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Constructing Tychonoff G-spaces which are not G-Tychonoff

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Abstract

Jan de Vries' compactification problem is whether every Tychonoff G-space can be equivariantly embedded in a compact G-space. In such a case, we say that G is a *V-group*. De Vries showed that every locally compact group G is a V-group. The first example of a non-V-group was constructed in 1988 by the first author. Until now, this was the only known counterexample. In this paper, we give a systematic method of constructing noncompactifiable G-spaces. We show that the class of non-V-groups is large and contains all second countable (even $\aleph_0$-bounded) nonlocally precompact groups. This establishes the existence of monothetic (even cyclic) non-V-groups, answering a question of the first author. As a related result, we obtain a characterization of locally compact groups in terms of "G-normality". © 1998 Elsevier Science B.V.

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1. Introduction

A *topological transformation group*, or a *G-space*, is a triple $\langle G, X, \alpha\rangle$, where G is a topological group, X is a topological space, and $\alpha : G \times X \to X$ is a continuous action. For basic information on G-spaces, see de Vries [16,21]. If X is Tychonoff (respectively, normal, compact, etc.), then $\langle G, X, \alpha\rangle$ (or just X, for short) is called a Tychonoff (respectively, normal, compact, etc.) G-space. A G-space is *G-Tychonoff* if it can be equivariantly embedded into a compact Hausdorff G-space. We call a group G a *V-group* if every Tychonoff G-space is G-Tychonoff.

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In [17], Jan de Vries posed the "compactification problem" in its full generality, which in our terms becomes: is every topological group G a V-group? See Carlson [4] for the case $G = \mathbb{R}$. In [3], Brook investigated a more general class of groups and showed that for *any* topological group G, the G-space $\langle G, G, \alpha_L \rangle$ ($\alpha_L(g_1, g_2) = g_1 g_2$) is G-Tychonoff.

Recall de Vries' well-known result [20] which states that every locally compact group is a V-group. (For the case where G is a compact Lie group, see [12]. If G is compact, see [1,18].) The following are examples of G-spaces which are G-Tychonoff:

(a) [17] Every coset G-space $\langle G, G/H, \alpha_L^* \rangle$, where $\alpha_L^*(g_1, g_2H) = (g_1g_2)H$. (If $H = \{e\}$, this gives Brook's result mentioned above.)
(b) [7] Every G-space X under an equicontinuous action.
(c) [10] Every G-group X (and, hence, every linear G-space X).
(d) [10] Every metric G-space (X, d), where G is second category and $\alpha^g : X \to X$ is d-uniformly continuous for every $g \in G$.
(e) [15] Every Baire G-space X, where G is $\aleph_0$-bounded and acts transitively on X.

In 1988, Megrelishvili [9] answered de Vries' question negatively. He found a continuous action α of a separable, complete metrizable group G on $J(\aleph_0)$, the so-called metrizable hedgehog of spininess $\aleph_0$, such that $\langle G, J(\aleph_0), \alpha \rangle$ is not G-Tychonoff. Therefore, the group G of this example is not a V-group. By [11], no dense subgroup of G is a V-group.

It is an open question whether there are nonlocally compact V-groups. In this paper, we prove the following

Main Theorem. *If G is an $\aleph_0$-bounded topological group which is not locally precompact, then G is not a V-group.*

Recall that a topological group G is called $\aleph_0$-*bounded* [2,6] if for every $V \in N_e(G)$ there exists a countable subset S of G such that $SV = G$. Guran [6] proved that G is $\aleph_0$-bounded iff G is a topological subgroup of a product of second countable topological groups. If G is separable, Lindelöf, or satisfies the countable chain condition, then G is $\aleph_0$-bounded.

In particular, the Main Theorem provides an example of a cyclic group G which is not a V-group, answering a question of Megrelishvili about monothetic groups [11]. As a related result, we characterize locally compact groups in terms of G-normality.

The compactification problem is still open in the case of a (monothetic) *precompact* group, even for a dense cyclic subgroup of the circle group $\mathbb{T}$. It is also an intriguing question whether the additive group $\mathbb{Q}$ of rational numbers is a V-group.

Our approach is to adapt techniques for constructing regular spaces which are not completely regular. For example, Tree [14] has found a method which turns any regular topological space which is not normal into a regular space which is not completely regular. We have adapted his technique to the context of G-spaces, and we have constructed a method which turns a normal G-space which is not G-normal in some sense into a Tychonoff G-space which is not G-Tychonoff.

