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MINIMALITY OF THE SEMIDIRECT PRODUCT

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Abstract. A topological group is minimal if it does not admit a strictly coarser Hausdorff group topology. We provide a sufficient and necessary condition for the minimality of the semidirect product $G \rtimes P$, where G is a compact topological group and P is a topological subgroup of $\mathrm{Aut}(G)$. We prove that $G \rtimes P$ is minimal for every closed subgroup P of $\mathrm{Aut}(G)$. In case G is abelian, the same is true for every subgroup $P \subseteq \mathrm{Aut}(G)$. We show, in contrast, that there exist a compact two-step nilpotent group G and a subgroup P of $\mathrm{Aut}(G)$ such that $G \rtimes P$ is not minimal. This answers a question of Dikranjan. Some of our results were inspired by a work of Gamarnik [12].

1. Introduction

A Hausdorff topological group G is *minimal* ([10], [24]) if it does not admit a strictly coarser Hausdorff group topology or, equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding. For information on minimal groups we refer to the surveys [6], [7], [8] and the book [9].

In [20] the two first-named authors study the minimality of the group $\mathrm{H}_+(X)$, where X is a compact linearly ordered space and $\mathrm{H}_+(X)$ is the topological group of all order-preserving homeomorphisms of X. In general, $\mathrm{H}_+(X)$ need not be minimal. The first result in the present paper is Theorem 3.1, which shows that for a compact (partially) ordered space X the compact-open topology on $\mathrm{H}_+(X, \leq)$ is minimal within the class

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of π-*uniform topologies* (in the sense of [16]). This result was inspired by results of Gamarnik [12] and of the first-named author [16]. Following Nachbin [22], by a *partially ordered topological space* we mean a topological space X equipped with a partial order which is closed in $X \times X$.

Let G be a compact topological group and denote by $\mathrm{Aut}(G)$ its group of automorphisms. With the compact-open topology, $\mathrm{Aut}(G)$ becomes a topological group (which is not necessarily minimal). We denote its compact subgroup of inner automorphisms by $\mathrm{Inn}(G)$. The center of G is denoted by $Z(G)$.

One of the main objectives of this paper is to prove that the semidirect product $G \rtimes P$ is minimal for every closed subgroup P of $\mathrm{Aut}(G)$ (Theorem 4.2). Using this result, as well as the Minimality Criterion, the minimal groups $G \rtimes P$ are fully characterized (Theorem 5.2). This characterization shows, in particular, that if G is abelian then $G \rtimes P$ is minimal for every (not necessarily closed) subgroup P of $\mathrm{Aut}(G)$ (Corollary 5.3). Furthermore, when G is not abelian, the condition that P is a closed subgroup of $\mathrm{Aut}(G)$ is essential to ensure the minimality of $G \rtimes P$. Indeed, negatively answering a question of Dikranjan, in Example 5.6 we show that there exist a compact two-step nilpotent group G and a subgroup P of $\mathrm{Aut}(G)$ such that $G \rtimes P$ is not minimal.

Note also that the compactness of G cannot be replaced by precompactness. Indeed, there exist a minimal precompact group G and a two-element subgroup P of $\mathrm{Aut}(G)$ such that $G \rtimes P$ is not minimal. See Eberhardt-Dierolf-Schwanengel [11, Example 10] and also [8, Example 4.6]. The latter example also demonstrates that, in general, for two arbitrary minimal groups G and H the topological semidirect product $G \rtimes H$ may fail to be minimal. However, adding the requirement of completeness of G, one has the following:

Lemma 1.1. [8, Theorem 4.3] *If G is complete (with respect to its two-sided uniformity), then the semidirect product $G \rtimes H$ is minimal for minimal groups G and H.*

Remark 1.2. Let G be a complete minimal topological group and assume that $\mathrm{Aut}(G)$ is minimal. Then $G \rtimes \mathrm{Aut}(G)$ is minimal by Lemma 1.1. So, in view of Theorem 4.2, it is important to note that there are compact groups G such that $\mathrm{Aut}(G)$ is not minimal. Indeed, as it was explained in [8, Section 5], one may take $G = (\mathbb{Q}, \mathrm{discrete})^*$, that is the Pontryagin dual of the discrete group $\mathbb{Q}$ of the rational numbers.

For additional results concerning the minimality of $\mathrm{Aut}(G)$ see [8]. For the minimality of semidirect products see for example [9, Section 7.2] and [8, Section 4]. More results about minimality of the homeomorphism groups can be found in [13, 21, 8, 4].

2. Preliminaries

All topological spaces are assumed to be Hausdorff and completely regular unless stated otherwise. The closure of a subset A in a topological space will be denoted by $\overline{A}$. In what follows, every compact topological space will be considered as a uniform space with respect to its natural (unique) uniformity. For a topological space X we denote by $\mathrm{H}(X)$ its group of homeomorphisms and, if X is ordered, $\mathrm{H}_+(X)$ denotes the group of all order-preserving homeomorphisms of X. With the compact-open topology τ_{co}, $\mathrm{H}(X)$ becomes a topological group for every compact space X.

For a topological group (G,γ) and its subgroup H denote by γ/H the natural quotient topology on the coset space G/H.

The following useful lemma can be found, for example, in [9, Lemma 7.2.3].

Lemma 2.1. (Merson's Lemma) *Let (G,γ) be a not necessarily Hausdorff topological group and H be a not necessarily closed subgroup of G. If $\gamma_1 \subseteq \gamma$ is a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1/H = \gamma/H$, then $\gamma_1 = \gamma$.*

2.1. **π-uniform actions.** Let $\pi : G \times X \to X$ be an action of a topological group G on a topological space X. Define two maps:

(1) *g-translation*: $\pi^g : X \to X$, $\pi^g(x) = gx$;
(2) *x-orbit*: $\pi_x : G \to X$, $\pi_x(g) = gx$.

For a topological group (G,τ) we denote by e_G (or simply e when G is understood) the identity element of G, and by $N_{g_0}(\tau)$ we denote the local base of G at $g_0 \in G$.

