Topology and its Applications 159 (2012) 2497–2505

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Notes on non-archimedean topological groups

Michael Megrelishvili *, Menachem Shlossberg

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

ARTICLE INFO

Dedicated to Professor Dikran Dikranjan on his 60th birthday

Keywords:
Boolean group
Heisenberg group
Isosceles
Minimal group
Non-archimedean group
Stone duality
Stone space
Ultra-metric

ABSTRACT

We show that the Heisenberg type group $H_X = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*$, with the discrete Boolean group $V := C(X, \mathbb{Z}_2)$, canonically defined by any Stone space X, is always minimal. That is, H_X does not admit any strictly coarser Hausdorff group topology. This leads us to the following result: for every (locally compact) non-archimedean G there exists a (resp., locally compact) non-archimedean minimal group M such that G is a group retract of M. For discrete groups G the latter was proved by S. Dierolf and U. Schwanengel (1979) [6]. We unify some old and new characterization results for non-archimedean groups.

1. Introduction and preliminaries

A topological group is *non-archimedean* if it has a local base at the identity consisting of open subgroups. This class of groups coincides with the class of topological subgroups of the homeomorphism groups Homeo(X), where X runs over *Stone spaces* (= compact zero-dimensional spaces) and Homeo(X) carries the usual compact open topology. Recall that by Stone's representation theorem, there is a duality between the category of Stone spaces and the category of Boolean algebras. The class $\mathcal{NA}$ of non-archimedean groups and the related class of ultra-metric spaces have many applications. For instance, in non-archimedean functional analysis, in descriptive set theory, computer science, etc. See, e.g., [36,3,22,21,43] and references therein.

In the present paper we provide some applications of generalized Heisenberg groups, with emphasis on minimality properties, in the theory of $\mathcal{NA}$ groups and actions on Stone spaces.

Recall that a Hausdorff topological group G is *minimal* (Stephenson [38] and Doïchinov [12]) if it does not admit a strictly coarser Hausdorff group topology, or equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding.

If otherwise is not stated all topological groups and spaces in this paper are assumed to be Hausdorff. We say that an additive topological group $(G, +)$ is a *Boolean group* if $x + x = 0$ for every $x \in G$. As usual, a G-*space* X is a topological space X with a continuous group action $\pi : G \times X \to X$ of a topological group G. We say that X is a G-*group* if, in addition, X is a topological group and all g-translations, $\pi^g : X \to X,\ x \mapsto gx := \pi(g, x)$, are automorphisms of X. For every G-group X we denote by $X \leftthreetimes G$ the corresponding topological semidirect product.

To every Stone space X we associate a (locally compact 2-step nilpotent) Heisenberg type group

$$H_X = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*,$$

* Corresponding author.
E-mail addresses: megereli@math.biu.ac.il (M. Megrelishvili), shlosbm@macs.biu.ac.il (M. Shlossberg).
URLs: http://www.math.biu.ac.il/~megereli (M. Megrelishvili), http://www.math.biu.ac.il/~shlosbm (M. Shlossberg).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.06.069

where $V := C(X, \mathbb{Z}_2)$ is a discrete Boolean group which can be identified with the group of all clopen subsets of X (symmetric difference is the group operation). $V^* := \operatorname{Hom}(V, \mathbb{Z}_2)$ is the compact group of all group homomorphisms into the two element cyclic group $\mathbb{Z}_2$. V^* acts on $\mathbb{Z}_2 \oplus V$ in the following way: every $(f, (a, x)) \in V^* \times (\mathbb{Z}_2 \oplus V)$ is mapped to $(a + f(x), x) \in \mathbb{Z}_2 \oplus V$. The group operation on H_X is defined as follows: for

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2) \in H_X$$

we define

$$u_1 u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big).$$

In Section 4 we study some properties of H_X and show in particular (Theorem 4.1) that the (locally compact) Heisenberg group $H_X = (\mathbb{Z}_2 \times V) \leftthreetimes V^*$ is minimal and non-archimedean for every Stone space X.

Every Stone space X is naturally embedded into $V^* := \operatorname{Hom}(V, \mathbb{Z}_2)$ by the natural map $\delta : X \to V^*$, $x \mapsto \delta_x$ where $\delta_x(f) := f(x)$. Every δ_x can be treated as a 2-valued measure on X. Identifying X with $\delta(X) \subset V^*$ we get a restricted evaluation map $V \times X \to \mathbb{Z}_2$ which in fact is the evaluation map of the Stone duality. Note that the role of $\delta : X \to V^*$ for a compact space X is similar to the role of the Gelfand map $X \to C(X)^*$, representing X via the point measures.

For every action of a group $G \subset \operatorname{Homeo}(X)$ on a Stone space X we can deal with a G-space version of the classical Stone duality. The map $\delta : X \to V^*$ is a G-map of G-spaces. Furthermore, a deeper analysis shows (Theorem 4.4) that every topological subgroup $G \subset \operatorname{Homeo}(X)$ induces a continuous action of G on H_X by automorphisms such that the corresponding semidirect product $H_X \leftthreetimes G$ is a minimal group.

We then conclude (Corollary 4.5) that every (locally compact) non-archimedean group is a group retract of a (resp., locally compact) minimal non-archimedean group. It covers a result of Dierolf and Schwanengel [6] (see also Example 3.5 below) which asserts that every discrete group is a group retract of a locally compact non-archimedean minimal group.

Section 2 contains additional motivating results and questions. Several interesting applications of generalized Heisenberg groups can be found in the papers [25–27,11,28,8,9,37].