2. Preliminaries and conventions

All spaces are Tychonoff, and all cardinals are assumed to be infinite. The filter of all neighborhoods of an element x of a space X is denoted by $N_x(X)$. The neutral element of a group is denoted by e. A group G is *locally precompact* if it is a subgroup of a locally compact group, or, equivalently, if its sup-completion (the completion with respect to its two-sided uniformity) is locally compact. Recall [19,20,1] that a continuous function $f: X \to \mathbb{R}$ defined on a G-space X with an action α is called *α-uniform* if for every $\varepsilon > 0$ there exists $U \in N_e(G)$ such that for all $g \in U$, $x \in X$, we have $|f(x) - f(gx)| < \varepsilon$. Denote by $C^*_\alpha(X)$ (respectively, $C^*(X)$) the set of all bounded α-uniform (respectively, continuous bounded) functions on X.

Lemma 2.1.
(i) *$C^*_\alpha(X)$ is a closed sub-algebra of $C^*(X)$.*
(ii) *If f is in $C^*_\alpha(X)$ and $r \in \mathbb{R}$, then the functions*

$$\mu(x) := \min\{f(x), r\} \quad \textit{and} \quad h(x) := |f(x)|$$

are also in $C^*_\alpha(X)$.

The compactification of X which corresponds to the algebra $C^*_\alpha(X)$ is the *maximal G-compactification* [19] and is denoted by $\beta_G X$.

Fact 2.2 [19,20]. *A Tychonoff G-space $\langle G, X, \alpha\rangle$ is G-Tychonoff iff $\beta_G X$ is a proper G-compactification of X iff $C^*_\alpha(X)$ separates the points and the closed subsets of X.*

As mentioned above, the G-space of all left translations $\langle G, G, \alpha_L\rangle$ is G-Tychonoff. In this case, the maximal (proper) G-compactification $\beta_G G$ is called the *greatest ambit*. (For further information, see [21].)

Let $\langle G, X, \alpha\rangle$ be a G-space. We say that subsets A and B of X are *α-disjoint* if there exists $U \in N_e(G)$ such that $UA \cap UB = \emptyset$. Two disjoint invariant subsets are obviously α-disjoint. Moreover, if A and B are separated by an α-uniform function, then A and B are α-disjoint. In fact, we have the following stronger result.

Lemma 2.3. *Let $\langle G, X, \alpha\rangle$ be a G-space. Let C and D be subsets of X which are separated by an α-uniform function. Then there are sequences $\{U_n\}$, $\{O_n\}$ and $\{O'_n\}$ of neighborhoods of e, C and D, respectively, such that for all $n \in \mathbb{N}$,*

$$\mathrm{cl}(U_n O_n) \subseteq O_{n+1}, \quad \mathrm{cl}(U_n O'_n) \subseteq O'_{n+1}, \quad \textit{and} \quad O_n \cap O'_n = \emptyset.$$

We say that a G-space X is *G-normal* (*equivariantly normal* in the terminology of [8]) if every pair of α-disjoint closed subsets of X has α-disjoint neighborhoods. Equivalently, X is G-normal iff $C^*_\alpha(X)$ separates the closed α-disjoint subsets of X. The continuity of the action α guarantees that for every closed subset F of X and every point $x \in X \setminus F$, the subsets F and $\{x\}$ are α-disjoint. Therefore, every G-normal G-space is G-Tychonoff. Every coset G-space $\langle G, G/H, \alpha_L\rangle$ is G-normal for arbitrary G. If G is locally compact, then every normal G-space is G-normal [8]. For the converse, see Theorem 5.2 below.

Definition 2.4. A G-space $\langle G, X, \alpha\rangle$ is called *weakly G-normal* if $C^*_\alpha(X)$ separates closed invariant subsets of X.

Proposition 2.5. *Let G be an arbitrary topological group which is not sup-complete. Then there exists a normal G-space X of weight $w(X) = w(G)$ which is not G-normal.*

Proof. Denote by $\widehat{G}$ the sup-completion of G. Let $\mathbb{B}$ be a base of neighborhoods of e in G. We may suppose that the cardinality of $\mathbb{B}$ is not greater than $w(G)$. By Brook's theorem [3] the $\widehat{G}$-space $\langle \widehat{G}, \widehat{G}, \alpha_L\rangle$ is $\widehat{G}$-Tychonoff. By [11], there is a compact $\widehat{G}$-extension Y of $\langle \widehat{G}, \widehat{G}, \alpha_L\rangle$ such that $w(Y) \leqslant w(\widehat{G}) \cdot w(\widehat{G}) = w(\widehat{G}) = w(G)$. We now form the topological $\widehat{G}$-sum $X = \bigoplus\{Y_U\colon U \in \mathbb{B}\}$, where each $Y_U := Y \times \{U\}$ is a copy of Y. Let α denote the action of $\widehat{G}$ on X restricted to G. Clearly, $\langle G, X, \alpha\rangle$ is a normal G-space and $w(X) = w(G)$. In order to show that it is not G-normal, we will construct two closed α-disjoint subsets C and D of X which are not separated by $C^*_\alpha(X)$.

