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Minimality of topological matrix groups and Fermat primes $\stackrel{\Rightarrow}{\Rightarrow}$

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ARTICLE INFO

Article history: Received 14 April 2022 Received in revised form 9 August 2022 Accepted 5 October 2022 Available online 6 October 2022

Dedicated to Prof. D. Dikranjan on the occasion of his 70th birthday

 $\begin{array}{c} MSC: \\ 20H20 \\ 20G25 \\ 54H11 \\ 11Sxx \\ 11F85 \\ 54H13 \end{array}$

Keywords: Iwasawa decomposition Local field Fermat primes Matrix group Minimal topological group Projective linear group Special linear group

Contents

1.	Introduction	2
2.	Preliminaries	4
3.	Minimality of $ST^+(n, \mathbb{F})$	6
4.	Minimality properties of $SL(n, \mathbb{F})$ and $PGL(n, \mathbb{F})$	5
5.	Fermat primes and minimality of special linear groups 1	8
Data	availability	0
Refere	ences	0

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ABSTRACT

Our aim is to study topological minimality of some natural matrix groups. We show that the special upper triangular group $ST^+(n, \mathbb{F})$ is minimal for every local field \mathbb{F} of characteristic $\neq 2$. This result is new even for the field \mathbb{R} of reals and it leads to some important consequences. We prove criteria for the minimality and total minimality of the special linear group $SL(n, \mathbb{F})$, where \mathbb{F} is a subfield of a local field. This extends some known results of Remus–Stoyanov (1991) and Bader–Gelander (2017).

One of our main applications is a characterization of Fermat primes, which asserts that for an odd prime p the following conditions are equivalent:

(1) p is a Fermat prime;

- (2) $\operatorname{SL}(p-1,\mathbb{Q})$ is minimal, where \mathbb{Q} is the field of rationals equipped with the *p*-adic topology;
- (3) $\operatorname{SL}(p-1,\mathbb{Q}(i))$ is minimal, where $\mathbb{Q}(i) \subset \mathbb{C}$ is the Gaussian rational field.

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 $^{^{\}pm}$ This research was supported by a grant of the Israel Science Foundation (ISF 1194/19) and also by the Gelbart Research Institute at the Department of Mathematics, Bar-Ilan University.

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https://doi.org/10.1016/j.topol.2022.108272

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1. Introduction

It is a well-known phenomenon that many natural topological groups in analysis and geometry are minimal [11,35,30,21,37,15,9,6,13]. For a survey, regarding minimality in topological groups, we refer to [9].

All topological spaces in the sequel are Hausdorff. A topological group G is minimal [12,34] if every continuous isomorphism $f: G \to H$, with H a topological group, is a topological isomorphism (equivalently, if G does not admit a strictly coarser group topology). If every quotient of G is minimal, then G is called totally minimal [10]. Recall also [24,8] that a subgroup H of G is said to be relatively minimal (resp., cominimal) in G if every coarser group topology on G induces on H (resp., on the coset set G/H) the original topology.

Let \mathbb{F} be a topological field. Denote by $\operatorname{GL}(n, \mathbb{F})$ the group of $n \times n$ invertible matrices over the field \mathbb{F} with the natural pointwise topology inherited from \mathbb{F}^{n^2} . Consider the following topological subgroups of $\operatorname{GL}(n, \mathbb{F})$:

- $SL(n, \mathbb{F})$ Special Linear Group matrices with determinant equal to 1.
- $T^+(n, \mathbb{F})$ Upper Triangular invertible matrices.
- $\operatorname{ST}^+(n, \mathbb{F}) := \operatorname{T}^+(n, \mathbb{F}) \cap \operatorname{SL}(n, \mathbb{F})$ Special Upper Triangular group.
- $\mathsf{N} := \mathrm{UT}(n, \mathbb{F})$ -Upper unitriangular matrices.
- D Diagonal invertible matrices.
- $A := D \cap SL(n, \mathbb{F})$. Note that $NA = ST^+(n, \mathbb{F})$.

Consider also the following projective linear groups (equipped with the quotient topology):

- $\operatorname{PGL}(n, \mathbb{F}) = \operatorname{GL}(n, \mathbb{F}) / Z(\operatorname{GL}(n, \mathbb{F})).$
- $\operatorname{PSL}(n, \mathbb{F}) = \operatorname{SL}(n, \mathbb{F})/Z(\operatorname{SL}(n, \mathbb{F})).$

1.1. Main results

In this paper, we study minimality conditions in topological matrix groups over local fields and their subfields. We thank D. Dikranjan whose kind suggestions led us to the following question which hopefully opens several fruitful research lines.

Question 1.1. Let G be a subgroup of $GL(n, \mathbb{F})$. Under which conditions is G (totally) minimal?

We prove in Theorem 3.19 that the solvable group $ST^+(n, \mathbb{F})$ is minimal for every local field \mathbb{F} of characteristic distinct from 2 and every $n \in \mathbb{N}$. This result is new even for the field \mathbb{R} of reals.

Using Iwasawa decomposition, our results on $ST^+(n, \mathbb{F})$ lead (see Theorem 4.3) to the total minimality of $SL(n, \mathbb{F})$ for local fields \mathbb{F} of characteristic distinct from 2. According to an important result of Remus and Stoyanov [30], $SL(n, \mathbb{R})$ is totally minimal. More generally, a connected semi-simple Lie group is totally minimal if and only if its center is finite. Recent results of Bader and Gelander [3], obtained in a different way, imply that $SL(n, \mathbb{F})$ is totally minimal for every local field \mathbb{F} with any characteristic.

We provide criteria for the minimality and total minimality of $SL(n, \mathbb{F})$, where \mathbb{F} is a subfield of a local field (see Theorem 4.7 and Proposition 5.1). Corollary 4.8(1) shows that $SL(2, \mathbb{F})$ is totally minimal, while $SL(2^k, \mathbb{F})$ is minimal for every $k \in \mathbb{N}$, by Corollary 5.2. It also turns out that $SL(n, \mathbb{F})$ is totally minimal for every topological subfield \mathbb{F} of \mathbb{R} (see Corollary 4.8(2)).

Sometimes for the same field, according to the parameter $n \in \mathbb{N}$, we have all three possibilities: minimality, total minimality and the absence of minimality. Indeed, see Corollary 5.3 which gives a trichotomy concerning

the group $SL(n, \mathbb{Q}(i))$, where $\mathbb{Q}(i) := \{a + bi : a, b \in \mathbb{Q}\}$ is the Gaussian rational field. If n is not a power of 2, then $ST^+(n, \mathbb{Q}(i))$ and $SL(n, \mathbb{Q}(i))$ are not minimal.

By Corollary 5.4, if p-1 is not a power of 2, then $\operatorname{SL}(p-1, (\mathbb{Q}, \tau_p))$ is not minimal, where (\mathbb{Q}, τ_p) is the field of rationals with the *p*-adic topology (treating it as a subfield of the local field \mathbb{Q}_p of all *p*-adic numbers). Furthermore, for every subfield \mathbb{F} of \mathbb{Q}_p , the groups $\operatorname{SL}(n, \mathbb{F})$ and $\operatorname{PSL}(n, \mathbb{F})$ are totally minimal for every *n* which is coprime to p-1.

It is known that if $p = 2^k + 1$ is an odd prime then k is a power of 2. These are the famous *Fermat* primes $F_n = 2^{2^n} + 1$. As of 2021, the only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$. The following theorem is one of our main applications (proved in Theorem 5.5), which characterizes Fermat primes in terms of topological minimality.

Theorem 1.2. For an odd prime p the following conditions are equivalent:

- (1) p is a Fermat prime;
- (2) $\operatorname{SL}(p-1,(\mathbb{Q},\tau_p))$ is minimal;
- (3) $SL(p-1, \mathbb{Q}(i))$ is minimal.

We prove in Theorem 4.11 that the projective general linear group $PGL(n, \mathbb{F})$ is totally minimal for every local field \mathbb{F} and every $n \in \mathbb{N}$. The same holds for topological subfields \mathbb{F} of \mathbb{R} as long as n is odd (see Theorem 4.12).

1.2. Some known results

One of the first examples (due to Dierolf and Schwanengel [7]) of a minimal locally compact group which is not totally minimal is

$$\mathbb{R} \rtimes \mathbb{R}_+ \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{R}_+, b \in \mathbb{R} \right\}.$$

Compact groups are totally minimal. Minimal abelian groups are necessarily precompact by a theorem of Prodanov–Stoyanov [29]. An interesting and useful generalization of this classical result has been found by T. Banakh [4]. For every minimal group G its center Z(G) is precompact. So, if G is, in addition, *supcomplete* (i.e., complete with respect to its two-sided uniformity) then Z(G) must be compact. For this reason, the group $GL(n, \mathbb{R})$ is not minimal. However, there are closed nonminimal subgroups of $GL(n, \mathbb{R})$ with compact (even, trivial) center. Indeed, the rank-two discrete free group F_2 is embedded into $SL(2, \mathbb{Z})$. Now recall that F_2 , being residually finite, admits a precompact group topology.

The minimality of Lie groups has been studied by many authors. Among others, we refer to van Est [14], Omori [26], Goto [16], Remus–Stoyanov [30] and the references therein. By Omori [26], connected nilpotent Lie groups with compact center are minimal. In particular, the classical Weyl–Heisenberg group $(\mathbb{T} \oplus \mathbb{R}) \rtimes_{\alpha} \mathbb{R}$ is minimal, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Moreover, as it was proved in [8], the *Generalized Weyl–Heisenberg groups* $H_0(V) = (\mathbb{T} \oplus V) \rtimes V^*$, defined for every Banach space V, are minimal.

The affine groups $\mathbb{R}^n \rtimes \operatorname{GL}(n, \mathbb{R})$ are minimal (Remus–Stoyanov [30]). Every closed matrix subgroup $G \leq \operatorname{GL}(n, \mathbb{R})$ is a retract of a minimal Lie group of dimension $2n + 1 + \dim(G)$. For every locally compact abelian group G and its dual G^* , the generalized Heisenberg group $(\mathbb{T} \oplus G^*) \rtimes G$ is minimal (see [22]). Therefore, every locally compact abelian group is a group retract of a locally compact minimal group. By [25], every topological group is a group retract of a minimal group.

By a result of Mayer [21], a locally compact connected group is totally minimal if and only if for every closed normal subgroup N of G the center Z(G/N) is compact. In addition to $SL(n, \mathbb{R})$, the following

concrete classical groups are totally minimal: the Euclidean motion group $\mathbb{R}^n \rtimes \mathrm{SO}(n, \mathbb{R})$ and the Lorentz group $\mathbb{R}^n \rtimes \mathrm{SL}(n, \mathbb{R})$.

The unitary group of an infinite-dimensional Hilbert space is one of the most influential examples of a totally minimal group (see Stoyanov [35]). By a result of Duchesne [13], the isometry group of the infinite dimensional separable hyperbolic space is minimal. The locally compact solvable groups having all subgroups minimal were characterized recently in [39].

Acknowledgment

We thank the referee for the valuable report. We also thank U. Bader, D. Dikranjan, A. Elashvili, B. Kunyavskii and G. Soifer for their useful suggestions. After reading a preliminary version of this work, Dikranjan pointed out the connection between our results and Fermat primes.

