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Abstract

We show that for every groupoid of partial homeomorphisms on a topological space X there exists a *topological embedding* $X \to \overline{X}$ and a *universal* group action on the space $\overline{X}$ which extends the given groupoid action. We also show that the construction is useful for *homogenizations* of topological spaces.

1 Some Useful Categories

Denote by $\mathcal{GR}$ and $\mathcal{TOP}$ the categories of all groups and of all topological spaces respectively.

An *action* of a group G on a topological space X is a triple $< G, X, \pi >$, where $\pi : G \to Homeo(X)$ is a homomorphism of G into the group $Homeo(X)$ of all homeomorphisms $X \to X$. Denote by $\mathcal{A}$ the category of all actions. A morphism from $< G_1, X_1, \pi_1 >$ to $< G_2, X_2, \pi_2 >$ is a pair of maps (H, f), where $H : G_1 \to G_2$ is a homomorphism and $f : X_1 \to X_2$ is a continuous function such that f is an $H-$*equivariant map,* that is, $f(\pi_1(g)x) = \pi_2(H(g))f(x)$ for any pair of elements $(g, x) \in G_1 \times X_1$.

Recall that a small category P is said to be a *groupoid* (see, for example, [Br, We]) if all of its morphisms are isomorphisms. A groupoid Q is a *subgroupoid* of P if Q is a (not necessarily, *full*) subcategory of P.

A category and its class of all morphisms sometimes will be denoted by the same symbol. We write $Ob(C)$ for the class of all objects in the category C.

For a topological space X denote by $T(X)$ the groupoid of all *partial homeomorphisms* of X. The *objects* in this category are precisely all topological subspaces of X. A morphism σ between two objects A, B is an arbitrary homeomorphism $A \to B$.

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[0]This work is dedicated to my friends in A. Razmadze Mathematical Institute (Tbilisi, Georgia).

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We use the notation $im(\sigma) = B, coim(\sigma) = A$. Let σ and δ be two morphisms of the category $T(X)$. If $im(\delta) = coim(\sigma)$ then the *composition* $\omega(\sigma, \delta)$ of these morphisms in the category $T(X)$ is exactly the usual composition $\sigma \circ \delta$ of maps. Besides this *partial* operation ω on $T(X)$, define also a binary operation ω^*. If $\sigma, \delta \in T(X)$ then $\omega^*(\sigma, \delta) = \sigma * \delta$ is defined as the map $\delta^{-1}(A) \to \sigma(A)$, where $A = im(\delta) \cap coim(\sigma)$ and $(\sigma * \delta)(x) = \sigma(\delta(x))$ for every $x \in A$. If $A = \emptyset$ then $\omega^*(\sigma, \delta) = \emptyset$, the *empty bijection*. Clearly, w is a suboperation of ω^* and $(T(X), \omega^*)$ is an inverse semigroup, a subsemigroup of a *symmetric inverse semigroup* of all partial bijections on X.

Definition 1.1. *A preaction of a groupoid P on a topological space X is a triple $< P, X, \pi >$, where $\pi : P \to T(X)$ is a functor.*

Every group G is a groupoid with a single object (and *vice versa*). Then, an action of a group G on X we can think of a functor $\pi : G \to T(X)$ with $\pi(G) = X$.

Definition 1.2. *We say that a preaction $\pi : P \to T(X)$ is effective if P is a subgroupoid of $T(X)$ and π is the corresponding inclusion. Such preaction sometimes will be denoted simply by $< P, X >$. More specifically, let Γ be a family of subspaces of X. Denote by P_Γ (or, $P_\Gamma(X)$) the full subcategory of $T(X)$ whose class of objects is exactly Γ. For instance, if Γ denotes:*

- $B = \{singletons\}$
- $\Sigma = \{finite\ subspaces\}$
- $C = \{compact\ subspaces\}$

then P_B, P_Σ, P_C will denote respectively the corresponding subgroupoids of $T(X)$. The groupoid P_B is the so-called tree groupoid [BH] (or, simplicial groupoid, in terms of [Hi]).

Definition 1.3. *Let P be a subgroupoid of $T(X)$. We say that a topological (metric) space X is $P-$homogeneous (resp., metrically $P-$homogeneous) if for every homeomorphism (isometry) $\sigma : A \to B$ from P there exists a homeomorphism (resp., isometry) $X \to X$ which extends σ. If P denotes one of the following groupoids: P_B, P_Σ, or, P_C, then, X will be called: (metrically) homogeneneous, (metrically) $\Sigma-$homogeneous, (metrically) $C-$homogeneous, respectively.*

For metrically homogeneous spaces, see [Sh, Ok, OP]. Metrically $\Sigma-$homogeneous spaces appear in [Us2] under the name *metrically $\omega-$homogeneous.*

We will deal with the following questions:

Question 1.4. *(Q1) Let $< P, X >$ be a preaction. Is it true that there exist a topological embedding $i : X \to Y$ and a group action $< G, Y >$ such that every partial homeomorphism $\sigma \in P$ of X can be realized as the trace of a $g-$transition $Y \to Y$ for a certain $g \in G$?*

(Q2) If such an embedding is possible, can we choose it to be universal ?

(Q3) What kind of topological properties of X may preserve Y ?

(Q4) Can a topological space X be embedded into a $C-$homogeneous ($\Sigma-$homogeneous) space preserving original topological properties of X ?

The main aim of the present paper is to answer in affirmative to the concrete questions (Q1), (Q2). We answer partially to the general questions (Q3), (Q4).

In order to formulate our main theorem in categorical terms we define a category $\mathcal{PA}$ of all preactions.