Let $U \in \mathbb{B}$, and let $\mathrm{cl}_{\widehat{G}}(U)$ be the closure of U in $\widehat{G}$. Since G is not sup-complete, we can pick $g_U \in \mathrm{cl}_{\widehat{G}}(U) \setminus G$. Set

$$C = \{(g_U, U)\colon U \in \mathbb{B}\} \quad \text{and} \quad D = \{(e, U)\colon U \in \mathbb{B}\}.$$

Clearly, C and D are closed α-disjoint subsets of X. It is easy to show, however, that for all $V \in \mathbb{B}$

$$(g_V, V) \in \mathrm{cl}_X(VD) \cap C.$$

By Lemma 2.3, C and D are not separated by $C^*_\alpha(X)$. □

For a compact space X, $H(X)$ denotes the (topological) group of all homeomorphisms of X, endowed with the compact-open topology. Let $\langle G, (X, \mu), \alpha\rangle$ be an arbitrary G-space, where μ is a compatible uniformity for X, and let $A \subseteq G$. We say that A acts *μ-uniformly equicontinuously* if for every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that for all $g \in A$ and for all $x, y \in X$, $(x, y) \in \delta$ implies $(gx, gy) \in \varepsilon$. We say that G acts *locally uniformly equicontinuously* if there exists $U \in N_e(G)$ such that U acts μ-uniformly equicontinuously.

Lemma 2.6. *Let (G, τ) be a topological group. For $g \in G$, let $\widetilde{g}\colon \beta_G G \to \beta_G G$ be a continuous extension of the transition map $\alpha^g : G \to G$. Then the map $\varphi\colon G \to H(\beta_G G)$ defined by $\varphi(g) = \widetilde{g}$ is a topological group embedding.*

Proof. Straightforward. See, for example, Theorem 3.2 of [3]. □

Lemma 2.7. *The action of G on the greatest ambit $\beta_G G$ is locally uniformly equicontinuous iff G is locally precompact.*

Proof. Suppose the action $\widetilde{\alpha}\colon G \times \beta_G G \to \beta_G G$ is locally uniformly equicontinuous. That is, there is a neighborhood U of e which acts μ-uniformly equicontinuously, where μ is the unique uniformity for $\beta_G G$. Then, the Ascoli–Arzela theorem implies that the closure of $\varphi(U)$ in $H(\beta_G G)$ is compact. Therefore, the closure of $\varphi(G)$ in $H(\beta_G G)$ is a locally compact group containing the group $G = \varphi(G)$ (Lemma 2.6).

Conversely, suppose G is locally precompact. By the equivariant completion theorem [11], there is a continuous extending action of the sup-completion $\widehat{G}$ of G on $\beta_G G$. By our assumption, there exists a compact neighborhood V of e in $\widehat{G}$. Easy compactness arguments now imply that V acts μ-uniformly equicontinuously on $\beta_G G$. Hence the neighborhood $V \cap G$ of e in G also acts μ-uniformly equicontinuously on $\beta_G G$. □

We will also need the following result.

Fact 2.8 (Equivariant Approximation Theorem) [10]. *Let G be an $\aleph_0$-bounded group, and let X be a compact G-space. Then X can be represented as a G-limit of an inverse G-system $\{\langle G, X_i, \alpha_i\rangle \colon i \in I\}$ of compact metrizable G-spaces X_i such that* $\dim X_i \leqslant \dim X$.

3. From non-G-normality to non-G-Tychonoff-ness

We now adapt the original construction of Tree [14] to the context of G-spaces. Let $\langle G, X, \alpha\rangle$ be a regular G-space. We describe the construction of a related G-space $\langle G, X^+, \alpha^+\rangle$. We assume that there are disjoint closed G-subspaces $C, D \subseteq X$.

Let $\omega = \mathbb{N} \cup \{0\}$ carry the discrete topology. Let Y be the quotient space formed from $X \times \omega$ by identifying the pairs $(c, 2i+1)$ and $(c, 2i+2)$ for $c \in C$, $i \in \omega$, and the pairs $(d, 2i)$ and $(d, 2i+1)$ for $d \in D$, $i \in \omega$. Let $p\colon X \times \omega \to Y$ be the quotient map. For $n \in \omega$, let $i_n : X \to X \times \omega$ be the canonical injection $x \mapsto (x, n)$.

