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SEMIGROUP ACTIONS: PROXIMITIES, COMPACTIFICATIONS AND NORMALITY

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This work is dedicated to the memory of Yu.M. Smirnov.

Abstract. We study equivariant compactifications of continuous actions of topological semigroups. We give a transparent description of such compactifications in terms of S-proximities, special action compatible proximities on X. This leads us to a dynamical generalization of classical Smirnov's theorem. As a strictly related concept we investigate equivariant normality of actions and give some relevant examples.

Contents

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1. Introduction

A *topological transformation semigroup* (*S-space*, or an *S-flow*) is a continuous action of a topological semigroup S on a topological space X. For the particular case of a topological group action we reserve the symbol G.

In the present work we investigate the following problems:
(A) *Characterize equivariant compactifications of semigroup actions. For instance what about corresponding proximities and a possible equivariant generalization of Smirnov's theorem?*
(B) *When a given action satisfies the normality conditions in the spirit of the classical Urysohn and Tietze theorems?*

One of our main objects is the dynamical analogue of the compactification concept. Compactifiability of topological spaces means the existence of topological embeddings into compact Hausdorff spaces. For the compactifiability of flows we require in addition the continuous extendability of the original action. Compactifiable G-spaces are known also as *G-Tychonoff spaces.* See Fact 4.3 below and also a recent review paper [18].

The compactifications of a Tychonoff space X can be described in several ways:

- Banach subalgebras of $C(X)$ (Gelfand-Kolmogoroff 1-1 correspondence [14]);
- Completion of totally bounded uniformities on X (Samuel compactifications);
- Proximities on X (Smirnov's Theorem).

It is well known (see for example [9, 37, 3, 22, 19]) that the first two correspondences admit dynamical generalizations in the category of G-flows. Instead of continuous bounded functions, we should use special subalgebras of π-uniform functions $C_\pi(X)$ (Definition 5.9) and instead of precompact uniformities, we need now precompact *equiuniformities* (Definition 4.4).

Now about the third case – the description by proximities. In the case of G-spaces, group actions, it was initiated by Smirnov himself in [5]. We will see that there exists a natural generalization of Smirnov's Theorem for semigroup actions in terms of S-proximities, special action compatible proximities on X.

Our second direction is a closely related theme of equivariant normality of actions. The existing functional and topological characterizations of usual normality admits a dynamical generalization for group actions; see [21, 26, 24]. However full proofs never have been published before. Later some natural generalizations for semigroup case appeared in [7].

We give here a self-contained and unified exposition of some new and old results around two problems (A) and (B) mentioned above. We obtain several results which concentrate on similarities between group and semigroup cases. On the other hand we give also relevant contrasting counterexamples.

2. Main results

For every subset A of a topological space X we denote by N_A or $N_A(X)$ the set of all neighborhoods (in short: nbds) of A in X.

Definition 2.1. Let X be an S-space where S is a topological semigroup.

(1) Subsets $A, B \in X$ are π-*disjoint* if for every $s_0 \in S$ there exists $U \in N_{s_0}(S)$ such that $U^{-1}A \cap U^{-1}B = \emptyset$. Notation: $A \overline{\triangle}_\pi B$. We write $A \triangle_\pi B$ if A and B are not π-disjoint. We write $A \ll_\pi B$ if $A \ \overline{\triangle}_\pi \ (X \setminus B)$.

(2) Assume that δ is a continuous proximity on X. We say that δ is an S-*proximity* if the following condition is satisfied: for every pair $A\bar{\delta}B$ of δ-far subsets A, B in X and every $s_0 \in S$ there exists a nbd $U \in N_{s_0}$ such that $U^{-1}A\bar{\delta}U^{-1}B$.

Theorem 2.2. (Generalized Smirnov's theorem for semigroup actions) *In the canonical 1-1 correspondence between continuous S-proximities on X and compactifications of X the S-compactifications are in 1-1 correspondence with continuous S-proximities.*

Definition 2.3. Let X be an S-space where S is a topological monoid and the action is monoidal. We say that X is *S-normal* if for every pair $A, B \subseteq X$ of closed π-disjoint subsets, there are open disjoint nbds $O_A \in N_A$ and $O_B \in N_B$, such that $A \ll_\pi O_A$ and $B \ll_\pi O_B$.

If S is discrete then this is equivalent to the usual (topological) normality of X.

Theorem 2.4. (Normality for monoidal actions) *Let X be an S-space. The following are equivalent:*

1. *X is S-normal.*
2. *The relation*
$$A\delta_\pi B \Longleftrightarrow cl(A)\triangle_\pi cl(B)$$
is an S-proximity on the set X.
3. *For every pair A, B of closed π-disjoint subsets in X there exists $f \in C_\pi(X)$ such that $f(A) = 0$, $f(B) = 1$.*
4. *For every pair $A, B \subseteq X$ of closed π-disjoint subsets, there are π-disjoint open nbds $O_A \in N_A$ and $O_B \in N_B$, such that $A \ll_\pi O_A$ and $B \ll_\pi O_B$.*
5. (Urysohn's Small Lemma for S-spaces) *For every closed subset A and its open nbd O such that $A \ll_\pi O$ there exists an open nbd O_1 of A such that $A \ll_\pi O_1$ and $cl(O_1) \ll_\pi O$.*

Furthermore, if one of these equivalent conditions is satisfied then $\delta_\pi = \beta_\pi$, the greatest continuous S-proximity on the S-space X.

3. Topological Background

3.1. **Compactifications of topological spaces.** Let X be a topological space, Y be a compact Hausdorff space and let $f : X \to Y$ be a function such that $f(X)$ is dense in Y. If f is continuous, then Y (and sometimes a pair (Y, f)) is called a *compactification of X*. If f is a homeomorphic embedding, then Y is called a *proper compactification of X*. Denote by $(\mathscr{C}(X), \leq)$ the partially ordered set of all compactifications of X up to the standard equivalence.

Let $C(X)$ be the algebra of all real valued bounded continuous functions on X with respect to the supremum norm. Recall that the *unital* (containing the constants) closed subalgebras of $C(X)$ determine the compactifications of X.

Fact 3.1. (Gelfand-Kolmogoroff [14]) *There exists a natural order preserving bijective correspondence between $\mathscr{C}(X)$ (different compactifications of X) and closed unital subalgebras of $C(X)$. In particular, $C(X)$ determines the greatest compactification $\beta : X \to \beta X$.*

3.2. **Uniform spaces.**

Definition 3.2. *A uniformity* μ on a set X is a subset of the set of all relations on X which satisfies the following conditions (*axioms of uniformity*):

(U1) If $\varepsilon \in \mu$ and $\varepsilon \subseteq \delta$, then $\delta \in \mu$;
(U2) If $\varepsilon_1, \varepsilon_2 \in \mu$, then $\varepsilon_1 \cap \varepsilon_2 \in \mu$;
(U3) If $\varepsilon \in \mu$, then $\varepsilon^{-1} \in \mu$;
(U4) For every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that $\delta^2 := \delta \circ \delta \subseteq \varepsilon$.

An element ε of the uniformity μ is called an *entourage* of μ and the pair (X, μ) is called a *uniform structure*. If in addition it satisfies the following axiom:

(U5) $\bigcap\limits_{\varepsilon \in \mu} \varepsilon = \triangle_X$ (where $\triangle_X$ is the diagonal of $X \times X$);

then μ will be referred to as a *separated* (or, *Hausdorff*) uniformity.

Let x_0 be a point of X and $\varepsilon \in \mu$; the set

$$B(x_0, \varepsilon) := \{y \in X : (y, x_o) \in \varepsilon\}$$

is called the *ball with center x_0 and radius ε, or ε-ball about x_0*. The family

$$\tau_\mu = top(\mu) := \{O \subseteq X : \forall x \in O \ \exists \varepsilon \in \mu \text{ such that } B(x, \varepsilon) \subseteq O\}$$

is a completely regular topology on X. Every entourage $\varepsilon \in \mu$ yields the cover $\{B(x, \varepsilon)\}_{x \in X}$ of X which is said to be ε-*uniform cover* with respect to μ. About equivalent approach to the theory of uniform spaces through coverings see [16].

Definition 3.3. Let (X, τ) be a topological space and μ be a uniformity on the set X. Then we say that μ is *continuous* if $\tau_\mu \subseteq \tau$ and *compatible* if $\tau_\mu = \tau$.

A finite subset $A \subseteq X$ is an *ε-net* with respect to an entourage $\varepsilon \in \mu$ if $\{B(x,\varepsilon) : \ x \in A\}$ still is a covering of X. If ε admits an ε-net then we say that ε is *totally bounded*. A uniform space (X,μ) is *totally bounded* or *precompact* if every $\varepsilon \in \mu$ is totally bounded. A Hausdorff uniform space (X,μ) is precompact iff the completion is a compact Hausdorff space. On a compact Hausdorff space X there exists a unique compatible uniformity μ_X. The family of all neighborhoods of the diagonal constitutes a uniform basis of μ_X.

Remark 3.4. (**Initial uniformity**) Let $f : X \to (Y,\mu)$ be an arbitrary map from the set X into a uniform space (Y,μ). The system of entourages

$$\{(f \times f)^{-1}(\varepsilon) \subseteq X \times X : \quad \varepsilon \in \mu\}$$

is a basis of some uniformity μ_f on X, the corresponding *initial uniformity*. Then

(1) μ_f is the coarsest uniformity on X such that $f : (X,\mu_f) \to (Y,\mu)$ is uniform.
(2) The uniform topology τ_{μ_f} coincides with the initial topology on X with respect to the map $f : X \to (Y,\tau_\mu)$. Therefore if τ is a topology on the set X such that $f : (X,\tau) \to (Y,\tau_\mu)$ is continuous then τ_{μ_f} is τ-continuous. That is, $\tau_{\mu_f} \subseteq \tau$.