2. Preliminaries

A subset $B \subseteq \mathbb{F}$ of a topological field \mathbb{F} is *bounded* if for every neighborhood U(0) there exists a neighborhood V(0) such that $VB \subseteq U$. A subset U of \mathbb{F} that contains zero is *retrobounded* if $(\mathbb{F} \setminus U)^{-1}$ is bounded.

If retrobounded neighborhoods of zero form a fundamental system of neighborhoods, then \mathbb{F} is said to be *locally retrobounded*. It is equivalent (see [38, Theorem 19.12]) to say that all neighborhoods of zero are retrobounded.

Remark 2.1. The completion $\widehat{\mathbb{F}}$ of a locally retrobounded field \mathbb{F} is again a locally retrobounded field (see [38, Theorems 13.9 and 8.3]). In general, the completion of a topological field is only a commutative topological ring and not always a field (see [38, p. 439]).

Following Nachbin, a topological field \mathbb{F} is said to be *strictly minimal* (or, *straight*, [38]) if \mathbb{F} is a minimal \mathbb{F} -module over \mathbb{F} . Any non-discrete *locally retrobounded field* K is strictly minimal. It is still unknown if any strictly minimal topological field is necessarily locally retrobounded (see [38, p. 487]). By [23], a topological field \mathbb{F} is strictly minimal if and only if the semidirect product $\mathbb{F} \rtimes \mathbb{F}^{\times}$ is a minimal topological group. Compare with the case of the group $\mathbb{R} \rtimes \mathbb{R}_+$ (Dierolf and Schwanengel) mentioned above. For another similar result, see Theorem 3.4 below.

A topological field is locally retrobounded if, for example, it is linearly ordered or topologized by an absolute value.

Definition 2.2. (for example, [20, p. 26]) A *local field* is a non-discrete locally compact field.

Every local field \mathbb{F} admits an absolute value (induced by the Haar measure). Therefore, any subfield of a local field is locally retrobounded. If the set $\{|n \cdot 1_{\mathbb{F}}| : n \in \mathbb{N}\}$ is unbounded, then \mathbb{F} is called *archimedean*. Otherwise, \mathbb{F} is a *non-archimedean* local field (see [33]). A subset of a local field is compact if and only if it is closed and bounded.

2.1. Roots of unity

Denote by $\mu_n(\mathbb{F})$ the finite subgroup of \mathbb{F}^{\times} consisting of all *n*-th roots of unity. Then SL(*n*, \mathbb{F}) has finite center (e.g., see [31, 3.2.6])

$$Z = Z(\mathrm{SL}(n, \mathbb{F})) = \{\lambda I : \lambda \in \mu_n\}$$

which, sometimes, will be denoted in Sections 3 and 4 simply by Z.

The following known lemma will be used in the sequel. We prove it for the sake of completeness.

Lemma 2.3. If $z^n = 1$ and $z \in \mathbb{Q}(i)$, then $z \in \{\pm 1, \pm i\}$.

Proof. If $z^n = 1$, then $(\bar{z})^n = 1$, where \bar{z} is the complex conjugate of z. It follows that both z and \bar{z} are algebraic integers. By [17, Proposition 6.1.5], $z + \bar{z}$ is an algebraic integer. Since $z \in \mathbb{Q}(i)$, the algebraic integer $z + \bar{z}$ is also rational. By [17, Proposition 6.1.1], $z + \bar{z}$ is an integer. As |z| = 1 and $z \in \mathbb{Q}(i)$, we deduce that $z \in \{\pm 1, \pm i\}$. \Box

The following result about the simplicity of $PSL(n, \mathbb{F}) = SL(n, \mathbb{F})/Z(SL(n, \mathbb{F}))$ is due to Jordan and Dickson (see [31, 3.2.9]).

Fact 2.4. Let \mathbb{F} be a field. If either n > 2 or n = 2 and $|\mathbb{F}| > 3$, then $PSL(n, \mathbb{F})$ is algebraically simple.

2.2. G-minimality and semidirect products

The following result is known as *Merson's Lemma* ([11, Lemma 7.2.3] or [9, Lemma 4.4]).

Fact 2.5. Let (G, γ) be a (not necessarily Hausdorff) topological group and H be a subgroup of G. If $\gamma_1 \subseteq \gamma$ is a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1/H = \gamma/H$, then $\gamma_1 = \gamma$.

As a corollary, one has:

Fact 2.6. [8, Corollary 3.2] A topological group G is minimal if and only if it contains a subgroup H which is both relatively minimal and co-minimal in G.

By $X \rtimes_{\pi} G$, we mean the (topological) semidirect product of the (topological) groups X, G, where $\pi: G \times X \to X$ is a given (continuous) action by group automorphisms. We denote by \mathbb{F}^{\times} the multiplicative group $\mathbb{F} \setminus \{0\}$. Given a semidirect product $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$, we identify \mathbb{F} with $\mathbb{F} \rtimes_{\alpha} \{1\}$ and \mathbb{F}^{\times} with $\{0\} \rtimes_{\alpha} \mathbb{F}^{\times}$.

If a topological group G continuously acts on a topological group X by group automorphisms, then X is called a *G*-group. Assuming that the *G*-group X has no strictly coarser Hausdorff group topology such that the action of G on X remains continuous, then X is *G*-minimal.

Fact 2.7. [8, Proposition 4.4] Let (G, σ) be a topological group and (X, τ) be a G-group. The following are equivalent:

(1) X is G-minimal.

(2) X is relatively minimal in the topological semidirect product $M := (X \rtimes G, \gamma)$.

2.3. $ST^+(n, \mathbb{F})$ as a topological semidirect product

Recall the following topological matrix group

$$ST^{+}(n, \mathbb{F}) := \left\{ \bar{x} = (x_{ij}) | x_{ij} \in \mathbb{F}, x_{ij} = 0 \ \forall i > j, \ \prod_{i=1}^{n} x_{ii} = 1 \right\},\$$

and its topological subgroups

$$\mathsf{N} := \{ \bar{x} | x_{ii} = 1 \ \forall i \},\$$

$$\mathsf{A} := \left\{ \bar{x} | x_{ij} = 0 \ \forall i \neq j, \ \prod_{i=1}^{n} x_{ii} = 1 \right\}.$$

Lemma 2.8. $ST^+(n, \mathbb{F})$ is a topological semidirect product of the subgroups N and A. That is, $ST^+(n, \mathbb{F}) \cong N \rtimes_{\alpha} A$, where α is the action by conjugations.

Proof. Clearly, $G := ST^+(n, \mathbb{F}) = NA$ and $N \cap A$ is trivial. Moreover, N is a closed normal subgroup in G and A is a closed subgroup in G. So, algebraically, G is isomorphic to the semidirect product $N \rtimes_{\alpha} A$, where α is the action by conjugations. The corresponding isomorphism is the map

$$i: \mathbb{N} \rtimes_{\alpha} \mathbb{A} \to \mathrm{ST}^+(n, \mathbb{F}) \quad (\bar{b}, \bar{a}) \mapsto \bar{b} \cdot \bar{a}.$$

Observe that $\bar{c} = \bar{b} \cdot \bar{a}$ satisfies $c_{ij} = b_{ij}a_{jj}$ for every i < j and $c_{ii} = a_{ii}$ for every i.

Using the definition of the pointwise topology (and the fact that \mathbb{F} is a topological field), it is easy to see that i is a homeomorphism. \Box

Remark 2.9.

- (1) Unless otherwise stated, below we assume that all fields are of characteristic distinct from 2.
- (2) By $\widehat{\mathbb{F}}$ we always mean the completion of a locally retrobounded field \mathbb{F} which always exists (Remark 2.1(1)). If \mathbb{F} is a subfield of a local field P, then the completion $\widehat{\mathbb{F}}$ can be identified with the closure of \mathbb{F} in P. In case \mathbb{F} is infinite then $\widehat{\mathbb{F}}$ is also a local field, as the local field P contains no infinite discrete subfields (see [20, p. 27]).

3. Minimality of $ST^+(n, \mathbb{F})$

By Prodanov–Stoyanov's theorem, if \mathbb{F} is an infinite topological field, then neither $\mathsf{N} := \mathrm{UT}(n, \mathbb{F})$ nor $\mathrm{T}^+(n, \mathbb{F})$ are minimal as their centers are not precompact. In this section, we study the minimality of $\mathrm{ST}^+(n, \mathbb{F})$. This case is also the key for further investigation.

3.1. Minimality of $ST^+(2, \mathbb{F})$

Fixing n = 2 in Lemma 2.8, we obtain the following subgroups of $SL(2, \mathbb{F})$ in a more explicit form:

$$ST^{+}(2, \mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}^{\times}, b \in \mathbb{F} \right\},$$
$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}^{\times} \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F} \right\}.$$

Lemma 3.1. The group $ST^+(2, \mathbb{F})$ is topologically isomorphic to the semidirect product $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$, where the action $\alpha : \mathbb{F}^{\times} \times \mathbb{F} \to \mathbb{F}$ is defined by $\alpha(a, b) = a^2b$.

Proof. As we already know G is topologically isomorphic to $N \rtimes_{\beta} A$, where β is the action by conjugations. Explicitly, we have the following topological group isomorphism:

$$\mathsf{N}\rtimes_{\beta}\mathsf{A}\to\mathrm{ST}^+(2,\mathbb{F})\quad (\bar{b},\bar{a})\mapsto \begin{pmatrix} a&\frac{b}{a}\\ 0&a^{-1} \end{pmatrix},$$

where $\bar{a} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Since

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix},$$

it follows that $\mathsf{N} \rtimes_{\beta} \mathsf{A} \cong \mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$ which completes the proof. \Box

Proposition 3.2. $ST^+(2,\mathbb{F})$ is a minimal topological group for every non-discrete locally retrobounded complete field \mathbb{F} .

Proof. By Lemma 3.1, it is equivalent to prove that $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$ is minimal, where the action $\alpha \colon \mathbb{F}^{\times} \times \mathbb{F} \to \mathbb{F}$ is defined by $\alpha(a, b) = a^2 b$. We will show that the subgroup \mathbb{F} is both relatively minimal and co-minimal in $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$. This will prove the minimality of the latter by Fact 2.6. Denote by τ and τ^{\times} the given topologies on \mathbb{F} and \mathbb{F}^{\times} , respectively, and let γ be the product of these topologies on $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$.

Relative minimality of \mathbb{F}

To establish the relative minimality of \mathbb{F} in $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$, it is equivalent to show by Fact 2.7 that \mathbb{F} is \mathbb{F}^{\times} -minimal. For this purpose, let $\sigma \subseteq \tau$ be a coarser Hausdorff group topology on \mathbb{F} such that

$$\alpha \colon \mathbb{F}^{\times} \times (\mathbb{F}, \sigma) \to (\mathbb{F}, \sigma) \quad \alpha(a, b) = a^2 b \tag{3.1}$$

remains continuous. We have to show that $\sigma = \tau$.

