tx, f).$$

Clearly, π defines a continuous representation. Therefore, the topological semidirect product $M(\omega) \leftthreetimes_\pi \mathbb{R}_+$ is well-defined. Denote this group by $M_+(\omega)$ and call it the *induced group* of the bilinear form ω. In the case of the canonical bilinear form $\langle \ , \ \rangle : E \times E^* \to \mathbb{R}$ we will use the notation $M_+(E)$ instead of $M_+(\langle \ , \ \rangle)$.

The proof of the following result is straightforward.

Proposition 3.7. *Let E be a normed space. Then the following mappings:*

$$i_1 : E^* \times \mathbb{R}_+ \to GL(\mathbb{R} \times E) \quad , \quad i_1(f, t)(a, x) = (ta + tf(x), tx),$$

$$i_2 : E \times \mathbb{R}_+ \to GL(\mathbb{R} \times E^*) \quad , \quad i_2(x, t)(a, f) = (ta + tf(x), tf)$$

are topological group embeddings.

Lemma 3.8.

(i) *If* $u = (a, x, f, t),\ v = (b, y, \varphi, s) \in M_+(\omega)$ *then*

$$uvu^{-1} = (a - sa + tf(y) - s\varphi(x) + tb,\ x - sx + ty, \varphi, s),$$

(ii) $Z(M_+(\omega)) = \{e\}$, *for every separated bilinear form* ω.

Proof (ii). By (i), it is clear that $Z(M_+(\omega)) \subset Z(M(\omega))$. According to Lemma 2.1(ii), $Z(M(\omega)) = \mathbb{R}$. Therefore, every element from $Z(M_+(\omega))$ has the form $v = (b, o_E, o_F, 1)$. The equality $uvu^{-1} = v$ for $u = (o, o_E, o_F, 2)$ implies $2b = b$. Thus, $b = o$. $\square$

Theorem 3.9. *Let E, F be normed spaces and let $\omega : E \times F \to \mathbb{R}$ be a bilinear form. If ω is minimal as a biadditive mapping (in particular if ω is a strong duality), then the induced group $M_+(\omega)$ is perfectly minimal.*

Proof. By Theorem 1.14 and Lemma 3.8(ii) it suffices to verify the minimality of $M_+(\omega)$. Denote by γ the given topology of $M_+(\omega)$ and suppose that $\gamma_1 \subset \gamma$ is a Hausdorff group topology. The minimal group $\mathbb{R} \leftthreetimes \mathbb{R}_+$ (Introduction (d)) is naturally embedded in $M_+(\omega)$. Therefore, $\gamma_1|_{\mathbb{R}} = \gamma|_{\mathbb{R}}$. From Proposition 2.9 immediately follows $\gamma_1|_{M(\omega)} = \gamma|_{M(\omega)}$. Now observe that the natural retraction $q : M(\omega) \to \mathbb{R}$ is central (moreover, by Lemma 2.1(ii), $\mathbb{R} = Z(M(\omega)))$ and the action of $\mathbb{R}_+$ on $\mathbb{R}$ is t-exact. By Proposition 2.7 (in the situation: $G := \mathbb{R}_+,\ X := M(\omega),\ Y := \mathbb{R}$) we get $\gamma_1 = \gamma$. $\square$

Theorem 3.10. *For every normed space E the induced group $M_+(E) = (\mathbb{R} \times E \leftthreetimes E^*) \leftthreetimes_\pi \mathbb{R}_+$ of the canonical duality $E \times E^* \to \mathbb{R}$ is perfectly minimal.*

Proof. This is immediate from Lemma 3.3 and Theorem 3.9. $\square$

In the sequel the left uniformity of a group G is denoted by $\mathcal{U}_\ell(G)$.

Corollary 3.11. *Every (metrizable) uniform space X is uniformly embedded as a closed $\mathcal{U}_\ell(M)$-uniform subspace into a (metrizable) perfectly minimal group M.*

Proof. Every metric space is isometric to a closed subspace of a normed space (Arens - Eells [3]). In the general case we consider a closed uniform embedding $X \to \prod E_i$ into a product of normed spaces and apply Theorem 1.15. $\square$

Recall that the dual space of the Banach space c_0 is ℓ_1. Consider the induced group $M(c_0) = (\mathbb{R} \times c_0 \ \lambda \ \ell_1)\lambda \ \mathbb{R}_+$. By a result of Aharoni [1] every separable metrizable uniform space is uniformly embedded in c_0. Thus, we obtain the following:

Theorem 3.12. *Every separable metrizable uniform space is uniformly embedded into the perfectly minimal separable metrizable group $M(c_0)$.*

Now we give a counterexample to a conjecture of Stoyanov ([27], [12, p. 263]).

