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Free topological groups over (semi)group actions

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ABSTRACT. We study equivariant embeddability into G-groups. A new regionally proximal type relation introduced in the paper gives a necessary condition providing some counterexamples. We establish also some sufficient conditions (for locally compact acting semigroups G, for instance) improving results of M.Eisenberg and J. de Vries.

Introduction.

Our aim is to investigate the following question.

Question. Let a topological (semi)group G act continuously on a space X. When can X be equivariantly embedded (or at least, G-mapped non-trivially) into a topological group P in such a way that G continuously acts on P by endomorphisms (hence, by automorphisms if G is a group)?

For the particular case when P is a *linear* G-space see de Vries [11] and the references there.

The question leads us to the definition of the free topological G-group over a (semi)group action (see Definition 1). Our main result is Theorem 6 which enubles us to find compact coset G-spaces G/H such that the free topological G-group over G/H is trivial. Roughly speaking, this means that every continuous G-map of G/H into a G-group P would be "collapsed" into a point. This happens, for example, when $G/H = \mathbb{S}^n$ is the n-dimensional sphere where G is the group of all autohomeomorphisms of $\mathbb{S}^n$ endowed with the compact open topology. The main tool will be a new "regionally proximal type" relation (Definition 2) which generalizes the classical notion from topological dynamics.

Eisenberg [3] has shown that if a locally compact group G acts continuously on a Tychonoff space X then the induced "lifted" action $G \times A(X) \to A(X)$ (which is

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separately continuous for arbitrary G) on the free Abelian topological group $A(X)$ is jointly continuous. A similar result, if $A(X)$ is replaced by the free locally convex space $L(X)$ is also true. This was remarked without proof in [3]. De Vries [11] proved it by categorical methods. We establish that analogous results remain true for the free topological group $F(X)$ considering an arbitrary locally compact topological *semigroup* G.

Conventions.

Recall that a *G-space* or alternatively, a *topological transformation (semi)group* (abbreviated: *tts, ttg*) is a system $\langle G, X, \alpha \rangle$ in which G is a topologized (semi)group, X is a topological space and $\alpha : G \times X \to X$, $\alpha(g,x) = gx$ is a continuous action. As usual, this means that $(gh)x = g(hx)$ for every $g, h \in G$ and every $x \in X$. If G has an identity e (i.e., if G is a monoid) then we require $ex = x$ for every $x \in X$. A *g-transition* is the mapping $\alpha^g : X \to X$, $\alpha^g(x) = gx$ and an *x-orbit mapping* is the mapping $\alpha_x : G \to X$, $\alpha_x(g) = gx$. A G-space X will be called a *G-group* or*G-endomorphic* if X is a topological group and each α^g is a group endomorphism. If G is a group we use the term: *G-automorphic*. In the case of a linear space X and linear $\alpha^g - s$, we obtain the known definition of a *linear G-space* [11].

The filter of all neighborhoods (nbd's) at a point x in a space X is denoted by $N_x(X)$. If μ is a compatible uniformity on a topological space X, then for every $\varepsilon \in \mu$ and $A \subset X$ denote by $\varepsilon(A)$ the set $\{y \in X \mid (x,y) \in \varepsilon,\ x \in A\}$. Subsets A, B will be called *ε-near* if $\varepsilon(A) \cap \varepsilon(B) \neq \emptyset$.

We denote the greatest compatible uniformity by $\mu_{\max}$.

Due to [10], the *left, right* and *upper* uniformities on a topological group will be denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L} \vee \mathcal{R}$ respectively.

We say that an action $\alpha : G \times X \to X$ is *locally uniformly equicontinuous* if for every $g \in G$ there exists $V \in N_g(G)$ such that $\{\alpha^g\}_{g \in V}$ is uniformly equicontinuous.

Main results.

As usual, for a topological space X denote by $F(X), A(X), L(X)$ the free topological group, the free Abelian topological group and the free locally convex space respectively.

Definition 1. Let $\langle G, X, \alpha \rangle$ be a *tts*. We will say that an endomorphic triple $\langle G, F_\alpha(X), \tilde{\alpha} \rangle$ with a continuous G-mapping $i_\alpha \colon X \to F_\alpha(X)$ is the *free topological G-group* over X, if for every continuous G-mapping $\varphi \colon X \to P$ to an endomorphic G-space P there exists a unique continuous G-homomorphism $\tilde{\varphi} \colon F_\alpha(X) \to P$ such that $\tilde{\varphi} \circ i_\alpha = \varphi$. If μ is a uniformity on X, then considering uniform G-mappings and the upper uniformities on topological groups, we obtain the definition of the *uniform free topological G-group* over (X, μ). The corresponding universal morphism is denoted by $i_\alpha \colon (X, \mu) \to F_\alpha(X, \mu)$.