Definition 1.5. *morphism from a preaction* $< P_1, X_1, \pi_1 >$ *to* $< P_2, X_2, \pi_2 >$ *is a pair* $m = (F, f)$, *where* $F : P_1 \to P_2$ *is a functor and* $f : X_1 \to X_2$ *is a continuous function such that the following two conditions are satisfied:*

(M1) $f(\pi_1(A)) \subseteq \pi_2(F(A))$ *for every object* A *in* P_1;

(M2) $f(\pi_1(\sigma)x) = \pi_2(F(\sigma))f(x)$ *for every* σ *from* P_1 *and every* $x \in coim(\pi_1(\sigma))$.

Observe the fact that $f(x) \in coim(\pi_2(F(\sigma)))$ is guaranteed by (M1). The composition of morphisms is defined naturally. Clearly, the category of all actions $\mathcal{A}$ is a full subcategory of $\mathcal{PA}$. Denote by $inc : \mathcal{A} \to \mathcal{PA}$ the corresponding inclusion. We will show that this functor has a *left adjoint*; or in other terms, $\mathcal{A}$ is a *reflective* subcategory of $\mathcal{PA}$ (for these and other categorical concepts see, for example [HS, Ma, Hi]).

For a category $\mathcal{C}$ denote by $E(\mathcal{C})$, or simply by E, the class of all identities. We say that a functor $F : C_1 \to C_2$ is *strictly faithful* if F maps each pair of distinct non-identical morphisms into distinct morphisms. That is, if F is injective on $C_1 \backslash E$.

2 Universal Actions

We will say that a morphism $m = (F, f)$ in the category $\mathcal{PA}$ is an *embedding* if F is strictly faithful and the map f is a topological embedding. The following result is our main result.

Theorem 2.1. *The functor* $inc : \mathcal{A} \to \mathcal{PA}$ *has a left adjoint* $\mathcal{A} \leftarrow \mathcal{PA}$. *That is, for every preaction* $< P, X, \pi >$ *there exists a universal action* $< G(P), \overline{X}, \overline{\pi} >$. *The corresponding morphism*

$$q = (u, \mathbf{i}) : \quad < P, X, \pi > \quad \longrightarrow \quad < G(P), \overline{X}, \overline{\pi} >$$

is an embedding.

As we will see, the most difficult part is to prove that the map $\mathbf{i} : X \to \overline{X}$ is a topological embedding. As to just the *existence* of the universal action, it can alternatively be obtained by purely categorical methods; namely, by *Freyd's Adjoint Functor Theorem* [Ma].

The proof of Theorem 2.1 is devided into several parts.

♦ *Description of the functor* $u : P \to G(P)$

Actually we construct a *universal group* of a groupoid P in terms of generators and relations. The group $G(P)$ is generated by the set P of all morphisms, taking into account the following identities:

$$e_A = e_B, \quad \sigma_1\sigma_2 = \sigma_1 \circ \sigma_2$$

where $e_A, e_B \in E(P)$ and $\sigma_1 \circ \sigma_2$ is defined in P. In other words, we have elementary reductions of two types: deleting identity morphisms and multiplying when it possible in P.

Proposition 2.2. *Every element $g \in G(P)$ has a unique reduced representation*

$$g = \sigma_1 \sigma_2 \cdots \sigma_n$$

where $\sigma_i \in P$.

Proof. The proof easily follows from results of [Hi, ch.10], where a method of van der Waerden (on group free products) is adapted for groupoids and graphs. For convenience we present here a short direct proof based on J. Stallings well-known notion of *pregroups* [St], representing $G(P)$ as a *universal group* of a certain pregroup P_0. In order to obtain P_0, we identify all identities in P. That is, we consider an equivalence relation on P, where the unique nontrivial equivalence class is E. Denote this class by $\mathbf{1}$. Now we can define a natural partial binary operation ω_0 on P_0 as follows:

1) $\omega_0(\sigma, \delta) = \sigma \circ \delta$, if the latter is defined in P;

2) $\omega_0(\sigma, \mathbf{1}) = \omega_0(\mathbf{1}, \sigma) = \sigma$;

3) Not defined, otherwise.

It is easy to show that the pair (P_0, ω_0) is a pregroup in the sense of Stallings. Therefore the pregroup P_0 defines a *universal morphism* $j : P_0 \to U(P_0)$, where $U(P_0)$ is the so-called, *universal group* of P_0. It follows from the main theorem on pregroups that the two P_0-*reduced representations*

$$g = \sigma_1 \sigma_2 \cdots \sigma_n, \qquad g = \delta_1 \delta_2 \cdots \delta_m$$

of the same element g necessarily have the same length, $m = n$, and there exists a word $(a_0, a_1, \cdots a_n)$ such that $a_0 = a_n = \mathbf{1}$ and $\delta_i = a_{i-1}^{-1} \sigma_i a_i$ in (P_0, ω_0). In our situation it suffices to show that every a_i is $\mathbf{1}$. Assuming the contrary, let $a_i \neq \mathbf{1}$ for some i. Then $\sigma_i \circ a_i$ and $a_i^{-1} \circ \sigma_{i+1}$ are defined in P. Therefore, $im(\sigma_{i+1}) = coim(\sigma_i)$. This means that $\sigma_i \circ \sigma_{i+1}$ is defined too. Hence the word $g = \sigma_1 \sigma_2 \cdots \sigma_n$ is not reduced, a contradiction. □

In view of Proposition 2.2, we may naturally define the *length* $l(g) = n$ for every $g \in U(P_0)$. As usual, we assume $l(\mathbf{1}) = 0$. Denote the group $U(P_0)$ by G(P) or, simply by G. Then the canonical map $u : P \to G(P)$ actually is a functor between groupoids. By Proposition 2.2 it is clear that $l(g) = 1$ iff $g \in P \setminus E$. Therefore u is strictly faithful.