Fix a point $a \notin Y$, and let $X^+ = Y \cup \{a\}$. Topologize X^+ by setting Y to be an open subset with its quotient topology and the nth basic nbd of a to be

$$N_n(a) = \{a\} \cup p\big(i_{2n}(X \setminus C)\big) \cup \bigcup \big\{p\big(i_m(X)\big)\colon m > 2n\big\}.$$

It is trivial to check that this generates a topology τ on X^+. We remark only that if O is an open subset of Y, then $O \cap N_n(a)$ is also open in Y. To see this, it is enough, by the definition of the quotient topology, to show that $p^{-1}(O \cap N_n(a))$ is open. Observe that $p^{-1}(O)$ is open by the continuity of p, and $p^{-1}(N_n(a)) = N_n(a) \setminus \{a\}$ is open by the definition of $N_n(a)$.

We now define a function $\alpha^+ : G \times X^+ \to X^+$. For any $g \in G$, set $\alpha^+(g, a) = a$. For $p((x, n)) \in Y$, set $\alpha^+(g, p((x, n))) = p(i_n(\alpha(g, x)))$. Note that α^+ is well-defined because C and D are invariant.

Claim 3.1. *$\langle G, X^+, \alpha^+\rangle$ is a G-space.*

Proof. It is easy to see that α^+ is an action on X^+. It remains to show that α^+ is continuous. The continuity of α^+ at points (g, y) for $y \in Y$ is easily proved using the continuity of α. For the continuity of α^+ at the point (g, a), note that for any $n \in \mathbb{N}$, $GN_n(a) = N_n(a)$. □

Theorem 3.2. *Let $\langle G, X, \alpha\rangle$ be a normal G-space with closed disjoint invariant subsets $C, D \subseteq X$ which are not separated by $C^*_\alpha(X)$. Then $\langle G, X^+, \alpha^+\rangle$ is a normal (and, hence, Tychonoff) G-space which is not G-Tychonoff.*

Proof. First we show that X^+ is normal.

Claim 1. *If $A \subseteq Y$, U is an open subset of $X \times \omega$, and $p^{-1}(A) \subseteq U$, then there is an open subset U' of $X \times \omega$ such that $p^{-1}(A) \subseteq U' \subseteq U$, and $p(U')$ is open in the space Y.*

Proof. For each $(x, n) \in p^{-1}(A)$, we will define an open set $U_{x,n} \subseteq X \times \{n\}$ such that $(x, n) \in U_{x,n} \subseteq U$. If $x \notin C \cup D$ (or if $n = 0$ and $x \notin D$), then there is an open set $U' \subseteq X$ such that $x \in U'$ and $U' \cap (C \cup D) = \emptyset$. Since $(x, n) \in U$, there is an open set $U'' \subseteq X \times \{n\}$ such that $(x, n) \in U'' \subseteq U$. Let $U_{x,n} := (U' \times \{n\}) \cap U''$. Then $U_{x,n}$ is open, $(x, n) \in U_{x,n}$, $U_{x,n} \subseteq U$, and $p(U_{x,n})$ is open in Y, since $p^{-1}(p(U_{x,n})) = U_{x,n}$. If $x \in C$, then assume without loss of generality that n is odd. So both (x, n) and $(x, n+1)$ belong to $p^{-1}(A)$. Since $(x, n) \in U$, there is an open set $U_n \subseteq X \times \{n\}$ such that $(x, n) \in U_n \subseteq U$. Similarly, there is an open set $U_{n+1} \subseteq X \times \{n+1\}$ such that $(x, n+1) \in U_{n+1} \subseteq U$. Let

$$U_{x,n} := \{(y, n) \in U_n\colon\ (y, n+1) \in U_{n+1}\},$$
$$U_{x,n+1} := \{(y, n+1) \in U_{n+1}\colon\ (y, n) \in U_n\}.$$

The case $x \in D$ is handled similarly.

Finally, define $U' = \bigcup_{(x,n)\in p^{-1}(A)} U_{x,n}$. This proves Claim 1. □

Claim 2. *If $C \subseteq X^+$ is closed and $a \notin C$, then there exists an $n \in \omega$ such that*

$$C \subseteq \bigcup_{k<n} p(X \times \{k\}).$$

Proof. If not, then for every $n \in \omega$, there are $m \geqslant n$ and $x_n \in p(X \times \{m\})$ such that $x_n \in C$. Then $x_n \to a$, so $a \in C$.