Studying the properties of the Heisenberg group H_X, we get a unified approach to several (mostly known) equivalent characterizations of the class $\mathcal{NA}$ of non-archimedean groups (Lemma 3.2 and Theorem 5.1). In particular, we show that the class of all topological subgroups of $\operatorname{Aut}(K)$, for compact abelian groups K, is precisely $\mathcal{NA}$.

2. Minimality and group representations

Clearly, every compact topological group is minimal. Trivial examples of nonminimal groups are: the group $\mathbb{Z}$ of all integers (or any discrete infinite abelian group) and $\mathbb{R}$, the topological group of all reals. By a fundamental theorem of Prodanov and Stoyanov [32] every abelian minimal group is precompact. For more information about minimal groups see review papers of Dikranjan [7] and Comfort, Hofmann and Remus [5], a book of Dikranjan, Prodanov and Stoyanov [10] and a recent book of Lukacs [23].

Unexpectedly enough many non-compact naturally defined topological groups are minimal.

Remark 2.1. Recall some nontrivial examples of minimal groups.

(1) Prodanov [31] showed that the p-adic topologies are the only precompact minimal group topologies on $\mathbb{Z}$.
(2) Symmetric topological groups S_X (Gaughan [15]).
(3) $\operatorname{Homeo}(\{0,1\}^{\aleph_0})$ (see Gamarnik [14] and also Uspenskij [42] for a more general case).
(4) $\operatorname{Homeo}[0,1]$ (Gamarnik [14]).
(5) The semidirect product $\mathbb{R} \leftthreetimes \mathbb{R}_+$ (Dierolf and Schwanengel [6]). More general cases of minimal (so-called *admissible*) semidirect products were studied by Remus and Stoyanov [35]. By [26], $\mathbb{R}^n \leftthreetimes \mathbb{R}_+$ is minimal for every $n \in \mathbb{N}$.
(6) Every connected semisimple Lie group with finite center, e.g., $SL_n(\mathbb{R})$, $n \geqslant 2$ (Remus and Stoyanov [35]).
(7) The full unitary group $U(H)$ (Stoyanov [39]).

One of the immediate difficulties is the fact that minimality is not preserved by quotients and (closed) subgroups. See for example item (5) with minimal $\mathbb{R} \leftthreetimes \mathbb{R}_+$ where its canonical quotient $\mathbb{R}_+$ (the positive reals) and the closed normal subgroup $\mathbb{R}$ are nonminimal. As a contrast note that in a minimal *abelian* group every closed subgroup is minimal [10].

In 1983 Pestov raised the conjecture that every topological group is a group retract of a minimal group. Note that if $f : M \to G$ is a group retraction then necessarily G is a quotient of M and also a closed subgroup in M. Arhangel'skiĭ asked the following closely related questions:

Question 2.2. *([2,30]) Is every topological group a quotient of a minimal group? Is every topological group a closed subgroup of a minimal group?*

By a result of Uspenskij [41] every topological group is a subgroup of a minimal group M which is Raikov-complete, topologically simple and Roelcke-precompact.

Recently a positive answer to Pestov's conjecture (and hence to Question 2.2 of Arhangel'skiĭ) was obtained in [28]. The proof is based on methods (from [25]) of constructing minimal groups using group representations on Banach spaces and involving generalized Heisenberg groups.

According to [25] every locally compact *abelian* group is a group retract of a minimal locally compact group. It is an open question whether the same is true in the non-abelian case.

Question 2.3. *([25,28,5]) Is it true that every locally compact group G is a group retract (at least a subgroup or a quotient) of a locally compact minimal group?*

A more general natural question is the following:

Question 2.4. *([25]) Let $\mathcal{K}$ be a certain class of topological groups and* $\underline{\text{min}}$ *denotes the class of all minimal groups. Is it true that every $G \in \mathcal{K}$ is a group retract of a group $M \in \mathcal{K} \cap \underline{\text{min}}$?*

So Corollary 4.5 gives a partial answer to Questions 2.3 and 2.4 in the class $\mathcal{K} := \mathcal{NA}$ of non-archimedean groups.

Remark 2.5. Note that by [28, Theorem 7.2] we can present any topological group G as a group retraction $M \to G$, where M is a minimal group having the same weight and character as G. Furthermore, if G is Raikov-complete then M also has the same property. These results provide in particular a positive answer to Question 2.4 in the following basic classes: second countable groups, metrizable groups, Polish groups.

2.1. Minimality properties of actions

Definition 2.6. Let $\alpha : G \times X \to X$, $\alpha(g, x) = gx$ be a continuous action of a Hausdorff topological group (G, σ) on a Hausdorff topological space (X, τ). The action α is said to be:

(1) *Algebraically exact* if $\ker_\alpha := \{g \in G\colon\ gx = x\ \forall x \in X\}$ is the trivial subgroup $\{e\}$.
(2) *Topologically exact* (*t-exact*, in short) if there is no strictly coarser, not necessarily Hausdorff, group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ', τ, τ)-continuous.

Remark 2.7.

(1) Every topologically exact action is algebraically exact. Indeed, otherwise $\ker_\alpha$ is a nontrivial subgroup in G. Then the preimage group topology $\sigma' \subset \sigma$ on G induced by the onto homomorphism $G \to G/\ker_\alpha$ is not Hausdorff (in particular, it differs σ) and the action remains (σ', τ, τ)-continuous.
(2) On the other hand, if α is algebraically exact then it is topologically exact if and only if for every strictly coarser *Hausdorff* group topology $\sigma' \subsetneq \sigma$ on G the action α is not (σ', τ, τ)-continuous. Indeed, since α is algebraically exact and (X, τ) is Hausdorff then every coarser group topology σ' on G which makes the action (σ', τ, τ)-continuous must be Hausdorff.