The uniqueness of reduced representations allows us to define the following partial order " $\preceq$ " on G(P)=G. We write $g_1 \preceq g_2$ if $g_1 = \mathbf{1}$ or $g_1 = g_2$, or if in the corresponding reduced representations we have

$$g_1 = \sigma_1 \sigma_2 \cdots \sigma_n, \quad g_2 = \sigma_1 \sigma_2 \cdots \sigma_n \sigma_{n+1} \cdots \sigma_m$$

For every pair $g, h \in G$ there exists the *greatest lower bound* denoted by $g \wedge h$.

We introduce also a special function $* : G \to T(X)$. If $g \in G$ and $g = \sigma_1 \sigma_2 \cdots \sigma_n$ is a reduced representation then we define $g^* = \pi(\sigma_1) * \pi(\sigma_2) * \cdots * \pi(\sigma_n)$. Then $\sigma^* = \pi(\sigma)$ for every $\sigma \in P \backslash E$. For $\mathbf{1}$, assume $\mathbf{1}^* = I_X$ (the identity function $X \to X$). The function $*$ is very important below in our construction. Using this function, roughly speaking, we may go back to the inverse subsemigroup generated by $\pi(P)$ in $(T(X), *)$.

Lemma 2.3. *(1)* $(g^{-1})^* = (g^*)^{-1}, \;\; im(g)^* = coim(g^{-1})^*.$

(2) The map $g_1^* * g_2^*$ *is a restriction of* $(g_1 g_2)^*$*.*

(3) If $g_1 \preceq g_2$ *then* $hg_1 \preceq hg_2$ *for every* $h \in G$*.*

(4) If $g_1 \preceq g_2$ *then* $im(g_2)^* \subseteq im(g_1)^*$*.*

Proof. Straightforward.

♦ *Construction of the universal action*

Now we are ready to construct the phase space $\overline{X}$ of the universal action. Denote by $\widetilde{X}$ the topological product $G \times X$, where G carries the discrete topology. Define the "first-coordinate action" as follows :

$$\widetilde{\pi} : G \to Homeo(\widetilde{X}), \qquad \widetilde{\pi}(g)(h, x) = (gh, x) = g(h, x).$$

Note that here, and sometimes in the sequel, instead of $\widetilde{\pi}(g)(u)$ we simply write gu or $g(u)$, and similarly for other actions. Define now an equivalence relation Ω on the set $\widetilde{X}$. We write $(g_1, x_1)\Omega(g_2, x_2)$ if there exists an element h of G such that $g_2 = g_1 h$ and $h^*(x_2) = x_1$. The following result easily follows from Lemma 2.3.

Lemma 2.4. *The relation* Ω *is a* $G-$*invariant equivalence relation on* $\widetilde{X}$*.*

Consider the corresponding quotient space $\widetilde{X}/\Omega$. This space will serve as the desired space $\overline{X}$. Since the relation Ω is $G-$invariant, there exists a unique action $\overline{\pi}$ of G on $\overline{X}$ such that the canonical projection $p : \widetilde{X} \to \overline{X}$ is equivariant. Thus, $p(gu) = gp(u)$ for every $u \in \widetilde{X}$.

The subspace $\{g\} \times X$ of the space $\widetilde{X}$ will be denoted by $X_g^{\upharpoonleft}$. For its image $p(X_g^{\upharpoonleft})$ in $\overline{X}$, we reserve the symbol X_g. The map

$$\widetilde{i}_g : X \to X_g^{\upharpoonleft}, \qquad \widetilde{i}_g(x) = (g, x)$$

is a homeomorphism. Define also the composition

$$i_g = p \circ \widetilde{i}_g : X \to X_g$$

and the restriction

$$p_g : X_g^{\upharpoonleft} \to X_g, \quad p_g(u) = p(u).$$

In the case of $g = \mathbf{1}$ we write $\widetilde{i}$ and $\mathbf{i}$, respectively.

The proof of the following lemma is straightforward.

Lemma 2.5. *For every* $g \in G$ *and* $u \in \overline{X}$*:*

(1) $gX_h = X_{gh}$.

(2) $X_g \cap X_h = g(X_{\mathbf{1}} \cap X_{g^{-1}h})$.

(3) $G\mathbf{i}(X) = GX_{\mathbf{1}} = \overline{X}$.

(4) $\widetilde{i}_g(x) = g\widetilde{i}(x), \quad i_g(x) = g\mathbf{i}(x)$.

(5) $p_g : X_g^{\upharpoonleft} \to X_g$ *and* $i_g : X \to X_g$ *are continuous injective maps.*

(6) $g(i_g^{-1}(u)) = p_g^{-1}(u)$.

In fact, as it follows from Proposition 2.10, every i_g is a topological embedding.

Lemma 2.6. *The triple* $< G(P), \overline{X}, \overline{\pi} >$ *is an action and the pair* $(u, \mathbf{i})$ *is a morphism of the category* $\mathcal{PA}$.