Claim 3. *X^+ is normal (and, hence, Tychonoff).*

Proof. Let $A, B \subseteq X^+$ be closed and disjoint.

Case 1. $a \notin A \cup B$. In this case, A and B are closed subsets of Y. So $p^{-1}(A)$ and $p^{-1}(B)$ are closed disjoint subsets of $X \times w$. Since $X \times \omega$ is normal, there are disjoint open sets $U, V \subseteq X \times \omega$ such that $p^{-1}(A) \subseteq U$ and $p^{-1}(B) \subseteq V$. By Claim 1, let $U', V' \subseteq X \times \omega$ be open sets such that $p^{-1}(A) \subseteq U' \subseteq U$, $p^{-1}(B) \subseteq V' \subseteq V$, and $p(U')$, $p(V')$ are open in Y. Note that U' and V' are disjoint. Now

$$A = p(p^{-1}(A)) \subseteq p(U') \quad \text{and} \quad B = p(p^{-1}(B)) \subseteq p(V').$$

Subclaim. $p(U') \cap p(V') = \emptyset$.

Proof. Suppose $z \in p(U') \cap p(V')$. If $z = p((x,n))$, where $x \notin C \cup D$, then $(x,n) \in U' \cap V' = \emptyset$, which is a contradiction. Hence we must have $x \in C \cup D$. Without loss of generality, assume that $x \in C$, n is odd, and $(x,n) \in U'$, $(x,n+1) \in V'$. From the construction of U' in Claim 1, we see that there must exist $c \in C$ such that $(x,n) \in U_{c,n}$. So $(x,n+1) \in U_{c,n+1}$. Hence $(x,n+1) \in U' \cap V'$, a contradiction.

Case 2. $a \in A \cup B$. Without loss of generality, assume that $a \in A \setminus B$. So $A \cap Y = A \setminus \{a\}$ is closed in Y. So we apply Case 1 to separate $A \cap Y$ and B by disjoint open sets U and V, respectively. By Claim 2 there is $n \in \omega$ such that

$$B \subseteq \bigcup_{k<n} p(X \times \{k\}).$$

Assume without loss of generality that n is odd. Define

$$V' := V \cap \left[\left(\bigcup_{k<n} p(X \times \{k\})\right) \cup p((X \setminus C) \times \{n\})\right].$$

Let $U' = U \cup N_n(a)$. Then $A \subseteq U'$, $B \subseteq V'$, $U' \cap V' = \emptyset$, and U' and V' are open. This completes the proof that X^+ is normal. $\square$

Next, we will show that $\langle G, X^+, \alpha^+\rangle$ is *not* G-Tychonoff. Our proof follows closely the proof in [14]. For the sake of completeness, we have included many of the details.

Notation. For a subset $A \subseteq X$ and $n \in \omega$, let $A_n^+ = p(i_n(A))$.

We claim that the point a and the set C_0^+ are not separated by a bounded α^+-uniform function. The proof is by contradiction. Suppose $f : X^+ \to \mathbb{R}$ is a bounded α^+-uniform function such that $f(C_0^+) = 0$ and $f(a) = 1$. As in Tree [14], there is an $n \in \omega$ such that for all $m \geqslant n$ and for all $x \in C_m^+$, $|f(x)| \geqslant 1/n$. Fix such an n. Note that

$$f(C_n^+) \cap \left(-\frac{1}{n}, \frac{1}{n}\right) = \emptyset.$$

Assume without loss of generality that n is odd. Let $n = 2r + 1$. Define a function $\mu : X^+ \to \mathbb{R}$ by $\mu(x) = \min\{n|f(x)|, 1\}$. By Lemma 2.1, μ is α^+-uniform. Note that $\mu(C_0^+) = 0$ and $\mu(C_n^+) = 1$. We now define $n+1$ functions $h_0, h_1, \ldots, h_n : X \to \mathbb{R}$ by restricting the domain of μ to X_k^+ for $k = 0, 1, \ldots, n$. More precisely,

$$h_k(x) = \mu(p(i_k(x))).$$

Claim 4. *Each h_k is α-uniform.*

Proof. Fix $\varepsilon > 0$. Let U be a nbd of e in G such that if $x \in X^+$ and $g \in U$, then $|\mu(x) - \mu(gx)| < \varepsilon$. Fix $x \in X$, $g \in U$ and $k \in \{0, 1, \ldots, n\}$. Then

$$\begin{aligned} |h_k(x) - h_k(gx)| &= |\mu(p(i_k(x))) - \mu(p(i_k(gx)))| = |\mu(p(x,k)) - \mu(p(gx,k))| \\ &= |\mu(p(x,k)) - \mu(gp(x,k))| < \varepsilon. \end{aligned}$$

This proves Claim 4.