Counterexample 3.13. The (perfectly) minimal group $M(c_0)$ is not topologically isomorphic to a subgroup of the unitary group $U(\mathbb{H})$ of a Hilbert space $\mathbb{H}$.

Proof. In [15] Enflo constructs a countable metric space which is not uniformly embeddable into a Hilbert space. Hence, by Theorem 3.12, $M(c_0)$ is not uniformly embedded in a Hilbert space. Therefore, it suffices to show that every topological subgroup G of $U(\mathbb{H})$, such that G has countable weight, is $\mathcal{U}_\ell(G)$-embedded in ℓ_2. Suppose that G is such a group. Since the left uniformity on $U(\mathbb{H})$ is induced by the strong operator topology (see [12, p. 246]), the system $\{\alpha_x : G \to \mathbb{H}\}_{x\in\mathbb{H}}$ of all orbit mappings $(\alpha_x(g) = gx)$ induces $\mathcal{U}_\ell(G)$. Since G has countable weight, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of $\mathbb{H}$ such that $\{\alpha_{x_n}\}_{n\in\mathbb{N}}$ generates $\mathcal{U}_\ell(G)$. From this fact it follows that $(G,\mathcal{U}_\ell(G))$ is uniformly embedded in the uniform space $\mathbb{H}^{\aleph_0}$. For every natural n the Hilbert subspace $\langle Gx_n\rangle$ generated by the orbit of x_n is separable. Thus, without loss of generality, we can assume that $\mathbb{H}$ is separable. Hence, in order to complete our proof it suffices to establish that $\ell_2^{\aleph_0}$ is uniformly embedded in ℓ_2. By a result of Aharoni [2] ℓ_2 is uniformly embedded into a bounded subset of itself. Hence, $\ell_2^{\aleph_0}$ is uniformly embedded into the product $\prod\{B_{2^{-n}} : n \in \mathbb{N}\}$ of 2^{-n}-balls. By definition, this product is a uniform subspace of the Hilbert sum $\bigoplus_{n\in\mathbb{N}}(\ell_2)_n$ being isomorphic to ℓ_2. $\square$

§4. Group representations in biadditive mappings.

Let G be a topological group and let $\omega : E \times F \to A$ be a biadditive mapping. A continuous *birepresentation* of G (or: *G-birepresentation*) in ω is a pair (α_1, α_2) of continuous actions $\alpha_1 : G \times E \to E$ and $\alpha_2 : G \times F \to F$ such that ω is G-invariant (i.e., $\omega(gx, gf) = \omega(x, f)$).

In the case of normed spaces E and F, a bilinear form $\omega : E \times F \to \mathbb{R}$ and linear actions α_1, α_2 we obtain the definition of a *linear birepresentation.*

We will say that a linear representation $\alpha : G \times E \to E$ in a normed space E is *bicontinuous* if the dual action $\alpha^* : G \times E^* \to E^*$, $\alpha^*(g, \varphi)(x) = \varphi(g^{-1}x)$ is continuous. In other words, a linear action is bicontinuous iff the induced canonical birepresentation is continuous. Not every continuous representation is bicontinuous even in the case of norm-invariant actions of a compact group.

Remark 4.1. Every continuous representation of a topological group G by unitary operators on a Hilbert space $\mathbb{H}$ is bicontinuous. It is also clear that for every normed space E the natural representation $GL(E) \times E \to E$ is bicontinuous.

Let $\psi = \{\psi_i\}_{i \in I}$ be a system of continuous G-birepresentations:

$$\psi_i = (\omega_i : E_i \times F_i \to A_i, \quad \alpha_{1i} : G \times E_i \to E_i, \quad \alpha_{2i} : G \times F_i \to F_i).$$

By the *induced group* $M(\psi)$ of a system ψ we mean the topological semidirect product $\prod M(\omega_i) \leftthreetimes_\pi G$, where the action

$$\pi : G \times \prod M(\omega_i) \to \prod M(\omega_i)$$

is defined coordinate-wise by means of the following system $\{\pi_i\}_{i \in I}$ of actions:

$$\pi_i : G \times M(\omega_i) \to M(\omega_i), \quad \pi_i(g, (a, x, f)) = (a, gx, gf),$$

where $gx = \alpha_{1i}(g, x), \quad gf = \alpha_{2i}(g, f)$.