Definition 2.2. [17, 16] Let $\pi : G \times X \to X$ be an action of a topological group (G,τ) on a Hausdorff uniform space $(X,\mathcal{U})$. The uniformity (or, the action) is said to be:

(1) *π-uniform at e* or *quasibounded* if for every $\varepsilon \in \mathcal{U}$ there exist $\delta \in \mathcal{U}$ and $U \in N_e(\tau)$ such that $(gx, gy) \in \varepsilon$ for every $(x,y) \in \delta$ and $g \in U$.
(2) *π-uniform* if for every $g_0 \in G$ and for every $\varepsilon \in \mathcal{U}$ there exist $\delta \in \mathcal{U}$ and $U \in N_{g_0}(\tau)$ such that $(gx, gy) \in \varepsilon$ for every $(x,y) \in \delta$ and $g \in U$.

The notion of a π-uniform action, defined in [17, 16], was originally used to study compactifications of G-spaces. Later it was employed by Gamarnik [12] to prove that for a compact space X, the compact-open topology on $\mathrm{H}(X)$ is minimal within the class of π-uniform topologies. More applications of π-uniformity can be found in [15] and in [5].

Definition 2.3. Let X be a compact space and G a subgroup of $\mathrm{H}(X)$. A Hausdorff group topology τ on G is said to be *π-uniform* if the natural action $(G,\tau)\times X \to X$ is π-uniform with respect to the unique compatible uniformity on X.

For a topological group X denote by $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}_{l\vee r}$ the left, right and two-sided uniform structures on X, respectively. We give here some simple but useful facts for further use.

Lemma 2.4. *Let G be a topological group and X is a uniform space.*

(1) If $\pi : G\times X \to X$ is a π-uniform action and all orbit maps $\pi_x : G \to X$ are continuous, then π is continuous.

(2) [18, Theorem 1.2] *If X is a topological group and $\pi : G \times X \to X$ is an action by continuous automorphisms, then the action is π-uniform with respect to $\mathcal{U} \in \{\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}_{l\vee r}\}$ if and only if π is continuous at (e_G, e_X).*

Lemma 2.5. [1, Theorem 2] *Let X be a compact space and let G be a subgroup of $\mathrm{H}(X)$. If τ is a group topology on G such that the action $G \times X \to X$ is continuous, then $\tau_{co} \subseteq \tau$.*

2.2. **Ordered Spaces.** By *order* we mean a reflexive, antisymmetric and transitive relation.

Definition 2.6. (Nachbin [22]) A *topological ordered space* is a triple $(X, \leq, \tau)$ where X is a set, $\leq$ is an order on X, τ is a topology on X and the graph of the order $Gr(\leq) = \{(x, y) : x \leq y\}$ is closed in $X\times X$. In particular, a *compact topological ordered space* is a topological ordered space that is compact. Since in this paper all ordered spaces are topological, we will sometimes omit the term "topological".

Remark 2.7. Every Hausdorff topological space X is a topological ordered space with respect to the trivial order (equality). Indeed, the diagonal is closed in $X \times X$ exactly when X is Hausdorff.

A subset $Y \subseteq X$ is said to be *decreasing* if $x \leq y \in Y$ implies $x \in Y$. Similarly one defines an *increasing* subset.

Lemma 2.8. [22, Prop. 1] *Let $(X, \leq)$ be an ordered set and let τ be a topology on X. The following conditions are equivalent:*

(1) $(X, \leq, \tau)$ is a topological ordered space (that is, $Gr(\leq)$ is τ-closed in $X \times X$);

(2) if $x \leq y$ is false, then there exist: an increasing neighborhood W of x and a decreasing neighborhood V of y such that $V \cap W = \varnothing$.

Lemma 2.9. *Let $(X, \leq, \tau)$ be a compact partially ordered space. Denote by $\mathrm{H}_+(X)$ the group of all order-preserving homeomorphisms of X. Then $\mathrm{H}_+(X)$ is a closed subgroup of the topological group $\mathrm{H}(X)$.*

Proof. Since $Gr(\leq)$ is τ-closed in $X \times X$, the subgroup $\mathrm{H}_+(X)$ is even pointwise closed in $\mathrm{H}(X)$. □

2.3. **Limit points and ultrafilters.** All definitions and results of this subsection can be found, for example, in [3, Chapter 1, Section 7]. Let X be a topological space and $\mathcal{J}$ is a filter on X. A point $x \in X$ is said to be a *limit point of a filter* $\mathcal{J}$, if $\mathcal{J}$ is finer than the neighborhood filter $\mathcal{N}_x$ of x. We also say that $\mathcal{J}$ is convergent to x. A point x is called a *limit point of a filter base* $\mathcal{B}$ on X, if the filter whose base is $\mathcal{B}$ converges to x. Let f be a mapping from a set X to a topological space Y, and let $\mathcal{J}$ be a filter on X. A point $y \in Y$ is a *limit point of f with respect to the filter* $\mathcal{J}$ if y is a limit point of the filter base $f(\mathcal{J})$.

Proposition 2.10. [3]

(1) If $\mathcal{B}$ is an ultrafilter base on a set X and if f is a mapping from X to Y, then $f(\mathcal{B})$ is an ultrafilter base on Y.
(2) Let f be a mapping from a set X into a topological space Y, and let $\mathcal{J}$ be a filter on X. A point $y \in Y$ is a limit point of f with respect to the filter $\mathcal{J}$ if and only if $f^{-1}(V) \in \mathcal{J}$ for each neighborhood V of y in Y.
(3) If X is a compact Hausdorff space, then every ultrafilter on X converges to a unique point.