Observation.

(1) For $i = 0, 1, \ldots, r-1$, $h_{2i+1} \restriction C = h_{2i+2} \restriction C$.

(2) For $i = 0, 1, \ldots, r$, $h_{2i} \restriction D = h_{2i+1} \restriction D$.

(3) $h_0 \restriction C = 0$.

(4) $h_n \restriction C = 1$.

Finally, define $h : X \to \mathbb{R}$ by

$$h(x) = \sum_{k=0}^{n} (-1)^{k+1} h_k(x).$$

By Lemma 2.1, h is α-uniform. Now $c \in C \Rightarrow h(c) = -h_0(c) + h_n(c) = 0 + 1 = 1$ by (1), (3) and (4) above, while $d \in D \Rightarrow h(d) = 0$ by (2).

This contradicts the assumption that C and D cannot be separated by an α-uniform function. This completes the proof of Theorem 3.2. $\square$

4. Proof of the Main Theorem

Definition 4.1. Let $\langle G, X, \alpha \rangle$ be a G-space and $S \subseteq G$. Subsets A and B of X are said to be *S-near* if for every pair O_1, O_2 of neighborhoods of A and B, respectively, there exist $g_1, g_2 \in S$ such that $g_1 O_1 \cap g_2 O_2 \neq \emptyset$, i.e., $SO_1 \cap SO_2 \neq \emptyset$.

Lemma 4.2. *Let $\langle G, X, \alpha \rangle$ be a G-space with a fixed point $z \in X$ and $S \subseteq G$. Let $x \in X \setminus \{z\}$ be such that x and z are S-near. Set*

$$Y = X^2 \setminus \{(z, z)\}, \quad C = (\{z\} \times X) \setminus \{(z, z)\}, \quad \text{and}$$
$$D = (X \times \{z\}) \setminus \{(z, z)\}.$$

Define $\alpha' : G \times Y \to Y$ as the "one-coordinate" action $\alpha'(g, (x, y)) = (\alpha(g, x), y)$. Then $\langle G, Y, \alpha' \rangle$ is a G-space, and C and D are closed disjoint invariant S-near subsets of Y.

Proof. Only the statement about the S-nearness of C and D is nontrivial. Fix neighborhoods O_1 and O_2 of C and D, respectively. Since $(x, z) \in D$, we have $(x, z) \in O_2$, and hence there are open neighborhoods $O_x(x)$, $O_z(z)$ of x and z, respectively, such that

$$(x, z) \in \big(O_x(x) \times O_z(z)\big) \setminus \{(z, z)\} \subseteq O_2.$$

It is easy to see that z is not isolated in X. Hence there exists $y \in O_z(z)$ such that $y \neq z$. Then $(z, y) \in O_1$, so there are open neighborhoods $O'_z(z)$, $O_y(y)$ of z and y, respectively, such that

$$(z, y) \in \big(O'_z(z) \times O_y(y)\big) \setminus \{(z, z)\} \subseteq O_1.$$

By hypothesis, x and z are S-near, so there are $g_1, g_2 \in S$ such that $g_2 O_x(x) \cap g_1 O'_z(z) \neq \emptyset$. Therefore,

$$g_2(O_x \times O_z) \cap g_1(O'_z \times O_y) \neq \emptyset.$$

This easily implies that $g_1 O_1 \cap g_2 O_2 \neq \emptyset$. $\square$

Main Theorem 4.3. *Let G be a nonlocally precompact $\aleph_0$-bounded topological group. Then G is not a V-group, i.e., there is a Tychonoff G-space which is not G-Tychonoff.*

Proof. Since G is not locally precompact, Lemma 2.7 implies that G does not act locally uniformly equicontinuously on $X = \beta_G G$.

We now construct a G-space X_U for every U in a collection $\mathbb{B}$ (of cardinality $\chi(G)$) of basic neighborhoods of e in G. By Approximation Theorem 2.8, X is a G-limit of an inverse G-system of compact metrizable G-spaces X_i ($i \in I$). Let μ and μ_i denote the unique compatible uniformity on X and X_i, respectively.