Analogously, in the case of linear birepresentations, we define the *induced group* $M_+(\psi)$ as the semidirect product $\prod M_+(\omega_i) \leftthreetimes_\pi G$ considering

$$\pi_i : G \times M_+(\omega_i) \to M_+(\omega_i), \quad \pi_i(g, (a, x, f, t)) = (a, gx, gf, t).$$

Definition 4.2. A system $\psi = \{\psi_i\}_{i\in I}$ is called *(ht-exact) t-exact* if the system $\{\alpha_{ki} : k \in \{1,2\}, i \in I\}$ of actions is (ht-exact) t-exact in the sense of Definition 1.8.

Theorem 4.3. *Assume that $\psi = \{\psi_i\}_{i\in I}$ is a t-exact system of G-birepresentations such that every $\omega_i : E_i \times F_i \to A_i$ is a minimal biadditive mapping and $\prod A_i$ is (perfectly) minimal. Then $M(\psi)$ is (perfectly) minimal.*

Proof. By our construction, $M(\psi) = \prod M(\omega_i)\ \lambda_\pi\ G$. Let $\gamma_1 \subset \gamma$ be a coarser Hausdorff group topology, and let X denote the subgroup $\prod M(\omega_i)$. The minimality of X (Theorem 2.15) implies $\gamma_1|_X = \gamma|_X$. By Lemma 2.5, the retractions $M(\omega_i) \to E_i, M(\omega_i) \to F_i$ are central. Then the natural projections $X \to E_i, X \to F_i$ are central, too. Using Proposition 2.7, we obtain $\gamma_1 = \gamma$. This proves the minimality of $M(\psi)$.

In the "perfectly minimal case," due to Theorem 1.14, it remains to show that $Z(M(\psi)) = \prod A_i$. Let $u = ((a_i)_{i\in I}, (x_i)_{i\in I}, (f_i)_{i\in I}, g) \in Z(M(\psi))$ and $k \in I$. Now, if $y \in E_k \subset M(\psi)$ and $f \in F_k \subset M(\psi)$, then the equalities $uy = yu, uf = fu$ and the elementary computations (see Lemma 2.1(i)) show that $\alpha_{1k}(g,y) = y,\ \alpha_{2k}(g,f) = f$. By exactness of ψ, we conclude that $g = e$. This means that $Z(M(\psi)) \subset Z(X)$. It follows from Lemma 2.1(ii) that $Z(X) = \prod A_i$. Therefore, $Z(M(\psi)) \subset \prod A_i$. Since the action $\pi|_{G\times\prod A_i}$ is trivial, then $Z(M(\psi)) = \prod A_i$. $\square$

Theorem 4.4. *Let (G,τ) be an LCA group and P be a topological subgroup of* $(\mathrm{Aut}\, G, \tau_B)$, *where τ_B is the Birkhoff topology. Consider the action $\alpha\colon P \times G \to G,\ \alpha(\varphi, g) = \varphi(g)$ and the dual action $\alpha^*\colon P \times G^* \to G^*,\ \alpha^*(\varphi,\chi)(g) = \chi(\varphi^{-1}(g))$. Denote by ψ the P-birepresentation in $\Delta\colon G^* \times G \to \mathbb{T}$ defined by the pair (α^*, α). Then the induced group $M(\psi) = M(G)\ \lambda_\pi\ P$ is perfectly minimal.*

Proof. Follows from Remark 1.9(ii), Proposition 1.10 and Theorem 4.3. $\square$

Corollary 4.5. *Every closed subgroup G of $GL(n,\mathbb{R})$ is a group retract of a minimal Lie group M of dimension* $\dim G + 2n + 1$.