This fact is a particular case of a more general assertion about initial uniformities. See for example [32, Proposition 0.17].

Remark 3.5. (**Associated Hausdorff uniformity**) For every pseudometric d on X one can define a natural metric space (X^*,d^*) defining X^* as the quotient set X/Ω with respect to the equivalence relation $x\Omega y$ iff $d(x,y) = 0$. Then d^* is defined as $d^*([x],[y]) = d(x,y)$. Analogously if a uniform space (X,μ) is not necessarily separated then the associated Hausdorff uniform space (X^*,μ^*) is defined in a similar way, [32, p.18]. More precisely, $X^* := X/\Omega$ where $\Omega := \cap\{\varepsilon : \ \varepsilon \in \mu\}$ is always an equivalence relation on X. Denote by $q : X \to X^*, \ x \mapsto [x]$ the natural map. Now the following conditions are satisfied:

(1) The family $\{(q \times q)(\varepsilon)\}_{\varepsilon \in \mu}$ is a base of some *Hausdorff* uniformity μ^* on X^*.
(2) $\varepsilon \subseteq (q \times q)^{-1}(q \times q)(\varepsilon) \subseteq \varepsilon \circ \varepsilon \circ \varepsilon \quad$ for every $\varepsilon \in \mu$.
(3) μ is an initial uniformity w.r.t. the map $q : (X,\mu) \to (X^*,\mu^*)$.

Remark 3.6. (**Samuel Compactification**) Now we recall the definition of *Samuel compactification* of a uniform space (X, μ). We consider separately two cases:

(a) μ is Hausdorff. Every uniform structure μ contains the *precompact replica* of μ. It is the finest uniformity among all coarser totally bounded uniformities. In particular, $\mu_{fin} \subseteq \mu$. It is well known that μ_{fin} is just the family of all totally bounded entourages of μ. Denote by $i_{fin} : (X, \mu) \to (X, \mu_{fin}),\ x \mapsto x$ the corresponding uniform map. This map is a homeomorphism because $top(\mu) = top(\mu_{fin})$. The uniformity μ_{fin} is separated and hence the corresponding completion $(X, \mu_{fin}) \to (\widehat{X}, \widehat{\mu_{fin}}) = (uX, \mu_u)$ (or simply uX) is a proper compactification of the topological space $(X, top(\mu))$. The compactification

$$u_X = u_{(X,\mu)} : X \to uX$$

is the well known *Samuel compactification* (or, *universal uniform compactification*) of (X, μ) (see [33, 16]).

(b) μ is not Hausdorff.

Then first we pass to the associated Hausdorff uniformity. That is consider the map $q : (X, \mu) \to (X^*, \mu^*)$. Now we can apply part (a). That is, consider the usual Samuel compactification of (X^*, μ^*). The resulting composition map $u \circ q : (X, \mu) \to (uX^*, \mu^*_u)$ is the Samuel compactification of (X, μ).

Clearly the uniformity μ is Hausdorff if and only if the corresponding Samuel compactification $u : X \to uX$ is proper. In both cases (a) or (b) the corresponding algebra $\mathcal{A}_u \subseteq C(X)$ consists with all μ-uniformly continuous real valued bounded functions on X.

Fact 3.7. *There exists a natural order preserving 1-1 correspondence between continuous (Hausdorff) totally bounded uniformities on X and (resp., proper) compactifications of X.*

Remark 3.8. This principal result is standard for Hausdorff uniformities and proper compactifications. In general this fact, due to Gal [13], requires a more careful analysis. For the sake of completeness we include a sketch of the proof.

Proof. (of Fact 3.7) Let $\nu : X \to Y$ be a compactification of (X, τ). The system of entourages

$$\{(\nu \times \nu)^{-1}(\varepsilon) \subseteq X \times X : \quad \varepsilon \in \mu_Y\}$$

is a base of a uniformity μ_ν on X (the corresponding initial uniformity). The compactness of Y implies that μ_Y is precompact. Then μ_ν is also precompact. It is also easy (Remark 3.4) that μ_ν is a continuous uniformity on the topological space (X, τ).

Conversely, let μ be a totally bounded τ-continuous uniformity on X. The corresponding Samuel compactification

$$\nu : (X, \mu) \to (\widehat{X^*}, \widehat{\mu^*})$$

is defined as the composition of two uniform maps: $q : (X, \mu) \to (X^*, \mu^*)$ (the associated Hausdorff uniformity) and the usual completion $j : (X^*, \mu^*) \to (\widehat{X^*}, \widehat{\mu^*})$. Denote by Y the compact Hausdorff space $\widehat{X^*}$ with its unique uniformity $\mu_Y := \widehat{\mu^*}$. Now observe that the initial uniformity on X w.r.t. map $\nu : X \to (Y, \mu_Y)$ is just μ (see Remarks 3.5 and 3.6 above).

Alternative proof can be derived also making use Fact 3.1. □

Example 3.9. Let G be a topological group and H is a subgroup. The following system of entourages

$$\tilde{U} := \{(xH, yH) \in G/H \times G/H : \quad xy^{-1} \in U\} \quad (U \in N_e(G))$$

is a basis of a compatible uniformity μ_R on the coset space G/H, the so-called *right uniformity*.

3.3. **Proximities and proximity spaces.** F. Riesz in 1908 first formulated a set of axioms to describe the notion of closeness of a pair of sets. The most useful version of proximity was introduced by V.A. Efremovich [11] (see also [29] and [6, Chapter 1.5]).

Definition 3.10. Let X be a nonempty set and δ be a relation in the set of all its subsets. We write $A\delta B$ if A and B are δ-related and $A\overline{\delta}B$ if not. The relation δ will be called a *proximity* on X provided that the following conditions are satisfied:

(P1) $A \cap B \neq \emptyset$ implies $A\delta B$.
(P2) $A\delta B$ implies $B\delta A$;
(P3) $A\delta B$ implies $A \neq \emptyset$;
(P4) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;
(P5) If $A\overline{\delta}B$ then there exist $C \subseteq X$ such that $A\overline{\delta}C$ and $(X \setminus C)\overline{\delta}B$.

A pair (X, δ) is called a *proximity space*. Two sets $A, B \subseteq X$ are *near* (or *proximal*) in (X, δ) if $A\delta B$ and *remote* (or, *far*) if $A\overline{\delta}B$. Every proximity space (X, δ) induces a topology $\tau := top(\delta)$ on X by the closure operator:

$$\mathrm{cl}_\delta(A) := \{x \in X : \ x\delta A\}.$$

The topology $top(\delta)$ is Hausdorff iff the following condition is satisfied

(P6) If $x, y \in X$ and $x\delta y$ then $x = y$.

Every (separated) proximity space (X, δ) is completely regular (resp., Tychonoff) with respect to the topology $top(\delta)$ (can be derived from Remark 3.13).

Let (X, τ) be a topological space. A proximity δ of X is called *continuous* (or, more precisely, *τ-continuous proximity*) if $top(\delta) \subseteq \tau$. In the case of $top(\delta) = \tau$, we say that δ is a *compatible* proximity on the topological space (X, τ).

We say a subset $A \subseteq X$ is *strongly contained* in $B \subseteq X$ with respect to δ, if $A\overline{\delta}(X \setminus B)$ and write: $A \Subset B$. A finite cover $\mathscr{A} = \{A_i\}_{i=1}^n$ is called a *δ-cover* if and only if there exists another finite cover $\mathscr{B} = \{B_i\}_{i=1}^n$, such that $B_i \Subset A_i$ for every $i = 1, \dots, n$.

Lemma 3.11. (1) *In Definition 3.10 one can replace* (P5) *by one of the following axioms:*

(P5.a) *If $A\overline{\delta}B$ then there exist $C, D \subseteq X$ such that $C \cup D = X$ and $A\overline{\delta}C$, $B\overline{\delta}D$.*

(P5.b) *If $A\overline{\delta}B$ then there exist subsets A_1 and B_1 of X such that $A \Subset A_1$, $B \Subset B_1$ and $A_1 \cap B_1 = \emptyset$.*

(P5.c) *If $A\overline{\delta}B$ then there exist subsets A_1 and B_1 of X such that $A \Subset A_1$, $B \Subset B_1$ and $A_1\overline{\delta}B_1$.*

3.4. **Uniform spaces and the corresponding proximity.**

Definition 3.12. Let μ be a uniformity on X. Then the corresponding relation δ_μ defined by

$$A\delta_\mu B \Longleftrightarrow \varepsilon \cap (A \times B) \neq \varnothing \quad \forall \varepsilon \in \mu$$

is a proximity on X which is called the *proximity induced by the uniformity μ*.

Always, $top(\mu) = top(\delta_\mu)$. Proximity δ_μ is Hausdorff iff μ is Hausdorff.

Conversely every proximity δ on a topological space defines canonically a totally bounded compatible uniformity μ_δ. It is well known for Hausdorff proximities (cf. for example Engelking [12]). For not necessarily Hausdorff case see Gal [13].

Remark 3.13. Let (X, δ) be a proximity space. The collection $\mathscr{B}$ of all sets V of the form:

$$V := \bigcup_{i=1}^{k} (A_i \times A_i),$$

where $\{A_i\}_{i=1}^k$ is a δ-cover of X, defines a basis of the uniformity μ_δ. Uniformity μ_δ is totally bounded and $\delta = \delta_{\mu_\delta}$. Moreover, $top(\delta) = top(\mu_\delta)$. We call μ_δ the *induced uniformity* of δ. Its induced proximity in the sense of Definition 3.12 gives back the original proximity δ.