The *(uniform) free locally convex G-space* $L_\alpha(X)$ (respectively : $L_\alpha(X, \mu)$)can be analogously defined.

An obvious *equivariant* generalization of the standard product procedure shows that the just defined free G-objects always exist. However, it turns out that the *embedding problem* for i_α is much more complicated. We start with the well known definition from topological dynamics.

Let $\langle G, X, \alpha\rangle$ be a *tts*, μ be a uniformity on X and $S \subset G$. A pair $(a,b) \in X \times X$ is called *regionally S-proximal* [1] and is indicated: $(a,b) \in Q_S$, if for every $\varepsilon \in \mu$ and arbitrary *nbd*'s $O_1 \in N_a(X), O_2 \in N_b(X)$ there exists $g \in S$ such that gO_1 and gO_2 are ε-near. Otherwise, (a,b) is said to be regionally S-distal. X is called regionally S-distal if $Q_S = \Delta_X :=\{(x,x) \mid x \in X\}$. The following definition seems to be new.

Definition 2. (i) We say that a pair $(a,b) \in X \times X$ is *regionally S-pseudoproximal* and write : $(a,b) \in Q^p_S$ (or: $(a,b) \in Q^p_S(X,\mu)$) if there exists a finite set $\{a = x_0, x_1, \dots, x_n = b\}$ with the following property:

$(*^S)$ for every $\varepsilon \in \mu$ and arbitrary *nbd*'s $O_i \in N_{x_i}(X)$, $i \in \{0,1,\dots,n\}$ there exists $g \in S$ such that gO_i and gO_{i+1} are ε-near, for every $i \in \{0,1,\dots,n-1\}$.

(ii) Let G be a monoid.A pair (a,b) will be called *regionally $*$-pseudoproximal* if $(a,b) \in Q^p_V$ for every $V \in N_e(G)$. This defines a relation $Q^p_* = \cap\{Q^p_V \mid V \in N_e(G)\}$. If $Q^p_* = X \times X$ or $Q^p_* = \Delta_X$, then we say that X is regionally$*$-pseudoproximal, or regionally $*$-pseudodistal respectively.

Obviously, Q^p_S and Q^p_* are reflexive symmetric relations on X and always $Q_S \subset Q^p_S$, $Q^p_* \subset Q^p_S$. In general, $Q_S \neq Q^p_S$, and $Q^p_* \neq Q^p_S$.

Example 3. Let $G_n = \{h \in H(I) \mid h(x_i) = x_i, x_i = \frac{i}{n},\ i \in \{0,1,\dots,n\}\}$ be the topological subgroup of $H(I)$. Consider the *ttg* $\langle G_n, I, \alpha\rangle$ and the canonical uniformity on I. Then, for every natural $n \geq 3$, the elements 0 and 1 are regionally G_n-distal. On the other hand, every pair $(a,b) \in I \times I$ is regionally $*$-pseudoproximal. In particular, Q_{G_n} is a proper subset of $Q^p_{G_n}$ for each $n \geq 3$.

Example 4. Define the homeomomorphism $h : I \to I$ by the rule

$$h(x) = \begin{cases} 3x^2, & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} + \frac{1}{3}\sqrt{3x-1}, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x^2 - 4x + 2, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Consider the cyclic group $G = \{h^n\}_{n\in\mathbb{Z}}$ and the natural action $G \times I \to I$. Since $0, \frac{1}{3}, \frac{2}{3}, 1$ are fixed then clearly $(0,1) \notin Q_G$. On the other hand, elementary computations show that $Q^p_G = I \times I$. Note also that $Q^p_* = \Delta_X$ if G is discrete.

Lemma 5. *If $f\colon (X_1,\mu_1) \to (X_2,\mu_2)$ is a uniform G-mapping, then $(f \times f)(Q^p_S(X_1,\mu_1)) \subset Q^p_S(X_2,\mu_2)$ and $(f \times f)(Q^p_*(X_1,\mu_1)) \subset Q^p_*(X_2,\mu_2)$.In particular, if (X,μ) is regionally $*$-pseudodistal, then every uniform G-subspace $(Y,\mu|_Y)$ is regionally $*$-pseudodistal.*

Theorem 6. *Let G be a topologized monoid.Then every G-group $\langle G,(X,\mu),\alpha\rangle$ is regionally$*$-pseudodistal for each $\mu\in\{\mathcal{L},\mathcal{R},\mathcal{L}\vee\mathcal{R}\}$.*