Proof. We have already shown that $u : P \to G(P)$ is a functor and $\mathbf{i} : X \to \overline{X}$ is continuous. We need only to show that two conditions (M1), (M2) of Definition 1.5 are satisfied. For every object $A \in Ob(P)$ the corresponding object $\overline{\pi}(u(A))$ is just the set $\overline{X}$. Therefore, $\mathbf{i}(\pi(A))$ always is a subset of $\overline{X}$. This proves (M1). In order to check (M2), consider a morphism σ of P. For every $x \in coim(\pi(\sigma))$ we have

$$\mathbf{i}(\pi(\sigma)(x)) = p(\sigma(\mathbf{1}, x)) = \sigma p(\mathbf{1}, x) = \overline{\pi}(\sigma)\mathbf{i}(x)$$

This proves (M2). □

♦ $\mathbf{i} : X \to \overline{X}$ *is a topological embedding*

Let β be a cover of a space Y. We will say that a topological space Y is a *free union* of β if a subset $A \subseteq Y$ is closed (open) in Y iff $A \cap B$ is closed (open) in the subspace B for every $B \in \beta$.

Lemma 2.7. *(1) Every* $X_g^{\natural}$ *is a clopen subset of* $\widetilde{X}$ *homeomorphic to* X *and* $\widetilde{X}$ *is a free disjoint union of the cover* $\{X_g^{\natural} | g \in G\}$.

(2) $\overline{X}$ *is a free union of the cover* $\{X_g | g \in G\}$.

Proof. Since G is discrete, the first assertion is trivial.

Let A be a subset of $\overline{X}$ such that $A \cap X_g$ is open in the subspace X_g for every $g \in G$. Then $p^{-1}(A) = \cup\{p^{-1}(A \cap X_g) | g \in G\} = \cup\{p_g^{-1}(A \cap X_g) | g \in G\}$. Since $p_g^{-1}(A \cap X_g)$ is open in $X_g^{\natural}$ then $p^{-1}(A)$ is open in $\widetilde{X}$. By the definition of the quotient topology we obtain that A is open in $\overline{X}$. □

Lemma 2.8. *(1)* $\mathbf{i}(A) \cap X_h = \mathbf{i}(A \cap im(h^*)) = \mathbf{i}(A \cap coim(h^{-1})^*)$.

(2) $X_{\mathbf{1}} \cap X_h = \mathbf{i}(im(h^*)) = \mathbf{i}(coim(h^{-1})^*)$;

(3) $X_g \cap X_h = g\mathbf{i}(im(g^{-1}h)^*) = g\mathbf{i}(coim(h^{-1}g)^*)$.

(4) In particular, $X_g \cap X_h \neq \emptyset$ *iff* $(g^{-1}h)^* \neq \emptyset$.

Proof. By the definition of Ω, $(\mathbf{1}, a)\Omega(h, x)$ iff $h^*(x) = a$, or equivalently, iff $a \in im(h^*)$. This proves (1). Now other assertions easily follow by Lemmas 2.5 and 2.3. □

The following lemma means, in particular, that $\{X_g |\ g \in G\}$ is a "treelike cover" of $\overline{X}$.

Lemma 2.9. *Let* $g \wedge h \preceq g_1 \preceq g, \quad g \wedge h \preceq h_1 \preceq h$. *Then* $X_g \cap X_h \subseteq X_{g_1} \cap X_{h_1}$. *In particular,* $X_g \cap X_h \subseteq X_{g \wedge h}$.

Proof. By Lemma 2.3, we have $im(h^{-1}g)^* \subseteq im(h^{-1}g_1)^*$. On the other hand, by Lemma 2.8:

$$X_h \cap X_g = h\mathbf{i}(im(h^{-1}g)^*), \qquad X_h \cap X_{g_1} = h\mathbf{i}(im(h^{-1}g_1)^*).$$

Therefore we can conclude that $X_g \cap X_h \subseteq X_{g_1} \cap X_h$. Similarly we can check that $X_{g_1} \cap X_h = X_h \cap X_{g_1} \subseteq X_{h_1} \cap X_{g_1}$. □

One more notation. Let $g = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n$ be a reduced representation. We denote $\sigma_1\sigma_2\cdots\sigma_{n-1}$ by $\breve{g}$.

Proposition 2.10. *The map* $\mathbf{i} : X \to \overline{X}$ *is a topological embedding.*

Proof. The map $\mathbf{i}$ is injective and continuous. We need to show that the induced map $X \to X_{\mathbf{1}}$ is closed. Let F be a closed subset of X. We will show that there exists a closed subset M of $\overline{X}$ such that $M \cap X_{\mathbf{1}} = \mathbf{i}(F)$. A subset A is closed in $\overline{X}$ iff $p^{-1}(A)$ is closed in $\widetilde{X}$. By Lemma 2.7 it is equivalent to say that $p_g^{-1}(M \cap X_g)$ is closed in $X_g^{\urcorner}$ for every $g \in G$. By Lemma 2.5 (6) it suffices to show that $i_g^{-1}(M \cap X_g)$ is closed in X. We will construct such M inductively, by building its intersections $M_g = M \cap X_g$ with each X_g, using induction on the length of the word $g \in G$. Set $L_n = \{g \in G | l(g) \leq n\}$.

For $g = \mathbf{1}$ we set $F_{\mathbf{1}} = \mathbf{i}(F)$ and $M_0 = \{F_{\mathbf{1}}\}$.

Let us suppose that for every $k \in \{0, 1, \cdots, n\}$ we have constructed a system $M_k = \{F_g | g \in L_n\}$, such that for every $g, h \in L_n, g_1 \preceq g, k \leq n-1$ the following conditions are satisfied:

(1) $M_k \subseteq M_{k+1}$;

(2) $F_{\mathbf{1}} = \mathbf{i}(F)$;

(3) $F_g \subseteq X_g$;

(4) Every $A_g = i_g^{-1}(F_g)$ is closed in X;

(5) $F_g \cap X_{\breve{g}} = F_{\breve{g}} \cap X_g$;

(6) $F_g \cap X_h \subseteq F_{g\wedge h}$;

(7) $F_{g_1} \cap X_g \subseteq F_g$.