Example 3.14. Let (X, d) be a metric space. The corresponding proximity on X is defined as follows

$$A\delta_d B \iff d(A, B) = 0.$$

Example 3.15. Let $(G, \cdot)$ be a topological group with a subgroup H and μ_R is the right uniformity (Example 3.9). The corresponding compatible proximity δ_R on G/H is:

$$A\delta_R B \iff \forall\, U \in N_e(G) :\ UA \cap B \neq \varnothing.$$

Example 3.16.
(1) Let X be a Tychonoff space. The relation δ_β defined by

$$A\delta_\beta B \Leftrightarrow \nexists f \in C(X) \text{ such that } f(A) = 0 \text{ and } f(B) = 1$$

is a proximity which corresponds to the greatest compatible uniformity on X. The proximity $\delta_\beta B$ comes from the Stone-Ĉech compactification $\beta : X \to \beta X$.

(2) A Hausdorff topological space X is normal iff the relation

$$A\delta_n B \quad \text{iff} \quad cl(A) \cap cl(B) \neq \emptyset$$

defines a proximity relation on the set X.

(3) Let Y be a compact Hausdorff space. Then there exists a unique compatible proximity on the space Y defined by

$$A\delta B \iff cl(A) \cap cl(B) \neq \emptyset.$$

3.5. **Proximity mappings.**

Definition 3.17. [29, Definition 4.1] Let (X, δ_1) and (Y, δ_2) be two proximity spaces. A mapping $f : X \to Y$ is called a *proximity mapping* if one of the following equivalent conditions are satisfied:

(C1) $A\delta_1 B \Longrightarrow f(A)\delta_2 f(B)$.
(C2) $C\overline{\delta}_2 D \Longrightarrow f^{-1}(C)\overline{\delta}_1 f^{-1}(D)$.
(C3) $C \Subset D \Longrightarrow f^{-1}(C) \Subset f^{-1}(D)$.

Fact 3.18. (1) *$f : (X, \delta_1) \to (Y, \delta_2)$ is a proximity mapping if and only if it is uniformly continuous with respect to the uniformities μ_{δ_1} and μ_{δ_2}.*
(2) *In a proximity space (X, δ), $A\bar{\delta}B$ if and only if there exists a proximity mapping $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*
(3) *Let A be any subspace of a proximity space (X, δ) and let $f : A \to [a, b]$ be a proximity mapping. Then f can be extended to a proximity mapping $\overline{f} : X \to [a, b]$.*

Remark 3.19. Let (Y, δ) be a proximity space and $f : X \to Y$ be an arbitrary mapping. Define the following natural relation ξ on $X \times X$ by the following rule:

$$A\xi B \Longleftrightarrow f(A)\delta f(B).$$

Then $\xi := \delta_f$ is the smallest proximity on X such that the map f is proximity. We call it the *initial proximity*. There exists a natural link between initial proximity and initial uniformity. Precisely, let μ_δ be the induced uniformity of δ and $(\mu_\delta)_f$ be the initial uniformity on X. Then the induced proximity of $(\mu_\delta)_f$ is just $\xi = \delta_f$.

3.6. **Smirnov's Theorem.** Let $\alpha : X \to Y$ be a compactification. Denote by δ_α the corresponding initial proximity on X defined via the canonical admissible proximity δ_Y of Y. More precisely for subsets A, B of X we define $A\bar{\delta}_\alpha B$ if $\alpha(A)\overline{\delta}_Y\alpha(B)$, i.e., if $cl(\alpha(A)) \cap cl(\alpha(B)) = \emptyset$.

Conversely every continuous proximity δ on a topological space induces a totally bounded uniformity μ_δ (Remark 3.13). Now completion gives a compactification which we denote by c_δ. It is equivalent to the Samuel compactification with respect to the uniformity μ_δ. This leads to a description of compactifications in terms of proximities. This approach is the so-called *Smirnov's compactification* (see for example [34, 12, 29]).

Theorem 3.20. *(Smirnov's classical theorem) Let X be a topological space. By assigning to any compactification $\alpha : X \to Y$ the proximity δ_α on X gives rise to a natural one-to-one order preserving correspondence between all compactifications of X and all continuous proximities on the space X.*

Note that originally Smirnov's theorem deals only with proper compactifications and compatible proximities. The present general version is due to Gal [13].

4. Semigroup actions

Let $(S, \cdot)$ be a semigroup and assume that also S has a topological structure τ. We say that $(S, \cdot)$ is a *topological semigroup* if the multiplication map: $S \times S \to S$ is continuous.

Topological S-flow (or an *S-space*) is a triple $\langle S, X, \pi \rangle$ where

$$\pi : S \times X \to X, \ \pi(s, x) = sx = \tilde{s}(x) = \tilde{x}(s)$$

is a jointly continuous left action of a topological semigroup S on a topological space X; we write it also as a pair (S, X), or simply, X (when π and S are understood). *Action* means that the following condition is satisfied:

$$s_1(s_2 x) = (s_1 s_2)x \quad \forall (s_1, s_2, x) \in S \times S \times X.$$

We define for every $x \in X$ the *corresponding orbit map* $\breve{x} : S \to X$, by $\breve{x}(s) := sx$ and also for every $s \in S$, *$\tilde{s}$-translations* $\tilde{s} : X \to X$, by $\tilde{s}(x) := sx$.

If S is a monoid and e is the identity of S then the action is *monoidal* will mean that $\tilde{e}$ is the identity mapping of X. In particular, for topological group actions we always require that the action is monoidal. For acting group we reserve the symbol G. As usual 'topological group' means that in addition the inverse mapping is also continuous.

For $U \subseteq S$ and $A \subseteq X$ define $UA := \{ua : \ (u, a) \in U \times A\}$. For every $(s, x) \in S \times X$ we define $s^{-1}x := \{y \in X : sy = x\}$ and for $A \subseteq X$,

$$s^{-1}A := \bigcup \{s^{-1}a : a \in A\} = \{y \in X : sy \in A\},$$

$$U^{-1}A := \bigcup \{u^{-1}A : \ u \in U\} = \{x \in X : Ux \cap A \neq \emptyset\},$$

$$U \star A := \bigcap \{u^{-1}A : \ u \in U\}.$$

Examples 4.1. We recall some natural ways getting topological monoids and monoidal actions (see for example [1], [19]).

(1) Let (Y, μ) be a uniform space. Denote by μ_{sup} the uniformity of uniform convergence on the set $Unif(Y, Y)$ of all uniformly continuous self-maps $Y \to Y$. Then under the corresponding topology $top(\mu_{sup})$ on $Unif(Y, Y)$ and the usual composition we get a topological monoid. For every subsemigroup $S \subseteq Unif(Y, Y)$ the induced action $S \times Y \to Y$ defines a topological flow.
(2) For instance, for every compact space Y the semigroup $C(Y, Y)$ endowed with the compact open topology is a topological monoid. The subset $Homeo(Y)$ in $C(Y, Y)$ of all selfhomeomorphisms $Y \to Y$ is a topological group.
(3) For every metric space (M, d) the semigroup $\Theta(M, d)$ of all *d-contractive maps* $f : X \to X$ (that is, $d(f(x), f(y)) \leq d(x, y)$) is a topological monoid with respect to the topology of pointwise convergence. Furthermore, $\Theta(M, d) \times M \to M$ is a continuous monoidal action.
(4) For every normed space $(V, ||\cdot||)$ the semigroup $\Theta(V)$ of all contractive linear operators $V \to V$ endowed with the strong operator topology (being a topological submonoid of $\Theta(V, d)$ where $d(x, y) := ||x - y||$) is a topological monoid.
(5) For every normed space V and a subsemigroup $S \subseteq \Theta(V)^{op}$, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$, the induced action $S \times B^* \to B^*$ on the weak star compact unit ball B^* of the dual space V^* is continuous (see [19, Lemma 2.4]).
(6) Every normed algebra A treated as a multiplicative monoid is a topological monoid. The subset B_A is a topological submonoid. In particular, for every normed space V the monoids $L(V)$ and $B_{L(V)}$ of all bounded and, respectively, of all *contractive* linear operators $V \to V$ are topological monoids endowed with the norm topology.

Let X, Y be S-flows. A function $f : X \to Y$ is an *S-map* if $f(sx) = sf(x)$ for all $(s, x) \in S \times X$. An *S-compactification* of X is an S-map $\alpha : X \to Y$ where α is a compactification of X.

We define an analogue of Tychonoff spaces in the class of S-spaces.

Definition 4.2. A flow (S, X) is said to be *compactifiable*, or *S-Tychonoff* if there exist a compact S-flow Y and a *proper S-*compactification $\alpha : X \hookrightarrow Y$.

In [36], de Vries posed the 'compactification problem': is it true that for arbitrary Hausdorff topological group G every Tychonoff G-space is compactifiable? This question is negatively answered in [23], even for Polish G-spaces.

Fact 4.3. *Recall some useful sufficient conditions (compare* [18]*) when G-spaces are compactifiable:*

(1) *if G is locally compact then every Tychonoff G-space is G-Tychonoff* (de Vries [38])*;*
(2) *every coset G-space G/H* (de Vries [36])*;*
(3) *every metric G-space (X, d) with the G-invariant metric d* (Ludescher and de Vries [17])*;*
(4) *every metric G-space (X, d), where G is second category and $\tilde{g} \colon X \to X$ is d-uniformly continuous for every $g \in G$, and also every linear G-space X* (Megrelishvili [22])*;*
(5) *every G-space X, where X is Baire, G is uniformly Lindelöf and acts transitively on X* (Uspenskij [35])*;*
(6) *every first countable G-space (more generally, every G-space with the b_f-property), where G is locally pseudocompact* (S. Antonyan and M. Sanchis [4, Theorem 5]).