We can sum these propositions as follows:

Corollary 2.11. *Let $\mathcal{J}$ be an ultrafilter on a set E and let f be a mapping from E to a compact space X. Then there exists a unique point $\widetilde{x} \in X$ such that each neighborhood O of $\widetilde{x}$ satisfies $f^{-1}(O) \in \mathcal{J}$. That is, $\widetilde{x}$ is the limit point of f with respect to $\mathcal{J}$.*

3. π-UNIFORM TOPOLOGIES ON H_+ AND Aut_+

The following theorem is an extended version of a result of Gamarnik [12, Prop. 2.1].

Theorem 3.1. *Let $(X, \tau_{\leq})$ be a compact partially ordered space and let P be a closed subgroup of $\mathrm{H}_+(X)$, the group of all order-preserving homeomorphisms of X. Then the compact-open topology τ_{co} is minimal within the class of π-uniform topologies on P.*

Proof. Assuming the contrary, suppose that there exists a π-uniform group topology τ on P such that $\tau \subsetneqq \tau_{co}$. Let $\pi : P \times X \to X$ be the natural action of P on X. If all orbit maps are continuous, then, by Lemma 2.4.1, π is continuous and, by Lemma 2.5, $\tau_{co} \subseteq \tau$. So we can assume that there exists an orbit map that is not continuous (at the identity). That is, there exists $x_0 \in X$ such that $\pi_{x_0} : P \to X$ is not continuous

at $e \in P$. Thus, denoting by $\mathcal{U}$ the natural uniformity on X, there exists $\varepsilon_0 \in \mathcal{U}$ such that for all $U \in N_e(\tau)$ there exists $g_U \in U$ for which

$$(g_U x_0, x_0) \notin \varepsilon_0. \tag{3.1}$$

For a given $U \in N_e(\tau)$ define $F(U) = \{V \in N_e(\tau) : V \subseteq U\}$. Denote by $\mathcal{F}$ the filter on the set $N_e(\tau)$ generated by the filter base $\{F(U)\}_{U \in N_e(\tau)}$. Since every filter is contained in an ultrafilter, choose an ultrafilter $\mathcal{J}$ on $N_e(\tau)$ that contains $\mathcal{F}$.

For each $x \in X$ define a map $f_x : N_e(\tau) \to X$ by $f_x(U) = g_U x$ for g_U that satisfies (3.1). Let $\widetilde{x}$ be the limit point of f_x with respect to the ultrafilter $\mathcal{J}$ given by Corollary 2.11. Define the following transformation

$$h : X \to X, \quad h(x) = \widetilde{x}. \tag{3.2}$$

In the rest of the proof we show that h is a nontrivial order-preserving homeomorphism that belongs to every neighborhood of the identity element in (P, τ), in contradiction to τ being a Hausdorff group topology.

Claim 3.2. *The map h defined by* (3.2) *is a nontrivial homeomorphism in P.*

Proof. We break the proof into five steps.

Step **1**. In order to prove that h is one-to-one, assume for a contradiction that there exist $x, y, z \in X$ such that $h(x) = h(y) = z$ and $x \neq y$. Choose an entourage $\varepsilon \in \mathcal{U}$ such that $(x, y) \notin \varepsilon$. The action is π-uniform at the identity, and thus there exist $U_\varepsilon \in N_e(\tau)$ and $\delta_\varepsilon \in \mathcal{U}$ such that $(gx, gy) \in \varepsilon$ for every $(x, y) \in \delta_\varepsilon$ and $g \in U_\varepsilon$. Choose a symmetric $\delta \in \mathcal{U}$ satisfying $\delta^2 \subseteq \delta_\varepsilon$.

By assumption $z = h(x)$ is the limit point of f_x with respect to $\mathcal{J}$. That is, for every entourage in the uniformity, and in particular for δ, we have:

$$A(x, \delta) = \{U \in N_e(\tau) : \ (g_U x, z) \in \delta\} \in \mathcal{J}.$$

Similarly,

$$A(y, \delta) = \{U \in N_e(\tau) : \ (g_U y, z) \in \delta\} \in \mathcal{J}.$$

Also, since $F(U_\varepsilon^{-1}) \in \mathcal{J}$, the intersection $A(x, \delta) \cap A(y, \delta) \cap F(U_\varepsilon^{-1})$ is not empty. If $U_0 \in A(x, \delta) \cap A(y, \delta) \cap F(U_\varepsilon^{-1})$ and $g_{U_0} \in U_0$, then $g_{U_0} \in U_\varepsilon^{-1}$ (and thus $g_{U_0}^{-1} \in U_\varepsilon$), $(g_{U_0} x, z) \in \delta$, $(g_{U_0} y, z) \in \delta$ (and thus $(g_{U_0} x, g_{U_0} y) \in \delta^2 \subseteq \delta_\varepsilon$). By the choice of δ_ε and U_ε we have $(g_{U_0}^{-1} g_{U_0} x, g_{U_0}^{-1} g_{U_0} y) = (x, y) \in \varepsilon$, and this contradicts the choice of ε. Therefore h is one-to-one.

Step **2**. To prove that h is onto, for a given $y \in X$ we will find $x \in X$ such that $h(x) = y$. Fix $y \in X$ and consider the map $N_e(\tau) \to X$ given by $U \mapsto g_U^{-1} y$. Let x be the limit point of this map with respect to the

ultrafilter $\mathcal{J}$. To show that $h(x) = y$ we will show that y is the limit point of f_x with respect to $\mathcal{J}$. Let $\varepsilon \in \mathcal{U}$ be an arbitrary entourage and choose U_ε, δ_ε from the definition of π-uniform topology. Since x is defined as the limit point of $g_U^{-1}y$, we know that

$$B(y, \delta_\varepsilon) = \{U \in N_e(\tau) : \ (g_U^{-1}y, x) \in \delta_\varepsilon\} \in \mathcal{J},$$

and since $F(U_\varepsilon)$ is also an element of $\mathcal{J}$, the intersection $B(y, \delta_\varepsilon) \cap F(U_\varepsilon)$ is not empty. Let $U \in B(y, \delta_\varepsilon) \cap F(U_\varepsilon)$. Then for $g_U \in U \subseteq U_\varepsilon$ and $(g_U^{-1}y, x) \in \delta_\varepsilon$ we have $(y, g_U x) \in \varepsilon$ (by the choice of $U_\varepsilon, \delta_\varepsilon$). This last condition is satisfied by all $U \in B(y, \delta_\varepsilon) \cap F(U_\varepsilon)$ and, therefore,