Let $l(g) = n+1$ and $g = \sigma_1\sigma_2\cdots\sigma_{n+1}$ be its reduced representation. If the set $F_{\breve{g}} \cap X_g$ is empty then define $F_g = \emptyset$. Otherwise, consider the partial homeomorphism $\sigma_{n+1} : coim(\sigma_{n+1}) \to im(\sigma_{n+1})$. By the induction assumption (4), $A_{\breve{g}}$ is a closed subset of X. Therefore there exists a closed subset A_g of X such that

$(\star)$ $\boxed{\sigma_{n+1}(A_g \cap coim(\sigma_{n+1})) = A_{\breve{g}} \cap im(\sigma_{n+1})}$

Define F_g as $i_g(A_g)$ and $M_{n+1} = \{F_g | \ \ g \in L_{n+1}\}$. We will show that all conditions are again satisfied for L_{n+1}. Indeed, the assertions (1), (2), (3) and (4), are trivial. In order to check (5), we use $(\star)$ and Lemmas 2.8 (1) and 2.5 :

$$F_g \cap X_{\breve{g}} = g\mathbf{i}(A_g) \cap gX_{\sigma_{n+1}^{-1}} = \breve{g}\sigma_{n+1}(\mathbf{i}(A_g) \cap X_{\sigma_{n+1}^{-1}}) =$$
$$= \breve{g}\mathbf{i}(\sigma_{n+1}(A_g \cap coim(\sigma_{n+1})) = \breve{g}\mathbf{i}(A_{\breve{g}} \cap im(\sigma_{n+1})) =$$
$$= \breve{g}(\mathbf{i}(A_{\breve{g}}) \cap X_{\sigma_{n+1}}) = F_{\breve{g}} \cap X_g$$

Let us check the assertion (6) for L_{n+1}. Let $g, h \in L_{n+1}$. The proof is nontrivial only when $g \neq h$ and $g \neq \mathbf{1}$. We can assume that $l(g \wedge h) \preceq n$. Observe that

$$F_g \cap X_h = (F_g \cap X_g) \cap X_h = F_g \cap (X_g \cap X_h).$$

By our assumptions, $g \wedge h \preceq \breve{g} < g$. Therefore by Lemma 2.9 we have $X_g \cap X_h \subseteq X_{\breve{g}} \cap X_h$. Taking into account (5) and again Lemma 2.9, we get

$$F_g \cap X_h \subseteq F_g \cap X_{\breve{g}} \cap X_h = F_{\breve{g}} \cap X_g \cap X_h \subseteq F_{\breve{g}} \cap X_{g\wedge h}.$$

On the other hand, since $\breve{g}, g \wedge h \in L_n$, by our induction assumption (6), we have

$$F_{\breve{g}} \cap X_{g \wedge h} \subseteq F_{\breve{g} \wedge g \wedge h} = F_{g \wedge h}$$

Therefore, eventually we obtain $F_g \cap X_h \subseteq F_{g \wedge h}$.

Finally, we have to prove (7) for every $g_1 \preceq g$ where $l(g) = n+1$. Since $g_1 \preceq \breve{g} < g$, by Lemma 2.9 we have $X_{g_1} \cap X_g \subseteq X_{\breve{g}} \cap X_g$. Thus,

$$F_{g_1} \cap X_g = F_{g_1} \cap X_{g_1} \cap X_g \subseteq F_{g_1} \cap X_{\breve{g}} \cap X_g$$

By induction assumption (7) for $g_1, \breve{g} \in L_n$, we can replace $F_{g_1} \cap X_{\breve{g}}$ by $F_{\breve{g}}$. Hence, taking into account (5), we obtain

$$F_{g_1} \cap X_g \subseteq F_{g_1} \cap X_{\breve{g}} \cap X_g \subseteq F_{\breve{g}} \cap X_g = F_g \cap X_{\breve{g}} \subseteq F_g$$

as desired.

Thus by induction we obtain a sequence $\{M_n | n \in N\}$, where $M_n = \{F_g | g \in L_n\}$ satisfies all assertions (1)-(7) for every $g \in G$. Define $M = \cup\{F_g | g \in G\}$. Then by our construction, $M \cap X_{\mathbf{1}} = \mathbf{i}(F)$. By the assertions (6) and (7),

$$F_g \cap X_h \subseteq F_{g \wedge h}, \quad F_{g \wedge h} \cap X_h \subseteq F_h.$$

Hence, $F_g \cap X_h \subseteq F_h$ for every $g, h \in G$. Taking into account the assertion (3), we obtain $i_h^{-1}(M \cap X_h) = i_h^{-1}(F_h) = A_h$. Since A_h is closed in X, the proof is completed. □

Corollary 2.11. *The morphism $p = (u, \mathbf{i})$ is an embedding.*

♦ *The universality*

Now in order to complete the proof of our main theorem, we have only to check the *universality* of the morphism $q = (u, \mathbf{i}) :< P, X, \pi > \to < G(P), \overline{X}, \overline{\pi} >$.

