(To appear in: Rendiconti del Seminario Mathematico della Universita di Padova, vol 97, 1997)

Minimal non-totally minimal topological rings

M.G. Megrelishvili (Levy)*
Department of Mathematics and Computer Science
Bar-Ilan University, 52 900 Ramat-Gan, ISRAEL

Abstract. We establish the existence of minimal non-totally minimal topological rings with a unit answering a question of Dikranjan. The Pontryagin duality and a generalization of Ursul's "semidirect product type" construction play major roles in the construction.

Introduction. A Hausdorff topological ring R is called *minimal* if its topology is minimal in the sense of Zorn among all Hausdorff ring topologies on R. If R/J is minimal for every closed ideal J, then R is called *totally minimal* [2].

The induced topology of a nontrivial valuation on a field is (totally) minimal (see [10, 6]). Some generalizations and related results in the context of fields or divisible rings may be found in [11, 13, 14]. For more general cases we refer to [1, 2, 3, 9]. Recall [2, 3] for instance that the class of all minimal rings with a unit is closed under forming topological products, direct sums and matrix rings. If P is a non-zero prime ideal of finite index in a Dedekind ring, then the P-adic topology is minimal.

The question about existence of minimal non-totally minimal rings with a unit is discussed by Dikranjan in [2, 3].

Conventions and preliminaries. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of all natural, integer and real numbers, respectively. The unit circle group $\mathbb{R}/\mathbb{Z}$ will be denoted by $\mathbb{T}$ and the n-element cyclic ring by $\mathbb{Z}_n$.

All rings are assumed to be associative. A ring R is *unital* if it has a unit. The zero-element will be denoted by 0. By $\mathrm{char}(R)$ we indicate the minimal natural number (if it exists) n such that $nx = 0$ for every $x \in R$. Otherwise we write $\mathrm{char}(R) = 0$. Clearly, $\mathrm{char}(R) = n > 0$ iff R is a (left) $\mathbb{Z}_n$-algebra in a natural way $(k, x) \mapsto x + x + \cdots + x$ (k terms) for each $(k, x) \in \mathbb{Z}_n \times R$.

---

*Partially supported by the Israel Ministry of Sciences, Grant No. 3505.
1991 *Mathematics Subject Classification.* 16W80, 54H13, 13J99, 22B99.
Key words and phrases. Minimal ring, locally compact ring, Pontryagin duality.

For a locally compact Abelian group G, denote by G^* the dual group $H(G,\mathbb{T})$ of all continuous characters endowed with the compact open topology. If R is a locally compact ring, then R^* is a topological (R,R)-bimodule [12].

If P is a subgroup of a topological group (G,τ), then $\tau\mid_P$ will denote the relative topology on P, and τ/P will be the quotient topology on the left coset space G/P . The following useful result is well known.

Merzon's Lemma. [8] *(See also* [4, Lemma 7.2.3] *for a proof).*

Let P be a subgroup of a group G, and let τ' and τ be (not necessarily Hausdorff) group topologies on G with the properties : $\tau'\subseteq\tau$, $\tau'\mid_P=\tau\mid_P$ and $\tau'/P=\tau/P$. Then $\tau'=\tau$.

Main results. Recall a construction from [12]. Let R be a topological ring and X a topological (R,R)-bimodule. On the product $R\times X$ of topological groups R and X, consider the multiplication

$$(r_1,x_1)\;(r_2,x_2)=(r_1r_2,r_1x_2+x_1r_2),\quad r_1,r_2\in R\;\;,\;\;x_1,x_2\in X.$$

Then $R\times X$ becomes a topological ring which is denoted by $R\leftthreetimes X$. For details and a particular case of $R\leftthreetimes R^*$ see Ursul [12].

Now we generalize this construction in two directions. The first change is minor. Let K be a commutative unital Hausdorff topological ring, (R,τ) a topological K-algebra, and (S,ν) be a topological K-module. Instead of $R^*=H_{\mathbb{Z}}(R,\mathbb{T})$, consider the K-module $H_K(R,S)$ of all continuous K-homomorphisms $R\to S$. As in the case of R^*, the left and right multiplications in R induce the (R,R)-bimodule structure in $H_K(R,S)$. The second modification is more essential. We add to $R\leftthreetimes H_K(R,S)$ a supplementary coordinate. Denote by $M_K(R,S)$ the product $R\times H_K(R,S)\times S$ of K-modules. The multiplication we define by the rule:

$$(r_1,f_1,s_1)\;(r_2,f_2,s_2)=(r_1r_2,r_1f_2+f_1r_2,f_2(r_1)+f_1(r_2))$$

where $r_1,r_2\in R$, $f_1,f_2\in H_K(R,S)$ and $s_1,s_2\in S$. Simple computations show that $M_K(R,S)$ becomes a K-algebra. Let $H_K(R,S)$ carry a K-module topology σ such that its (R,R)-bimodule structure is topological too. Moreover, suppose that the evaluation mapping

$$\omega\colon H_K(R,S)\times R\longrightarrow S,\quad \omega(f,r)=f(r)$$

is continuous with respect to the triple (σ,τ,ν) of Hausdorff topologies. Then $(M_K(R,S),\gamma)$ is a Hausdorff topological K-algebra with respect to the product topology γ. In particular, if R is a locally compact ring, $S=\mathbb{T}$ and $K=\mathbb{Z}$, then one gets a topological ring $M_{\mathbb{Z}}(R,\mathbb{T})=R\times R^*\times\mathbb{T}$ which will be denoted by $M(R)$.

Furthermore, we identify R, $H_K(R,S)$ and $H_K(R,S)\times S$ with the corresponding subsets of $M_K(R,S)$. We will keep below our assumptions about $(M_K(R,S),\gamma)$.

Proposition 1. *Let γ' be a new ring topology on $M_K(R,S)$ such that the canonical group retraction $q\colon H_K(R,S)\times S\longrightarrow S$ is continuous for the topologies $\gamma'\mid_{H_K(R,S)\times S}$ and ν. Then the evaluation mapping ω is continuous with respect to the triple of topologies $\gamma'\mid_{H_K(R,S)}$, $\gamma'/H_K(R,S)\times S$ and ν.*

Proof. Fix $\varphi_0\in H_K(R,S), r_0\in R$ and a ν-neighborhood O at $\varphi_0(r_0)$ in S. By the continuity of q, we may choose a γ'-neighborhood U of the element $z_0=(0,\varphi_0 r_0,\varphi_0(r_0))\in M_K(R,S)$, such that $q(U\cap(H_K(R,S)\times S))\subseteq O$.

By our assumption, the ring multiplication is γ'-continuous. Therefore, there exist γ'-neighborhoods V,W of the elements $(0,\varphi_0,0)$ and $(r_0,0,0)$ respectively, such that $V\cdot W$ is contained in the chosen γ'-neighborhood U of $z_0=(0,\varphi_0,0)\,(r_0,0,0)$.

For every $\varphi\in V\cap H_K(R,S)$ and every $(r,f,s)\in W$, we have

$$(0,\varphi,0)\ (r,f,s)=(0,\varphi r,\varphi(r))\in U\cap(H_K(R,S)\times S).$$

Clearly, $\varphi(r)=\omega(\varphi,r)\in\omega(V\cap H_K(R,S),\mathrm{pr}(W))$, where pr denotes the projection $M_K(R,S)\longrightarrow R$ on the first coordinate.

Then,

$$\varphi(r)\in\omega(V\cap H_K(R,S),\mathrm{pr}(W))\subseteq q(U\cap(H_K(R,S)\times S))\subseteq O.$$

Since $\mathrm{pr}(W)$ is a $\gamma'/H_K(R,S)\times S$-neighborhood of the point r_0 and $V\cap H_K(R,S)$ is a $\gamma'\mid_{H_K(R,S)}$-neighborhood of the point φ_0, then the continuity of ω at (φ_0,r_0) is proved. $\square$

Let (F,σ). (E,τ), (S,ν) be Abelian Hausdorff groups. A continuous mapping $\omega\colon F\times E\longrightarrow S$ is called biadditive if the induced mappings $\omega_x\colon F\longrightarrow S,\omega_f\colon E\longrightarrow S$ are homomorphisms for every $x\in E$ and every $f\in F$. We say that a coarser pair $(\sigma',\tau')\leq(\sigma,\tau)$ of group topologies is ω-compatible if ω remains continuous with respect to the triple (σ',τ',ν). If ω is separated (i.e., if the annihilators of E and F are both zero), then the Hausdorff property of ν implies that every ω-compatible pair (σ',τ') is necessarily Hausdorff. Following [7], we say that ω is *minimal* if for every ω-compatible pair $(\sigma',\tau')\leq(\sigma,\tau)$, we have necessarily $\sigma'=\sigma,\tau'=\tau$.