Let U be an arbitrary τ -neighborhood of 0. We will show that U is a σ -neighborhood of 0 and thus $\sigma = \tau$. Since σ is a Hausdorff group topology and the field \mathbb{F} has characteristic distinct from 2 (Remark 2.9), there exists a σ -neighborhood Y of 0 such that $4 \notin Y - Y$. By the continuity of α and since \mathbb{F}^{\times} is open in \mathbb{F} , there exist a symmetric τ -neighborhood V of 0 and a σ -neighborhood W of 0 such that

$$\alpha((1+V) \times W) \subseteq Y. \tag{3.2}$$

Since \mathbb{F} is locally retrobounded, U is retrobounded. That is, $(\mathbb{F} \setminus U)^{-1}$ is bounded in (\mathbb{F}, τ) . So, there exists a τ -neighborhood M_1 of zero in \mathbb{F} such that $(\mathbb{F} \setminus U)^{-1}M_1 \subseteq V$. Choose another τ -neighborhood M_2 of zero such that $M_2M_2 \subseteq M_1$. Since \mathbb{F} is not discrete, M_2 contains a nonzero element λ . It follows that

$$(\mathbb{F} \setminus U)^{-1} \lambda^2 \subseteq V.$$

By the continuity of α (see (3.1)), we obtain that $\lambda^2 W$ is a σ -neighborhood of 0. We claim that $\lambda^2 W \subseteq U$ (this will imply that U is a σ -neighborhood of 0 and $\sigma = \tau$). Assume by contradiction that there exists $\mu \in W$ such that $\lambda^2 \mu \notin U$. Then

$$\mu^{-1} = (\mu^{-1}\lambda^{-2})\lambda^2 \in (\mathbb{F} \setminus U)^{-1}\lambda^2 \subseteq V.$$

By (3.2), we have

$$\alpha(1+\mu^{-1},\mu) - \alpha(1-\mu^{-1},\mu) = (1+\mu^{-1})^2\mu - (1-\mu^{-1})^2\mu = 4 \in Y - Y,$$

a contradiction.

Co-minimality of \mathbb{F}

Next, we prove that \mathbb{F} is co-minimal in $(\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}, \gamma)$. Let $\mu \subseteq \gamma$ be a coarser Hausdorff group topology on $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$. We have to show that the coset topology μ/\mathbb{F} on \mathbb{F}^{\times} is just the original topology τ^{\times} (which is equal to γ/\mathbb{F}). It is equivalent to show that the projection $q: (\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}, \mu) \to (\mathbb{F}^{\times}, \tau^{\times})$ is continuous. Since $\gamma/\mathbb{F} = \tau^{\times}$, this will imply that $\gamma/\mathbb{F} = \mu/\mathbb{F}$, establishing the co-minimality of \mathbb{F} . It suffices to show that the homomorphism q is continuous at the identity (0, 1). Let U be a τ^{\times} -neighborhood of 1. We will find a μ -neighborhood V of (0, 1) such that $q(V) \subseteq U$. Since \mathbb{F}^{\times} is open in (\mathbb{F}, τ) , it follows that there exists a τ -neighborhood O of 0 such that $1 + O \subseteq U$. Being complete and relatively minimal in $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$, the subgroup \mathbb{F} is also μ -closed. Hence, the group topology μ/\mathbb{F} is Hausdorff. Taking into account also the fact that char(\mathbb{F}) $\neq 2$, we find μ/\mathbb{F} -neighborhood V_1 of (0, 1) such that $q(V_1) = W_1$. Using the fact that \mathbb{F}^{\times} is open in (\mathbb{F}, τ) and since $\mu/\mathbb{F} \subseteq \gamma/\mathbb{F} = \tau^{\times}$, we obtain that $M = 1 + W_2$ is a τ -neighborhood of 0. The definitions of V_1 and M together with the fact that $W_1 \cap W_2 = \emptyset$ imply that

$$q(V_1) + 1 \subseteq \mathbb{F} \setminus M. \tag{3.3}$$

Since \mathbb{F} is locally retrobounded, $(\mathbb{F} \setminus M)^{-1}$ is bounded. So, there exists a τ -neighborhood B of 0 such that

$$(\mathbb{F} \setminus M)^{-1} B \subseteq O. \tag{3.4}$$

By the relative minimality of \mathbb{F} , there exists a μ -neighborhood V_2 of (0, 1) such that $V_2 \cap \mathbb{F} = B$. Since μ is a group topology, there exists a μ -neighborhood V_3 of (0, 1) such that the commutator $[(b, a), (1, 1)] \in V_2$ for every $(b, a) \in V_3$. Computing this commutator, we obtain

$$[(b,a),(1,1)] = (b,a)(1,1)(b,a)^{-1}(1,1)^{-1} = (a^2 - 1,1) \in V_2 \cap \mathbb{F} = B.$$
(3.5)

Now we show that $q(V) \subseteq U$ for $V = V_1 \cap V_3$, which is a μ -neighborhood of (0, 1). Fix an arbitrary $(b, a) \in V$. By (3.3) and since $V \subseteq V_1$, we obtain

$$(a+1)^{-1} \in (q(V)+1)^{-1} \subseteq (\mathbb{F} \setminus M)^{-1}.$$

Moreover, V is also a subset of V_3 . So, (3.5) implies that $a^2 - 1 \in B$. Using (3.4), we now have

$$q(b,a) - 1 = a - 1 = (a+1)^{-1}(a^2 - 1) \in (\mathbb{F} \setminus M)^{-1}B \subseteq O.$$

Finally, we get $q(b, a) = 1 + (q(b, a) - 1) \in 1 + O \subseteq U$, as needed.

Now we can conclude that the topological group $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$ is minimal. \Box

Let *H* be a subgroup of a topological group *G*. Recall that *H* is *essential* in *G* if $H \cap L \neq \{e\}$ for every non-trivial closed normal subgroup *L* of *G*. The following minimality criterion of dense subgroups is well-known (for compact *G* see also [28,34]).

Fact 3.3. [5, Minimality Criterion] Let H be a dense subgroup of a topological group G. Then H is minimal if and only if G is minimal and H is essential in G.

The following theorem deals with the minimality of $ST^+(n, \mathbb{F})$ only for n = 2 in case \mathbb{F} is a non-discrete locally retrobounded field. However, if \mathbb{F} is a local field, then $ST^+(n, \mathbb{F})$ is minimal for every $n \in \mathbb{N}$ (see Theorem 3.19 below).

Theorem 3.4. $ST^+(2,\mathbb{F})$ is minimal for every non-discrete locally retrobunded field \mathbb{F} .

Proof. The completion $\widehat{\mathbb{F}}$ of a locally retrobunded field \mathbb{F} is a locally retrobunded field (Remark 2.1(1)). According to Lemma 3.1, $\operatorname{ST}^+(2,\widehat{\mathbb{F}})$ is isomorphic to $\widehat{\mathbb{F}} \rtimes_{\alpha}(\widehat{\mathbb{F}})^{\times}$, where the action $\alpha : (\widehat{\mathbb{F}})^{\times} \times \widehat{\mathbb{F}} \to \widehat{\mathbb{F}}$ is defined by $\alpha(a,b) = a^2b$. Clearly, $\operatorname{ST}^+(2,\widehat{\mathbb{F}})$ contains $\operatorname{ST}^+(2,\mathbb{F}) \cong \mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$ as a dense subgroup. By Proposition 3.2, $\widehat{\mathbb{F}} \rtimes_{\alpha} (\widehat{\mathbb{F}})^{\times}$ is minimal. To establish the minimality of $\operatorname{ST}^+(2,\widehat{\mathbb{F}})$ it is sufficient to prove, in view of Fact 3.3, that the subgroup $\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$ is essential in $\widehat{\mathbb{F}} \rtimes_{\alpha} (\widehat{\mathbb{F}})^{\times}$.

Let *L* be a closed non-trivial normal subgroup of $\widehat{\mathbb{F}} \rtimes_{\alpha} (\widehat{\mathbb{F}})^{\times}$. We have to show that $L \cap (\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times})$ is non-trivial. Let (m, n) be a non-trivial element of *L*. If $n \neq \pm 1$, then $1 - n^2 \neq 0$. Letting $a = (1 - n^2)^{-1}$ and computing the commutator [(a, 1), (m, n)], we obtain

$$[(a,1),(m,n)] = (1,1) \in L \cap (\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}).$$

Now assume that $n \in \{1, -1\}$. Since (m, n) is non-trivial and

$$\{(0,-1),(-1,1),(-1,-1),(1,1),(1,-1)\} \subseteq \mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times}$$

we may assume that $m \notin \{0, -1, 1\}$. Moreover, without loss of generality, n = 1. Indeed, this follows from the fact that $(m, n)^2 = (2m, 1)$. So, $(m, n) = (m, 1) \in L$, where $m \notin \{0, -1, 1\}$. For every $a, b \in \mathbb{F}^{\times}$, we have

$$(0,a)(m,1)(0,a)^{-1}(0,b)(m,1)^{-1}(0,b)^{-1} = ((a^2 - b^2)m,1) \in L,$$

as L is normal in G. In particular, letting $a = 2^{-1}(1 + m^{-1})$ and b = a - 1, we conclude that

$$((a^2 - b^2)m, 1) = ((a - b)(a + b)m, 1) = (1, 1) \in L \cap (\mathbb{F} \rtimes_{\alpha} \mathbb{F}^{\times})$$

This proves that $ST^+(2,\mathbb{F})$ is essential in $ST^+(2,\widehat{\mathbb{F}})$. \Box

Theorem 3.4 is not true for an arbitrary n. Indeed, in Example 3.5 below we prove that $ST^+(n, \mathbb{Q}(i))$ is not minimal in case n is not a power of 2.

Example 3.5. Let *n* be a natural number that is not a power of 2. Then the group $ST^+(n, \mathbb{Q}(i))$ is not minimal. Indeed, by our assumption on *n*, there exists an odd prime *p* that divides *n*. We claim that the finite (hence, closed) central subgroup $L = \{\lambda I : \lambda^p = 1\}$ of $ST^+(n, \mathbb{C})$ trivially intersects $ST^+(n, \mathbb{Q}(i))$. To see this, observe that by Lemma 2.3 if $\lambda^p = 1$ and $\lambda \in \mathbb{Q}(i)$, then $\lambda = 1$. This means that $ST^+(n, \mathbb{Q}(i))$ is not essential in $ST^+(n, \mathbb{C})$. By the Minimality Criterion (Fact 3.3), $ST^+(n, \mathbb{Q}(i))$ is not minimal.

In view of Theorem 3.4 and Example 3.5, the following natural questions arise:

Question 3.6. Let $k \in \mathbb{N}$ and \mathbb{F} be a non-discrete locally retrobounded field. Is $ST^+(2^k, \mathbb{F})$ minimal? What if, in addition, \mathbb{F} is complete?

3.2. Minimality of $ST^+(n, \mathbb{F})$ and $ST^+(n, \mathbb{F})/Z(SL(n, \mathbb{F}))$

Let \mathbb{F} be a topological field. Recall that by Lemma 2.8, $\mathrm{ST}^+(n, \mathbb{F}) \cong \mathbb{N} \rtimes_{\alpha} A$, where $\mathbb{N} = \mathrm{UT}(n, \mathbb{F})$, A is the group of diagonal matrices with determinant 1 and α is the action by conjugations. In the sequel, we sometimes identify $\mathrm{ST}^+(n, \mathbb{F})$ with $\mathbb{N} \rtimes_{\alpha} A$.