For some results related to Fact 4.3.5 see Chatyrko and Kozlov [10]. For some new directions in the theory of G-compactifications we refer to van Mill [27, 28] and the references thereof.

Several well known results for G-spaces cannot be generalized to the case of S-spaces. As a typical example note that there exists a discrete S-space X with a compact monoid S such that X is not S-Tychonoff [19].

4.1. **Actions and uniformities.**

Definition 4.4. Let μ be a uniformity on an S-space X. We call the action:

(1) *μ-saturated* if every s-translation $\tilde{s} : X \to X$ is μ-uniform;
(2) *μ-bounded at s_0* if for every $\varepsilon \in \mu$ there exists a neighborhood $U \in N_{s_0}$ such that $(s_0x, sx) \in \varepsilon$ for each $x \in X$ and $s \in U$. If this condition holds for every $s_0 \in S$ then we simply say *μ-bounded* or μ is a *bounded* uniformity;

(3) μ-*equiuniform* if it is μ-saturated and μ-bounded. Sometimes we say also that μ is an *equiuniformity.*

For topological group actions this concept is well known. See for instance [9, 38, 25].

Proposition 4.5. *Let $\langle S, (X, \mu), \pi \rangle$ be a uniform S-space, then the following conditions are equivalent:*

(1) *μ is an equiuniformity;*
(2) *μ is saturated and the corresponding homomorphism*

$$h_\pi : S \to Unif(X, X), \quad s \mapsto \tilde{s}$$

is continuous;
(3) *For every $\varepsilon \in \mu$ and $s_0 \in S$, there exist $\delta \in \mu$ and $U \in N_{s_0}(S)$, such that if $(x, y) \in \delta$, then $(s_1 x, s_2 y) \in \varepsilon \ \ \forall s_1, s_2 \in U$;*
(4) *For every $\varepsilon \in \mu$ and $s_0 \in S$, there exist $\varepsilon' \in \mu$ and $U \in N_{s_0}(S)$, such that if A and B are subsets in X with the property $(A \times B) \cap \varepsilon = \varnothing$, then $(U^{-1}A \times U^{-1}B) \cap \varepsilon' = \varnothing$ (that is if A and B are ε-far then $U^{-1}A$ and $U^{-1}B$ are ε'-far).*

Proof. $(1) \Leftrightarrow (2)$: Is trivial.

$(1) \Rightarrow (3)$: We choose $s_0 \in S$ and $\varepsilon \in \mu$. There exists $\varepsilon' \in \mu$ such that $\varepsilon' \circ \varepsilon' \circ \varepsilon' \subseteq \varepsilon$.

Since the action is μ-saturated for ε' we can choose $\delta \in \mu$ such that

$$(x, y) \in \delta \implies (s_0 x, s_0 y) \in \varepsilon'.$$

Also from the boundedness of the action we can choose $U \in N_{s_0}(S)$ such that

$$(s_1 x, s_0 x) \in \varepsilon', \quad (s_0 y, s_2 y) \in \varepsilon' \quad \forall s_1, s_2 \in U.$$

Now for $(x, y) \in \delta$ we obtain $(s_1 x, s_2 y) \in \varepsilon' \circ \varepsilon' \circ \varepsilon' \subseteq \varepsilon$.

$(3) \Rightarrow (4)$: The condition $(A \times B) \cap \varepsilon = \varnothing$, means that

$$\forall \ (a, b) \in A \times B : (a, b) \notin \varepsilon.$$

Let $s_0 \in S$. By (3) for $\varepsilon \in \mu$ we can choose $\varepsilon' \in \mu$ and $U \in N_{s_0}(S)$ such that

$$(x, y) \in \varepsilon' \Longrightarrow (s_1 x, s_2 y) \in \varepsilon, \quad \forall \ s_1, s_2 \in U. \tag{4.1}$$

Now we claim that $(U^{-1}A\times U^{-1}B)\cap\varepsilon' = \varnothing$. Assuming the contrary we get

$\varepsilon'\cap(U^{-1}A\times U^{-1}B) \neq \varnothing \Longrightarrow \exists\,(x,y)\in(U^{-1}A\times U^{-1}B) : (x,y)\in\varepsilon'$.

Therefore by definition of $U^{-1}A$ and $U^{-1}B$ we conclude:

$$\exists\ s', s''\in U :\quad (s'x, s''y)\in A\times B.$$

On the other hand by Formula 4.1 for $s', s''\in U$ we have:

$$(x,y)\in\varepsilon' \Longrightarrow (s'x, s''y)\in\varepsilon.$$

This means

$$\forall\ \varepsilon\in\mu :\ (s'x, s''y)\in(A\times B)\cap\varepsilon.$$

Hence $(A\times B)\cap\varepsilon\neq\varnothing$, a contradiction.

(4) $\Rightarrow$ (3) : Choose $\varepsilon'\in\mu$ and $U(s_0)$ such that (4) is satisfied. Then we claim that $(s_1x, s_2y)\in\varepsilon\ \ \forall s_1, s_2\in U$; whenever $(x,y)\in\varepsilon'$. Assuming the contrary let $(s_1x, s_2y)\notin\varepsilon$. Denote $A := \{s_1x\}$ and $B := \{s_2x\}$. Then $(A\times B)\cap\varepsilon = \varnothing$. Hence necessarily $(U^{-1}A\times U^{-1}B)\cap\varepsilon' = \varnothing$. On the other hand $x\in U^{-1}A$ and $y\in U^{-1}B$. Therefore $(x,y)\notin\varepsilon'$, a contradiction.

(3) $\Rightarrow$ (1) : Is trivial. □

The following simple lemma provides two important examples. In fact the second assertion can be derived from the first. The first assertion follows from Proposition 4.5. Alternatively they can also be easily veryfied directly.

Lemma 4.6. (1) *For every separated uniform space (X,μ) the natural action of the topological semigroup $S := Unif(X,X)$ on X is μ-equiuniform.*

(2) *For every compact S-space Y the action is equiuniform with respect to the canonical uniformity μ_Y.*

Proposition 4.7. *Let $f : (X,\xi)\to(Y,\mu)$ be a uniform S-map. Suppose that ξ is the initial uniformity (Remark 3.4). Then if μ is an equiuniformity then the same is true for ξ.*

Proof. It is straightforward using the fact that the system of entourages

$$\{(f\times f)^{-1}(\varepsilon)\subseteq X\times X :\quad \varepsilon\in\mu\}$$

is a base of the uniformity ξ. □

Proposition 4.8. *Let* (X^*, μ^*) *be the associated uniform space (Remark 3.5) of the uniform space* (X, μ)*. Assume that* μ *is equiuniform. Then* μ^* *is also equiuniform.*

Proof. It is straightforward using the fact that the system of entourages $\{(q \times q)(\varepsilon)\}_{\varepsilon \in \mu}$ is a base of the uniformity μ^* on X^*. □

The following theorem is well known for group actions and separated uniformities (see for example, [9, 25]). For semigroup actions and separated uniformities it appears in [19].

Theorem 4.9. *Let* X *be an* S*-space.*

(1) *Assume that* $\pi : S \times X \to X$ *is a (separated)* μ*-equiuniform semigroup action. Then the induced action* $\pi_u : S \times uX \to uX$ *on the Samuel compactification* $uX := u(X, \mu)$ *is a (resp., proper)* S*-compactification of* X*.*
(2) *There exists a natural one-to-one correspondence between* S*-compactifications of* X *and continuous totally bounded equiuniformities on* X*.*

Proof. (2) follows from (1).

For separated equiuniformities (1) is exactly [19, Prop. 4.9]. It is easy to extend this result for a not necessarily separated case by Propositions 4.7 and 4.8 □

5. Generalized Smirnov's Theorem for semigroup actions

5.1. **Proximities for semigroup actions.**

Definition 5.1. Let X be an S-space.

(1) The subsets A, B of X are π-*disjoint at* $s_0 \in S$ if there exists $U \in N_{s_o}(S)$ such that $U^{-1}A \cap U^{-1}B = \emptyset$. If this condition holds for every $s_0 \in S$ then we simply say: π-*disjoint sets.* Notation: $A \overline{\triangle}_\pi B$.
(2) We write $A \triangle_\pi B$ if A and B are not π-disjoint.
(3) We write $A \ll_\pi B$ if sets A and B^c are π-disjoint (where $B^c := X \setminus B$).

Lemma 5.2. (1) $A \triangle_\pi B$ *iff there exists* $s_0 \in S$ *such that for every neighborhood* U *of* s_0 *one may choose* $x_0 \in X$ *such that* $Ux_0 \cap A \neq \emptyset$ *and* $Ux_0 \cap B \neq \emptyset$*.*

(2) *Let S be a monoid and $A \cap B \neq \emptyset$. Then $A \triangle_\pi B$. Hence π-disjoint subsets are disjoint.*
(3) *If S is discrete and A and B are disjoint then they are π-disjoint. If in addition S is a monoid then the converse is also true.*
(4) *$A \triangle_\pi B$ iff $B \triangle_\pi A$;*
(5) *$\emptyset \overline{\triangle}_\pi X$;*
(6) *$A \triangle_\pi (B \cup C)$ iff $A \triangle_\pi B$ or $A \triangle_\pi C$;*
(7) *The relation $\triangle_\pi$ is a proximity on the set X for every topological group action.*
(8) *$A \ll_\pi B$ iff for every $s_0 \in S$ there exists $U \in N_{s_0}(G)$ such that $s^{-1}A \subseteq t^{-1}B$ for every $s, t \in U$. It is also equivalent to saying that $U^{-1}A \subseteq U \star B$, where $U \star B := \cap \{u^{-1}B : \ u \in U\}$.*

Proof. The proof is straightforward. For example, for (6) use the equality $U^{-1}B \cup U^{-1}C = U^{-1}(B \cup C)$. □

The following definition is a generalized version of Smirnov's concept from [5].