Let X be a locally compact group and $\mathrm{Aut}(X)$ be the group of all automorphisms endowed with the *Birkhoff topology* (see [16, §26] and [10, p. 260]). Some authors use the name *Braconnier topology* (see [4]).

The latter is a group topology on $\mathrm{Aut}(X)$ and has a local base formed by the sets

$$\mathcal{B}(K, O) := \left\{ f \in \mathrm{Aut}(X)\colon\ f(x) \in Ox \text{ and } f^{-1}(x) \in Ox\ \forall x \in K \right\}$$

where K runs over compact subsets and O runs over neighborhoods of the identity in X. In the sequel $\mathrm{Aut}(X)$ is always equipped with the Birkhoff topology. It equals to the Arens *g-topology* [1,4]. If X is compact then the Birkhoff topology coincides with the usual compact-open topology. If X is discrete then the Birkhoff topology on $\mathrm{Aut}(X) \subset X^X$ coincides with the pointwise topology.

Lemma 2.8. *In each of the following cases the action of G on X is t-exact*:

(1) *([25]) Let X be a locally compact group and G be a subgroup of* $\mathrm{Aut}(X)$.
(2) *Let G be a topological subgroup of* $\mathrm{Homeo}(X)$, *the group of all autohomeomorphisms of a compact space X with the compact open topology.*
(3) *Let G be a subgroup of $Is(X, d)$ the group of all isometries of a metric space (X, d) with the pointwise topology.*

Proof. Straightforward. □

Remark 2.9. Every locally compact abelian group G can be embedded into the group $\mathrm{Aut}(X)$, where X is a locally compact abelian group (with $X := \mathbb{T} \times G^*$, [25, Prop. 2.3]). Note that for locally compact subgroups of $\mathrm{Aut}(X)$ [25, Theorem 4.4] positively resolves Question 2.3. See also Remark 5.2(3).

2.2. From minimal dualities to minimal groups

In this subsection we recall some definitions and results from [25,28].

Let E, F, A be abelian additive topological groups. A map $w : E \times F \to A$ is said to be *biadditive* if the induced mappings

$$w_x : F \to A, \qquad w_f : E \to A, \qquad w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$.

A biadditive mapping $w : E \times F \to A$ is *separated* if for every pair (x_0, f_0) of nonzero elements there exists a pair (x, f) such that $f(x_0) \neq 0_A$ and $f_0(x) \neq 0_A$.

A continuous separated biadditive mapping $w : (E, \sigma) \times (F, \tau) \to A$ is *minimal* if for every coarser pair (σ_1, τ_1) of Hausdorff group topologies $\sigma_1 \subseteq \sigma,\ \tau_1 \subseteq \tau$ such that $w : (E, \sigma_1) \times (F, \tau_1) \to A$ is continuous, it follows that $\sigma_1 = \sigma$ and $\tau_1 = \tau$.

Let $w : E \times F \to A$ be a continuous biadditive mapping. Consider the action: $w^{\triangledown} : F \times (A \oplus E) \to A \oplus E, \quad w^{\triangledown}(f, (a, x)) = (a + w(x, f), x)$. Denote by $H(w) = (A \oplus E) \rightthreetimes F$ the topological semidirect product of F and the direct sum $A \oplus E$. The group operation on $H(w)$ is defined as follows: for a pair

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2)$$

we define

$$u_1 u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big)$$

where, $f_1(x_2) = w(x_2, f_1)$. Then $H(w)$ becomes a Hausdorff topological group which is said to be a *generalized Heisenberg group* (induced by w).

Let G be a topological group and let $w : E \times F \to A$ be a continuous biadditive mapping. A continuous *birepresentation* of G in w is a pair (α_1, α_2) of continuous actions by group automorphisms $\alpha_1 : G \times E \to E$ and $\alpha_2 : G \times F \to F$ such that w is G-invariant, i.e., $w(gx, gf) = w(x, f)$.

The birepresentation ψ is said to be *t-exact* if $\ker(\alpha_1) \cap \ker(\alpha_2) = \{e\}$ and for every strictly coarser *Hausdorff* group topology on G the birepresentation does not remain continuous. For instance, if one of the actions α_1 or α_2 is t-exact then clearly ψ is t-exact.

Let ψ be a continuous G-birepresentation

$$\psi = (w : E \times F \to A,\ \alpha_1 : G \times E \to E,\ \alpha_2 : G \times F \to F).$$

The topological semidirect product $M(\psi) := H(w) \rightthreetimes_\pi G$ is said to be the *induced group*, where the action $\pi : G \times H(w) \to H(w)$ is defined by

$$\pi\big(g, (a, x, f)\big) = (a, gx, gf).$$

Fact 2.10. *Let $w : E \times F \to A$ be a minimal biadditive mapping and A is a minimal group. Then*

(1) *([9, Corollary 5.2]) The Heisenberg group $H(w)$ is minimal.*
(2) *([25, Theorem 4.3] and [28]) If ψ is a t-exact G-birepresentation in w then the induced group $M(\psi)$ is minimal.*

Fact 2.11. *([25]) Let G be a locally compact abelian group and $G^* := \mathrm{Hom}(G, \mathbb{T})$ be the dual (locally compact) group. Then the canonical evaluation mapping*

$$G \times G^* \to \mathbb{T}$$

is minimal and the corresponding Heisenberg group $H = (\mathbb{T} \oplus G) \rightthreetimes G^*$ is minimal.