Proof. First we consider the case $\mu=\mathcal{R}$. Assuming the contrary, take a pair $(a,b)\in Q^p_*$ of distinct elements. Since X is a Hausdorff topological group and α is continuous, then we can choose *nbd*'s $V_0\in N_e(X)$, $U\in N_e(G)$ such that

$$V_0\cap g(V_0ab^{-1})=\emptyset \qquad \forall\, g\in U \tag{1}$$

Since $Q^p_*\subset Q^p_U$, then $(a,b)\in Q^p_U$. Consider a finite set $\{x_0,x_1,\ldots,x_n\}$ satisfying Definition 2. Choose symmetric *nbd*'s $V_1,V_2\in N_e(X)$ with the properties:

$$x_0x_i^{-1}V_2^2\subset V_1x_0x_i^{-1} \qquad \forall\, i\in\{0,1,\ldots,n\} \tag{2}$$

$$V_1^{n+1}\subset V_0 \tag{3}$$

Due to Definition 2(i), we pick for $\varepsilon\colon=\{(x,y)\in X\times X \mid xy^{-1}\in V_2\}$ an element $g\in U$ such that $g(V_2x_i)$ and $g(V_2x_{i+1})$ are V_2-near with respect to the right uniformity $\mathcal{R}$ on X. More precisely, there exist finite sequences $\{p_0,p_1,\ldots,p_{n-1}\},\{q_1,q_2,\ldots,q_n\}$ in V_2 such that $g(p_ix_i)(g(q_{i+1}x_{i+1}))^{-1}\in V_2$ for every $i\in\{0,1,\ldots,n-1\}$.

Since α^g is an endomorphism, then

$$g(p_ix_ix_{i+1}^{-1}q_{i+1}^{-1})\in V_2 \qquad \forall\, i\in\{0,1,\ldots,n-1\} \tag{4}$$

Consider the element

$$z=g(p_0x_0x_1^{-1}q_1^{-1})\;g(p_1x_1x_2^{-1}q_2^{-1})\cdots g(p_{n-1}x_{n-1}x_n^{-1}q_n^{-1}).$$

Since $V_2\subset V_1$ by (2), then (4) and (3) imply $z\in V_2^n\subset V_1^n\subset V_0$. Clearly,

$$z=g(p_0x_0x_1^{-1}(q_1^{-1}p_1)x_1x_2^{-1}(q_2^{-1}p_2)\cdots(q_{n-1}^{-1}p_{n-1})x_{n-1}x_n^{-1}q_n^{-1})$$

and, $q_i^{-1}p_i\in V_2^{-1}V_2=V_2^2$ for each $i\in\{1,\ldots,n-1\}$. Using (2) and the trivial cancellations of the form $x_0x_i^{-1}x_ix_{i+1}^{-1}=x_0x_{i+1}^{-1}$, $(1\le i\le n-1)$ after $n-1$ steps we get

$$z\in g(p_0V_1^{n-1}x_0x_n^{-1}q_n^{-1})\subset g(V_1^nx_0x_n^{-1}q_n^{-1}).$$

Using (2) (for $i=n$), we obtain

$$z\in g(V_1^{n+1}x_0x_n^{-1})=g(V_1^{n+1}ab^{-1})\subset g(V_0ab^{-1}).$$

Thus, $z\in V_0\cap g(V_0ab^{-1})$, which contradicts (1). This proves the case $\mu=\mathcal{R}$.

For $\mu=\mathcal{L}$, use the G-unimorphism $(X,\mathcal{L})\to(X,\mathcal{R}),x\to x^{-1}$ and if $\mu=\mathcal{L}\vee\mathcal{R}$, use Lemma 5 for the uniform G-mapping $f=1_X\colon(X,\mathcal{L}\vee\mathcal{R})\to(X,\mathcal{R})$. $\square$

Theorem 7. *Let G be a topologized monoid and let (X,μ) be a $*$-pseudoproximal G-space. Then every uniform G-mapping $(X,\mu) \to (Y,\xi)$ into a G-group Y is constant for each $\xi \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$.In particular, the free uniform G-group $F_\alpha(X,\mu)$ is cyclic and discrete.*

Proof. Combine Lemma 5 and Theorem 6. □

Example 8. Let $X = I^n$ be the n-dimensional cube, or let $X = \mathbb{S}^n$ be the n-dimensional sphere (in both cases $n \in \mathbb{N}$). Denote by $H(X)$ the group of all autohomeomorphisms of X endowed with the compact open topology. Then $\langle H(X), X, \alpha \rangle$ is a regionally $*$-pseudoproximal *ttg* with respect to the unique uniformity on X. Then, by Theorem 7, the free topological G-group $F_\alpha(X)$ is cyclic and discrete. It is remarkable that by Effros's Theorem [2], $\mathbb{S}^n$ is a coset space of $H(\mathbb{S}^n)$. If $X = I$ then the example answers the question posed by the author in [4, Problem 1.14].