Definition 5.3. Let X be an S-space where S is a topological semigroup. Assume that δ is a proximity on X. We say that δ is an *S-proximity* if for every pair $A\bar{\delta}B$ of δ-far subsets A, B in X and every $s_0 \in S$ there exists a nbd $U \in N_{s_0}$ such that $U^{-1}A\overline{\delta}U^{-1}B$.

If S is discrete then this condition simply means that the translations $\tilde{s} : X \to X$ are δ-proximity mapping (see Definition 3.17).

Proposition 5.4. *Let μ be an equiuniformity on an S-space X. Then δ_μ is an S-proximity.*

Proof. Let $A\overline{\delta}_\mu B$, i.e. by Definition 3.12 there exists an entourage $\varepsilon \in \mu$, such that $(A \times B) \cap \varepsilon = \varnothing$. Fix $s_0 \in S$. Then by Proposition 4.5 there exist $\varepsilon' \in \mu$ and a neighborhood U of s_0 in S such that $U^{-1}A$ and $U^{-1}B$ are ε'-far. This means that $U^{-1}A\overline{\delta}_\mu U^{-1}B$. □

Proposition 5.5. (1) *Let $\alpha : X \to Y$ be an S-compactification. The corresponding initial proximity δ_α on X is a (continuous) S-proximity on X.*
(2) *For every compact S-space Y the canonical proximity δ_Y is an S-proximity.*

Proof. (1): By Lemma 4.6.2 the unique compatible uniformity μ of Y is equiuniform. By Proposition 4.7 the corresponding initial uniformity μ_α on X is equiuniform, too. Then the induced proximity by Proposition 5.4 is an S-proximity. Finally observe that by Remark 3.19 this proximity is just $\delta_{\mu_\alpha} = \delta_\alpha$.

(2): Easily follows from (1). □

5.2. **Generalized Smirnov's theorem.** Our aim here is to prove an equivariant generalization of the classical Smirnov's Theorem 3.20 for the case of semigroups actions. For group actions it was done (in a different but equivalent form) by Smirnov himself in [5].

Theorem 5.6. (Smirnov's theorem for semigroup actions) *In the canonical 1-1 correspondence between continuous proximities on X and compactifications of X the S-compactifications are in 1-1 correspondence with continuous S-proximities.*

Proof. Let $\alpha : X \to Y$ be an S-compactification. Then by Proposition 5.5 the corresponding proximity δ_α on X is an S-proximity on X. Converse direction will follow by Proposition 5.8 below which states that if δ is a continuous S-proximity, then α_δ is an S-compactification. □

Lemma 5.7. *Let X be an S-space and δ be a proximity on X. The following are equivalent :*

1. *δ is an S-proximity on X (for every pair $A\bar{\delta}B$ of δ-far subsets A, B in X and every $s_0 \in S$ there exists $U \in N_{s_0}$ such that $U^{-1}A\overline{\delta}U^{-1}B$).*
2. *The following two conditions are satisfied:*
   - (i) *every s-translation $\tilde{s} : X \to X$ is a proximity mapping.*
   - (ii) *for every pair $A\bar{\delta}B$ of δ-far subsets A, B in X we have $A\overline{\triangle}_\pi B$ (that is for every $s_0 \in S$ there exists $U \in N_{s_0}$ such that $U^{-1}A \cap U^{-1}B = \emptyset$).*
3. *The induced uniformity μ_δ is an equiuniformity on X.*

Proof. (1) $\Rightarrow$ (2):
(i) Since $s_0 \in U(s_0)$ we have

$$A\overline{\delta}B \Rightarrow s_0^{-1}(A)\overline{\delta}s_0^{-1}(B).$$

This condition means by Definition 3.17 that $\tilde{s_0} : X \to X$ is a proximity mapping. (ii) is trivial because δ-far subsets are always disjoint by axiom (P1) of Definition 3.10.

(2) $\Rightarrow$ (3): Consider the corresponding induced precompact uniformity μ_δ on X (Remark 3.13). We have to show that (X, μ_δ) is *equiuniform*. Observe that the action on X is μ_δ-saturated (because every translation $\tilde{s} : X \to X$ is μ_δ-uniform by (i)). Hence we have only to show that it is also μ_δ-bounded. If not then there exist: $\varepsilon \in \mu_\delta$ and $s_0 \in S$ such that for every $U \in N_{s_0}$ we can choose $(u, x) \in U \times X$ with the property $(s_0 x, ux) \notin \varepsilon$. Then $ux \notin \varepsilon(s_0 x)$. On the other hand, by the properties of μ_δ (see for example [12, Theorem 8.4.8]) there exist two finite covers $\mathcal{P} := \{A_1, A_2, \cdots, A_n\}$ and $\mathcal{Q} := \{B_1, B_2, \cdots, B_n\}$ such that, $B_i \Subset A_i$ and $\bigcup(A_i \times A_i) \subseteq \varepsilon$. Then by condition (ii) there exists a nbd V of s_0 such that

$$V^{-1}B_i \cap V^{-1}(X \setminus A_i) = \emptyset \qquad \forall i = 1, \dots, n.$$

By our assumption on the pair (ε, s_0) there exists a pair $(v, x) \in V \times X$ such that $vx \notin \varepsilon(s_0 x)$. Choose i_0 such that $s_0 x \in B_{i_0}$. We get $s_0 x \in B_{i_0} \Subset A_{i_0} \subseteq \varepsilon(s_0 x)$. Then clearly $vx \notin A_{i_0}$. Equivalently, $vx \in X \setminus A_{i_0}$. Therefore, $x \in V^{-1}(X \setminus A_{i_0})$. On the other hand, $x \in V^{-1}B_{i_0}$ because $s_0 x \in B_{i_0}$. Thus, $x \in V^{-1}B_{i_0} \cap V^{-1}(X \setminus A_{i_0})$. This contradicts the fact $V^{-1}B_{i_0} \cap V^{-1}(X \setminus A_{i_0}) = \emptyset$.

(3) $\Rightarrow$ (1): Directly follows from Proposition 5.4. □

Proposition 5.8. *Let δ be a continuous S-proximity on an S-flow X. Then the associated Smirnov's compactification $\alpha_\delta : X \to Y$ is an S-compactification of X.*

Proof. Let δ be a continuous S-proximity on X. Consider the corresponding continuous precompact uniformity μ_δ on X. Then the Smirnov's compactification of X defined by the proximity δ is just the Samuel compactification

$$\alpha_\delta := u_{(X,\mu_\delta)} : X \to u_\delta X$$

of μ_δ. By Lemma 5.7, μ_δ is an equiuniformity on X. Now Proposition 4.9 says that α_δ is an S-compactification of X. □

5.3. Algebras of π-uniform functions for semigroup actions.

Definition 5.9. Let X be an S-space. A function $f : X \to \mathbb{R}$ is π-*uniform* if f is continuous, bounded and for every $\varepsilon > 0$ and $s_0 \in S$ there exists a nbd $U(s_0)$ such that

$$|f(s_1 x) - f(s_2 x)| < \varepsilon \quad \forall\ (x, s_1, s_2) \in X \times U \times U.$$

Family of all π-uniform functions on X constitutes Banach unital S-invariant subalgebra of $C(X)$ which is denoted by $C_\pi(X)$.

By the standard compactness argument we have:

Lemma 5.10. *For every compact S-space X we have $C_\pi(X) = C(X)$.*

There exists a natural 1-1 correspondence between S-compactifications of X and closed unital S-invariant subalgebras of $C_\pi(X)$. In particular, $C_\pi(X)$ determines the maximal S-compactification $\beta_S : X \to \beta_S X$. These facts and Definition 5.9 are well known for group actions [38]. For semigroup actions see for example [7, 19].

We say that a bounded function $f : X \to \mathbb{R}$ *weakly separates* subsets A, B of X if $cl(f(A))$ and $cl(f(B))$ are disjoint. If $f(A) = a \neq b = f(B)$ for some different points $a, b \in \mathbb{R}$ then we simply say that f *separates* A and B. For a uniformly continuous function $\phi : \mathbb{R} \to \mathbb{R}$ we have $f \in C_\pi(X)$ iff the composition $\phi \circ f \in C_\pi(X)$. It follows that A and B are weakly separated by $C_\pi(X)$ iff they are separated by $C_\pi(X)$. Furthermore we can suppose that $a = 0$ and $b = 1$.

Lemma 5.11. *If $C_\pi(X)$ (weakly) separates sets A and B in X, then $cl(A)$ and $cl(B)$ are π-disjoint. In particular, if a π-uniform function separates A and $X \setminus B$, then $cl(A) \ll_\pi int(B^c)$.*

Proof. If not then by Lemma 5.2.1 there exists $s_0 \in S$ such that for every $U(s_0)$ we can choose $x_0 \in X$ and $u_1, u_2 \in U$ with the property

$$u_1 x_0 \in cl(A), \quad u_2 x_0 \in cl(B).$$

If $C_\pi(X)$ (weakly) separates A and B then we can assume that $f(cl(A)) = 0, \ f(cl(B)) = 1$ for some $f \in C_\pi(X)$. For s_0 choose $V \in N_{s_0}$ such that

$$|f(v'x) - f(v''x)| < 1 \quad \forall (v', v'', x) \in V \times V \times X.$$

On the other hand $v_1 x_0 \in cl(A)$ and $v_2 x_0 \in cl(B)$ for certain $v_1, v_2 \in V$. Therefore we get

$$|f(v_1 x_0) - f(v_2 x_0)| = 1$$

which contradicts the previous fact. □

Remark 5.12. Compactification β_S is an analogue of the standard maximal (Stone-Ĉech) compactification for S-spaces. Clearly, X is S-Tychonoff iff the maximal S-compactification β_π is proper iff $C_\pi(X)$ separates points and closed subsets.