Proof. Using the selfduality $(\mathbb{R}^n)^* = \mathbb{R}^n$ we get from Theorem 4.4 that the group

$M = (\mathbb{T} \times \mathbb{R}^n \ \lambda_\alpha \ \mathbb{R}^n) \ \lambda_\pi \ G$ is minimal. In order to show that M is a Lie group, it suffices to check the analyticity of α and π. Let $\langle \ , \ \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the canonical scalar product and let $q \colon \mathbb{R} \to \mathbb{T}$ be the natural homomorphism. Then $\alpha(u,(t,v)) = (q(\langle u, v\rangle) + t, v)$. This description shows that α is analytic. The action π is defined by the rule $\pi(g,(t,v,u)) = (t, \nu(g,v), \nu(g,u))$, where ν is a restriction of the action $GL(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ to the submanifold $G \times \mathbb{R}^n$. Thus, π is analytic, too. $\square$

Theorem 4.6. *Let for every $i \in I$, E_i, F_i be normed spaces, $\omega_i : E_i \times F_i \to \mathbb{R}$ be strong dualities and $\psi = \{\psi_i\}_{i \in I}$ be a t-exact system of linear G-birepresentations. Then $M_+(\psi)$ is perfectly minimal.*

Proof. Let X_+ and X denote $\prod M_+(\omega_i)$ and $\prod M(\omega_i)$ respectively. Every $Z(M_+(\omega_i))$ is trivial (Lemma 3.8(ii)). Therefore, $Z(X_+) = \{e\}$. As in the proof of Theorem 4.3, elementary computations (using Lemma 3.8(i)) show that $Z(M_+(\psi)) \subset Z(X_+)$. Thus, $Z(M_+(\psi)) = \{e\}$. By Theorem 1.14 it suffices to check the minimality of $M_+(\psi)$. Let $\gamma_1 \subset \gamma$ be a coarser Hausdorff group topology. Since each $M_+(\omega_i)$ is minimal (Theorem 3.9), then Theorem 1.15 establishes the minimality of X_+. Therefore, $\gamma_1|_{X_+} = \gamma|_{X_+}$. In particular, $\gamma_1|_X = \gamma|_X$.

Now, as in the proof of Theorem 4.3, we use central retractions. It follows from Lemma 2.5 that the natural projections $X \to E_i, X \to F_i$ are central. Using Proposition 2.7 we obtain that γ_1 agrees with γ on $X \ \lambda \ G$. Arbitrary powers of the group $\mathbb{R} \ \lambda \ \mathbb{R}_+$ are minimal (see Introduction (d) or Theorem 1.15). Hence, the "product" action of $\prod(\mathbb{R}_+)_i$ on $\prod \mathbb{R}_i$ is t-exact. Taking this fact into account, we use Proposition 2.7 in the following situation: $G := \prod(\mathbb{R}_+)_i$, $X := \prod M(\omega_i) \ \lambda \ G$, $Y := \prod \mathbb{R}_i$ and $q : X \to Y$ is the natural projection. Hence we obtain $\gamma_1 = \gamma$. $\square$

Definition 4.7. We say that a topological group G is an (*HBR-group*) *BR-group* if there exists an (ht-exact) t-exact system $\{\psi_i\}_{i \in I}$ of linear G-birepresentations, such that every $\omega_i : E_i \times F_i \to \mathbb{R}$ is a strong duality.

Theorem 4.8. *For every BR-group G there exists a continuous group retraction $p: M \to G$ such that M and $\ker p$ are perfectly minimal.*

Proof. Take the perfectly minimal group $M := M_+(\psi)$ from Theorem 4.6 and consider the projection $p: M_+(\psi) \to G$. Then $\ker p$ coincides with $\prod M_+(\psi_i)$ which is perfectly minimal by Theorem 3.9, Lemma 3.8(ii) and Theorem 1.15. $\square$

Let $\omega: L^1(G) \times \mathcal{K}(G) \to \mathbb{R}$ be the bilinear form from Proposition 3.4. The pair of actions:

$$\alpha_1: G \times L^1(G) \to L^1(G), \qquad \alpha_1(g,f)(x) = f(g^{-1}x)$$
$$\alpha_2: G \times \mathcal{K}(G) \to \mathcal{K}(G), \qquad \alpha_2(g,\varphi)(x) = \varphi(g^{-1}x)$$

defines a continuous birepresentation in ω. Moreover, since α_2 induces a topological group embedding $G \to Is(\mathcal{K}(G))$, then this birepresentation is ht-exact. Thus, LC is a subclass of HBR.