Question 9. Under which conditions is the G-space G/H automorphizable (= G-subspace of an automorphic G-space) ?

This is so, for example, if H is a *neutral* [10] subgroup. Indeed, in such cases, Theorem 5.8 and Proposition 7.7 from [10] imply that the action α_ℓ of G on G/H is uniformly equicontinuous with respect to the quotient uniformity $\mathcal{L}/H$. Therefore, by [7, Th.1.2] (or by our Proposition 12) G/H is even G-linearizable.

Question 10. Under which conditions does the free uniform G-group $F_\alpha(X,\mu)$ coincide with the free uniform group $F(X,\mu)$ over X?

Lemma 11. *(For a stronger version for groups, see* [5, Lemma 2.1]*). Let an action $\alpha \colon G \times X \to X$ be locally uniformly equicontinuous with respect to a uniformity μ on X and let orbit mapping $\alpha_y \colon G \to X$ be continuous for each $y \in Y$, where Y is dense in X. Then α is continuous.*

Proposition 12. *Let $\alpha \colon G \times X \to X$ be a continuous and locally uniformly equicontinuous action on a uniform space (X,μ). Then $F_\alpha(X,\mu) = F(X,\mu)$ and $L_\alpha(X,\mu) = L(X,\mu)$.*

Proof. Let $\tilde{\alpha} \colon G \times F(X,\mu) \to F(X,\mu)$ be the lifted action. Clearly, each g-transition $\tilde{\alpha}^g$ is continuous. Since X algebraically generates $F(X,\mu)$, then the continuity of orbit mappings $\alpha_x \colon G \to X$ and of group operations in $F(X,\mu)$ imply that for each $w \in F(X,\mu)$ the orbit mapping $\tilde{\alpha}_w \colon G \to F(X,\mu)$ is continuous. From the constructive description of a neighborhood system of the identity in $F(X,\mu)$ [8] follows that V acts $\mathcal{L} \vee \mathcal{R}$-uniformly equicontinuously on $F(X,\mu)$,provided that V acts uniformly equicontinuous on (X,μ). By Lemma 11, $\tilde{\alpha}$ is continuous.Obviously, this implies that $F_\alpha(X,\mu) = F(X,\mu)$. Essentially the same proof works for $L(X,\mu)$ using [9]. □

Pestov [6] proved the continuity of the associated action $\tilde{\alpha}\colon G \times F_\nu^b(X) \to F_\nu^b(X)$ for the uniformly equicontinuous group action $\alpha\colon G \times X \to X$, where $F_\nu^b(X)$ denotes the free uniform *balanced* (i.e., $\mathcal{L} = \mathcal{R}$) group in a variety ν. For an analogous "lifting" Theorem for a modification of the free locally convex spaces, see [7, Th. 1.2].

Lemma 13. *Every continuous action α of a locally compact topological semigroup G on a Tychonoff space X is locally $\mu_{\max}$-uniformly equicontinuous.*

Proof. Let a system $S = \{d_k\}_{k\in K}$ of pseudometrics generate $\mu_{\max}$ and let $\mathbb{B}$ be the system of all compact subsets in G. Consider the family $S^{\mathbb{B}} = \{d_k^C \mid k \in K,\ C \in \mathbb{B}\}$ where

$$d_k^C(x,y) = \sup\{d_k(gx,gy) \mid g \in C\}.$$

The compactness of C and the continuity of α imply that the system $\{\alpha^g \mid g \in C\}$ is d_k-equicontinuous for every $k \in K$. Then it is easy to see that uniformity θ generated by the system $S^{\mathbb{B}} \cup S$ is compatible with the original topology. If $A, B \in \mathbb{B}$ then $A \cdot B \in \mathbb{B}$. Eventually, the given action is locally θ-uniformly equicontinuous. Finally, observe that the maximality of $\mu_{\max}$ and the inclusion $\mu_{\max} \subset \theta$ imply $\mu_{\max} = \theta$. $\square$

Theorem 14. *For every continuous action α of a locally compact topological semigroup G on a Tychonoff space X holds $F_\alpha(X) = F(X), A_\alpha(X) = A(X), L_\alpha(X) = L(X)$.*

Proof. It is well known that $F(X,\mu_{\max}) = F(X), A(X,\mu_{\max}) = A(X)$ and $L(X,\mu_{\max}) = L(X)$. So we can apply Proposition 12 and Lemma 13. $\square$

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