6. Equivariant normality of semigroup actions

In this section we study S-normality of monoidal actions applying S-proximities.

First recall that the topological normality condition can be reformulated in terms of proximity spaces. Namely, a Hausdorff topological space is normal iff the relation

$$A\delta_n B \quad \text{iff} \quad cl(A) \cap cl(B) \neq \emptyset$$

defines a proximity relation on X. Moreover then this proximity is exactly the proximity of the Stone-Ĉech compactification (Example 3.16) of X (compare Theorem 6.2 below).

In the present section we assume that S is a monoid with a neutral element e. All actions are monoidal, that is, e acts as the identity mapping.

Definition 6.1. Let S be a monoid. An S-space X is *S-normal* (or, *equinormal*) if for every pair $A, B \subseteq X$ of closed π-disjoint subsets, there are open disjoint nbds $O_A \in N_A(X)$ and $O_B \in N_B(X)$, such that $A \ll_\pi O_A$ and $B \ll_\pi O_B$.

If S is discrete then S-normality is equivalent to the usual (topological) normality of X.

In the case of group actions, G-normality was introduced in [26, 24]. It is also studied in [21]. Note that this definition (in fact in a different but equivalent form) also appears in a work by Ball and Hagler [7].

Theorem 6.2. *Let X be an S-space. The following are equivalent:*

1. *X is S-normal.*
2. *The relation*
   $$A\delta_\pi B \Leftrightarrow cl(A) \triangle_\pi cl(B)$$
   defines an S-proximity on the set X. [1]
3. *$C_\pi(X)$ separates closed π-disjoint subsets in X.*
4. *For every pair $A, B \subseteq X$ of closed π-disjoint subsets, there are open π-disjoint nbds $O_A \in N_A(X)$ and $O_B \in N_B(X)$, such that $A \ll_\pi O_A$ and $B \ll_\pi O_B$.*

---

[1] Recall that $cl(A) \triangle_\pi cl(B)$ means by Definition 5.1 that there exists $s_0 \in S$ such that for every neighborhood U of s_0 we have $U^{-1}cl(A) \cap U^{-1}cl(B) \neq \emptyset$.

(5) *(*Urysohn's Small Lemma for S-spaces*) For every closed subset A and its open nbd O such that $A \ll_\pi O$ there exists an open nbd O_1 of A such that $A \ll_\pi O_1$ and $cl(O_1) \ll_\pi O$.*

Furthermore, if one of these equivalent conditions is satisfied then $\delta_\pi = \beta_\pi$, the greatest continuous S-proximity on the space X.

Proof. (1) $\Rightarrow$ (2): The axioms $(P1)-(P4)$ of Definition 3.10 easily follow from Lemma 5.2 (namely by the assertions (2), (4), (5) and (6)). For (P1) we use the assumption that S is a monoid. By Lemma 3.11 instead of (P5) it suffices to check (P5.b). This directly follows by the definition of S-normality because if C is closed and O is its open nbd then $C \ll_\pi O$ if and only if $C \Subset O$ with respect to δ_π.

Now we check that δ_π is an S-proximity. By Lemma 5.7 we have to show

(i) every s-translation $\tilde{s} : X \to X$ is a δ_π-proximity mapping.
(ii) for every pair $A\bar{\delta_\pi}B$ of δ_π-far subsets A, B in X we have $A\overline{\triangle}_\pi B$.

The first condition is straightforward by Definition 3.17 (C2) using the inclusion

$$cl(t^{-1}C) \subseteq t^{-1}cl(C)$$

for every $C \subseteq X$ and $t \in S$.

If $A\bar{\delta}_\pi B$ then $cl(A)\overline{\triangle}_\pi cl(B)$. Hence necessarily $A\overline{\triangle}_\pi B$. This means that δ_π is an S-proximity.

(2) $\Rightarrow$ (3): We show first that δ_π is a continuous proximity on the space X. Let a be a point and B a subset in X such that $a \in cl(B)$. Since S is a monoid then the definition of δ_π for $s_0 := e$ implies that $\{a\}\delta_\pi B$. Thus, $top(\delta_\pi) \subseteq \tau$.

By generalized Smirnov's theorem (see Theorem 5.6) the proximity δ_π corresponds to an S-compactification of X. In particular, $\delta_\pi \leq \beta_\pi$. Let A, B be closed π-disjoint subsets in X. Then they are δ_π-far. Then they are also β_π-far. There exists a β_π-uniform bounded function separating A and B. Now observe that β_π-uniform function means exactly that it lies in $C_\pi(X)$ (see Remark 5.12).

(3) $\Rightarrow$ (4): Easily follows from Lemma 5.11. Take $f \in C_\pi(X)$ with $f(A) = 0$, $f(B) = 1$ $O_A := \{x \in X : f(x) < \frac{1}{3}\}$ and define $O_B := \{x \in X : f(x) > \frac{2}{3}\}$.

(4) $\Rightarrow$ (5): Define $B := X \backslash O$. Then $A \overline{\triangle}_\pi B$. By (4) we can choose π-disjoint open nbds $O_A \in N_A(X)$ and $O_B \in N_B(X)$, such that $A \ll_\pi O_A$ and $B \ll_\pi O_B$. Then $(X \backslash O_B) \overline{\triangle}_\pi B$. Since the open subsets O_A and O_B are disjoint we have $cl(O_A) \subseteq X \backslash O_B$. Therefore we get $cl(O_A) \overline{\triangle}_\pi B$. This means that $cl(O_A) \ll_\pi X \backslash B = O$.

(5) $\Rightarrow$ (1): Use (5) twice.

Thus, we see that all four conditions are equivalent. We already established above that δ_π is a continuous proximity on the space X and also $\delta_\pi \leq \beta_\pi$. By the characterization of β_π if two subsets A and B are β_π-far then A and B are separated by a function from $C_\pi(X)$. Then $cl(A)$ and $cl(B)$ are separated by the same function. By Lemma 5.11 we get that $cl(A) \bar{\triangle}_\pi cl(B)$. This means that $A \bar{\delta}_\pi B$. Thus $\delta_\pi \geq \beta_\pi$. So we get $\delta_\pi = \beta_\pi$, as desired. □

Corollary 6.3. *For monoidal actions every compact S-space is S-normal.*

Proof. Use the fact that by Lemma 5.10 we have $C_\pi(X) = C(X)$. □

Remark 6.4. Since $\delta_\pi = \beta_\pi$ we get that δ_π is compatible with the topology of X iff X is S-Tychonoff iff every singleton $\{a\}$ and a closed subset B with $a \notin B$ are π-disjoint. If $S := G$ is a topological group then the latter condition always holds. Therefore, every G-normal space is G-Tychonoff. It is not always true in general for semigroup actions (Example 7.5).

6.1. **Urysohn's Theorem for semigroup actions.** In this subsection we give a dynamical generalization of Urysohn's classical topological result about extensions of functions. We deal with π-uniform functions and monoidal actions. For group actions a similar result was obtained first in [24] (see also [2]).

Definition 6.5. Let $\Sigma = \{\Gamma_r : r \in R\}$ be a family of nonempty open subsets on a topological space X where R is a dense subset of the closed unit interval $I := [0, 1]$. We say that Σ is a *Urysohn system* (*u-system*, in short) if it satisfies the following condition:

$$(r_1, r_2) \in R \times R, \quad r_1 < r_2 \ \Rightarrow \ \mathrm{cl}(\Gamma_{r_1}) \subseteq \Gamma_{r_2}.$$

For every u-system Σ naturally defined function f_Σ from X to $[0, 1]$, by:

$$f_\Sigma(x) := \inf\{r \in R : \ x \in \Gamma_r\} \tag{6.1}$$

is continuous on X. We call it a *u-function* of the system Σ. Conversely, for every continuous function $f : X \to [0,1]$ and a countable dense set $R \subseteq [0,1]$ the family $\Sigma_f := \{\Gamma_r = f^{-1}[0,r) : \ r \in R\}$ is a u-system and $f_{\Sigma_f} = f$.

Definition 6.6. Let $\pi : S \times X \to X$ be a continuous monoidal action. Assume that $\Sigma = \{\Gamma_r : r \in R\}$ is a u-system. We say that Σ is *stable* (with respect to the action π) if it satisfies the following condition:

$$r_1 < r_2 \ \Rightarrow \ \forall s_0 \in S\ \exists U \in N_{s_0}(S) : \ U^{-1}\Gamma_{r_1} \subseteq U{\star}\Gamma_{r_2} \text{ or } \Gamma_{r_1} \ll_\pi \Gamma_{r_2}$$

(see Lemma 5.2.8).

Theorem 6.7. *Let $\pi : S \times X \to X$ be a continuous monoidal action. Then the u-function f_Σ of a u-system $\Sigma = \{\Gamma_r : \ r \in R\}$ is in $C_\pi(X)$ if and only if Σ is stable.*

Proof. $\Rightarrow$
u-system $\Sigma = \{\Gamma_r : r \in R\}$ is stable. We have to show that the u-function f_Σ (Formula 6.1) is π-uniform at every $s_0 \in S$. Let $\varepsilon > 0$. Then we need to choose a neighborhood $U \in N_{s_0}$, such that

$$f_\Sigma(s_0x) - \varepsilon < f_\Sigma(sx) < f_\Sigma(s_ox) + \varepsilon : \quad \forall s \in U, \ \forall x \in X. \tag{6.2}$$

Without restriction of generality we assume that $R := \mathbb{Q}_2$, the set of rational dyadic numbers and for every $n \in \mathbb{N}$ define: $\mathbb{Q}_2^{(n)} = \{\frac{m}{2^n} : m = 0,1,2,\dots,2^n\}$, then $R = \mathbb{Q}_2 = \bigcup\limits_{n\in\mathbb{N}} \mathbb{Q}_2^{(n)}$.