Lemma 4.9. *Let $\{\omega_i: E_i \times F_i \to \mathbb{R}\}_{i\in I}$ be a system of strong dualities, $\{(G_i, \tau_i)\}_{i\in I}$ be a system of topological groups, and for every $i \in I$ let ψ_i be a continuous linear G_i-birepresentation $\psi_i = (\omega_i: E_i \times F_i \to \mathbb{R},\ \alpha_{1i}: G_i \times E_i \to E_i,\ \alpha_{2i}: G_i \times F_i \to F_i)$. Suppose that for every $i \in I$ the family $\{(\alpha_{1i})_x: G_i \to E_i,\ (\alpha_{2i})_f: G_i \to F_i\}_{x\in E_i,\ f\in F_i}$ of all orbit mappings generates the topology τ_i. Then the product $(G,\tau) = \prod(G_i,\tau_i)$ is an HBR-group.*

Proof. For every $i \in I$ define the actions:

$$\alpha_{1i}^{\Pi}: G \times E_i \to E_i\ , \quad \alpha_{1i}^{\Pi}((g_s)_{s\in I}, x_i) = \alpha_{1i}(g_i, x_i)$$
$$\alpha_{2i}^{\Pi}: G \times F_i \to F_i\ , \quad \alpha_{2i}^{\Pi}((g_s)_{s\in I}, f_i) = \alpha_{2i}(g_i, f_i).$$

Consider the corresponding continuous linear G-birepresentations:

$$\psi_i^{\Pi} = (\omega_i: E_i \times F_i \to \mathbb{R},\ \alpha_{1i}^{\Pi}: G \times E_i \to E_i,\ \alpha_{2i}^{\Pi}: G \times F_i \to F_i).$$

Then the family $\{(\alpha_{1i}^{\Pi})_x: G \to E_i,\ (\alpha_{2i}^{\Pi})_f: G \to F_i\}_{x\in E_i, f\in F_i, i\in I}$ of all orbit mappings generates the topology τ. Clearly, this implies that (G,τ) is an HBR-group. $\square$

Proposition 4.10. *For every family $\{E_i\}_{i\in I}$ of normed spaces, the topological group product $\prod GL(E_i)$ is an HBR-group.*

Proof. For each normed space E_i from the given family, denote by $\mathrm{End}(E_i)$ the normed space of all continuous endomorphisms. The action $\nu_i\colon GL(E_i)\times \mathrm{End}(E_i)\to \mathrm{End}(E_i)$, defined by $\nu_i(A,B)=A\cdot B$ (multiplication in $\mathrm{End}(E_i)$), induces a topological group embedding $\tilde{\nu}_i\colon GL(E_i)\to GL(\mathrm{End}(E_i))$. By Remark 4.1, the corresponding representation is bicontinuous. Next, the system $\{O_\varepsilon\}_{\varepsilon>0}$, where

$$O_\varepsilon=\{A\in GL(E_i)\colon \|A\cdot \mathrm{Id}_{E_i}-\mathrm{Id}_{E_i}\|<\varepsilon\}$$

is a local base of the natural topology of $GL(E_i)$. Hence, $GL(E_i)$ carries the strong operator topology with respect to $\tilde{\nu}_i$. Now, we can use Lemma 4.9 in the situation: $E_i\colon=\mathrm{End}(E_i)$, $F_i\colon=(\mathrm{End}(E_i))^*, G_i\colon=GL(E_i)$. □

Proposition 4.11. *For every locally convex vector space E, the group $(E,+)$ belongs to HBR.*

Proof. Since $(E,+)$ is an additive topological subgroup of a product $\prod E_i$ of normed spaces, then by Proposition 3.7 the group $\prod E_i$ is a topological subgroup of the product $\prod GL(\mathbb{R}\times E_i^*)$. Now our assertion follows from Proposition 4.10. □

Corollary 4.12. *Let $A(X)$ be the free Abelian topological group of X. Then $A(X)$ belongs to HBR.*

Proof. By a result of Tkačhenko [28], (see also Uspenskii [29, p. 657]), $A(X)$ is a topological subgroup of the free locally convex space $L(X)$ of X. □

The following result solves for Abelian groups (in a slightly stronger form) the above-mentioned problem (Question 2.13(ii)) of Arkhangel'skii.

Theorem 4.13. *Every Abelian topological group is a factor group of a perfectly minimal group.*

Proof. Every Abelian topological group G is a factor group of its free Abelian topological group $A(G)$. On the other hand, by Theorem 4.8 and Corollary 4.12, $A(G)$ is a group retract of a perfectly minimal group. $\square$

Question 4.14. Is every free topological group $F(X)$ a BR-group?

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