For $1 \leq i < j \leq n$, let $G_{i,j}$ be the 1-parameter subgroup of N such that for every matrix $X \in G_{i,j}$ we have $p_{k,l}(X) = x_{k,l} = 0$ if $k \neq l$ and $(k,l) \neq (i,j)$, where $p_{k,l} : \operatorname{GL}(n,\mathbb{F}) \to \mathbb{F}$, $p_{k,l}(X) = x_{k,l}$ is the canonical coordinate projection.

Denote by $H(n, \mathbb{F})$ the 2n + 1-dimensional Heisenberg group over a field \mathbb{F} . More precisely, define $H(n, \mathbb{F})$ as the following subgroup of $UT(n + 2, \mathbb{F})$

$$\mathbf{H}(n,\mathbb{F}) := \left\{ \left. \begin{pmatrix} 1 & a & b \\ 0 & I_n & c \\ 0 & 0 & 1 \end{pmatrix} \right| \, a, c \in \mathbb{F}^n, b \in \mathbb{F} \right\},\$$

where I_n is the identity matrix of size n. As a corollary of [22, Proposition 2.9] we have the following.

Corollary 3.7. Let G be a topological subgroup of $GL(n+2,\mathbb{F})$ containing $H(n,\mathbb{F})$. If the corner 1-parameter subgroup $G_{1,n+2}$ of $H(n,\mathbb{F})$ is relatively minimal in G, then $H(n,\mathbb{F})$ is relatively minimal in G.

The proof of the following proposition heavily relies on the algebraic structure of the matrix groups involved.

Proposition 3.8. Let \mathbb{F} be a non-discrete locally retrobounded complete field. Then the subgroup $\mathsf{N} = \mathrm{UT}(n, \mathbb{F})$ is relatively minimal in $\mathrm{ST}^+(n, \mathbb{F})$.

Proof. By Theorem 3.4, $ST^+(2, \mathbb{F})$ is minimal. In particular, its subgroup $UT(2, \mathbb{F})$ is relatively minimal in $ST^+(2, \mathbb{F})$. The corner 1-parameter group $G_{1,3}$ is a subgroup of

$$P := \left\{ \left. \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \right| \, a, b \in \mathbb{F} \right\}.$$

Observe that P is topologically isomorphic to the minimal group $ST^+(2, \mathbb{F})$. So, $G_{1,3}$ is relatively minimal in P and hence also in the larger group $ST^+(3, \mathbb{F})$. By Corollary 3.7, the Heisenberg group $UT(3, \mathbb{F}) = H(1, \mathbb{F})$ is relatively minimal in $ST^+(3, \mathbb{F})$.

Continuing by induction on n and assuming that $UT(n, \mathbb{F})$ is relatively minimal in $ST^+(n, \mathbb{F})$, we will prove that $UT(n + 2, \mathbb{F})$ is relatively minimal in $ST^+(n + 2, \mathbb{F})$. Fix $n \ge 2$ and observe that $H(n, \mathbb{F})$ is a normal subgroup of $ST^+(n + 2, \mathbb{F})$. In particular, $H(n, \mathbb{F})$ is a normal subgroup of $UT(n + 2, \mathbb{F})$.

Moreover, we have

$$\mathrm{UT}(n+2,\mathbb{F}) = \widetilde{\mathrm{UT}(n,\mathbb{F})} \operatorname{H}(n,\mathbb{F}), \tag{3.6}$$

where

$$\widetilde{\mathrm{UT}(n,\mathbb{F})} = \left\{ \left(\begin{array}{ccc} 1 & 0_{1\times n} & 0\\ 0_{n\times 1} & X & 0_{n\times 1}\\ 0 & 0_{1\times n} & 1 \end{array} \right) \middle| X \in \mathrm{UT}(n,\mathbb{F}) \right\}$$

Indeed, if $X \in UT(n, \mathbb{F})$, $a, c \in \mathbb{F}^n$ and $b \in \mathbb{F}$, then

$$\begin{pmatrix} 1 & 0_{1\times n} & 0\\ 0_{n\times 1} & X^{-1} & 0_{n\times 1}\\ 0 & 0_{1\times n} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & b\\ 0_{n\times 1} & X & c\\ 0 & 0_{1\times n} & 1 \end{pmatrix} \in \mathcal{H}(n, \mathbb{F}).$$

Claim 1. $UT(n, \mathbb{F})$ and $H(n, \mathbb{F})$ are relatively minimal in $ST^+(n+2, \mathbb{F})$.

Proof. Denote by $\psi: \widetilde{\mathrm{ST}^+(n,\mathbb{F})} \to \mathrm{ST}^+(n,\mathbb{F})$ the natural topological group isomorphism from

$$\widetilde{\mathrm{ST}^+(n,\mathbb{F})} = \left\{ \left(\begin{array}{ccc} 1 & 0_{1\times n} & 0\\ 0_{n\times 1} & X & 0_{n\times 1}\\ 0 & 0_{1\times n} & 1 \end{array} \right) \mid X \in \mathrm{ST}^+(n,\mathbb{F}) \right\}$$

onto $\operatorname{ST}^+(n, \mathbb{F})$. Since $\psi(\operatorname{UT}(n, \mathbb{F})) = \operatorname{UT}(n, \mathbb{F})$, we deduce by the induction hypothesis that $\operatorname{UT}(n, \mathbb{F})$ is relatively minimal in $\operatorname{ST}^+(n, \mathbb{F})$ and hence also in the larger group $\operatorname{ST}^+(n+2, \mathbb{F})$.

The corner 1-parameter group $G_{1,n+2}$ is a subgroup of

$$P := \left\{ \left(\begin{array}{ccc} a & 0_{1 \times n} & b \\ 0_{n \times 1} & I_n & 0_{n \times 1} \\ 0 & 0_{1 \times n} & a^{-1} \end{array} \right) \middle| a \in \mathbb{F}^{\times}, b \in \mathbb{F} \right\}$$

and P is topologically isomorphic (Theorem 3.4) to the minimal group $ST^+(2, \mathbb{F})$. So, $G_{1,n+2}$ is relatively minimal in P and also in the larger group $ST^+(n+2,\mathbb{F})$. Now by Corollary 3.7, the Heisenberg group $H(n,\mathbb{F})$ is relatively minimal in $ST^+(n+2,\mathbb{F})$. \Box

Let $\sigma \subseteq \tau_p$ be a coarser Hausdorff group topology on $\mathrm{ST}^+(n+2,\mathbb{F})$, where τ_p is the given (pointwise) topology. Clearly, $\mathrm{H}(n,\mathbb{F}) \cap \widetilde{\mathrm{UT}(n,\mathbb{F})}$ is trivial. So by (3.6), we deduce that $\mathrm{UT}(n+2,\mathbb{F})$ is algebraically isomorphic to $\mathrm{H}(n,\mathbb{F}) \rtimes \widetilde{\mathrm{UT}(n,\mathbb{F})}$.

Claim 2. $(\mathrm{UT}(n+2,\mathbb{F}),\sigma|_{\mathrm{UT}(n+2,\mathbb{F})})$ is topologically isomorphic to

$$(\mathrm{H}(n,\mathbb{F}),\sigma|_{\mathrm{H}(n,\mathbb{F})})\rtimes(\widetilde{\mathrm{UT}(n,\mathbb{F})},\sigma|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}).$$

Proof. Consider the quotient map

$$q: (\mathrm{ST}^+(n+2,\mathbb{F}),\sigma) \to \mathrm{ST}^+(n+2,\mathbb{F})/\mathrm{H}(n,\mathbb{F}).$$

From Claim 1 we obtain that $\sigma|_{\mathrm{H}(n,\mathbb{F})} = \tau_p|_{\mathrm{H}(n,\mathbb{F})}$. So, the completeness of \mathbb{F} implies that $\mathrm{H}(n,\mathbb{F})$ is σ -closed in $\mathrm{ST}^+(n+2,\mathbb{F})$. This means that $\sigma/\mathrm{H}(n,\mathbb{F})$ is Hausdorff. Clearly, $\mathrm{ST}^+(n,\mathbb{F}) \cap \mathrm{H}(n,\mathbb{F})$ is trivial. Hence, the restriction

$$q|_{\widetilde{\operatorname{ST}^+(n,\mathbb{F})}} \colon (\widetilde{\operatorname{ST}^+(n,\mathbb{F})},\sigma|_{\widetilde{\operatorname{ST}^+(n,\mathbb{F})}}) \to q(\widetilde{\operatorname{ST}^+(n,\mathbb{F})})$$

is a continuous isomorphism into a Hausdorff group. By the induction hypothesis, $UT(n, \mathbb{F})$ is relatively minimal in $\widetilde{ST^+(n, \mathbb{F})}$ and

$$q|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}} \colon (\widetilde{\mathrm{UT}(n,\mathbb{F})},\sigma|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}) \to q(\widetilde{\mathrm{UT}(n,\mathbb{F})})$$

is a topological group isomorphism. Since $q(UT(n, \mathbb{F})) = UT(n+2, \mathbb{F})/H(n, \mathbb{F})$ and using [32, Proposition 6.17], we deduce that $(UT(n+2, \mathbb{F}), \sigma|_{UT(n+2, \mathbb{F})})$ is topologically isomorphic to the semidirect product

$$(\mathrm{H}(n,\mathbb{F}),\sigma|_{H(n,\mathbb{F})})\rtimes (\widetilde{\mathrm{UT}(n,\mathbb{F})},\sigma|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}). \quad \Box$$

By Claim 1, $(\sigma|_{\mathrm{UT}(n+2,\mathbb{F})})|_{\mathrm{H}(n,\mathbb{F})} = (\tau_p|_{\mathrm{UT}(n+2,\mathbb{F})})|_{\mathrm{H}(n,\mathbb{F})}$. By Claim 2,

$$(\sigma|_{\mathrm{UT}(n+2,\mathbb{F})})/\mathrm{H}(n,\mathbb{F}) = \sigma|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}$$

and

$$(\tau_p|_{\mathrm{UT}(n+2,\mathbb{F})})/\mathrm{H}(n,\mathbb{F}) = \tau_p|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}.$$

Using Claim 1 again, we obtain that $\sigma|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}} = \tau_p|_{\widetilde{\mathrm{UT}(n,\mathbb{F})}}$. It follows that

$$(\sigma|_{\mathrm{UT}(n+2,\mathbb{F})})/\mathrm{H}(n,\mathbb{F}) = (\tau_p|_{\mathrm{UT}(n+2,\mathbb{F})})/\mathrm{H}(n,\mathbb{F})$$

and by Merson's Lemma (Fact 2.5) we deduce that $\sigma|_{\mathrm{UT}(n+2,\mathbb{F})} = \tau_p|_{\mathrm{UT}(n+2,\mathbb{F})}$, as needed. \Box

Lemma 3.9. For every $1 \le i \le n-1$, let $E_{i,i+1} \in G_{i,i+1}$ with $p_{i,i+1}(E_{i,i+1}) = 1$. Then for every $B \in A$, we have

$$p_{i,i+1}(\alpha(B, E_{i,i+1})) = p_{i,i+1}(BE_{i,i+1}B^{-1}) = p_{i,i}(B)(p_{i+1,i+1}(B))^{-1}.$$
(3.7)

Proof. Easy calculations. \Box

Lemma 3.10. Let \mathbb{F} be a topological field and $n \geq 2$ be a positive number. Suppose that τ is a group topology on A such that all n - 1 actions

$$\alpha_i: (\mathsf{A}, \tau) \times (G_{i,i+1}, \tau_p) \to (G_{i,i+1}, \tau_p), \quad i \in \{1, \cdots, n-1\}$$

are continuous, where τ_p is the pointwise topology and $\alpha_i = \alpha|_{\mathsf{A} \times G_{i,i+1}}$. Then

(1) the homomorphism

$$t_i: \mathsf{A} \to \mathbb{F}^{\times}, \ t_i(B) = (p_{1,1}(B))(p_{i+1,i+1}(B))^{-1}$$

is continuous for every $1 \le i \le n-1$;

(2) the homomorphism $m_i: A \to \mathbb{F}^{\times}, m_i(B) = (p_{i,i}(B))^n$ is continuous for every $1 \le i \le n$.