Let $m = (H, \phi) :< P, X, \pi > \to < K, Y, \mu >$ be a morphism in the category $\mathcal{A}$, where $< K, Y, \mu >$ is an action. By the universality of the functor $u : P \to G(P)$ it follows that there exists a unique group homomorphism $\overline{H} : G \to K$ such that $\overline{H} \circ u = H$. Define

$$\widetilde{\phi} : \widetilde{X} \to Y, \quad \widetilde{\phi}(g, x) = \overline{H}(g)\phi(x).$$

If $(g_1, x_1)\Omega(g_2, x_2)$ then by the definition, $g_1^{-1}g_2 = h$ and $h^*(x_2) = x_1$. Let $h = \sigma_1\sigma_2 \cdots \sigma_n$ be the reduced word. Taking into account that $\overline{H}$ is a homomorphism and m is a morphism, we obtain

$$\overline{H}(g_1^{-1}g_2)\phi(x_2) = \overline{H}(\sigma_1)\overline{H}(\sigma_2) \cdots \overline{H}(\sigma_n)\phi(x_2)$$

$$= \overline{H}(\sigma_1)\overline{H}(\sigma_2) \cdots \overline{H}(\sigma_{n-1})\phi(\pi(\sigma_n)(x_2)) = \cdots = \phi(h^*(x_2)) = \phi(x_1)$$

Thus, $\overline{H}(g_2)\phi(x_2) = \overline{H}(g_1)\phi(x_1)$. Then $\widetilde{\phi}(g_1, x_1) = \widetilde{\phi}(g_2, x_2)$. This means that the map $\widetilde{\phi}$ preserves the equivalence relation Ω. Therefore $\widetilde{\phi}$ induces on $\overline{X}$ a unique map $\overline{\phi} : \overline{X} \to Y$ such that $\overline{\phi} \circ p = \widetilde{\phi}$. It is easy to show that $\widetilde{\phi}$ is $\overline{H}-$equivariant. Then this implies directly that $\overline{\phi}$ is also an $\overline{H}-$equivariant. Hence the pair $\overline{m} = (\overline{H}, \overline{\phi})$ is a morphism.

Next we show that $\overline{\phi} \circ \mathbf{i} = \phi$. Indeed,

$$\overline{\phi}(\mathbf{i}(x)) = \overline{\phi}(p(x)) = \widetilde{\phi}((\mathbf{1}, x)) = \phi(x).$$

As we already know, $\overline{H} \circ u = H$. Therefore, $m = \overline{m} \circ q$. The uniqueness of $\overline{H}$ is clear because the universality of $G(P)$. As to $\overline{\phi}$, note that $\overline{\phi}(p(g,x)) = \overline{\phi}(gp(\mathbf{1},x))$, $G\mathbf{i}(X) = \overline{X}$ and $\overline{\phi}$ must be equivariant. Therefore, the definition of $\overline{\phi}$ is the unique possible. Theorem 2.1 is proved.

We will identify X with the subspace $\mathbf{i}(X)$.

Proposition 2.12. *For every effective preaction $< P, X >$ each partial homeomorphism $\sigma \in P$ of X is the trace of the homeomorphism $\sigma \in G(P)$ of $\overline{X}$.*

Proof. Indeed, the trace of the map $\sigma : \overline{X} \to \overline{X}$ on $X = \mathbf{i}(X)$ is the following partial homeomorphism

$$X \cap \sigma^{-1}(X) \to \sigma(X) \cap X, \quad x \to p(\sigma, x) = \sigma(x).$$

Now, observe that by Lemma 2.8 (2) we have

$$X \cap \sigma^{-1}(X) = X \cap X_{\sigma^{-1}} = coim(\sigma)$$

$$\sigma(X) \cap X = X_\sigma \cap X = im(\sigma).$$

□

3 Homogenization of topological spaces

Definition 3.1. *We say that a preaction $< P, X, \pi >$ is transitive if for every pair $a, b \in X$ there exist a finite sequence $x_1, x_2, \cdots, x_n$ in X and a finite sequence of morphisms $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ in P such that*

$$x_1 = a, x_n = b, \qquad \pi(\sigma_i)(x_i) = x_{i+1}$$

for every $i \in \{1, 2, \cdots, n-1\}$, equivalently, if $h^*(a) = b$ for some $h \in G(P)$.

Lemma 3.2. *The universal action $< G(P), \overline{X}, \overline{\pi} >$ is transitive iff the given preaction $< P, X, \pi >$ is transitive.*

Proof. If the universal action is transitive and $a, b \in X = \mathbf{i}(X)$ then $ga = b$ for some $g \in G$. Therefore, $(g, a)\Omega(\mathbf{1}, b)$. Then by the definition of Ω it is clear that $g^*(a) = b$. Conversely, let the preaction be transitive and $u, v \in \overline{X}$. There exist $g_1, g_2 \in G$ and $a, b \in X$ such that $u = g_1 a, v = g_2 b$. By Definition 3.1 there exist $x_1 = a, x_2, \cdots, x_{n-1}, x_n = b$ and $\sigma_1, \sigma_2. \cdots, \sigma_{n-1}$ such that $\pi(\sigma_i)(x_i) = x_{i+1}$. Denote $h = \sigma_1\sigma_2 \cdots \sigma_n$. Then $g_2 h g_1^{-1}(u) = v$. □

Remark 3.3. *The preaction $< \overline{P}_B, X >$ defined by the tree groupoid P_B of all singletons, clearly is transitive. Therefore the corresponding universal action $< G(P_B), \overline{X}_B >$ is transitive. This provides us some "homogenization" $X \to \overline{X}$ of X. By comparing the universal properties, it is easy to show that $< G(P_B), \overline{X}_B >$ is isomorphic to the so-called free homogeneous space in the sense of Belnov [Be]. More information about homogenizations can be found in [AE, Ar, Mi, Ok, OP, Sh, Us1, Us2, Ya].*

We discuss now the question (Q4). We will prove that our construction frequently preserves the normal type properties and the dimension.