3. Some facts about non-archimedean groups and uniformities

3.1. Non-archimedean uniformities

For information on *uniform spaces*, we refer the reader to [13] (in terms of *entourages*) and to [19] (via *coverings*). If μ is a uniformity for X in terms of coverings, then the collection of elements of μ which are *finite* coverings of X forms a base for a topologically compatible uniformity for X which we denote by μ_{fin} (the precompact replica of μ).

A *partition* of a set X is a covering of X consisting of pairwise disjoint subsets of X. Due to Monna (see [36, p. 38] for more details), a uniform space (X, μ) is *non-archimedean* if it has a base consisting of partitions of X. In terms of entourages, it is equivalent to saying that there exists a base $\mathfrak{B}$ of the uniform structure such that every entourage $P \in \mathfrak{B}$ is an equivalence relation. Equivalently, iff its *large uniform dimension* (in the sense of Isbell [19, p. 78]) is zero.

A metric space (X,d) is said to be an *ultra-metric space* (or, *isosceles* [21]) if d is an *ultra-metric*, i.e., it satisfies *the strong triangle inequality*

$$d(x,z) \leqslant \max\bigl\{d(x,y), d(y,z)\bigr\}.$$

The definition of *ultra-semimetric* is the same as ultra-metric apart from the fact that the condition $d(x,y)=0$ need not imply $x=y$. For every ultra-semimetric d on X every ε-covering $\{B(x,\varepsilon)\colon x\in X\}$ by the open balls is a clopen partition of X.

Furthermore, a uniformity is non-archimedean iff it is generated by a system $\{d_i\}_{i\in I}$ of *ultra-semimetrics*. The following result (up to obvious reformulations) is well known. See, for example, [19] and [18].

Lemma 3.1. *Let (X,μ) be a non-archimedean uniform space. Then both (X,μ_{fin}) and the uniform completion $(\widehat{X},\widehat{\mu})$ of (X,μ) are non-archimedean uniform spaces.*

3.2. Non-archimedean groups

The class $\mathcal{NA}$ of all non-archimedean groups is quite large. Besides the results of this section see Theorem 5.1 below. The prodiscrete (in particular, the profinite) groups are in $\mathcal{NA}$. All $\mathcal{NA}$ groups are totally disconnected and for every locally compact totally disconnected group G both G and $\mathrm{Aut}(G)$ are $\mathcal{NA}$ (see Theorems 7.7 and 26.8 in [16]). Every abelian $\mathcal{NA}$ group is embedded into a product of discrete groups.

The minimal groups $(\mathbb{Z},\tau_p), S_X, \mathrm{Homeo}(\{0,1\}^{\aleph_0})$ (in items (1), (2) and (3) of Remark 2.1) are non-archimedean. By Theorem 4.1 the Heisenberg group $H_X=(\mathbb{Z}_2\oplus V)\leftthreetimes V^*$ is $\mathcal{NA}$ for every Stone space X. It is well known that there exist $2^{\aleph_0}$-many nonhomeomorphic metrizable Stone spaces.

Recall that, as it follows by results of Teleman [40], every topological group can be identified with a subgroup of $\mathrm{Homeo}(X)$ for some compact X and also with a subgroup of $Is(M,d)$, topological group of isometries of some metric space (M,d) endowed with the pointwise topology (see also [34]). Similar characterizations are true for $\mathcal{NA}$ with compact zero-dimensional spaces X and ultra-metric spaces (M,d). See Lemma 3.2 and Theorem 5.1 below.

We will use later the following simple observations. Let X be a Stone space (compact zero-dimensional space) and G be a topological subgroup of $\mathrm{Homeo}(X)$. For every finite clopen partition $P=\{A_1,\dots,A_n\}$ of X define the subgroup

$$M(P) := \{g\in G\colon\ gA_k=A_k\ \forall 1\leqslant k\leqslant n\}.$$

Then all subgroups of this form defines a local base (subbase, if we consider only two-element partitions P) of the original compact-open topology on $G\subset \mathrm{Homeo}(X)$. So for every Stone space X the topological group $\mathrm{Homeo}(X)$ is non-archimedean. More generally, for every non-archimedean uniform space (X,μ) consider the group $\mathrm{Unif}(X,\mu)$ of all uniform automorphisms of X (that is, the bijective functions $f: X\to X$ such that both f and f^{-1} are μ-uniform). Then $\mathrm{Unif}(X,\mu)$ is a non-archimedean topological group in the topology induced by the uniformity of uniform convergence.

Lemma 3.2. *The following assertions are equivalent*:

(1) *G is a non-archimedean topological group.*
(2) *The right* (*left*) *uniformity on G is non-archimedean.*
(3) $\dim\beta_G G=0$, *where $\beta_G G$ is the maximal G-compactification* [29] *of G.*
(4) *G is a topological subgroup of* $\mathrm{Homeo}(X)$ *for some Stone space X* (*where $w(X)=w(G)$*).
(5) *G is a topological subgroup of* $\mathrm{Unif}(Y,\mu)$ *for some non-archimedean uniformity μ on a set Y.*

Proof. For the sake of completeness we give here a sketch of the proof. The equivalence of (1) and (3) was established by Pestov [33, Prop. 3.4]. The equivalence of (1), (2) and (3) is [29, Theorem 3.3].

(1) ⇒ (2) Let $\{H_i\}_{i\in I}$ be a local base at e (the neutral element of G), where each H_i is an open (hence, clopen) subgroup of G. Then the corresponding decomposition of $G=\cup_{g\in G}H_i g$ by right H_i-cosets defines an equivalence relation Ω_i and the set $\{\Omega_i\}_{i\in I}$ is a base of the right uniform structure μ_r on G.