$$\{U \in N_e(\tau) : (y, g_U x) \in \varepsilon\} \in \mathcal{J}$$

(since $B(y, \delta_\varepsilon) \cap F(U_\varepsilon) \in \mathcal{J}$ and $B(y, \delta_\varepsilon) \cap F(U_\varepsilon) \subseteq \{U \in N_e(\tau) : (y, g_U x) \in \varepsilon\}$). This holds for all $\varepsilon \in \mathcal{U}$, which proves that y is the limit point of $f_x = g_U x$ with respect to $\mathcal{J}$. And that, in turn, proves that $h(x) = y$ and therefore h is onto.

Step **3**. In order to prove that h is (uniformly) continuous we will show that for every $\varepsilon \in \mathcal{U}$ there exists $\delta \in \mathcal{U}$ such that $(h(x), h(y)) \in \varepsilon$ for all $(x, y) \in \delta$. Let $\varepsilon_0 \in \mathcal{U}$ and choose a symmetric entourage $\varepsilon \in \mathcal{U}$ such that $\varepsilon^3 \subseteq \varepsilon_0$. Choose $\delta_\varepsilon, U_\varepsilon$ from Definition 2.2 of π-uniformity. We will show that if $(x, y) \in \delta_\varepsilon$ then $(h(x), h(y)) \in \varepsilon_0$. Let $(x, y) \in \delta_\varepsilon$ and assume for a contradiction that $(h(x), h(y)) \notin \varepsilon_0$. This means that if t_1, t_2 satisfy $(h(x), t_1) \in \varepsilon$, $(h(y), t_2) \in \varepsilon$, then (since $\varepsilon^3 \subseteq \varepsilon_0$) we have

$$(t_1, t_2) \notin \varepsilon. \tag{3.3}$$

Since $h(x)$ is the limit point of $g_U x$, $A(x, \varepsilon) = \{U \in N_e(\tau) : (g_U x, h(x)) \in \varepsilon\} \in \mathcal{J}$ and, similarly, $A(y, \varepsilon) = \{U \in N_e(\tau) : (g_U y, h(y)) \in \varepsilon\} \in \mathcal{J}$. Also, since $F(U_\varepsilon) \in \mathcal{J}$, the intersection $A(x, \varepsilon) \cap A(y, \varepsilon) \cap F(U_\varepsilon)$ is not empty. Let $V \in A(x, \varepsilon) \cap A(y, \varepsilon) \cap F(U_\varepsilon)$. In particular $(g_V x, h(x)) \in \varepsilon$ and $(g_V y, h(y)) \in \varepsilon$. Next, from (3.3) it follows that $(g_V x, g_V y) \notin \varepsilon$. But $V \subseteq U_\varepsilon$ and thus $g_V \in U_\varepsilon$. Since $(x, y) \in \delta_\varepsilon$ we know, by the definition of π-uniformity, that $(g_V x, g_V y) \in \varepsilon$ and this is the desired contradiction.

Step **4**. To see that h is not trivial, recall that from (3.1) we have $x_0 \in X$ and $\varepsilon_0 \in \mathcal{U}$ such that $(g_U x_0, x_0) \notin \varepsilon_0$ for every $U \in N_e(\tau)$. This implies that $h(x_0) \neq x_0$.

Step **5**. Finally, we show that $h \in P$. Denote by A the set of all g_U that satisfy (3.1), that is $A = \{g_U : U \in N_e(\tau)\}$. Since P is closed, we have $A \subseteq \overline{A} \subseteq P$, where $\overline{A}$ is the closure of A in $\mathrm{H}_+(X)$ with respect to the compact-open topology τ_{co}.

We are going to show that $h \in \overline{A}$. That is, we need to show that every neighborhood of h contains some g_U. Since X is compact, τ_{co} coincides with the topology of uniform convergence, and hence for $\varepsilon \in \mathcal{U}$ a basic neighborhood of h is of the form $\widetilde{\varepsilon}(h) = \{f \in P : (f(x), h(x)) \in \varepsilon \ \forall x \in X\}$. Therefore, for every $\varepsilon \in \mathcal{U}$ we will find $U \in N_e(\tau)$ such that $\forall x \in X : (g_U x, hx) \in \varepsilon$.

Fix an $\varepsilon \in \mathcal{U}$ and choose a symmetric entourage $\delta \in \mathcal{U}$ such that $\delta^3 \subseteq \varepsilon$. Since τ is π-uniform at the identity, for δ there exist $\eta \in \mathcal{U}$ and $U_0 \in N_e(\tau)$ such that

$$(\forall (x,y) \in \eta)(\forall g \in U_0) : (gx, gy) \in \delta.$$

Since $g_U \in U$ we obtain, in particular,

$$\forall (x,y) \in \eta \ \forall U \subseteq U_0 : (g_U x, g_U y) \in \delta. \tag{3.4}$$

Since h is uniformly continuous, for δ there exists $\kappa \in \mathcal{U}$ such that

$$\forall (x,y) \in \kappa : (hx, hy) \in \delta. \tag{3.5}$$

If necessary, we intersect κ with η to ensure that $\kappa \subseteq \eta$, which we will need later in the proof.

Now, since X is compact, for κ that satisfies (3.5) there exists a finite collection of points $x_1, \dots, x_n \in X$ such that $\kappa[x_1] \cup \dots \cup \kappa[x_n] = X$.