We will say that a preaction $< P, X, \pi >$ is *closed (open)* if $\pi(A)$ is a closed (resp., open) subset of X for every $A \in P$.

Lemma 3.4. *Let $< P, X, \pi >$ be a closed (open) preaction. Then every X_g is a closed (resp., open) subset of $\overline{X}$.*

Proof. Let $h = \sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i \in P$. By our assumption, the subsets $im(\pi(\sigma_i))$ and $coim(\pi(\sigma_i))$ all are closed (open) in X. Since every $\pi(\sigma_i)$ is a partial homeomorphism of X, it is easy to show by the definition of ω^* that the sets $coim(h^*)$ and $im(h^*)$ also are closed (open) in X. Now the rest follows by Lemmas 2.7 (2) and 2.8 (3). □

The following definitions are inspired by [Be] and [Wa].

(D1) A subset $A \subseteq \overline{X}$ is called an *$F-$set* if A, as a topological subspace, is a free union of the cover $\{A \cap X_g | g \in G\}$.

Clearly every closed (or open) subset A is an $F-$set.

(D2) Let X, Y be topological spaces. We write $X\tau Y$ if for any closed subset $S \subseteq X$, every continuous map $f_0 : S \to Y$ admits a continuous extension $f : X \to Y$.

Proposition 3.5. *Let $< P, X, \pi >$ be a closed preaction, let A be an $F-$set, and let $(A \cap X_g)\tau Y$ for every $g \in G$. Then $A\tau Y$.*

Proof. Let $f : M \to Y$ be a continuous map, where M is a closed subset of the space A. Denote by $f_0 : M \cap X_{\mathbf{1}} \to Y$ the corresponding restriction. Since $(A \cap X_{\mathbf{1}})\tau Y$ and $M \cap X_{\mathbf{1}}$ is closed in $A \cap X_{\mathbf{1}}$, there exists a continuous extension $\overline{f}_0 : A \cap X_{\mathbf{1}} \to Y$ of f_0. Suppose that $0 \leq n$ and for every k with $0 \leq k \leq n$ there exists a continuous map $\overline{f}_k : A_k \to Y$ such that: $A_k = \cup\{A \cap X_g : g \in L_k\}$, and the maps $\overline{f}_k$ and f agree on $M \cap A_k$.

Let $g = \sigma_1\sigma_2\cdots\sigma_n\sigma_{n+1}$ be a reduced word. Then $\check{g} = \sigma_1\sigma_2\cdots\sigma_n$. Define the map

$$\phi_g : (A \cap X_{\check{g}} \cap X_g) \cup (M \cap X_g) \to Y$$

$$\phi_g(x) = \begin{cases} f(x), & x \in M \cap X_g \\ \overline{f}_n(x), & x \in A \cap X_{\check{g}} \cap X_g \end{cases}$$

Clearly, $M \cap X_g$ is closed in $A \cap X_g$. By Lemma 3.4 the set $A \cap X_{\check{g}} \cap X_g$ is also closed in the space $A \cap X_g$. Hence, ϕ_g is continuous. Since $(A \cap X_g)\tau Y$, there exists a continuous extension $\overline{\phi}_g : A \cap X_g \to Y$. If $h \in L_{n+1}$ and $h \neq g$ then Lemma 2.9 implies that $X_g \cap X_h \subseteq X_{\check{g}} \cap X_h$. Therefore, the maps $\overline{\phi}_h$ and $\overline{\phi}_g$ agree on $A \cap X_h \cap X_g$. By this fact, and using the maps $\phi_g, \overline{f}_n$, we can define on the set $A_{n+1} = \cup\{A \cap X_g | g \in L_{n+1}\}$ a map $\overline{f}_{n+1} : A_{n+1} \to Y$ such that it extends $\overline{f}_n$ and coincides with f on the intersection $M \cap A_{n+1}$. By our assumption, A is an $F-$set. This implies that $\overline{f}_{n+1}$ is also continuous. The direct limit of the maps $\{\overline{f}_n : n \in N\}$ is the desired continuous extension of f. □

Proposition 3.6. *Let $< P, X, \pi >$ be a closed preaction. Then*

(i) If the space X is (hereditarily) normal then $\overline{X}$ is also (hereditarily) normal.

(ii) If X is normal and $dim X \leq n$ then $dim\overline{X} \leq n$.

(iii) If X is a Tychonoff space and every $A \in Ob(P)$ is a C^*-embedded subset of X then $\overline{X}$ is Tychonoff.

Proof. The normality, as well as the property $dim X \le n$, may be formulated in terms of $X \tau Y$ for $Y = [0,1]$ and $Y = \mathcal{S}_n$ (the $n-$dimensional sphere) respectively.

Hereditary normality is equivalent to the normality of all its *open* subspaces (see [En]). Hence, we can use the definition of $F-$sets.