Choose $n_0 \in \mathbb{N}$ big enough such that for every $t \in [0,1]$ there exists $m_t = 2^{n_t}$, $n_t \in \{1,\dots,n_0\}$ such that:

$$t - \varepsilon < \frac{m_t - 2}{2^{n_0}} < \frac{m_t - 1}{2^{n_0}} < t < \frac{m_t + 1}{2^{n_0}} < \frac{m_t + 2}{2^{n_0}} < t + \varepsilon, \tag{6.3}$$

where $\frac{m_t \pm 1}{2^{n_0}}, \frac{m_t \pm 2}{2^{n_0}} \in \mathbb{Q}_2^{(n_0)}$.

Since Σ is stable system at s_0, then for every pair $r_1 < r_2$ (with $r_1, r_2 \in \mathbb{Q}_2^{(n_0)}$) there exists a neighborhood $U_{r_1r_2} \in N_{s_0}$, such that $U_{r_1r_2}^{-1}\Gamma_{r_1} \subseteq U_{r_1r_2} \star \Gamma_{r_2}$. The finite intersection

$$U := \bigcap U_{r_ir_{i+1}}, \quad \forall r_i, r_{i+1} \in \mathbb{Q}_2^{(n_0)}$$

is a neighborhood of s_0. For every $r_1, r_2 \in \mathbb{Q}_2^{(n_0)}$ we have:

$$U^{-1}\Gamma_{r_1} \subseteq U \star \Gamma_{r_2}. \tag{6.4}$$

Let $x \in X$, $y := s_0 x \in X$ and $f_\Sigma(y) = t \in I$. By condition 6.3 we have

$$y \in \Gamma_{\frac{m_t+1}{2^{n_0}}} \bigcap (X \setminus \Gamma_{\frac{m_t-1}{2^{n_0}}}). \tag{6.5}$$

Then $x \in s_0^{-1}y \subseteq U^{-1}\Gamma_{\frac{m_t+1}{2^{n_0}}}$. By Formula 6.4 we have $U^{-1}\Gamma_{\frac{m_t+1}{2^{n_0}}} \subseteq U \star \Gamma_{\frac{m_t+2}{2^{n_0}}}$. Hence $x \in U \star \Gamma_{\frac{m_t+2}{2^{n_0}}}$. This implies $sx \in \Gamma_{\frac{m_t+2}{2^{n_0}}}$ for every $s \in U$. It follows that for every $s \in U$ we have

$$f_\Sigma(sx) \leq \frac{m_t+2}{2^{n_0}}. \tag{6.6}$$

By condition 6.5 we have $y \notin \Gamma_{\frac{m_t-1}{2^{n_0}}}$. Therefore, $s_0^{-1}y \bigcap U \star \Gamma_{\frac{m_t-1}{2^{n_0}}} = \varnothing$. In particular, $x \notin U \star \Gamma_{\frac{m_t-1}{2^{n_0}}}$. By Formula 6.4, $U^{-1}\Gamma_{\frac{m_t-2}{2^{n_0}}} \subseteq U \star \Gamma_{\frac{m_t-1}{2^{n_0}}}$. Hence $x \notin U^{-1}\Gamma_{\frac{m_t-2}{2^{n_0}}}$. We get $\forall s \in U$, $sx \notin \Gamma_{\frac{m_t-2}{2^{n_0}}}$. For every $s \in U$ we can conclude that

$$f_\Sigma(sx) \geq \frac{m_t-2}{2^{n_0}}. \tag{6.7}$$

Since $s_0 x = y$ and $f_\Sigma(s_0 x) = t$, from conditions 6.6 and 6.7 we conclude:

$$f_\Sigma(s_0x) - \varepsilon < \frac{m_t-2}{2^{n_0}} \leq f_\Sigma(sx) \leq \frac{m_t+2}{2^{n_0}} < f_\Sigma(s_0x) + \varepsilon \quad \forall s \in U.$$

We obtain condition 6.2. This means that the u-function f_Σ is π-uniform at s_0.

$\Leftarrow$

Similar to the proof of Lemma 5.11. □

7. Some examples

Example 7.1. Let X be a locally compact Hausdorff space. Then the following relation:

$$A\overline{\delta}_a B \Leftrightarrow cl(A) \cap cl(B) = \varnothing \text{ where } cl(A) \text{ or } cl(B) \text{ is compact}$$

defines a compatible proximity on X which defines the (one-point) Alexandroff compactification.

If we have a group action then this proximity is a G-proximity. It can be verified directly or using a result of de Vries [36], which states that the one point compactification is always a proper G-compactification. For semigroup actions it is not true in general. See for instance [8, Example 3.1.10], [7, Example 7.2], or Example 7.5 below, where the points are not π-disjoint.

Example 7.2. Let (X, d) be a metric space. Define a binary relation on subsets of X as follows:

$$A\delta_0 B \Leftrightarrow \begin{cases} A\delta_d B \\ \text{ or} \\ diam(A) = \infty \ \ \& \ \ diam(B) = \infty \end{cases}$$

where δ_d is the standard metric proximity: $A\delta_d B \Leftrightarrow d(A, B) = 0$ (Example 3.14).

For every metric space (X, d), δ_0 is a separated proximity, $\delta_d \geqslant \delta_0$ and $top(\delta_0) = top(\delta_d)$. In particular, we can consider the proximity δ_0 for Banach spaces.

Example 7.3. Let $X = V$ be a Banach space. Then the group of all linear continuous automorphisms $GL(V) \subseteq L(V, V)$ is a topological group with respect to the operator norm. The natural action

$$\pi : GL(V) \times V \to V, \quad \pi(L, v) = Lv$$

is continuous and δ_0 is a $GL(V)$-proximity. In particular, V is a $GL(V)$-Tychonoff space.

Example 7.4. Let V be a Banach space. Consider the action of the topological monoid $S := (\Theta(V), \|\cdot\|)$ (with the operator norm topology) on the unit ball $(B_V, \|\cdot\|)$. Then

(1) The norm uniformity μ_d on B_V is an equiuniformity.
(2) The norm proximity δ_d on B_V is an S-proximity.
(3) (S, B_V) is S-Tychonoff.

Example 7.5. (S-`normal which is not` S-`Tychonoff`). Consider the (linear) action of the compact multiplicative monoid $S := ([-1, 1], \cdot)$ on $X := \mathbb{R}$. It is easy to see that every pair of points are not π-disjoint. Indeed, let $a, b \in \mathbb{R}$. For every nbd U of 0 in S choose $z \in \mathbb{R}$ big enough such that $\frac{a}{z} \in U$ and $\frac{b}{z} \in U$. Then clearly, $z \in U^{-1}a \cap U^{-1}b \neq \emptyset$. Therefore, by Lemma 5.11, every $f \in C_\pi(X)$ is a constant function. The corresponding proximity δ_π on X is trivial (i.e., the proximity of the trivial compactification).

Recall in contrast that for group action case every G-normal is necessarily G-Tychonoff (Remark 6.4).

Remark 7.6. From Example 3.15 we know that the relation δ_R defined by

$$A\delta_R B \Leftrightarrow \forall U \in N_e(G): \ UA \cap B \neq \varnothing$$

is a compatible proximity on G/H for every topological group G and its subgroup H. In fact it is a G-proximity for the left action $\pi : G \times G/H \to G/H$. The proximity δ_R is separated if and only if H is closed in G.

8. Actions of topological groups

Lemma 8.1. *Let X be a G-space with respect to the continuous action $\pi : G \times X \to X$ of a topological group G. Let $N_e(G)$ be the set of all nbd's of the identity $e \in G$.*

(1) *The relation $\triangle_\pi$ (Definition 5.1) can be described in a simpler way:*

$$A\triangle_\pi B \ \text{ if and only if } UA \cap B \neq \varnothing \text{ for every } U \in N_e(G). \tag{8.1}$$

Furthermore $\triangle_\pi$ is a proximity on the set X and $top(\triangle_\pi) \supseteq top(X)$ holds.

In particular we have

(a) *A and B are* π-disjoint *if and only if $UA \cap B = \emptyset$ for some $U \in N_e(G)$.*

(b) *$A \ll_\pi D$ if and only if $UA \subseteq D$ for some $U \in N_e(G)$.*

(2) *The family of entourages $\{\varepsilon_U\}_{U \in N_e(G)}$ where*

$$\varepsilon_U := \{(u_1x, u_2x) \in X \times X |\ x \in X, \ u_1, u_2 \in U\}$$

is a base of a uniformity μ_π on the set X such that the corresponding proximity δ_{μ_π} is exactly $\triangle_\pi$.

Proof. Straightforward using some elementary properties of $N_e(G)$. □

Lemma 8.2. *Let $f : X \to \mathbb{R}$ be a continuous bounded function on a G-space X. The following are equivalent :*

(1) *$f : X \to \mathbb{R}$ is π-uniform.*
(2) *For every $\varepsilon > 0$ there exists a nbd $U \in N_e(G)$ such that*

$$|f(ux) - f(x)| < \varepsilon \quad \forall\ (u, x) \in U \times X.$$

(3) *$f : X \to \mathbb{R}$ is μ_π-uniformly continuous.*
(4) *$f : X \to \mathbb{R}$ is δ_{μ_π}-uniform mapping.*

Proof. For (3) $\Leftrightarrow$ (4) use Fact 3.18 and Lemma 8.1.2. □

Definition 8.3. Let A be a (not necessarily G-invariant) subspace of a G-space X.