Let $U \in \mathbb{B}$. Since U does not act μ-uniformly equicontinuously on $\beta_G G$, there exists an index $i \in I$ such that U does not act μ_i-uniformly equicontinuously on (X_i, μ_i). Therefore, there is $\varepsilon \in \mu_i$ such that for every $\delta \in \mu_i$ there exist $(x_\delta, y_\delta) \in \delta$ and $g_\delta \in U$ such that

$$(g_\delta x_\delta, g_\delta y_\delta) \notin \varepsilon. \tag{$*$}$$

Thus we obtain nets $\langle x_\delta \rangle, \langle y_\delta \rangle, \langle g_\delta x_\delta \rangle$, and $\langle g_\delta y_\delta \rangle$ in (X_i, μ_i), indexed by the elements δ of μ_i. Passing to subnets if necessary, we may assume that there exist $x^U, a^U, b^U \in X$ such that $x_\delta \to x^U$, $y_\delta \to x^U$, $g_\delta x_\delta \to a^U$, and $g_\delta y_\delta \to b^U$. By $(*)$ we have $a^U \neq b^U$. Hence $(x^U, x^U) \in \Delta$ and $(a^U, b^U) \notin \Delta$, where $\Delta = \{(x, x) \mid x \in X_i\}$. Clearly, (x^U, x^U) and (a^U, b^U) are U-near in the G-space $(X_i, \mu_i) \times (X_i, \mu_i)$, with the natural "two-coordinate" action.

We now form the quotient G-space $Y_i = (X_i \times X_i)/\Delta$. Consider the quotient G-map $p : X_i \times X_i \to Y_i$. Let $z := p(x, x)$. Then z and $p(a^U, b^U)$ are U-near in Y_i.

Since z is a fixed point of Y_i, we may apply Lemma 4.2, with $X = Y_i$, $S = U$, and $x = p(a^U, b^U)$. That is, let

$$\begin{aligned} X_U &= (Y_i \times Y_i) \setminus \{(z, z)\}, \\ C^U &= (\{z\} \times Y_i) \setminus \{(z, z)\}, \quad \text{and} \\ D^U &= (Y_i \times \{z\}) \setminus \{(z, z)\}. \end{aligned}$$

Define the "one-coordinate" action $\alpha' : G \times X_U \to X_U$ by

$$\alpha'(g, (x, y)) = (\alpha(g, x), y),$$

where α is the action of G on Y_i. Then, by Lemma 4.2, $\langle G, X_U, \alpha' \rangle$ is a G-space, and C^U and D^U are closed disjoint invariant U-near subsets of X_U. This completes the construction of X_U from U.

Now form the topological G-sum $S = \bigoplus \{X_U \colon U \in \mathbb{B}\}$. Let $\alpha^* : G \times S \to S$ be the natural action. Define

$$C = \bigcup_{U \in \mathbb{B}} C^U, \qquad D = \bigcup_{U \in \mathbb{B}} D^U.$$

Then C and D are closed disjoint invariant subsets of S.

Claim 1. *C and D are not separated by $C^*_{\alpha^*}(S)$.*

Proof. By Lemma 2.3, it is enough to show that C and D have no α^*-disjoint neighborhoods. Let O_C, O_D be neighborhoods of C and D, respectively. Let $U \in N_e(G)$, $U' \subseteq U$, with $U' \in \mathbb{B}$. Define $O_1 = O_C \cap X_{U'}$, $O_2 = O_D \cap X_{U'}$. Then $U'O_1 \cap U'O_2 \neq \emptyset$. Hence $UO_C \cap UO_D \neq \emptyset$.

Claim 2. *S is metrizable.*

Proof. It is sufficient to show that each X_U in the topological sum is metrizable. Each X_i is a metrizable compact space and Δ is closed in $X_i \times X_i$. Clearly, $Y_i = (X_i \times X_i)/\Delta$ is metrizable and compact. Hence every X_U is metrizable, being a subspace of $Y_i \times Y_i$.

By the above claims, $\langle G, S, \alpha^* \rangle$ is a normal G-space which is not weakly G-normal. Therefore, by Theorem 3.2, $\langle G, S^+, (\alpha^*)^+ \rangle$ is a Tychonoff G-space which is not G-Tychonoff. This completes the proof of the Main Theorem. □

We discuss here some topological properties of S^+. Observe that by our construction, $w(S^+) = w(S) \cdot \chi(G)$. If G is second countable, then we can choose S such that $w(S) = w(G)$. In this case, we get $w(S^+) = w(G) = \aleph_0$.

Note that the space S^+ has only one point, namely a, of nonlocal compactness. This is interesting because any locally compact G-space is G-Tychonoff.

Remark. We note that the situation for Polish groups G is totally clear. If G is locally compact, then G is a V-group, by de Vries' theorem. Now suppose that G is not locally compact. The completeness of G implies that G cannot be locally precompact. Since G is clearly $\aleph_0$-bounded, the Main Theorem implies that G is not a V-group.