Proof. (1) Since $\alpha_1 \colon (\mathsf{A}, \tau) \times (G_{1,2}, \tau_p) \to (G_{1,2}, \tau_p)$ is continuous and τ_p is the pointwise topology, (3.7) guarantees that t_1 is continuous. Now assume that t_{i-1} is continuous and let us see that t_i is continuous. Using (3.7) again, in view of the continuity of α_i , we deduce that the homomorphism

$$\psi \colon \mathsf{A} \to \mathbb{F}^{\times}, \ \psi(B) = p_{i,i}(B)(p_{i+1,i+1}(B))^{-1}$$

is continuous. The equality $t_i(B) = t_{i-1}(B)\psi(B)$ completes the proof.

(2) For every $B \in A$ we have $\prod_{i=1}^{n} p_{i,i}(B) = 1$. This implies that $\prod_{i=1}^{n-1} t_i = (p_{1,1})^n$. By item (1) and the fact that \mathbb{F} is a topological field, we deduce that $m_1 = (p_{1,1})^n$ is continuous. We use the equality $m_i = m_1(t_{i-1})^{-n}$ to establish the continuity of m_i for every $1 < i \leq n$. \Box

3.3. The action $\tilde{\alpha}$

Denote by τ_p the original pointwise topology on $\mathrm{ST}^+(n,\mathbb{F})$ and by $\tilde{\tau_p}$ the quotient topology on $\mathrm{ST}^+(n,\mathbb{F})/Z$ with respect to the homomorphism

$$q: \operatorname{ST}^+(n, \mathbb{F}) \to \operatorname{ST}^+(n, \mathbb{F})/Z,$$

where $Z = Z(\mathrm{SL}(n, \mathbb{F}))$. The continuous action $\alpha \colon (\mathsf{A}, \tau_p|_{\mathsf{A}}) \times (\mathsf{N}, \tau_p|_{\mathsf{N}}) \to (\mathsf{N}, \tau_p|_{\mathsf{N}})$ induces the action

$$\widetilde{\alpha} \colon (q(\mathsf{A}), \widetilde{\tau_p}|_{q(\mathsf{A})}) \times (q(\mathsf{N}), \widetilde{\tau_p}|_{q(\mathsf{N})}) \to (q(\mathsf{N}), \widetilde{\tau}|_{q(\mathsf{N})}).$$

The next lemma will be used to prove the continuity of $\tilde{\alpha}$.

Lemma 3.11. The map $q|_{\mathsf{N}} \colon \mathsf{N} \to q(\mathsf{N})$ is a topological isomorphism.

Proof. The homomorphism $q|_{\mathsf{N}}$ is a bijection because $\mathsf{N} \cap Z$ is trivial. It suffices to show that $q|_{\mathsf{N}} \colon \mathsf{N} \to q(\mathsf{N})$ is an open map. Observe that q is open and $q^{-1}(q(\mathsf{N})) = \mathsf{N}Z$. This implies that the restriction map

$$q|_{\mathsf{N}Z}: q^{-1}(q(\mathsf{N})) = \mathsf{N}Z \to q(\mathsf{N})$$

is also open. Having finite index in NZ, the closed subgroup N is open in NZ. It follows that $q|_N$ is an open map. \Box

Lemma 3.12. The action $\widetilde{\alpha}$: $(q(\mathsf{A}), \widetilde{\tau_p}|_{q(\mathsf{A})}) \times (q(\mathsf{N}), \widetilde{\tau_p}|_{q(\mathsf{N})}) \to (q(\mathsf{N}), \widetilde{\tau_p}|_{q(\mathsf{N})})$ is continuous.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathsf{A} \times \mathsf{N} & \xrightarrow{\alpha} & \mathsf{N} \\ q & & & & & \\ q & & & & & \\ q(\mathsf{A}) \times q(\mathsf{N}) & \xrightarrow{\tilde{\alpha}} & q(\mathsf{N}) \end{array} \tag{3.8}$$

Fix an arbitrary $(a, n) \in \mathsf{A} \times \mathsf{N}$ and let U be a $\widetilde{\tau_p}|_{q(\mathsf{N})}$ -neighborhood of

$$\widetilde{\alpha}(q(a), (q(n)) = q(\alpha(a, n))$$

By the continuity of $q|_{\mathsf{N}}$, there exists a $\tau_p|_{\mathsf{N}}$ -neighborhood V of $\alpha(a, n)$ such that $q(V) \subseteq U$. By the continuity of α , there exist a $\tau_p|_{\mathsf{A}}$ -neighborhood W of a and a $\tau_p|_{\mathsf{N}}$ -neighborhood of n such that $\alpha(W \times O) \subseteq V$. Since $q^{-1}(q(\mathsf{A})) = \mathsf{A}$, it follows that $q|_{\mathsf{A}}$ is open. By Lemma 3.11, also $q|_{\mathsf{N}}$ is open. So, q(W) is a $\tilde{\tau}_p|_{q(\mathsf{A})}$ neighborhood W of q(a) and q(O) is a $\tilde{\tau}_p|_{q(\mathsf{N})}$ -neighborhood of q(n). Then

$$q(\alpha(W \times O)) = \widetilde{\alpha}(q(W) \times q(\mathsf{A})) \subseteq q(V) \subseteq U$$

which proves the continuity of $\tilde{\alpha}$ in (q(a), q(n)). \Box

Proposition 3.13. Let \mathbb{F} be a non-discrete locally retrobounded complete field. Then q(N) is q(A)-minimal with respect to the action $\tilde{\alpha}$.

Proof. By Lemma 3.12, the action $\widetilde{\alpha}$ is $(\widetilde{\tau_p}|_{q(\mathsf{A})}, \widetilde{\tau_p}|_{q(\mathsf{N})})$ -continuous. Let $\sigma \subseteq \widetilde{\tau_p}|_{q(\mathsf{N})}$ be a coarser Hausdorff group topology such that $\widetilde{\alpha}$ is $(\widetilde{\tau_p}|_{q(\mathsf{A})}, \sigma, \sigma)$ -continuous. We have to show that $\sigma = \widetilde{\tau_p}|_{q(\mathsf{N})}$.

Let us see that α is $(\tau_p|_{\mathsf{A}}, (q|_{\mathsf{N}})^{-1}(\sigma), (q|_{\mathsf{N}})^{-1}(\sigma))$ -continuous. Indeed, this follows from the equality

$$q|_{\mathsf{N}} \circ \alpha = \widetilde{\alpha} \circ (q|_{\mathsf{A}} \times q|_{\mathsf{N}})$$

and the $(\tilde{\tau_p}|_{q(\mathsf{A})}, \sigma, \sigma)$ -continuity of $\tilde{\alpha}$. Since $q|_{\mathsf{N}}$ is an injection and σ is a Hausdorff group topology on $q(\mathsf{N})$, then clearly $(q|_{\mathsf{N}})^{-1}(\sigma) \subseteq \tau_p|_{\mathsf{N}}$ is a coarser Hausdorff group topology on N . By Proposition 3.8 and Fact 2.7, N is A-minimal with respect to the action α . In particular, we deduce that $(q|_{\mathsf{N}})^{-1}(\sigma) = \tau_p|_{\mathsf{N}}$. This implies that $\sigma = \tilde{\tau_p}|_{q(\mathsf{N})}$, which completes the proof. \Box Using Fact 2.7, we immediately obtain:

Corollary 3.14. Let \mathbb{F} be a non-discrete locally retrobounded complete field. Then the subgroup $q(\mathsf{N})$ is relatively minimal in $\mathrm{ST}^+(n,\mathbb{F})/Z$.

3.4. When \mathbb{F} is a local field

It is easy to see that for every $1 \le i \le n-1$, there exists a continuous *central retraction* r from $q(\mathsf{N})$ to its $q(\mathsf{A})$ -subgroup $q(G_{i,i+1})$. This means that $r(q(x)q(a)q(x)^{-1}) = q(a)$ for every $x \in \mathsf{N}$ and $a \in G_{i,i+1}$.

The following fact will be used to prove Theorem 3.17 which provides sufficient conditions for the minimality of $ST^+(n, \mathbb{F})/Z$.

Fact 3.15. [22, Proposition 2.7] Let $M = (X \rtimes_{\alpha} G, \gamma)$ be a topological semidirect product and $\{Y_i\}_{i \in I}$ be a system of G-subgroups in X such that the system of actions

$$\{\alpha|_{G\times Y_i}: G\times Y_i \to Y_i\}_{i\in I}$$

is t-exact (that is, there is no strictly coarser (not necessarily Hausdorff) group topology on G such that all actions remain continuous). Suppose that for each $i \in I$ there exists a continuous central retraction $q_i: X \to Y_i$. Then if $\gamma_1 \subseteq \gamma$ is a coarser group topology on M such that $\gamma_1|_X = \gamma|_X$, then $\gamma_1 = \gamma$.

The proof of the next proposition was inspired by the proof of the total minimality of $SL(2, \mathbb{R})$ given in [11, Theorem 7.4.1].

Proposition 3.16. Let \mathbb{F} be a local field and $n \geq 2$. Then the system of n-1 actions

$$\{\widetilde{\alpha_i}: (q(\mathsf{A}), \widetilde{\tau_p}) \times (q(G_{i,i+1}), \widetilde{\tau_p}) \to (q(G_{i,i+1}), \widetilde{\tau_p}) | \quad i \in \{1, \cdots, n-1\}\}$$

is t-exact.

Proof. Recall that \mathbb{F} admits an absolute value $|\cdot|$. Let $\sigma \subseteq \tilde{\tau}_p$ be a coarser group topology on $q(\mathsf{A})$ such that all n-1 actions

$$\widetilde{\alpha}_i \colon (q(\mathsf{A}), \sigma) \times (q(G_{i,i+1}), \widetilde{\tau}_p) \to (q(G_{i,i+1}), \widetilde{\tau}_p)$$

are continuous. This implies that the n-1 actions

$$\alpha_i \colon (\mathsf{A}, q^{-1}(\sigma)) \times (G_{i,i+1}, \tau_p) \to (G_{i,i+1}, \tau_p)$$

are continuous. By Lemma 3.10(2), the homomorphism

$$m_i = (p_{i,i})^n \colon (\mathsf{A}, q^{-1}(\sigma)) \to \mathbb{F}^{\times}$$

is continuous. If q(B) = q(C) then $B^{-1}C = \lambda I$, where $\lambda^n = 1$. It follows that the map $\widehat{m_i}: (q(\mathsf{A}), \sigma) \to \mathbb{F}^{\times}$ defined by $\widehat{m_i} \circ q = m_i$ is well-defined and continuous for every $1 \leq i \leq n$. Consider an arbitrary net $\{\varepsilon_{\alpha}\}_{\alpha}$ in A , such that $\lim q(\varepsilon_{\alpha}) = q(I)$ in σ . By the continuity of $\widehat{m_i}$, we deduce that $\lim (p_{i,i})^n(\varepsilon_{\alpha}) = 1$. In particular, the nets $\{p_{i,i}(\varepsilon_{\alpha})\}$, where $1 \leq i \leq n$, are bounded with respect to the absolute value. Hence, there exists a σ -neighborhood V of q(I) that is contained in a compact subset of $q(\mathsf{A})$. This implies that $\sigma = \widetilde{\tau_p}$. \Box

Theorem 3.17. Let \mathbb{F} be a local field. Then $ST^+(n, \mathbb{F})/Z$ is minimal for every $n \in \mathbb{N}$.