We will show that there exists $U_1 \subseteq U_0$ such that for all $i \in \{1, \dots, n\}$

$$(g_{U_1} x_i, hx_i) \in \delta. \tag{3.6}$$

For a fixed index $i \in \{1, \dots, n\}$, since $hx_i = \widetilde{x_i}$ is the limit of f_{x_i} with respect to $\mathcal{J}$, we have $A(x_i, \delta) = \{U \in N_e(\tau) : (g_U x_i, hx_i) \in \delta\} \in \mathcal{J}$. Recall that $F(U_0) \in \mathcal{J}$ and thus the intersection $F(U_0) \bigcap (\bigcap_{i=1}^n A(x_i, \delta))$ is not empty. Choose a set U_1 from this intersection. Then $U_1 \subseteq U_0$ and for all $i \in \{1, \dots, n\}$ we have $(g_{U_1} x_i, hx_i) \in \delta$, as required. We claim that U_1 is the desired neighborhood. That is, for every $x \in X$ we have $(g_{U_1} x, hx) \in \varepsilon$. Indeed, fix an $x \in X$. There exists $i \in \{1, \dots, n\}$ such that $x \in \kappa[x_i]$. At this point recall that $\kappa \subseteq \eta$ and $g_{U_1} \in U_1 \subseteq U_0$. Since $(x, x_i) \in \kappa \subseteq \eta$, by (3.4) we have

$$(g_{U_1} x, g_{U_1} x_i) \in \delta. \tag{3.7}$$

Also, by (3.5) we have

$$(hx, hx_i) \in \delta. \tag{3.8}$$

Now, combining (3.7), (3.6) and (3.8) we get

$$(g_{U_1} x, g_{U_1} x_i), (g_{U_1} x_i, hx_i), (hx_i, hx) \in \delta^3.$$

Therefore, for every $x \in X$ we have $(g_{U_1} x, hx) \in \delta^3 \subseteq \varepsilon$ and thus $h \in \overline{A}$, as required. □

The following claim shows that τ is not Hausdorff.

Claim 3.3. *For every $U \in N_e(\tau)$, $h \in U$.*

Proof. For $g \in P$ and $\varepsilon \in \mathcal{U}$ define

$$\tilde{\varepsilon}(g) = \{f \in P : (g(x), f(x)) \in \varepsilon \text{ for all } x \in X\}.$$

It can be easily verified that $\{\tilde{\varepsilon}(g)\}_{\varepsilon\in\mathcal{U}}$ is a local base of neighborhoods for every point $g \in P$, with respect to the compact-open topology on P. In order to prove the statement, it suffices to show that $h \in [\tilde{\varepsilon}^3(e)]^{-1}U_0$ for every $U_0 \in N_e(\tau)$ and for every $\varepsilon \in \mathcal{U}$. Indeed, for each $U \in N_e(\tau)$ we can find $U_0 \in N_e(\tau)$ such that $U_0^2 \subseteq U$ and $U_0^{-1} = U_0$. But $\tau \subseteq \tau_{co}$, and $\{\tilde{\varepsilon}(e)\}_{\varepsilon\in\mathcal{U}}$ is a local base at e, thus there exists $\varepsilon \in \mathcal{U}$ with $\tilde{\varepsilon}^3(e) \subseteq U_0$. Therefore $[\tilde{\varepsilon}^3(e)]^{-1}U_0 \subseteq U_0^{-1}U_0 \subseteq U$.

Let $\varepsilon \in \mathcal{U}$ and $U_0 \in N_e(\tau)$. Choose $\delta_\varepsilon \in \mathcal{U}$ and $U_\varepsilon \in N_e(\tau)$ from the definition of π-uniform topology. For $x \in X$ define $A(h^{-1}(x), \varepsilon) = \{U \in N_e(\tau) : (g_U h^{-1}(x), x) \in \varepsilon\}$. Since $h(h^{-1}(x)) = x$, from the definition of h we have $A(h^{-1}(x), \varepsilon) \in \mathcal{J}$. Indeed, $x = h(h^{-1}(x)) = \widetilde{h^{-1}(x)}$, x is the limit point of the map $f_{h^{-1}(x)} : N_e(\tau) \to X$ defined by $f_{h^{-1}(x)}(U) = g_U h^{-1}(x)$. Since h (and thus h^{-1}) is uniformly continuous, we can choose $\alpha \in \mathcal{U}$ such that $\alpha \subseteq \varepsilon$ and

$$(t_1, t_2) \in \alpha \Rightarrow (h^{-1}(t_1), h^{-1}(t_2)) \in \delta_\varepsilon. \tag{3.9}$$

Since X is compact, we can find a finite subset $\{x_1, x_2, ..., x_n\} \subseteq X$ such that for every $x \in X$ there exists $1 \le i \le n$ for which $(x, x_i) \in \alpha$. Let

$$U \in \left(\bigcap_{i=1}^{n} A(h^{-1}(x_i), \varepsilon)\right) \bigcap F\left(U_\varepsilon \cap U_0\right).$$

For every $x \in X$ there exists i such that $(x, x_i) \in \alpha$ and from (3.9) we have $(h^{-1}(x), h^{-1}(x_i)) \in \delta_\varepsilon$. Since $U \subseteq U_\varepsilon$, by the choice of U_ε and δ_ε we have $(g_U h^{-1}(x), g_U h^{-1}(x_i)) \in \varepsilon$. Since $U \in A(h^{-1}(x_i), \varepsilon)$, it follows that $(g_U h^{-1}(x_i), x_i) \in \varepsilon$. Recalling that $(x, x_i) \in \alpha \subseteq \varepsilon$ we obtain $(g_U h^{-1}(x), x) \in \varepsilon^3$, and therefore $g_U h^{-1} \in \tilde{\varepsilon}^3(e)$. But since $g_U \in U \subseteq U_0$, we get $h \in [\tilde{\varepsilon}^3(e)]^{-1}U_0$, and this completes the proof. □

Claims 3.2 and 3.3 complete the proof of Theorem 3.1. □

By Remark 2.7, every Hausdorff topological space can be viewed as an ordered topological space. Therefore, Theorem 3.1 directly yields the following:

Theorem 3.4. *Let (X, τ) be a compact topological space and let P be a closed subgroup of $\mathrm{H}(X)$, the group of all homeomorphisms of X. Then the compact-open topology τ_{co} is minimal within the class of π-uniform topologies on P.*

By Lemma 2.9 $\mathrm{H}_+(X)$ is a closed subgroup of $\mathrm{H}(X)$ for every partially ordered compact space X. So, if P is a closed subgroup of $\mathrm{H}_+(X)$ it is also a closed subgroup of $\mathrm{H}(X)$. Therefore, by Theorem 3.4 the compact-open topology τ_{co} is minimal within the class of π-uniform topologies on P. It follows that Theorem 3.1 can be derived back from Theorem 3.4.