(2) ⇒ (3) If the right uniformity μ is non-archimedean then by Lemma 3.1 the completion $(\widehat{X},\widehat{\mu_{\mathrm{fin}}})$ of its precompact replica (Samuel compactification of (X,μ)) is again non-archimedean. Now recall (see for example [29]) that this completion is just the greatest G-compactification $\beta_G G$ (the G-space analog of the Stone–Čech compactification) of G.

(3) ⇒ (4) A result in [24] implies that there exists a zero-dimensional proper G-compactification X of the G-space G (the left action of G on itself) with $w(X)=w(G)$. Then the natural homomorphism $\varphi: G\to \mathrm{Homeo}(X)$ is a topological group embedding.

(4) ⇒ (5) Trivial because $\mathrm{Homeo}(X)=\mathrm{Unif}(X,\mu)$ for compact X and its unique compatible uniformity μ.

(5) ⇒ (1) The non-archimedean uniformity μ has a base $\mathfrak{B}$ where each $P\in\mathfrak{B}$ is an equivalence relation. Then the subsets

$$M(P) := \bigl\{g\in G\colon\ (gx,x)\in P\ \forall x\in X\bigr\}$$

form a local base of G. Observe that $M(P)$ is a subgroup of G. □

$\mathcal{NA}$-ness of a dense subgroup implies that of the whole group. Hence the Raikov-completion of $\mathcal{NA}$ groups are again $\mathcal{NA}$. Subgroups, quotient groups and (arbitrary) products of $\mathcal{NA}$ groups are also $\mathcal{NA}$. Moreover the class $\mathcal{NA}$ is closed under group extensions.

Fact 3.3. *[17, Theorem 2.7] If both N and G/N are $\mathcal{NA}$, then so is G.*

For the readers convenience we reproduce here the proof from [17].

Proof. Let U be a neighborhood of e in G. We will find an open subgroup H contained in U. We choose neighborhoods U_0, V and W of e in G as follows. First let U_0 be such that $U_0^2 \subseteq U$. By the assumption, there is an open subgroup M of N contained in $N \cap U_0$. Let $V \subseteq U_0$ be open with $V = V^{-1}$ and $V^3 \cap N \subseteq M$. We denote by π the natural homomorphism $G \to G/N$. Since $\pi(V)$ is open in G/N, it contains an open subgroup K. We set $W = V \cap \pi^{-1}(K)$. We show that $W^2 \subseteq WM$. Suppose that $w_0, w_1 \in W$. Since $\pi(w_0), \pi(w_1) \in K$, we have $\pi(w_0 w_1) \in K$. So there is $w_2 \in W$ with $\pi(w_2) = \pi(w_0 w_1)$. Then $w_2^{-1} w_0 w_1 \in N \cap W^3 \subseteq M$, and hence $w_0 w_1 \in w_2 M$. Using this result and also the fact that M is a subgroup of N we obtain by induction that $W^k \subseteq WM$ $\forall k \in \mathbb{N}$. Now let H be the subgroup of G generated by W. Clearly, $H = \bigcup_{k=1}^{\infty} W^k$. Then H is open and

$$H \subseteq WM \subseteq U_0^2 \subseteq U$$

as desired. □

Corollary 3.4. *Suppose that G and H are non-archimedean groups and that H is a G-group. Then the semidirect product $H \leftthreetimes G$ is non-archimedean.*

Example 3.5. (Dierolf and Schwanengel [6]) Every discrete group H is a group retract of a locally compact non-archimedean minimal group.

More precisely, let $\mathbb{Z}_2$ be the discrete cyclic group of order 2 and let H be a discrete topological group. Let $G := \mathbb{Z}_2^H$ be endowed with the product topology. Then

$$\sigma : H \to \mathrm{Aut}(G), \qquad \sigma(k)\big((x_h)_{h\in H}\big) := (x_{hk})_{h\in H} \quad \forall k \in H,\ (x_h)_{h\in H} \in G$$

is a homomorphism. The topological semidirect (wreath) product $G \leftthreetimes_\sigma H$ is a locally compact non-archimedean minimal group having H as a retraction.

Corollary 4.5 below provides a generalization.

4. The Heisenberg group associated to a Stone space

Let X be a Stone space. Let $V = (V(X), \triangle)$ be the discrete group of all clopen subsets in X with respect to the symmetric difference. As usual one may identify V with the group $V := C(X, \mathbb{Z}_2)$ of all continuous functions $f : X \to \mathbb{Z}_2$.

Denote by $V^* := \hom(V, \mathbb{T})$ the Pontryagin dual of V. Since V is a Boolean group every character $V \to \mathbb{T}$ can be identified with a homomorphism into the unique 2-element subgroup $\Omega_2 = \{1, -1\}$, a copy of $\mathbb{Z}_2$. The same is true for the characters on V^*, hence the natural evaluation map $w : V \times V^* \to \mathbb{T}$ ($w(x, f) = f(x)$) can be restricted naturally to $V \times V^* \to \mathbb{Z}_2$. Under this identification $V^* := \hom(V, \mathbb{Z}_2)$ is a closed (hence compact) subgroup of the compact group $\mathbb{Z}_2^V$. Clearly, the groups V and $\mathbb{Z}_2$, being discrete, are non-archimedean. The group $V^* = \hom(V, \mathbb{Z}_2)$ is also non-archimedean since it is a subgroup of $\mathbb{Z}_2^V$.

In the sequel G is an arbitrary non-archimedean group. X is its associated Stone space, that is, G is a topological subgroup of $\mathrm{Homeo}(X)$ (see Lemma 3.2). V and V^* are the non-archimedean groups associated to the Stone space X we have mentioned at the beginning of this subsection. We intend to show using the technique introduced in Subsection 2.2, among others, that G is a topological group retract of a non-archimedean minimal group.