Lemma 2. [7, Proposition 1.10]. *For every Hausdorff locally compact Abelian group G, the evaluation mapping $G^* \times G \longrightarrow \mathbb{T}$ is minimal.*

Another example of a minimal biadditive mapping is the canonical duality $E^* \times E \longrightarrow \mathbb{R}$ for a normed space E.

Proposition 3. *Let the evaluation mapping*

$$\omega\colon (H_K(R,S),\sigma) \;\times\; (R,\tau) \longrightarrow (S,\nu)$$

be minimal, and let $\gamma' \subseteq \gamma$, be a coarser Hausdorff ring topology on $M_K(R,S)$ such that γ' and γ coincide on $H_K(R,S)\times S$. Then $\gamma' = \gamma$.

Proof. Because γ' and γ agree on $H_K(R,S)\times S$, then, in particular, the mapping

$$q\colon H_K(R,S)\times S \longrightarrow S$$

is continuous with respect to the pair $(\gamma' \,|_{H_K(R,S)\times S}\,,\;\; \nu)$. So, we can apply Proposition 1. Then $\gamma' \,|_{H_K(R,S)}\,,\; \gamma'/H_K(R,S)\times S$ is a ω-compatible pair of group topologies. The minimality of ω implies $\gamma'/H_K(R,S)\times S = \tau = \gamma/H_K(R,S)\times S$. Now Merzon's Lemma finishes the proof. $\square$

As a corollary we get

Proposition 4. *Let the evaluation mapping ω be minimal and let S and $H_K(R,S)$ be compact. Then $M_K(R,S)$ is a minimal ring.*

Theorem 5. *Let R be a discrete ring. Then the topological ring $M(R) = R \times R^* \times \mathbb{T}$ is minimal. Hence, every (commutative) discrete ring is a continuous ring retract of a minimal (commutative) locally compact ring.*

Proof. By Pontryagin's Theorem, R^* is compact iff R is discrete. Now the minimality of $M(R)$ follows from Lemma 2 and Proposition 4. The canonical retraction $\mathrm{pr}\colon M(R) \longrightarrow R$ is the desired one. $\square$

The ring $M(R)$ from Theorem 5 is not unital. In order to "improve" this, we use a well known unitalization procedure. Let R be a topological K-algebra. Consider a new K-algebra

$$R_+ = \{r + \alpha 1_+ \;|\; r \in R, \;\; \alpha \in K\}$$

adjoining a unit 1_+. More precisely, R_+ is a topological K-module sum $R \oplus K$, and we identify $(r,\alpha) = r + \alpha 1_+$. A multiplication on R_+ is defined in the following manner:

$$(a + \alpha 1_+)\,(b + \beta 1_+) = ab + \alpha b + \beta a + \alpha\beta 1_+$$

where $\alpha, \beta \in K$ and $a, b \in R$. The following lemma is trivial.

Lemma 6. *If J is a (closed) ideal in R, then J is a (closed) ideal in R_+ and $R_+/J = (R/J)_+$.*

In the following result we use a method familiar from the theory of minimal topological groups (see, for example, [5]).

Theorem 7. *Let R be a complete K-algebra such that (R,τ), (K,σ) are minimal topological rings. Then the K-unitalization R_+ is a minimal topological ring.*

Proof. Denote by γ the given product topology on R_+ and suppose that $\gamma' \subseteq \gamma$ is a new Hausdorff ring topology. Since (R,τ) is a minimal ring, $\gamma' \mid_R = \gamma \mid_R = \tau$. By our assumption, (R,τ) is complete. Therefore, R is a closed ideal in (R_+,γ'). Consider the *Hausdorff* ring topology γ'/R on K. Since $\gamma'/R \subseteq \gamma/R = \sigma$ and (K,σ) is a minimal ring, then $\gamma'/R = \gamma/R$. By Merzon's Lemma we get $\gamma' = \gamma$. $\square$

Corollary 8. *Let R be a minimal complete ring with* $\mathrm{char}(R) = n > 0$. *Then the $\mathbb{Z}_n$-unitalization R_+ of R is a minimal ring.*

Theorem 9. *Let R be a discrete ring with* $\mathrm{char}(R) = n > 0$. *Then the $\mathbb{Z}_n$-unitalization R_+ of R is a continuous ring retract of a minimal locally compact unital ring M_+.*