Let us extend Theorem 3.1 in some algebraic setting. Let $\omega : K \times K \to K$ be a binary operation on a compact space K. Denote by $\mathrm{Aut}(K)$ the group of all topological automorphisms of the structure (K, ω). If K is a compact ordered space, then we denote by $\mathrm{Aut}_+(K)$ the group of all order preserving automorphisms of $(K, \omega, \leq)$. Note that $\mathrm{Aut}_+(K) = \mathrm{Aut}(K) \cap \mathrm{H}_+(K)$. Since $\mathrm{Aut}_+(K)$ is a closed subgroup of $\mathrm{H}_+(K)$ by Theorem 3.1 we obtain:

Corollary 3.5. *If K is a compact ordered space with a binary operation ω and P is a closed subgroup of $\mathrm{Aut}_+(K)$, then the compact-open topology is minimal within the class of π-uniform topologies on P.*

By Remark 2.7, every topological group can be viewed as an ordered topological space equipped with a group operation. Therefore, by Corollary 3.5 we get the following:

Corollary 3.6. *If K is a compact topological group and P is a closed subgroup of $\mathrm{Aut}(K)$, then the compact-open topology is minimal within the class of π-uniform topologies on P.*

4. Minimality of $G \leftthreetimes P$ where P is closed

The main goal of this section is to prove that for every compact topological group G, the natural semidirect product $G \leftthreetimes P$ is a minimal topological group for every closed subgroup $P \leq \mathrm{Aut}(G)$ (Theorem 4.2).

We need the following technical result which is inspired by [19, Prop. 2.6] (see also [8, Theorem 4.13]).

Theorem 4.1. *Let (M, γ) be a topological group, X and G are subgroups of M such that M is* algebraically *a semidirect product $M = X \leftthreetimes_\alpha G$. Assume that the topological subgroup $(X, \gamma|_X)$ of (M, γ) is compact. Then the action*

$$\alpha : (G, \gamma/X) \times (X, \gamma|_X) \to (X, \gamma|_X)$$

is continuous at (e_G, e_X), where γ/X is the coset topology on G induced by γ.

Proof. Let $pr : M \to G = M/X$, $(x, g) \mapsto g$, denote the canonical projection. Algebraically $M/X = \{X \times \{g\}\}_{g \in G}$, which allows us to identify G with M/X, and thus the topological group $(G, \gamma/X)$ is well defined.

To show that α is continuous at (e_G, e_X) let $O \in \gamma|_X$ be a neighborhood of e_X. We will find neighborhoods P of e_G in $(G, \gamma/X)$ and U of e_X in $(X, \gamma|_X)$ such that $\alpha(P \times U) \subseteq O$.

Since X is a compact group, there exists a neighborhood O_1 of e_X such that for all $x \in X$ we have $x^{-1}O_1 x \subseteq O$. The restriction $M \times X \to X$, $(a, x) \mapsto axa^{-1}$ of the conjugation $M \times M \to M$ in the topological group (M, γ) is (well-defined, because X is a normal subgroup of M) continuous at (e_M, e_X). Therefore, for O_1 there exist a neighborhood U of e_X in $(X, \gamma|_X)$) and a neighborhood V of e_M in (M, γ) such that $vUv^{-1} \subseteq O_1$ for all $v \in V$.

Consider the canonical projection $pr : M \to G = M/X$. Then $P := pr(V) \in \gamma/X$ is a neighborhood of e_G in $(G, \gamma/X)$. We claim that P and U satisfy the needed conditions above. That is, we want to show that $\alpha(g, z) := g(z) \in O$ for all $(g, z) \in pr(V) \times U$. Indeed, if $g \in pr(V)$ there exists $x \in X$ such that $(x, g) \in V$, and recall that $z \in U$ is in fact (z, e_G). We know that $vzv^{-1} \in O_1$. Therefore,

$$vzv^{-1} = (x, g)(z, e_G)(x, g)^{-1} = (xg(z), g)(g^{-1}(x^{-1}), g^{-1}) =$$
$$= (xg(z)g(g^{-1}(x^{-1})), e_G) = (xg(z)x^{-1}, e_G) = xg(z)x^{-1} \in O_1.$$

Thus $\alpha(g, z) \in x^{-1}O_1 x \subseteq O$, which completes the proof. □

Theorem 4.2. *If G is a compact topological group, then $G \rtimes P$ is minimal for every closed subgroup P of* $\mathrm{Aut}(G)$.

Proof. Let τ be the given topology on G, and τ_{co} the compact-open topology on $P \subseteq \mathrm{Aut}(G)$. Denote by γ the product topology on $G \rtimes P$, and by $e = id_G \in P$ the identity automorphism. Assume that $\gamma_1 \subseteq \gamma$ is a coarser Hausdorff group topology on $G \rtimes P$. Since G is compact we have $\gamma_1|_G = \gamma|_G = \tau$.

The action

$$\alpha : (P, \gamma_1/G) \times (G, \gamma_1|_G) \to (G, \gamma_1|_G)$$

is continuous at the identity (id_G, e_G) by Theorem 4.1. Furthermore, γ_1/G is a Hausdorff topology on $\mathrm{Aut}(G)$ since G is a compact (hence, closed) subgroup of the Hausdorff group $(G \rtimes P, \gamma_1)$. Therefore γ_1/G is an α-uniform topology on P (Lemma 2.4.2 and Definition 2.3).