Theorem 4.1. *For every Stone space X the (locally compact 2-step nilpotent) Heisenberg group $H = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*$ is minimal and non-archimedean.*

Proof. Using Fact 2.11 (or, by direct arguments) it is easy to see that the continuous separated biadditive mapping

$$w : V \times V^* \to \mathbb{Z}_2$$

is minimal. Then by Fact 2.10.1 the corresponding Heisenberg group H is minimal. H is non-archimedean by Corollary 3.4. □

Lemma 4.2. *Let G be a topological subgroup of* Homeo(X) *for some Stone space X (see Lemma 3.2). Then $w(G) \leqslant w(X) = w(V) = |V| = w(V^*)$.*

Proof. Use the facts that in our setting V is discrete and V^* is compact. Recall also that (see e.g., [13, Theorem 3.4.16])

$$w\big(C(A,B)\big) \leqslant w(A) \cdot w(B)$$

for every locally compact Hausdorff space A (where the space $C(A,B)$ is endowed with the compact-open topology). □

The action of $G \subset \mathrm{Homeo}(X)$ on X and the functoriality of the Stone duality induce the actions on V and V^*. More precisely, we have

$$\alpha : G \times V \to V, \qquad \alpha(g, A) = g(A)$$

and

$$\beta : G \times V^* \to V^*, \qquad \beta(g, f) := gf, \qquad (gf)(A) = f\big(g^{-1}(A)\big).$$

Every translation under these actions is a continuous group automorphism. Therefore we have the associated group homomorphisms:

$$i_\alpha : G \to \mathrm{Aut}(V),$$
$$i_\beta : G \to \mathrm{Aut}\big(V^*\big).$$

The pair (α, β) is a birepresentation of G on $w : V \times V^* \to \mathbb{Z}_2$. Indeed,

$$w\big(gf, g(A)\big) = (gf)\big(g(A)\big) = f\big(g^{-1}\big(g(A)\big)\big) = f(A) = w(f, A).$$

Lemma 4.3.

(1) *Let G be a topological subgroup of* Homeo(X) *for some Stone space X. The action $\alpha : G \times V \to V$ induces a topological group embedding $i_\alpha : G \hookrightarrow \mathrm{Aut}(V)$.*
(2) *The natural evaluation map*

$$\delta : X \to V^*, \qquad x \mapsto \delta_x, \qquad \delta_x(f) = f(x)$$

is a topological G-embedding.
(3) *The action $\beta : G \times V^* \to V^*$ induces a topological group embedding $i_\beta : G \hookrightarrow \mathrm{Aut}(V^*)$.*
(4) *The pair $\psi := (\alpha, \beta)$ is a t-exact birepresentation of G on $w : V \times V^* \to \mathbb{Z}_2$.*

Proof. (1) Since V is discrete, the Birkhoff topology on Aut(V) coincides with the pointwise topology. Recall that the topology on G inherited from Homeo(X) is defined by the local subbase

$$H_A := \{g \in G : gA = A\}$$

where A runs over nonempty clopen subsets in X. Each H_A is a clopen subgroup of G. On the other hand the pointwise topology on $i_\alpha(G) \subset \mathrm{Aut}(V)$ is generated by the local subbase of the form

$$\big\{i_\alpha(g) \in i_\alpha(G) : gA = A\big\}.$$

So, i_α is a topological group embedding.

(2) Straightforward.

(3) Since V^* is compact, the Birkhoff topology on Aut(V^*) coincides with the compact open topology.

The action of G on X is t-exact. Hence, by (2) it follows that the action β cannot be continuous under any weaker group topology on G. Now it suffices to show that the action $\beta : G \times V^* \to V^*$ is continuous.

The topology on $V^* \subset \mathbb{Z}_2^V$ is a pointwise topology inherited from $\mathbb{Z}_2^V$. So it is enough to show that for every finite family $A_1, A_2, \ldots, A_m$ of nonempty clopen subsets in X there exists a neighborhood O of $e \in G$ such that $(g\psi)(A_k) = \psi(A_k)$ for every $g \in O$, $\psi \in V^*$ and $k \in \{1, \ldots, m\}$. Since $(g\psi)(A_k) = \psi(g^{-1}(A_k))$ we may define O as

$$O := \bigcap_{k=1}^{m} H_{A_k}.$$

(Another way to prove (3) is to combine (1) and [16, Theorem 26.9].)

(4) $\psi = (\alpha, \beta)$ is a birepresentation as we already noticed before this lemma. The t-exactness is a direct consequence of (1) or (3) together with Fact 2.8(1). □

Theorem 4.4. *The topological group*

$$M := M(\psi) = H(w) \leftthreetimes_\pi G = \big((\mathbb{Z}_2 \oplus V) \leftthreetimes V^*\big) \leftthreetimes_\pi G$$

is a non-archimedean minimal group.

Proof. By Corollary 3.4, M is non-archimedean. Use Theorem 4.1, Lemma 4.3 and Fact 2.10 to conclude that M is a minimal group. □

Corollary 4.5. *Every (locally compact) non-archimedean group G is a group retract of a (resp., locally compact) minimal non-archimedean group M where $w(G) = w(M)$.*

Proof. Apply Theorem 4.4 taking into account Fact 2.8(1) and the local compactness of the groups $\mathbb{Z}_2, V, V^*$ (resp., G). □

Remark 4.6. Another proof of Corollary 4.5 can be obtained by the following way. By Lemma 4.3 every non-archimedean group G can be treated as a subgroup of the group of all automorphisms $\mathrm{Aut}(V^*)$ of the compact abelian group V^*. In particular, the action of G on V^* is t-exact. The group V^* being compact is minimal. Since V^* is abelian one may apply [25, Corollary 2.8] which implies that $V^* \leftthreetimes G$ is a minimal topological group. By Lemmas 3.2 and 4.2 we may assume that $w(G) = w(V^* \leftthreetimes G)$.