To show (iii), first observe that $\overline{X}$ is a T_1-space iff X is T_1 (without any restrictions on P). This follows from Lemma 2.7. In order to prove that a point $u \in \overline{X}$ and a closed subset $A \subset \overline{X}$ are separated by a continuous real valued function, we can suppose that $u \in \mathbf{i}(X)$. Starting from a continuous function $f_0 : X_\mathbf{1} \to [0,1]$ which separates u and $A \cap X_\mathbf{1}$, we may construct $f : \overline{X} \to [0,1]$ by induction, slightly modifying the proof of Proposition 3.5. □

Proposition 3.7. *Let X be a normal space (and $dim X \le n$). Then X can topologically be embedded into a normal $C-$homogeneous space Y (resp., with $dim Y \le n$).*

Proof. The effective preaction on X defined by the groupoid $P_C(X)$ is transitive. The corresponding universal action $< G(P_C), \overline{X}_C >$ is transitive by Lemma 3.2. But we need much more than homogeneity. In order to achieve the C-homogeneity, we iterate our construction. In every step we "preserve old homeomorphisms" in the new groupoid preaction. More precisely, as a first step, we denote $X_0 = X, X_1 = \overline{X}_C$ and $P_1 = P_C(X_0)$. Let $i_0 : X_0 \to X_1$ be the corresponding topological embedding (Proposition 2.10). In order to build X_2, we define a groupoid P_2 as the minimal subgroupoid of $T(X_1)$ which contains both $G(P_1)$ and the set $P_C(X_1)$ of all homeomorphisms between compact subsets of X_1. Define now X_2 as the phase space of the universal action for the preaction $< P_2, X_1 >$. We will identify X_k with its image $i_k(X_k)$ in X_{k+1}. Continuing in this manner, by Propositions 2.10 and 3.6 we will obtain an increasing sequence of topological closed embeddings of *normal* spaces (with $dim \le n$)

$$X_0 = X \subset X_1 \subset X_2 \subset \cdots$$

The crucial property of these embeddings is the fact that every partial homeomorphism between compact subsets of X_n *canonically* can be extended to a homeomorphism of X_k for every $n+1 \le k$. If we consider the direct limit space $Y = \cup\{X_n | n \in N\}$ then every pair of compact subsets of Y are contained in some X_n. By the above mentioned *extension property*, Y is $C-$homogeneous. The normality of such direct limits is well-known; see, for example [FR, Proposition 3.4]. By the *countable sum theorem* [En], $dim Y \le n$. □

4 Concluding remarks and some perspectives

The present paper is a simplified version of results first presented in the dissertation [Me2, ch.3] and in a short form in [Me1]. Some of our results have already been used in [Pe].

There are several natural directions for possible developments:

- *More properties*

Find more topological properties that are inherited by $\overline{X}$. Note, for instance, that Proposition 3.6 is not true if we replace dim by Ind or ind. The existence of a counterexample follows easily from the fact that the finite sum theorem is not valid in general for Ind and ind in compact Hausdorff spaces (see [En, 7.4.15]). Nevertheless, if the objects of P are *finite* subspaces of X, then inductive constructions are still valid. Therefore, we can preserve Ind and ind substituting $\Sigma-$homogeneous for $C-$homogeneous. For these and some more results in this direction, see [Me1, Me2].

• *Varying the topology on* $\overline{X}$

If we restrict our attention primarily on topological properties of homogenizations, then we may consider *weaker topologies* on $\overline{X}$. Interesting results in this direction in the case of $\overline{X}_B$ can be found in [Ok, OP].

• *Topological groupoids*

It seems interesting to find an appropriate generalization of our construction for *topological groupoids* G (see, for example [Br, BH, HM, We]), for instance, in the case of the free topological group $G(P)$ of a topological groupoid P.

• *Varying the group* G

What happens if for a given preaction $< P, X, \pi >$ we fix a functor $\gamma : P \to G$ from a groupoid P into a *given group* G Γ The question about existence of a $G-$universal (with respect to γ) action admits an appropriate reformulation in categorical terms By constructing the spaces $\widetilde{X}$ and $\overline{X}$ in the same way as above (but for the given individual G and γ), it is easy to show that such a universal G-action $< G, \overline{X}, \overline{\pi} >$ always exists.

• *Partial actions of groups*

Our construction of $\overline{X}$ uses an equivalence relation naturally defined on $G \times X$ by a certain inverse semigroup of some partial homeomorphisms on X. The same construction was rediscovered recently by J. Kellendonk and M.V. Lawson [KL] in the context of universal globalizations of group partial actions on topological spaces.

• *Other categories*

A category $\mathcal{C}$ is *concrete* if any object $X \in Ob(\mathcal{C})$ is a set with some extra structure. It is clear how to define in such categories the groupoid $T(X)$ of all partial isomorphisms of the object X. Then a *preaction* of a groupoid P on X can be defined as a functor $P \to T(X)$. The definitions of actions and universal actions, can also be easily modified.

For instance, what happens if we replace $\mathcal{TOP}$ by other categories that are useful in topology, e. g. $\mathcal{UNIF}$ (uniform spaces), and $\mathcal{METR}$ (the category of all metric spaces and non-expanding maps) Γ In the latter case, important concrete results can be found in [OP, Us2]. In [OP], the authors found an interesting *metric* version of the free homogeneous space $\overline{X}_B$. In [Us2], Uspenskii establishes that every metric space can isometrically be embedded into a metrically $\Sigma-$homogeneous space.

The idea of using coproducts with amalgamations seems to be quite fruitful for several (not necessarily topological) categories. The central question is whether the corresponding morphism $X \to \overline{X}$ is a *monomorphism*. For example, it is well known that in the category $\mathcal{GR}$ of all groups, free products with amalgamated subgroups are very useful. In the classical work [HNN] the authors show how partial isomorphisms between subgroups of a group X can be realized as the trace of a conjugation in the suitable group Y. Further, Theorem 3 in [HNN] states that for every torsion-free group X there exists a group embedding $X \to Y$ such that in Y any two non-unit elements are conjugate. The proof uses a “group tower” with a suitable extension property similar to the “space tower” in the proof of Proposition 3.7.

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