Since $\gamma_1/G \subseteq \gamma/G = \tau_{co}$ and τ_{co} is minimal within the class of α-uniform topologies on P (Corollary 3.6), we have $\gamma_1/G = \gamma/G$. Finally, using Merson's Lemma 2.1, we deduce that $\gamma_1 = \gamma$ and that concludes the proof. □

5. Minimality of $G \rtimes P$

In this section we provide an equivalent condition (see Theorem 5.2) for the non-minimality of $G \rtimes P$, where G is a compact group and $P \leq \mathrm{Aut}(G)$.

As a corollary we obtain that if the compact group G is *abelian*, then Theorem 4.2 holds for all (thus not necessarily closed) subgroups P of $\mathrm{Aut}(G)$. Theorem 5.2 also allows us to construct relevant counterexamples (Example 5.6 and Theorem 5.5).

We use the well known Minimality Criterion which, for compact groups, can be traced back to Stephenson [24] and Prodanov [23]. Note that Banaschewski [2] generalized this criterion by proving it for minimal topological algebras.

First recall that a subgroup $H \leq G$ of a topological group G is said to be *essential* in G if $H \cap N$ is nontrivial for every nontrivial closed normal subgroup N of G.

The following theorem can be found, for example, in [9, Theorem 2.5.1].

Theorem 5.1 (Minimality Criterion)**.** *Let G be a topological group and H its dense subgroup. H is minimal if and only if G is minimal and H is essential in G.*

Theorem 5.2. *Let G be a compact group and $P \leq \mathrm{Aut}(G)$. Then the following two conditions are equivalent:*

(1) $G \rtimes P$ is not minimal.
(2) There exists a closed nontrivial subgroup H of G satisfying the following conditions:
  (a) $H \cap Z(G) = \{e_G\}$.
  (b) H is P-invariant (that is, $f(H) \subseteq H$ for every $f \in P$).
  (c) $\Gamma(H) \subseteq \overline{P}$ and $\Gamma(H) \cap P = \{e_P\}$, where $\Gamma : G \to \mathrm{Inn}(G)$ is the natural homomorphism defined by $\Gamma(g) = \gamma_g$.

Proof. (2) $\Rightarrow$ (1): Let H be a closed nontrivial subgroup of G satisfying conditions $(a)-(c)$. By Theorem 4.2 $G \rtimes \overline{P}$ is minimal. Clearly, $G \rtimes P$ is a dense subgroup of $G \rtimes \overline{P}$. We are going to construct a nontrivial closed normal subgroup N of $G \rtimes \overline{P}$ such that $N \cap (G \rtimes P)$ is trivial. Using the Minimality Criterion this will imply that $G \rtimes P$ is not minimal. Let $N := \{(h, \gamma_h^{-1}) |\ h \in H\}$. Since H is a compact nontrivial subgroup of G and $\Gamma(H)$ is a compact subgroup of $\overline{P}$, we obtain that N is a nontrivial compact (hence closed) subgroup of $G \rtimes \overline{P}$. Being P-invariant and closed the subgroup H is also $\overline{P}$-invariant. This implies that N is normal in $G \rtimes \overline{P}$. Indeed, $(g, f)(h, \gamma_h^{-1})(g, f)^{-1} = (f(h), \gamma_{f(h)}^{-1})$. Let (h, γ_h^{-1}) be a nontrivial element of N. Then we necessarily have $h \neq e_G$. By condition (a) this implies that γ_h^{-1} is a nontrivial element of $\Gamma(H)$. It follows from (c) that $\gamma_h^{-1} \notin P$. Therefore, $(h, \gamma_h^{-1}) \notin G \rtimes P$ and we conclude that $N \cap (G \rtimes P)$ is trivial as needed.

(1) $\Rightarrow$ (2) : Assume that $G \leftthreetimes P$ is not minimal. If follows from Theorem 4.2 and the Minimality Criterion that $G \leftthreetimes P$ is not essential in $G \leftthreetimes \overline{P}$. So, there exists a nontrivial closed normal subgroup N of $G \leftthreetimes \overline{P}$ such that $N \cap (G \leftthreetimes P)$ is trivial. We will prove (2) by showing that $N = \{(h, \gamma_h^{-1}) |\ h \in H\}$, where H is a nontrivial closed subgroup of G satisfying conditions (a) – (c). First, let us show that every element in N has the form (h, γ_h^{-1}), for some $h \in G$. Indeed, otherwise, there exists $u_1 = (h, f) \in N$ such that $f \neq \gamma_h^{-1}$. Choose $u_2 = (g, e_P)$, where $f(g) \neq \gamma_h^{-1}(g)$. Since N is normal in $G \leftthreetimes \overline{P}$ the commutator

$$[u_1, u_2] = (h \cdot f(g) \cdot h^{-1} \cdot g^{-1}, e_P)$$

is an element of N. The inequality $f(g) \neq \gamma_h^{-1}(g)$ means that $h \cdot f(g) \cdot h^{-1} \cdot g^{-1} \neq e_G$. It follows that $[u_1, u_2]$ is a nontrivial element of $G \leftthreetimes P$. This contradicts the fact that $N \cap (G \leftthreetimes P)$ it trivial. Therefore, all elements of N have the form (h, γ_h^{-1}). The group N is compact being a closed subgroup of the compact group $G \leftthreetimes (\overline{P} \cap \mathrm{Inn}(G))$. By the continuity of the canonical projection $pr : G \leftthreetimes \overline{P} \to G$, we conclude that the subgroup $H := pr(N)$ is closed (being compact) in G. Moreover, the structure of N implies that H is also nontrivial.

To prove (a) assume that there exists a nontrivial element $h \in H \cap Z(G)$. But then (h, γ_h^{-1}) is a nontrivial element of $N \cap (G \leftthreetimes P)$, and that is a contradiction.