5. More characterizations of non-archimedean groups

The results and discussions above lead to the following list of characterizations (compare Lemma 3.2).

Theorem 5.1. *The following assertions are equivalent:*

(1) *G is a non-archimedean topological group.*
(2) *G is a topological subgroup of the automorphisms group (with the pointwise topology) $\mathrm{Aut}(V)$ for some discrete Boolean ring V (where $|V| = w(G)$).*
(3) *G is embedded into the symmetric topological group S_κ (where $\kappa = w(G)$).*
(4) *G is a topological subgroup of the group $Is(X, d)$ of all isometries of an ultra-metric space (X, d), with the topology of pointwise convergence (where $w(X) = w(G)$).*
(5) *The right (left) uniformity on G can be generated by a system of right (left) invariant ultra-semimetrics.*
(6) *G is a topological subgroup of the automorphism group $\mathrm{Aut}(K)$ for some compact abelian group K (with $w(K) = w(G)$).*

Proof. (1) ⇒ (2) As in Lemma 4.3(1).

(2) ⇒ (3) Simply take the embedding of G into $S_V \cong S_\kappa$, with $\kappa = |V| = w(G)$.

(3) ⇒ (4) Consider the two-valued ultra-metric on the discrete space X with $|X| = \kappa$.

(4) ⇒ (5) For every $z \in X$ consider the left invariant ultra-semimetric

$$\rho_z(s, t) := d(sz, tz).$$

Then the collection $\{\rho_z\}_{z \in X}$ generates the left uniformity of G.

(5) ⇒ (1) Observe that for every right invariant ultra-semimetric ρ on G and $n \in \mathbb{N}$ the set

$$H := \{g \in G\colon\ \rho(g, e) < 1/n\}$$

is an open subgroup of G.

(3) ⇒ (6) Consider the natural (permutation of coordinates) action of S_κ on the usual Cantor additive group $\mathbb{Z}_2^\kappa$. It is easy to see that this action implies the natural embedding of S_κ (and hence, of its subgroup G) into the group $\mathrm{Aut}(\mathbb{Z}_2^\kappa)$.

(6) ⇒ (1) Let K be a compact abelian group and K^* be its (discrete) dual. By [16, Theorem 26.9] the natural map $\nu : g \mapsto \tilde{g}$ defines a topological anti-isomorphism of $\mathrm{Aut}(K)$ onto $\mathrm{Aut}(K^*)$. Now, K^* is discrete, hence, $\mathrm{Aut}(K^*)$ is non-archimedean as a subgroup of the symmetric group S_{K^*}. Since G is a topological subgroup of $\mathrm{Aut}(K)$ we conclude that G is also non-archimedean (because its opposite group $\nu(G)$ being a subgroup of $\mathrm{Aut}(K^*)$ is non-archimedean). □

Remark 5.2.

(1) Note that the universality of $S_{\mathbb{N}}$ among Polish groups was proved by Becker and Kechris (see [3, Theorem 1.5.1]). The universality of S_κ for $\mathcal{NA}$ groups with weight $\leqslant \kappa$ can be proved similarly. It appears in the work of Higasikawa, [17, Theorem 3.1]. For universal non-archimedean actions see [29].
(2) Isometry groups of ultra-metric spaces studied among others by Lemin and Smirnov [22]. Note for instance that [22, Theorem 3] implies the equivalence (1) ⇔ (4). Lemin [20] established that a metrizable group is non-archimedean iff it has a left invariant compatible ultra-metric.
(3) By item (6) of Theorem 5.1, the class of all topological subgroups of $\mathrm{Aut}(K)$, where K runs over all compact abelian groups K, is $\mathcal{NA}$. It would be interesting (see Remark 2.9) to characterize the corresponding classes of topological groups when K runs over all: a) locally compact abelian groups; b) compact groups; c) locally compact groups.
(4) In item (6) of Theorem 5.1 it is essential that the compact group K is *abelian*. For every connected non-abelian compact group K the group $\mathrm{Aut}(K)$ is not $\mathcal{NA}$ containing a nontrivial continuous image of K.

(5) Every non-archimedean group admits a topologically faithful unitary representation on a Hilbert space. It is straightforward for S_X (hence, also for its subgroups) via permutation of coordinates linear action.

Acknowledgements

We thank Dikranjan for many valuable ideas and concrete suggestions. We are indebted also to the referee for several improvements.