Proof. Apply our construction for the situation $S = K = \mathbb{Z}_n$ and consider the $\mathbb{Z}_n$-algebra $M\colon = M_{\mathbb{Z}_n}(R,\mathbb{Z}_n) = R \times H_{Z_n}(R,\mathbb{Z}_n) \times \mathbb{Z}_n$. Denote by M_+ the $\mathbb{Z}_n$-unitalization of M. Since $\mathrm{char}(R) = n > 0$, then every character $\xi\colon R \longrightarrow \mathbb{T}$ can actually be considered as a restricted homomorphism $R \longrightarrow \mathbb{Z}_n \subset \mathbb{T}$ identifying $\mathbb{Z}_n$ with the n-element cyclic subgroup of $\mathbb{T}$. It is also clear that every homomorphism $R \longrightarrow \mathbb{Z}_n$ is even a morphism of $\mathbb{Z}_n$-algebras. Therefore, $H_{\mathbb{Z}_n}(R,\mathbb{Z}_n)$ and $R^* = H(R,\mathbb{T})$ coincide algebraically. Endow $H_{\mathbb{Z}_n}(R,\mathbb{Z}_n)$ with the compact topology σ of R^*. Eventually, the mapping $\omega\colon (R,\tau) \times (H_{\mathbb{Z}_n}(R,\mathbb{Z}_n),\sigma) \longrightarrow \mathbb{Z}_n \subset \mathbb{T}$ is minimal, because of Lemma 2. By Proposition 4, the ring M is minimal. Since M is a $\mathbb{Z}_n$-algebra, then Corollary 8 and Lemma 6 complete the proof. $\square$

Corollary 10. *For every nonnegative integer n which is not equal to 1 there exists a minimal non-totally minimal separable metrizable locally compact unital ring with* $\mathrm{char}(R) = n$.

Proof. Fix a natural number $n \geq 2$. Let $F_i = \mathbb{Z}_n$ for every $i \in \mathbb{N}$. Consider the topological ring product $\left(\prod_{i\in\mathbb{N}} F_i, \sigma\right)$ and the dense countable topological subring

$\left(\sum_{i\in\mathbb{N}} F_i, \tau\right)$. Denote by τ_d the discrete topology on $R :=\sum_{i\in\mathbb{N}} F_i$. Clearly, the $\mathbb{Z}_n$-unitalization R_+ of (R, τ_d) is not a minimal ring because we can take on R_+ the (strictly coarser) ring topology of the $\mathbb{Z}_n$-unitalization for (R, τ). On the other hand, by Theorem 9, the discrete non-minimal ring R_+ is a continuous ring retract of a minimal ring M_+. Eventually, M_+ is the desired ring.

For the case $n = 0$, consider the ring product $\mathbb{R} \times M_+$, where M_+ is a minimal ring constructed for the case $n \geq 2$, and use the productivity of the class of minimal unital rings [3].

Acknowledgement. I would like to express my gratitude to D.Dikranjan for helpful comments and suggestions.

References

1. B. Banaschewski, *Minimal topological algebras*, Math. Ann. **211** (1974), 107-114.
2. D. Dikranjan, *Minimal ring topologies and Krull dimension*, Topology, vol. **23**, Budapest, Hungary, 1978, pp. 357-366.
3. D. Dikranjan, *Minimal topological rings*, Serdica **8** (1982), 139-165.
4. D.N. Dikranjan, I.R. Prodanov, L.N. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Marcel Dekker; Pure Appl. Math. **130** (1989).
5. V. Eberhardt, S. Dierolf, U. Schwanengel, *On products of two (totally) minimal topological groups and the three-space-problem*, Math. Ann. **251** (1980), 123-128.
6. H. Kowalsky, *Beitrage zur topologische algebra*, Math. Nachr. **11** (1954), 143-185.
7. M. Megrelishvili (Levy), *Group representations and construction of minimal topological groups*, Topology Appl. **62** (1995), 1-19.
8. A.E. Merzon, Uspehi Mat. Nauk **27, No. 4** (1972), 217, (in Russian).
9. A. Mutylin, *Completely simple topological commutative rings*, Mat. Zametki **5** (1969), 161-171, (in Russian).
10. L. Nachbin, *On strictly minimal topological division rings*, Bull. Amer. Math. Soc. **55** (1949), 1128-1136.
11. N. Shell, *Topological Fields and Near Valuations*, Dekker 135, New York and Basel, 1990.
12. M.I. Ursul, *Boundedness in locally compact rings*, Topology Appl. **55** (1994), 17-27.
13. W. Wieslaw, *Topological Fields*, Dekker 119, New York and Basel, 1988.
14. S. Warner, *Topological Fields, North Holland Mathematics Studies 157*, North-Holland-Amsterdam, New York, Oxford, Tokyo, 1989.

e-mail: megereli@bimacs.cs.biu.ac.il