(1) The subspace proximity on $A \subseteq X$ induced by $\triangle_\pi$ is denoted by $\triangle_\pi^A$.
(2) Let $f : A \to \mathbb{R}$ be a bounded continuous function on A. We say that f is *π-uniform* if it is a proximal mapping on the subspace (A, Δ_π^A). Precisely this means that for every $\varepsilon > 0$ there exists a nbd $U \in N_e(G)$ such that

$$|f(ua) - f(a)| < \varepsilon \quad \forall\ (u, a, ua) \in U \times A \times A.$$

Remark 8.4. If A is a G-invariant subspace of X then Definitions 8.3.2 and 5.9 agree.

Theorem 8.5. *Let X be a G-space such that X is a Tychonoff space. The following are equivalent:*

(1) *X is G-normal (that is for every pair of π-disjoint closed subsets A and B there exist disjoint open nbds O_1 and O_2 such that $A \ll_\pi O_1$ and $B \ll_\pi O_2$).*
(2) *For every pair of π-disjoint closed subsets A and B there exist π-disjoint open nbds O_1 and O_2 such that $A \ll_\pi O_1$ and $B \ll_\pi O_2$.*
(3) *For every pair of π-disjoint closed subsets A and B there exist π-disjoint nbds O_1 and O_2.*
(4) *The relation*

$$A\delta_\pi B \Leftrightarrow U cl(A) \cap cl(B) \neq \emptyset \qquad \forall\ U \in N_e(G)$$

is a proximity on X.
(5) *$C_\pi(X)$ separates closed π-disjoint subsets in X.*

(6) *Every π-uniform function $f : A \to \mathbb{R}$ (in the sense of Definition 8.3.2) on a closed subset $A \subseteq X$ is a restriction of a π-uniform function $F : X \to \mathbb{R}$.*

Furthermore, if one of these equivalent conditions is satisfied then δ_π is a compatible G-proximity on the space X and $\delta_\pi = \beta_\pi$.

Proof. (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Theorem 6.2 taking into account Lemma 8.1.

(2) $\Rightarrow$ (3): Is trivial.

(3) $\Rightarrow$ (1): There exist $U(e)$ such that $UO_1 \cap UO_2 = \emptyset$, where O_1 and O_2 are nbd's of A and B respectively. We can suppose that O_1 and O_2 are open (passing to the interiors if necessary). Take a symmetric nbd $V(e)$ s.t. $VV \subseteq U$. Define $O_1' := VO_1$ and $O_2' := VO_2$. Then still O_1' and O_2' are π-disjoint (because $VO_1' \cap VO_2' = \emptyset$) and also $A \ll_\pi O_1'$ and $B \ll_\pi O_2'$ by Lemma 8.1 (because, $VA \subseteq VO_1 = O_1'$ and $VB \subseteq VO_2 = O_2'$).

(4) $\Rightarrow$ (6): Use Fact 3.18.3 taking into account Remark 8.4.

(6) $\Rightarrow$ (5): Let A and B be π-disjoint closed subsets of X. Define the function $f : A \cup B \to [0,1]$ by $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$. It is easy to see that f is a π-uniform function on $A \cup B$ in the sense of Definition 8.3.2. By assumption (6), f is a restriction of a π-uniform function $F : X \to \mathbb{R}$. The latter function clearly separates A and B. This proves (5).

Finally the compatability of δ_π follows from the fact that every singleton $\{a\}$ and a closed subset B with $a \notin B$ are π-disjoint for every G-space (Remark 6.4). □

Some versions of the Urysohn lemma for G-spaces appears in [15, Theorem 3.9] and [2].

Proposition 8.6. *Every G-normal G-space X is G-Tychonoff.*

Proof. By Theorem 8.5 there exists a compatible G-proximity on X. □

Lemma 8.7. [24] *Suppose that a G-space X, as a topological space, is normal. The following are equivalent:*

(1) *Every pair of π-disjoint closed subsets A and B in the G-space X there exists $U \in N_e(G)$ such that $cl(UA) \cap cl(UB) = \emptyset$.*

(2) *X, as a G-space, is G-normal.*

Proof. (1) $\Rightarrow$ (2): By the normality there exist disjoint nbds O_1 and O_2 of the closed disjoint subsets $cl(UA)$ and $cl(UB)$. Then $A \ll_\pi O_1$ and $B \ll_\pi O_2$.

(2) $\Rightarrow$ (1): Use Theorem 6.2.5 and Lemma 8.1.1. □

According to a fundamental result of de Vries [36] every Tychonoff G-space is G-Tychonoff for any locally compact group G. If X is normal then we can prove a stronger result.

Proposition 8.8. [26] *Let G be a locally compact group and X is a G-space such that X (as a topological space) is normal. Then*

1. *X is G-normal.*
2. *For every closed G-subspace A in X the compactifications $A \to \beta_G A$ and $A \to cl(A) \subseteq \beta_G X$ are equivalent.*

Proof. (1): Let A and B be π-disjoint closed subsets. Then there exists a nbd $U(e)$ such that $UA \cap UB = \emptyset$. We can suppose that U is compact. Then the subsets UA and UB are closed. We can now apply Lemma 8.7.

(2): By (1) and Theorem 8.5.6 every π-uniform function on A can be extended to a π-uniform function on X. This implies that the G-compactifications $A \to \beta_G A$ and $A \to cl(A) \subseteq \beta_G X$ are equivalent (both correspond to the same algebra $C_\pi(A)$). □

Proposition 8.8.2 answers a question of Yu.M. Smirnov (private communication).

Proposition 8.9. [26] *For every Hausdorff topological group G and every closed subgroup H the corresponding coset G-space G/H is G-normal.*

Proof. Observe that μ_π is exactly the right uniformity on G/H and δ_π is the proximity associated to μ_π (see Remark 7.6). It follows that assertion (4) of Theorem 8.5 is satisfied. □

Since every G-normal is G-Tychonoff Proposition 8.9 strengthens a result of de Vries [36] which asserts that any coset G-space G/H is G-Tychonoff. There exists a G-normal G-space X which is not normal as a topological space. Indeed take a topological group G which is not normal and consider the G-space G under the left action.

Corollary 8.10. *The following actions, being coset spaces, are equinormal:*

1. $(U(H), \mathbb{S}_H)$, *where* $\mathbb{S}_H$ *is the unit sphere of a Hilbert space* H*;*
2. $(\mathrm{Is}\,(\mathbb{U}), \mathbb{U})$ *where* $\mathrm{Is}\,(\mathbb{U})$ *is the isometry group of the Urysohn space* [2] $\mathbb{U}$ *with the pointwise topology);*
3. $(GL(V), V \setminus \{0\})$ *for every normed space* V *(see* [20]*);*
4. $(GL(V), P_V)$ *for every normed space* V *and its projective space* P_V*.*

It follows in particular by (4) that P_V is $GL(V)$-Tychonoff. This fact was well known among experts and easy to prove using equiuniformities (cf. e.g. Pestov [31]).

By [20, Proposition 2.3], $(GL(V), V \setminus \{0\})$ is the coset space for every normed space V. In order to see that the projective space P_V for every normed space V is the coset space of the group $GL(V)$ observe that the projective space is an open G-quotient of the G-space $\mathbb{S}_V$, the sphere of V. It suffices to show that $\mathbb{S}_V$ is a coset G-space with respect to the action

$$(g, v) \mapsto \frac{g(v)}{||g(v)||}.$$

As in [20, Proposition 2.3] it is easy to see that for every $\varepsilon < \frac{1}{2}$ the set $P_\varepsilon z$ is a nbd of z in $\mathbb{S}_V$, where

$$P_\varepsilon := \{A_{f,y} |\ ||f|| \leq 1,\ ||y|| < \varepsilon\}$$

and $A_{f,y}(x) := x + f(x)y$ for every functional $f \in V^*$ and $y \in V$. Observe that $\|A_{f,y} - I\| = \|f\| \cdot \|y\|$.

Question 8.11. [18] *Is it true that the following (G-Tychonoff) actions are G-normal:* $(U(\ell_2), \ell_2)$, $(\mathrm{Is}\,(\ell_p), \mathbb{S}_{\ell_p}))$, $p > 1$, $(p \neq 2)$*?*

The following concrete example shows that there exist G-spaces X admitting a G-invariant metric (hence X is G-Tychonoff by Fact 4.3.3) such that X is not G-normal.

Example 8.12. [24, page 60] The action of the group $\mathbb{Q}$ of rational numbers on $\mathbb{R}$ by translations is G-Tychonoff but not G-normal.

The idea of this example leads to a generalized version.

[2] For the definition and properties of the Urysohn space see for example [31].

Fact 8.13. [21, Proposition 2.5] *Let G be an arbitrary topological group which is not Raikov complete. Then there exists a normal G-space X (of weight $w(X) = w(G)$) which is not G-normal.*

One can characterize locally compact groups in terms of G-normality.

Fact 8.14. [21, Theorem 5.2] *For every topological group G the following are equivalent:*

(1) *Every normal G-space is G-normal.*
(2) *G is locally compact.*

Recall that X is *weakly G-normal* (see [26, 24, 21]) if every pair of π-disjoint closed *G-invariant* subsets in X can be separated by a function from $C_\pi(X)$.

Question 8.15. *Is every second countable G-space weakly G-normal for the group $G := \mathbb{Q}$ of rational numbers?*

If not, then by [21, Theorem 3.2] one can construct for $G := \mathbb{Q}$ a Tychonoff G-space X which is not G-Tychonoff. That is, it will follow that $\mathbb{Q}$ is not a V-group (resolving the Question [18, Question 2.3]).

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