References

[1] R. Arens, Topologies for homeomorphism groups, Amer. J. Math. 68 (1946) 593–610.
[2] A.V. Arhangel'skiĭ, Topological homogeneity, topological groups and their continuous images, Russian Math. Surveys 42 (2) (1987) 83–131.
[3] H. Becker, A. Kechris, The Descriptive Set Theory of Polish Group Actions, London Math. Soc. Lecture Notes Ser., vol. 232, Cambridge Univ. Press, 1996.
[4] P.E. Caprace, N. Monod, Decomposing locally compact groups into simple pieces, arXiv:0811.4101v3, May, 2010.
[5] W.W. Comfort, K.H. Hofmann, D. Remus, A survey on topological groups and semigroups, in: M. Husek, J. van Mill (Eds.), Recent Progress in General Topology, North-Holland, Amsterdam, 1992, pp. 58–144.
[6] S. Dierolf, U. Schwanengel, Examples of locally compact non-compact minimal topological groups, Pacific J. Math. 82 (2) (1979) 349–355.
[7] D. Dikranjan, Recent advances in minimal topological groups, Topology Appl. 85 (1998) 53–91.
[8] D. Dikranjan, A. Giordano Bruno, Arnautov's problems on semitopological isomorphisms, Appl. Gen. Topol. 10 (1) (2009) 85–119.
[9] D. Dikranjan, M. Megrelishvili, Relative minimality and co-minimality of subgroups in topological groups, Topology Appl. 157 (2010) 62–76.
[10] D. Dikranjan, Iv. Prodanov, L. Stoyanov, Topological Groups: Characters, Dualities and Minimal Group Topologies, Pure Appl. Math., vol. 130, Marcel Dekker, 1998, pp. 289–300.
[11] D. Dikranjan, M. Tkachenko, Iv. Yaschenko, Transversal group topologies on non-abelian groups, Topology Appl. 153 (17) (2006) 3338–3354.
[12] D. Doïchinov, Produits de groupes topologiques minimaux, Bull. Sci. Math. 96 (1972) 59–64.
[13] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[14] D. Gamarnik, Minimality of the group Aut(C), Serdika 17 (4) (1991) 197–201.
[15] E. Gaughan, Topological group structures of infinite symmetric groups, Proc. Natl. Acad. Sci. USA 58 (1967) 907–910.
[16] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis I, Springer, Berlin, 1963.
[17] M. Higasikawa, Topological group with several disconnectedness, arXiv:math/0106105v1, 2000, pp. 1–13.
[18] J. Isbell, Zero-dimensional spaces, Tohoku Math. J. (2) 7 (1955) 1–8.
[19] J. Isbell, Uniform Spaces, American Mathematical Society, Providence, 1964.
[20] A.Yu. Lemin, Isosceles metric spaces and groups, in: Cardinal Invariants and Mappings of Topological Spaces, Izhevsk, 1984, pp. 26–31.
[21] A.Yu. Lemin, The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT*, Algebra Universalis 50 (2003) 35–49.
[22] A.Yu. Lemin, Yu.M. Smirnov, Groups of isometries of metric and ultrametric spaces and their subgroups, Russian Math. Surveys 41 (6) (1986) 213–214.
[23] G. Lukacs, Compact-Like Topological Groups, Res. Exp. Math., vol. 31, Heldermann Verlag, 2009.
[24] M. Megrelishvili, Compactification and factorization in the category of G-spaces, in: J. Adámek, S. Maclane (Eds.), Categorical Topology, Its Relation to Analysis, Algebra, Combinatorics, World Scientific, Singapore, 1989, pp. 220–237.
[25] M. Megrelishvili, Group representations and construction of minimal topological groups, Topology Appl. 62 (1) (1995) 1–19.
[26] M. Megrelishvili, G-minimal topological groups, in: Abelian Groups, Module Theory and Topology, in: Lecture Notes in Pure and Appl. Math., vol. 201, Marcel Dekker, 1998, pp. 289–300.
[27] M. Megrelishvili, Generalized Heisenberg groups and Shtern's question, Georgian Math. J. 11 (4) (2004) 775–782.
[28] M. Megrelishvili, Every topological group is a group retract of minimal group, Topology Appl. 155 (17–18) (2008) 2105–2127.
[29] M. Megrelishvili, T. Scarr, The equivariant universality and couniversality of the Cantor cube, Fund. Math. 167 (3) (2001) 269–275.
[30] J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, 1990.
[31] Iv. Prodanov, Precompact minimal group topologies and p-adic numbers, Annuaire Univ. Sofia Fac. Math. 66 (1971/1972) 249–266.
[32] Iv. Prodanov, L. Stoyanov, Minimal group topologies, in: Topology, Theory and Applications, Eger, 1983, in: Colloq. Math. Soc. János Bolyai, vol. 41, pp. 493–508.
[33] V.G. Pestov, On free actions, minimal flows, and a problem by Ellis, Trans. Amer. Math. Soc. 350 (1998) 4149–4175.
[34] V. Pestov, Topological groups: where to from here?, Topology Proc. 24 (1999) 421–502, http://arXiv.org/abs/math.GN/9910144.
[35] D. Remus, L. Stoyanov, Complete minimal and totally minimal groups, Topology Appl. 42 (1) (1991) 57–69.
[36] A.C.M. van Rooij, Non-Archimedean Functional Analysis, Monogr. Textb. Pure Appl. Math., vol. 51, Marcel Dekker, Inc., New York, 1978.
[37] M. Shlossberg, Minimality in topological groups and Heisenberg type groups, Topology Proc. 35 (2010) 331–344.
[38] R.M. Stephenson, Minimal topological groups, Math. Ann. 192 (1971) 193–195.
[39] L. Stoyanov, Total minimality of the unitary groups, Math. Z. 187 (2) (1984) 273–283.
[40] S. Teleman, Sur la représentation linéare des groupes topologiques, Ann. Sci. Ecole Norm. Super. 74 (1957) 319–339.
[41] V.V. Uspenskij, On subgroups of minimal topological groups, Topology Appl. 155 (14) (2008) 1580–1606.
[42] V.V. Uspenskij, The Roelcke compactification of groups of homeomorphisms, Topology Appl. 111 (2001) 195–205.
[43] S. Warner, Topological Fields, North-Holland Math. Stud., vol. 157, North-Holland, Amsterdam, London, New York, Tokyo, 1993.