5. Further results

The following corollary to the Main Theorem provides a negative answer to Megrelishvili's question [11] whether every monothetic group is a V-group.

Corollary 5.1. *There exists a cyclic (metrizable) topological group G which is not a V-group.*

Proof. Take, for example, a dense cyclic subgroup G of the monothetic nonlocally precompact group from [13]. □

It is interesting to note that local compactness of the acting group G can be characterized in terms of G-normality.

Theorem 5.2. *The following are equivalent for a topological group G:*

(i) *G is locally compact.*

(ii) *Every normal G-space is G-normal.*

Proof. (i) ⇒ (ii) see [8].

(ii) $\Rightarrow$ (i) First observe that by Proposition 2.5, G is sup-complete. Therefore, we may assume that G is not locally precompact. Then, by Lemma 2.7, the action of G on $X = \beta_G G$ is not locally uniformly equicontinuous. Let $U \in N_e(G)$, and let μ be the unique uniformity for X. Then U does not act μ-uniformly equicontinuously. As in the proof of the Main Theorem, we obtain nets $\langle x_\delta \rangle, \langle y_\delta \rangle$ in X and a net $\langle g_\delta \rangle$ in U, each indexed by the elements δ of μ. Without loss of generality, we may assume that there exist $x^U, a^U, b^U \in X$ such that $x_\delta \to x^U$, $y_\delta \to x^U$, $g_\delta x_\delta \to a^U$, $g_\delta y_\delta \to b^U$, and $a^U \neq b^U$. Form the topological G-sum

$$S = \bigoplus \{Y_U\colon\ U \in N_e(G)\},$$

where each Y_U is a copy of $X \times X$, with the "two-coordinate" action. Clearly, S is normal. Define

$$C = \{(x^U, x^U)_U\colon\ U \in N_e(G)\} \quad \text{and} \quad D = \{(a^U, b^U)_U\colon\ U \in N_e(G)\}.$$

Then C and D are closed and α-disjoint subsets of S. However, for every $U \in N_e(G)$ and for every pair of neighborhoods O_1, O_2 of C and D, respectively, we have

$$\mathrm{cl}(UO_1) \cap \mathrm{cl}(UO_2) \neq \emptyset.$$

Now Lemma 2.3 implies that C and D are not separated by $C^*_\alpha(X)$. This proves that S is not G-normal. $\square$

In a topological space X, points a and b are called *twins* if $f(a) = f(b)$ for every continuous function $f : X \to \mathbb{R}$. In a G-space $\langle G, X, \alpha \rangle$, points a and b are called *α-twins* if $f(a) = f(b)$ for every α-uniform function $f : X \to \mathbb{R}$.

In [14], Tree shows how to modify any regular space which is not completely regular in order to obtain a regular space with twins. We have adapted his technique to the context of G-spaces.

Let $\langle G, X, \alpha \rangle$ be a regular G-space. We construct a related G-space $\langle G, X^*, \alpha^* \rangle$. We assume that there is a proper closed invariant subset C of X. Let X^* be the quotient space formed from $X \times \{0, 1\}$ by identifying the pairs $(c, 0)$ and $(c, 1)$ for $c \in C$. Let $p : X \times \{0, 1\} \to X^*$ be the quotient map. Let $i_0, i_1 : X \to X \times \{0, 1\}$ be the canonical injections. Define $\alpha^* : G \times X^* \to X^*$ by $\alpha^*(g, p((x, n))) = p(i_n(\alpha(g, x)))$.

The action α^* is well-defined because C is invariant. It is easy to see that $\langle G, X^*, \alpha^* \rangle$ is a G-space. The corresponding proof in [14] works here because $C^*_\alpha(X)$ is an algebra. Thus we obtain

Theorem 5.3. *Let $\langle G, X, \alpha \rangle$ be a normal G-space with a fixed point z and a closed invariant subset C which are not separated by $C^*_\alpha(X)$. Then $\langle G, X^*, \alpha^* \rangle$ is a Tychonoff G-space with α^*-twins. (In fact, X^* is even normal.)*

Combining the Main Theorem 4.3 and Theorem 5.3, we see that for every nonlocally precompact, $\aleph_0$-bounded group G, there is a Tychonoff G-space with α-twins.

Questions.

(1) Let G be a locally precompact (nonlocally compact) topological group. Is G a V-group? In particular, is the additive group $\mathbb{Q}$ of rational numbers a V-group?

(2) In Theorem 5.2, can we replace "G-normal" by "weakly G-normal"?

(3) (Yu. Smirnov) Is there a Tychonoff G-space X on which every α-uniform function is constant?

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