Proof. Clearly, we may assume that $n \ge 2$. By Corollary 3.14, the subgroup q(N) is relatively minimal in $q(ST^+(n, \mathbb{F})) = ST^+(n, \mathbb{F})/Z$. By Proposition 3.16, the system of n-1 actions

$$\{\widetilde{\alpha}_i \colon (q(\mathsf{A}), \widetilde{\tau}_p) \times (q(G_{i,i+1}), \widetilde{\tau}_p) \to (q(G_{i,i+1}), \widetilde{\tau}_p)\}$$

is t-exact. Using Fact 3.15 we complete the proof. \Box

In case G/L is sup-complete for every closed normal subgroup L of G, then G is called *totally sup-complete*. In particular, if G is either a compact group or a sup-complete (topologically) simple group, then it is totally sup-complete.

Fact 3.18. [11, Theorem 7.3.1] Let G be a topological group and let L be a closed normal subgroup of G which is (totally) sup-complete. If L and G/L are both (totally) minimal, then G is (totally) minimal, too.

Since $Z = Z(SL(n, \mathbb{F}))$ is finite and using Fact 3.18 and Theorem 3.17, we obtain one of our main results:

Theorem 3.19. Let \mathbb{F} be a local field. Then $ST^+(n, \mathbb{F})$ is minimal for every $n \in \mathbb{N}$.

One can consider the topological group $T^{-}(n, \mathbb{F})$ of lower triangular $n \times n$ matrices over \mathbb{F} and its subgroup $ST^{-}(n, \mathbb{F}) = T^{-}(n, \mathbb{F}) \cap SL(n, \mathbb{F})$. It is easy to see that $ST^{+}(n, \mathbb{F})$ is topologically isomorphic to $ST^{-}(n, \mathbb{F})$. So, Theorem 3.19 immediately implies:

Corollary 3.20. Let \mathbb{F} be a local field. Then $ST^{-}(n, \mathbb{F})$ is minimal for every $n \in \mathbb{N}$.

4. Minimality properties of $SL(n, \mathbb{F})$ and $PGL(n, \mathbb{F})$

It is known that an archimedean local field is either the field of reals \mathbb{R} or the field of complex numbers \mathbb{C} .

The following Iwasawa decomposition of $SL(n, \mathbb{F})$ (see [1,5,27,36]) plays a key role in proving Theorem 4.7.

Fact 4.1. Let \mathbb{F} be a local field. Then there exists a compact subgroup K of $SL(n, \mathbb{F})$ such that $SL(n, \mathbb{F}) = ST^+(n, \mathbb{F})K$. In particular,

- (1) if $\mathbb{F} = \mathbb{R}$, then K is the orthogonal group $O(n, \mathbb{R})$;
- (2) if $\mathbb{F} = \mathbb{C}$, then K is the special unitary group $\mathrm{SU}(n, \mathbb{C})$;
- (3) if \mathbb{F} is non-archimedean, then $K = SL(n, \mathcal{O}_{\mathbb{F}})$, where $\mathcal{O}_{\mathbb{F}}$ is the ring of integers of \mathbb{F} , namely, $\mathcal{O}_{\mathbb{F}} = \{a \in \mathbb{F} : |a| \leq 1\}$.

Recall that a subgroup H of a topological group G is said to be *co-compact* if the coset space G/H is compact. If G = KH (equivalently, G = HK) for some compact subset K of G and a subgroup H, then His co-compact in G. Indeed, let $q: G \to G/H, x \mapsto xH$ be the natural projection. Then its restriction on Kis onto because G = KH. So, q(K) = G/H is also compact. Since Z is finite, we obtain the following as a corollary of Fact 4.1:

Corollary 4.2. Let \mathbb{F} be a local field. Then $\mathrm{ST}^+(n,\mathbb{F})/Z$ is co-compact in $\mathrm{PSL}(n,\mathbb{F})$.

A subgroup H of a Hausdorff topological group (G, τ) is called *strongly closed*, [8] if H is σ -closed for every Hausdorff group topology $\sigma \subseteq \tau$ on G.

Theorem 4.3. Let \mathbb{F} be a local field. Then $SL(n, \mathbb{F})$ and the projective special linear group $PSL(n, \mathbb{F}) = SL(n, \mathbb{F})/Z(SL(n, \mathbb{F}))$ are totally minimal for every $n \in \mathbb{N}$.

Proof. We may assume that $n \ge 2$. By Fact 2.4, $PSL(n, \mathbb{F})$ is simple so it suffices to prove that $PSL(n, \mathbb{F})$ is minimal. By Theorem 3.17, $G := ST^+(2, \mathbb{F})/Z(SL(n, \mathbb{F}))$ is minimal. So, in particular, G is relatively minimal in $PSL(n, \mathbb{F})$. Furthermore, G is also sup-complete since G is locally compact. So we obtain that G is strongly closed. Then the subgroup G is also co-minimal in $PSL(n, \mathbb{F})$, being co-compact by Iwasawa decomposition. It follows from Fact 2.6 that $PSL(n, \mathbb{F})$ is minimal. By Fact 3.18, $SL(n, \mathbb{F})$ is totally minimal. \Box

Remark 4.4. Bader and Gelander ([3, Corollary 5.3]) recently proved that every separable quasi-semisimple group is totally minimal. Then by [20, Ch. I, Proposition (1.2.1)] every Zariski-connected semi-simple group (e.g., $SL(n, \mathbb{F})$) over a local field \mathbb{F} is quasi-semisimple. It follows that for every local field \mathbb{F} (so also when char(\mathbb{F}) = 2) the groups $SL(n, \mathbb{F})$ and $PSL(n, \mathbb{F})$ are totally minimal.

The following concept has a key role in the Total Minimality Criterion.

Definition 4.5. A subgroup H of a topological group G is totally dense if for every closed normal subgroup L of G the intersection $L \cap H$ is dense in L.

Fact 4.6. [10, Total Minimality Criterion] Let H be a dense subgroup of a topological group G. Then H is totally minimal if and only if G is totally minimal and H is totally dense in G.

In the sequel we no longer assume that $char(\mathbb{F}) \neq 2$ in view of Remark 4.4.

Theorem 4.7. Let \mathbb{F} be a subfield of a local field. Then the following conditions are equivalent:

- (1) $PSL(n, \mathbb{F})$ is totally minimal;
- (2) $SL(n, \mathbb{F})$ is totally minimal;
- (3) $Z(\mathrm{SL}(n,\mathbb{F})) = Z(\mathrm{SL}(n,\widehat{\mathbb{F}}))$ (i.e., $\mu_n(\mathbb{F}) = \mu_n(\widehat{\mathbb{F}})$).

Proof. If \mathbb{F} is finite, then $\mathbb{F} = \widehat{\mathbb{F}}$ and conditions (1), (2), (3) are all satisfied. So we may assume that \mathbb{F} is infinite and $\widehat{\mathbb{F}}$ is a local field in view of Remark 2.9(2).

(1) \Rightarrow (2): Use Fact 3.18 with $G = SL(n, \mathbb{F})$ and its finite subgroup $L = Z(SL(n, \mathbb{F}))$.

 $(2) \Rightarrow (3)$: Let $G := \mathrm{SL}(n,\widehat{\mathbb{F}})$, $H := \mathrm{SL}(n,\mathbb{F})$ and suppose that $Z(\mathrm{SL}(n,\mathbb{F})) \neq Z(\mathrm{SL}(n,\widehat{\mathbb{F}}))$. Then $L := Z(\mathrm{SL}(n,\widehat{\mathbb{F}}))$ is a closed normal subgroup of G, and $L \cap H = Z(\mathrm{SL}(n,\mathbb{F}))$ is not dense in L, being a finite proper subgroup of L. So $\mathrm{SL}(n,\mathbb{F})$ is not totally dense in $\mathrm{SL}(n,\widehat{\mathbb{F}})$. By the Total Minimality Criterion, we deduce that $\mathrm{SL}(n,\mathbb{F})$ is not totally minimal.

 $(3) \Rightarrow (1)$: By Theorem 4.3, the group $\text{PSL}(n, \widehat{\mathbb{F}})$ is totally minimal. Since $Z(\text{SL}(n, \mathbb{F})) = Z(\text{SL}(n, \widehat{\mathbb{F}}))$, we deduce that $\text{PSL}(n, \mathbb{F})$ is dense in $\text{PSL}(n, \widehat{\mathbb{F}})$. As $\text{PSL}(n, \widehat{\mathbb{F}})$ is simple (Fact 2.4), its dense subgroup $\text{PSL}(n, \mathbb{F})$ is, in fact, totally dense. By the Total Minimality Criterion, $\text{PSL}(n, \mathbb{F})$ is also totally minimal. \Box

Corollary 4.8. Let \mathbb{F} be a local field. If $Z(SL(n, \mathbb{F})) \subseteq \{I, -I\}$, then for every topological subfield H of \mathbb{F} the groups PSL(n, H) and SL(n, H) are totally minimal. In particular,

- (1) SL(2, H) and PSL(2, H) are totally minimal for every topological subfield H of \mathbb{F} ;
- (2) SL(n, H) and PSL(n, H) are totally minimal for every topological subfield H of \mathbb{R} .

4.1. Total minimality of $PGL(n, \mathbb{F})$

The next result is probably known. We prove it for the sake of completeness. Perhaps it can be derived also from the results of [19, Section 26].

Lemma 4.9. Let \mathbb{F} be a local field, $n \in \mathbb{N}$, and $M_n = \{x^n | x \in \mathbb{F}^{\times}\}$. Then

(1) M_n is closed in \mathbb{F}^{\times} ;

(2) the group \mathbb{F}^{\times}/M_n is compact.

Proof. Recall that the local field \mathbb{F} admits an absolute value $|\cdot|$.

(1) Suppose that $\lim_{m\to\infty} (x_m)^n = y \in \mathbb{F}^{\times}$, where $\{x_m\}_{m\in\mathbb{N}}$ is a sequence contained in \mathbb{F}^{\times} . We have to show that $y \in M_n$. Clearly, the sequence $\{x_m\}_{m\in\mathbb{N}}$ is bounded with respect to the absolute value. Since \mathbb{F} is a local field, there exists a subsequence $\{x_{m_l}\}_{l\in\mathbb{N}}$ of $\{x_m\}_{m\in\mathbb{N}}$ such that $\lim_{l\to\infty} x_{m_l} = t$ for some $t \in \mathbb{F}$. Since \mathbb{F} is a Hausdorff topological field, it follows that $\lim_{l\to\infty} (x_{m_l})^n = t^n = y$. Clearly, $t \neq 0$ and we deduce that $y \in M_n$.