Property (b) follows from the normality of N. Indeed, if $f \in P$ and $h \in H$ we have $(e_G, f)(h, \gamma_h^{-1})(e_G, f)^{-1} = (f(h), \gamma_{f(h)}^{-1}) \in N$. Hence, $f(h) \in H$.

Finally, we prove property (c). Let $\gamma_h \in \Gamma(H)$, where $h \in H = pr(N)$. Then, $(h, \gamma_h^{-1}) \in N \subseteq G \leftthreetimes \overline{P}$. Therefore, $\gamma_h^{-1} \in \overline{P}$ and since $\overline{P}$ is a group we also have $\gamma_h \in \overline{P}$. This proves that $\Gamma(H) \subseteq \overline{P}$. Let us show that $\Gamma(H) \cap P$ is trivial. Otherwise, there exists a nontrivial $h \in H$ such that $\gamma_h^{-1} \in \Gamma(H) \cap P$. But then $(e_G, e_P) \neq (h, \gamma_h^{-1}) \in N \cap (G \leftthreetimes P)$, contradicting the triviality of $N \cap (G \leftthreetimes P)$. This completes the proof. □

Corollary 5.3. *Let G be a compact group and $P \leq \mathrm{Aut}(G)$. Then $G \leftthreetimes P$ is minimal in each of the following cases:*

(1) G is abelian;
(2) P is essential in $\overline{P}$ (e.g., P is closed);
(3) $P \cap \mathrm{Inn}(G)$ is essential in $\overline{P} \cap \mathrm{Inn}(G)$.

Proof. In each case at least one of the conditions of Theorem 5.2.2 does not hold.
(1): If G is an abelian group then $G = Z(G)$. Thus for every nontrivial $H \leq G$ the group $H \cap Z(G)$ is nontrivial and so (a) is impossible.
(2): Assume for a contradiction that H is a nontrivial closed subgroup of G satisfying conditions (a) – (c). It follows that $N = \Gamma(H)$ is a nontrivial

closed subgroup of $\overline{P}$ which trivially intersects P. This contradicts the assumption that P is essential in $\overline{P}$.
(3): Assume for a contradiction that H is a nontrivial closed subgroup of G satisfying conditions $(a)-(c)$. It follows that $N = \Gamma(H)$ is a nontrivial closed subgroup of $\overline{P} \cap \mathrm{Inn}(G)$ which trivially intersects $\overline{P} \cap \mathrm{Inn}(G)$. This contradicts the assumption that $P \cap \mathrm{Inn}(G)$ is essential in $\overline{P} \cap \mathrm{Inn}(G)$. □

Theorem 5.4. *Let G be a compact group with trivial center and $P \leq \mathrm{Inn}(G)$. Then $G \leftthreetimes P$ is minimal if and only if P is essential in $\overline{P}$.*

Proof. Sufficiency follows from Corollary 5.3.3.
To prove the necessity assume that P is not essential in $\overline{P}$. Then, there exists a closed normal subgroup N of $\overline{P}$ such that $N \cap P$ is trivial. Since the center of G is trivial, the continuous homomorphism $\Gamma : G \to \mathrm{Inn}(G)$ is in fact a topological isomorphism. It follows that $H = \Gamma^{-1}(N)$ is a closed nontrivial subgroup of G. Clearly, $H \cap Z(G) = \{e_G\}$. By the normality of N, one can show that H is P-invariant. Furthermore, $\Gamma(H) = N$ is a subgroup of $\overline{P}$ and $\Gamma(H) \cap P = N \cap P = \{e_P\}$. It follows from Theorem 5.2 that $G \leftthreetimes P$ is not minimal. □

Theorem 5.5. *Let G be a compact group containing a non-closed Boolean subgroup B and let $P = \Gamma(B)$. If in addition $\overline{B} \cap Z(G)$ is trivial, then $G \leftthreetimes P$ is not minimal.*

Proof. By Theorem 5.2, it suffices to show that there exists a closed nontrivial subgroup $H \leq G$ with properties $(a)-(c)$. Since B is a Boolean subgroup of G, its closure $\overline{B}$ is also Boolean. Fix $h \in \overline{B} \setminus B$ and let $H = \{h, e_G\}$. Then, H is a nontrivial closed subgroup of G with $H \cap Z(G) = \{e_G\}$. For every $b \in B$ we have $bhb^{-1} = h$, since $\overline{B}$ is abelian. This implies that H is P-invariant. Clearly, $\Gamma(H)$ is a subgroup of $\overline{P} = \Gamma(\overline{B})$. Finally, assume for contradiction that $\Gamma(H) \cap P \neq \{e_P\}$. Hence, there exists $b \in B$ such that $\gamma_b = \gamma_h$. This implies that $e_G \neq hb \in \overline{B} \cap Z(G)$, a contradiction. □

We use Theorem 5.5 in the following example, where we show that there exist a compact two-step nilpotent group G and a subgroup P of $\mathrm{Aut}(G)$ such that $G \leftthreetimes P$ is not minimal. This answers a question of Dikranjan.

Example 5.6. Let R be the compact ring $\mathbb{Z}_2^{\mathbb{N}}$ (with operations defined coordinatewise) and consider its dense subring

$$\tilde{R} := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{Z}_2 \wedge \ |n : x_n \neq 0| < \infty\}.$$

Let $G := (R \times R) \leftthreetimes R$ be the *generalized Heisenberg group* (see, for example, [8]) defined via the action

$$\pi : R \times (R \times R) \to R \times R, \ \pi(f, (a, x)) = (a + fx, x).$$

By [19, Lemma 2.1], $Z(G) = (R \times \{0_R\}) \rtimes \{0_R\}$. Let $B := (\{0_R\} \times \tilde{R}) \rtimes \{0_R\}$ and let $P = \Gamma(B)$. Then, B is a non-closed Boolean subgroup of G. Indeed, its closure $\overline{B}$ coincides with $(\{0_R\} \times R) \rtimes \{0_R\}$. Since $\overline{B} \cap Z(G)$ is trivial, it follows from Theorem 5.5 that $G \rtimes P$ is not minimal.

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