(2) It is easy to see that if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then \mathbb{F}^{\times}/M_n is a group with at most two elements. So, we may assume that \mathbb{F} is a non-archimedean local field. In this case, the value group (i.e., the set $\{|x| : |x| \neq 0\}$) is the infinite cyclic closed subgroup $\{a^k | k \in \mathbb{Z}\}$ of \mathbb{R}^{\times} , where $a := \max\{|x| : |x| < 1\}$ (see [33]). Let $r : \mathbb{F}^{\times} \to \mathbb{F}^{\times}/M_n$ be the quotient map. It suffices to show that for every sequence $\{x_m\}_{m \in \mathbb{N}} \subseteq \mathbb{F}^{\times}$ there exists a subsequence $\{x_{m_l}\}_{l \in \mathbb{N}}$ such that the sequence $\{r(x_{m_l})\}_{l \in \mathbb{N}}$ converges in \mathbb{F}^{\times}/M_n . For every $m \in \mathbb{N}$, we have $|x_m| = a^{t_m}$ for some $t_m \in \mathbb{Z}$. There exist $s_m \in \mathbb{Z}$ and $r_m \in [1 - n, n - 1] \cap \mathbb{Z}$ with $t_m = ns_m + r_m$. Choose $\lambda_m \in \mathbb{F}^{\times}$ with $|\lambda_m| = a^{-s_m}$. Letting $y_m = x_m \lambda_m^n$, we obtain a sequence $\{y_m\}_{m \in \mathbb{N}}$ with $r(x_m) = r(y_m)$ and $a^{n-1} \leq |y_m| \leq a^{1-n}$ for every $m \in \mathbb{N}$. As the sequence $\{y_m\}_{m \in \mathbb{N}}$ is bounded and \mathbb{F} is a local field, there exists a converging subsequence $\{y_{m_l}\}_{l \in \mathbb{N}}$. Since $a^{n-1} \leq |y_m|$, the limit is in \mathbb{F}^{\times} . As $r(x_m) = r(y_m)$ for every $m \in \mathbb{N}$ and using the continuity of r, we deduce that the sequence $\{r(x_{m_l})\}_{l \in \mathbb{N}}$ converges in \mathbb{F}^{\times}/M_n . \Box

Recall that the center $Z(\operatorname{GL}(n,\mathbb{F}))$ is $\{\lambda I : \lambda \in \mathbb{F}^{\times}\}$. Below we denote it by Z. In the sequel, $\operatorname{PSL}(n,\mathbb{F}) = q(\operatorname{SL}(n,\mathbb{F})) = (\operatorname{SL}(n,\mathbb{F}) \cdot Z)/Z$ is a normal subgroup of $\operatorname{PGL}(n,\mathbb{F})$, where

$$q: \operatorname{GL}(n, \mathbb{F}) \to \operatorname{GL}(n, \mathbb{F})/Z = \operatorname{PGL}(n, \mathbb{F})$$

is the quotient map. The map q induces a continuous isomorphism

$$\phi : \mathrm{PSL}(n, \mathbb{F}) \to \widetilde{\mathrm{PSL}(n, \mathbb{F})}.$$

It is worth noting that the second isomorphism theorem, which implies that ϕ is an algebraic isomorphism, does not hold in general for topological groups (see [18, p. 14]).

Proposition 4.10. Let \mathbb{F} be a local field. Then $PSL(n, \mathbb{F})$ is a totally minimal totally sup-complete group and the factor group $PGL(n, \mathbb{F})/PSL(n, \mathbb{F})$ is compact Hausdorff.

Proof. By Theorem 4.3, the group $PSL(n, \mathbb{F})$ is totally minimal. Being locally compact and simple (see Fact 2.4), $PSL(n, \mathbb{F})$ is also totally sup-complete. Consider the continuous isomorphism ϕ : $PSL(n, \mathbb{F}) \rightarrow PSL(n, \mathbb{F})$. Since $PSL(n, \mathbb{F})$ is totally minimal it follows that ϕ is in fact a topological group isomorphism, so $PSL(n, \mathbb{F})$ is a totally minimal totally sup-complete group.

As $SL(n, \mathbb{F}) \cdot Z = \det^{-1}(M_n)$, Lemma 4.9(1) implies that the factor group

$$\operatorname{GL}(n, \mathbb{F})/(\operatorname{SL}(n, \mathbb{F}) \cdot Z)$$

is Hausdorff. The continuous homomorphism

$$\psi \colon \mathbb{F}^{\times} \to \mathrm{GL}(n, \mathbb{F}), \ \psi(\lambda) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

induces a continuous isomorphism $\tilde{\psi}$ from \mathbb{F}^{\times}/M_n onto $\operatorname{GL}(n,\mathbb{F})/(\operatorname{SL}(n,\mathbb{F})\cdot Z)$.

By Lemma 4.9(2), \mathbb{F}^{\times}/M_n is compact. Hence $\tilde{\psi}$ is a topological group isomorphism. This proves that $\operatorname{GL}(n,\mathbb{F})/(\operatorname{SL}(n,\mathbb{F})\cdot Z)$ is compact. By the *third isomorphism theorem for topological groups* (which is easy to verify for all topological groups; see [2, Theorem 1.5.18] and [18, Proposition 3.6]), we have

$$\mathrm{PGL}(n,\mathbb{F})/\mathrm{PSL}(n,\mathbb{F}) = (\mathrm{GL}(n,\mathbb{F})/Z)/((\mathrm{SL}(n,\mathbb{F})\cdot Z)/Z) \cong \mathrm{GL}(n,\mathbb{F})/(\mathrm{SL}(n,\mathbb{F})\cdot Z)$$

which completes the proof. \Box

By Proposition 4.10 and Fact 3.18, we immediately obtain

Theorem 4.11. Let \mathbb{F} be a local field. Then $PGL(n, \mathbb{F})$ is totally minimal.

Very recently, U. Bader informed us that Theorem 4.11 follows also from [3, Theorem 3.4].

Theorem 4.12. Let \mathbb{F} be a topological subfield of \mathbb{R} and n be an odd number. Then $PGL(n, \mathbb{F})$ is totally minimal.

Proof. By Corollary 4.8(2), $PSL(n, \mathbb{F})$ is totally minimal. It follows that $PSL(n, \mathbb{F})$ is also totally minimal. To establish the total minimality of $PGL(n, \mathbb{F})$, it suffices to show, in view of Fact 4.6, that $PSL(n, \mathbb{F})$ is dense in $PGL(n, \mathbb{F})$. Let us see first that M_n is dense in \mathbb{F}^{\times} . If $a \in \mathbb{F}^{\times}$, then $\sqrt[n]{a} \in \mathbb{R}$. As $\mathbb{Q} \subseteq \mathbb{F}$, there exists a sequence $\{x_m\}_{m \in \mathbb{N}} \subseteq \mathbb{F}^{\times}$ converging to $\sqrt[n]{a}$. So $\lim_{m \to \infty} (x_m)^n = a$, which proves that M_n is dense in \mathbb{F}^{\times} . The continuous homomorphism

$$\psi \colon \mathbb{F}^{\times} \to \operatorname{GL}(n, \mathbb{F}), \ \psi(\lambda) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

induces a continuous homomorphism $\hat{\psi}$ from \mathbb{F}^{\times} onto $\operatorname{PGL}(n, \mathbb{F})$. Since $\hat{\psi}(M_n) = \operatorname{PSL}(n, \mathbb{F})$, we deduce that $\operatorname{PSL}(n, \mathbb{F})$ is dense in $\operatorname{PGL}(n, \mathbb{F})$, as needed. \Box

Question 4.13. Let \mathbb{F} be a topological subfield of \mathbb{R} and n be an even number. Is $PGL(n, \mathbb{F})$ (totally) minimal?

5. Fermat primes and minimality of special linear groups

The next proposition may be viewed as a counterpart of Theorem 4.7.

Proposition 5.1. Let \mathbb{F} be a subfield of a local field. Then the following conditions are equivalent:

- (1) $SL(n, \mathbb{F})$ is minimal;
- (2) any non-trivial central subgroup of $SL(n, \widehat{\mathbb{F}})$ intersects $SL(n, \mathbb{F})$ non-trivially.

Proof. If \mathbb{F} is finite, then $\mathbb{F} = \widehat{\mathbb{F}}$ and conditions (1), (2) are satisfied. So we may assume that \mathbb{F} is infinite and $\widehat{\mathbb{F}}$ is a local field in view of Remark 2.9(2).

(1) \Rightarrow (2): Immediately follows from the Minimality Criterion, as any non-trivial central subgroup of $SL(n, \widehat{\mathbb{F}})$ is normal and closed (being finite).

 $(2) \Rightarrow (1)$: Let us see that $\mathrm{SL}(n,\mathbb{F})$ is essential in the minimal group $\mathrm{SL}(n,\widehat{\mathbb{F}})$. To this aim, let L be a closed non-trivial normal subgroup of $\mathrm{SL}(n,\widehat{\mathbb{F}})$. If the central subgroup $Z(\mathrm{SL}(n,\widehat{\mathbb{F}})) \cap L$ is non-trivial, then by our assumption $L \cap \mathrm{SL}(n,\mathbb{F})$ is non-trivial. If L trivially intersects $Z(\mathrm{SL}(n,\widehat{\mathbb{F}}))$, then q(L) is a nontrivial normal subgroup of $\mathrm{PSL}(n,\widehat{\mathbb{F}})$, where $q \colon \mathrm{SL}(n,\widehat{\mathbb{F}}) \to \mathrm{PSL}(n,\widehat{\mathbb{F}})$ is the quotient map. Using the simplicity of $\mathrm{PSL}(n,\widehat{\mathbb{F}})$ (Fact 2.4), we deduce that $q(L) = \mathrm{PSL}(n,\widehat{\mathbb{F}})$. Choose $A \in \mathrm{SL}(n,\mathbb{F})$ such that A is not a root of I. Since $q(L) = \mathrm{PSL}(n,\widehat{\mathbb{F}})$, there exist $X \in L$ and nth-roots of unity $\lambda, \mu \in \widehat{\mathbb{F}}$ such that $\lambda X = \mu A$. Therefore, $X^n = A^n$ is a non-trivial element of $L \cap \mathrm{SL}(n,\mathbb{F})$. This proves that $\mathrm{SL}(n,\mathbb{F})$ is essential in $\mathrm{SL}(n,\widehat{\mathbb{F}})$. By the Minimality Criterion, $\mathrm{SL}(n,\mathbb{F})$ is minimal being a dense essential subgroup of the minimal group $\mathrm{SL}(n,\widehat{\mathbb{F}})$. \Box

Corollary 5.2. Let \mathbb{F} be a subfield of a local field. Then $SL(2^k, \mathbb{F})$ is minimal for every $k \in \mathbb{N}$. If char $(\mathbb{F}) = 2$, then $SL(2^k, \mathbb{F})$ is totally minimal.

Proof. If char(\mathbb{F}) = 2 and $\lambda^{2^k} = 1$, where $\lambda \in \widehat{\mathbb{F}}$, then $\lambda = 1$. It follows that $Z(\mathrm{SL}(2^k, \widehat{\mathbb{F}}))$ is trivial. By Theorem 4.7, $\mathrm{SL}(2^k, \mathbb{F})$ is totally minimal.

Now assume that $\operatorname{char}(\mathbb{F}) \neq 2$ and let L be a non-trivial central subgroup of $\operatorname{SL}(2^k, \widehat{\mathbb{F}})$. Then

$$I \neq -I \in \mathrm{SL}(2^k, \mathbb{F}) \cap L.$$

This proves the minimality of $SL(2^k, \mathbb{F})$, in view of Proposition 5.1. \Box

Since \mathbb{C} is a local field, $SL(n,\mathbb{C})$ is totally minimal. We have the following trichotomy for $\mathbb{Q}(i)$.

Corollary 5.3. Let n be a natural number.

- (1) If $n \in \{1, 2, 4\}$, then $SL(n, \mathbb{Q}(i))$ is totally minimal.
- (2) If $n = 2^k$ for k > 2, then $SL(n, \mathbb{Q}(i))$ is minimal but not totally minimal.
- (3) If n is not a power of 2, then $SL(n, \mathbb{Q}(i))$ is not minimal.

Proof. (1) If $n \in \{1, 2, 4\}$, then $Z(SL(n, \mathbb{Q}(i))) = Z(SL(n, \mathbb{C})) \subseteq \{\pm I, \pm iI\}$. By Theorem 4.7, $SL(n, \mathbb{Q}(i))$ is totally minimal.

(2) By Corollary 5.2, $SL(n, \mathbb{Q}(i))$ is minimal. Let $\rho_8 = e^{\frac{\pi i}{4}}$ be the 8-th primitive root of unity. As $n = 2^k$ for k > 2, we have $\rho_8 I \in Z(SL(n, \mathbb{C}))$ but $\rho_8 I \notin Z(SL(n, \mathbb{Q}(i)))$. So, $SL(n, \mathbb{Q}(i))$ is not totally minimal by Theorem 4.7.

(3) One can show using the same arguments from Example 3.5 that the finite central subgroup $L = \{\lambda I : \lambda^p = 1\}$ of $SL(n, \mathbb{C})$ trivially intersects $SL(n, \mathbb{Q}(i))$. This means that $SL(n, \mathbb{Q}(i))$ is not essential in $SL(n, \mathbb{C})$. By the Minimality Criterion, $SL(n, \mathbb{Q}(i))$ is not minimal. \Box

Now we consider the field of *p*-adic numbers \mathbb{Q}_p . It is known that \mathbb{Q}_p contains p-1 roots of unity in case p > 2 and that ± 1 are the only roots of unity in \mathbb{Q}_2 (see [33, p. 15]).

Corollary 5.4. Let \mathbb{F} be a topological subfield of \mathbb{Q}_p .

- (1) $SL(n, \mathbb{F})$ and $PSL(n, \mathbb{F})$ are totally minimal for every n which is coprime to p-1 (e.g., arbitrary n for p=2).
- (2) If p-1 is not a power of 2, then $SL(p-1, \mathbb{Q})$ is not minimal.

Proof. (1) Use Corollary 4.8.

(2) $\operatorname{SL}(p-1,\mathbb{Q})$ is not essential in $\operatorname{SL}(p-1,\mathbb{Q}_p)$. Indeed, let q be an odd prime dividing p-1. Then the finite central subgroup $L = \{\lambda I : \lambda^q = 1\}$ of $\operatorname{SL}(p-1,\mathbb{Q}_p)$ trivially intersects $\operatorname{SL}(p-1,\mathbb{Q})$. \Box

The next theorem is one of our main results. It provides a characterization of Fermat primes via the minimality of some topological matrix groups.

Theorem 5.5. For an odd prime p the following conditions are equivalent:

- (1) p is a Fermat prime;
- (2) $SL(p-1,(\mathbb{Q},\tau_p))$ is minimal, where (\mathbb{Q},τ_p) is the field of rationals with the p-adic topology;
- (3) $SL(p-1, \mathbb{Q}(i))$ is minimal, where $\mathbb{Q}(i) \subset \mathbb{C}$ is the Gaussian rational field.

Proof. If p is a Fermat prime, then $p-1 = 2^k$ for some positive integer k. By Corollary 5.2, $SL(p-1, (\mathbb{Q}, \tau_p))$ and $SL(p-1, \mathbb{Q}(i))$ are both minimal. If p is not a Fermat prime, then p-1 is not a power of 2. By Corollary 5.4(2), $SL(p-1, (\mathbb{Q}, \tau_p))$ is not minimal. By Corollary 5.3(3), $SL(p-1, \mathbb{Q}(i))$ is not minimal. \Box

Remark 5.6.

- (1) One cannot replace in item (3) of Theorem 5.5 the field $\mathbb{Q}(i)$ with its subfield \mathbb{Q} . Indeed, since \mathbb{Q} is also a subfield of \mathbb{R} it follows from Corollary 4.8(2) that $SL(n, \mathbb{Q})$ is (totally) minimal for every $n \in \mathbb{N}$.
- (2) If p is a Fermat prime then every odd n is coprime to p-1. Hence, $SL(n, \mathbb{F})$ and $PSL(n, \mathbb{F})$ are totally minimal for every subfield \mathbb{F} of \mathbb{Q}_p , in view of Corollary 5.4(1).

Data availability

No data was used for the research described in the article.

References

- [1] O. Ahlén, Global Iwasawa-decomposition of $SL(n, \mathbb{A}_{\mathbb{Q}})$, arXiv:1609.06621, 2016, 1–20.
- [2] A. Arhangel'skii, M. Tkachenko, Topological groups and related structures, in: J. van Mill (Ed.), Atlantis Studies in Math., World Scientific, 2008.
- [3] U. Bader, T. Gelander, Equicontinuous actions of semisimple groups, Groups Geom. Dyn. 11 (2017) 1003–1039.
- [4] T. Banakh, A quantitative generalization of Prodanov–Stoyanov Theorem on minimal Abelian topological groups, Topol. Appl. 271 (2020), 17 pp.
- [5] B. Banaschewski, Minimal topological algebras, Math. Ann. 211 (1974) 107-114.
- [6] I. Ben Yaacov, T. Tsankov, Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups, Trans. Am. Math. Soc. 368 (11) (2016) 8267–8294.
- [7] S. Dierolf, U. Schwanengel, Examples of locally compact non-compact minimal topological groups, Pac. J. Math. 82 (1979) 349–355.
- [8] D. Dikranjan, M. Megrelishvili, Relative minimality and co-minimality of subgroups in topological groups, Topol. Appl. 157 (2010) 62–76.
- [9] D. Dikranjan, M. Megrelishvili, Minimality conditions in topological groups, in: K.P. Hart, J. van Mill, P. Simon (Eds.), Recent Progress in General Topology III, Springer, Atlantis Press, 2014, pp. 229–327.
- [10] D. Dikranjan, Iv. Prodanov, Totally minimal topological groups, Annuaire Univ. Sofia Fat. Math. Méc. 69 (1974/1975) 5–11.
- [11] D. Dikranjan, Iv. Prodanov, L. Stoyanov, Topological Groups: Characters, Dualities and Minimal Group Topologies, Pure and Appl. Math., vol. 130, Marcel Dekker, New York-Basel, 1989.
- [12] D. Doïtchinov, Produits de groupes topologiques minimaux, Bull. Sci. Math. 97 (2) (1972) 59-64.

- [13] B. Duchesne, The Polish topology of the isometry group of the infinite dimensional hyperbolic space, arXiv:2005.12204, 2020.
- [14] W.T. van Est, Dense imbeddings of Lie groups, Indag. Math. 13 (1951) 321–328.
- [15] E. Glasner, The group $Aut(\mu)$ is Roelcke precompact, Can. Math. Bull. 55 (2012) 297–302.
- [16] M. Goto, Absolutely closed Lie groups, Math. Ann. 204 (1973) 337–341.
- [17] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, second ed., Grad. Texts in Math., vol. 84, Springer-Verlag, 1990.
- [18] K.B. Iturbe, Topological groups, Final degree dissertation, Universidad del Pais Vasco, 2015.
- [19] F. Lorenz, Algebra: Volume II: Fields with Structure, Algebras and Advanced Topics, Springer, 2008.
- [20] G.A. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1990.
- [21] M. Mayer, Asymptotics of matrix coefficients and closures of Fourier–Stieltjes algebras, J. Funct. Anal. 143 (1997) 42–54.
- [22] M. Megrelishvili, Group representations and construction of minimal topological groups, Topol. Appl. 62 (1995) 1–19.
- [23] M. Megrelishvili, G-minimal topological groups, in: Abelian Groups, Module Theory and Topology, in: Lecture Notes in Pure and Appl. Math., vol. 201, Marcel Dekker, 1998, pp. 289–300.
- [24] M. Megrelishvili, Generalized Heisenberg groups and Shtern's question, Georgian Math. J. 11 (4) (2004) 775–782.
- [25] M. Megrelishvili, Every topological group is a group retract of a minimal group, Topol. Appl. 155 (2008) 2105–2127.
- [26] H. Omori, Homomorphic images of Lie groups, J. Math. Soc. Jpn. 18 (1966) 97–117.
- [27] D. Prasad, A. Raghuram, Representation theory of GL(n) over non-Archimedean local fields, in: School on Automorphic Forms on GL(n), in: ICTP Lect. Notes, vol. 21, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008, pp. 159–205.
- [28] Iv. Prodanov, Precompact minimal group topologies and p-adic numbers, Annuaire Univ. Sofia Fac. Math. Méc. 66 (1971/1972) 249–266.
- [29] Iv. Prodanov, L. Stoyanov, Every minimal abelian group is precompact, C. R. Acad. Bulgare Sci. 37 (1984) 23-26.
- [30] D. Remus, L. Stoyanov, Complete minimal topological groups, Topol. Appl. 42 (1991) 57–69.
- [31] D.J.S. Robinson, A Course in the Theory of Groups, 2nd edn., Springer-Verlag, New York, 1996.
- [32] W. Roelcke, S. Dierolf, Uniform Structures on Topological Groups and Their Quotients, McGraw Hill, New York, 1981.
- [33] A.C.M. van Rooij, Non-Archimedean Functional Analysis, Monographs and Textbooks in Pure and Applied Math., vol. 51, Marcel Dekker, New York, 1978.
- [34] R.M. Stephenson Jr., Minimal topological groups, Math. Ann. 192 (1971) 193–195.
- [35] L. Stoyanov, Total minimality of the unitary groups, Math. Z. 187 (1984) 273–283.
- [36] J. Tits, Reductive groups over local fields, in: A. Borel, W. Casselman (Eds.), Automorphic Forms, Representations and L-Functions, in: Proc. Symp. Pure Math., vol. 33, 1977, pp. 29–71.
- [37] V.V. Uspenskij, On subgroups of minimal topological groups, Topol. Appl. 155 (2008) 1580–1605.
- [38] S. Warner, Topological rings, Math. Stud. 178 (1993).
- [39] W. Xi, D. Dikranjan, M. Shlossberg, D. Toller, Hereditarily minimal topological groups, Forum Math. 31 (3) (2019) 619–646.