TODORČEVIĆ' TRICHOTOMY AND A HIERARCHY IN THE CLASS OF TAME DYNAMICAL SYSTEMS

ELI GLASNER AND MICHAEL MEGRELISHVILI

ABSTRACT. Todorčević' trichotomy in the class of separable Rosenthal compacta induces a hierarchy in the class of tame (compact, metrizable) dynamical systems (X,T) according to the topological properties of their enveloping semigroups E(X). More precisely, we define the classes

$Tame_2 \subset Tame_1 \subset Tame$,

where Tame₁ is the proper subclass of tame systems with first countable E(X), and Tame₂ is its proper subclass consisting of systems with hereditarily separable E(X). We study some general properties of these classes and exhibit many examples to illustrate these properties.

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INTRODUCTION

In this work we continue our study of the following theme:

Problem 0.1. Let T be a topological group and (X,T) a compact dynamical system. Let E := E(X,T) be the enveloping semigroup of (X,T). How is the

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topology of the compact space E related to the dynamical properties of the system (X, T) ?

For the background and history of research in this direction see, for example, the survey papers [31], [40]. It is a well known phenomenon that enveloping semigroups of compact metrizable dynamical systems often are non-metrizable huge compacts. Recall that by [45], E(X) is metrizable if and only if (X, T) is hereditarily non-sensitive (HNS); see below the short explanation in Section 0.1. This is a quite restricted class of systems, although it contains the weakly almost periodic (WAP) systems.

A much larger class is that of dynamical systems (X, T) for which E(X) is a Rosenthal compactum. It coincides with the class of tame metrizable dynamical systems. This fact is a consequence of the dynamical analog of the Bourgain– Fremlin–Talagrand dichotomy theorem [37] (see Theorem 0.5 below). Thus tame systems play a principal role when we address the above problem. Many remarkable naturally defined systems coming from geometry, analysis and symbolic dynamics are tame but not HNS.

In the present paper we introduce two subclasses of tame metric dynamical systems.

Definition 0.2. Let (X, T) be a compact metrizable dynamical system and E(X, T) be its (necessarily separable) enveloping semigroup. We say that this system is:

- (1) tame₁ if E(X,T) is first countable;
- (2) tame₂ if E(X,T) is hereditarily separable.

The corresponding classes will be denoted by $Tame_1$ and $Tame_2$ respectively. As is shown in Proposition 0.6 below, we have

$$\text{Tame}_{\mathbf{2}} \subset \text{Tame}_{\mathbf{1}} \subset \text{Tame}.$$

This hierarchy arises naturally from deep results of Todorčević and Argyros– Dodos–Kanellopoulos about separable Rosenthal compacta.

Theorem 0.3 (Todorčević' Trichotomy [61], [4, Section 4.3.7]). Let K be a nonmetrizable separable Rosenthal compactum. Then K satisfies exactly one of the following alternatives:

- (0) K is not first countable (it then contains a copy of $A(\mathbf{c})$, the Alexandroff compactification of a discrete space of size continuum).
- K is first countable but K is not hereditarily separable (it then contains either a copy of D({0,1}^N), the Alexandroff duplicate of the Cantor set, or D(S({0,1}^N)), the extended duplicate of the split Cantor set).
- (2) K is hereditarily separable and non-metrizable (it then contains a copy of the split interval).

By results of R. Pol [57, Section 4, Theorem 3.3], every hereditarily separable Rosenthal compact space is first countable (see Debs [15]). (A topological space K is *perfectly normal* if it is normal and every closed subset of K is a G_{δ} set.)

Theorem 0.4. For a separable Rosenthal compactum K the following conditions are equivalent:

- (1) K is perfectly normal.
- (2) K is hereditarily separable.
- (3) K contains no discrete subspace of cardinality continuum.

We also observe that $A(\mathbf{c})$ is a continuous image of $D(\{0,1\}^{\mathbb{N}})$ (we refer to [15,61] for the definition and discussion of these spaces). Thus a first countable Rosenthal compactum can admit a quotient which is not first countable. We will have a similar situation among our tame classes; namely a tame₁ system can admit a factor which is not tame₁ (Examples 5.7, 5.8 and Lemma 7.6). Also, as we will see (Proposition 4.1), the product of two tame₂ systems need not be tame₂. Note that the class Tame₂ is closed under subsystems and factors. The class Tame₁ is closed under countable products but not under subsystems (see Example 5.9 below).

0.1. **Preliminaries.** Recall that for any topological group T and any dynamical T-system X, defined by a continuous homomorphism $j: T \to \text{Homeo}(X)$ into the group of homeomorphisms of the compact space X, the corresponding enveloping semigroup E(X,T) (or just E(X), E when T and X are understood) was defined by Robert Ellis as the pointwise closure of the subgroup j(T) of Homeo (X) in the product space X^X . One may easily modify this definition for semigroup actions.

E(X,T) is a compact right topological semigroup whose algebraic and topological structure often reflects dynamical properties of (X,T).

Let $\pi: X \to Y$ be a factor map of *T*-systems (i.e., $\pi(tx) = t\pi(x)$ for all $t \in T$ and $x \in X$). Then there exists a (unique) continuous surjective semigroup homomorphism $\pi_*: E(X) \to E(Y)$ such that $\pi \circ p = \pi_*(p) \circ \pi$ for every $p \in E(X)$ and such that $\pi_*(j_X(t)) = j_Y(t)$ for every $t \in T$.

For more information about dynamical systems and their enveloping semigroups we refer to [31] and [40]. Of special interest for us will be the successively larger classes of almost periodic (AP), weakly almost periodic (WAP), hereditarily nonsensitive (HNS), and tame dynamical systems. We include here some information about the class of tame systems.

Tame dynamical systems. A real valued bounded continuous function $f \in C(X)$ is said to be *tame* if the family $fT := \{f \circ t : t \in T\}$ of real valued functions has no *independent* infinite sequence (in the sense of H. Rosenthal [58]); we denote by Tame(X) the collection of these functions.

By [38] f is tame iff for every $p \in E(X,T)$ the function $f \circ p: X \to \mathbb{R}$ has the point of continuity property (PCP in short). The system (X,T) is tame iff Tame(X) = C(X). A metric dynamical system (X,T) is tame iff every element pof the enveloping semigroup E = E(X,T) is a limit of a sequence of elements from T, [37,45], iff p is of Baire class 1. Another equivalent condition is that card E(X)is $\leq 2^{\aleph_0}$ (Theorem 0.5).

Tame dynamical systems were introduced by A. Köhler [50] under the name *regular systems*. The term "tame" was proposed later in [32].

Tame dynamical systems form a class of low complexity dynamical systems where several remarkable dynamical and topological properties meet. This class is quite large and is closed under subsystems, products and factors. By the dynamical analog of the Bourgain–Fremlin–Talagrand theorem, a compact metrizable dynamical system is tame if and only if its (always, compact and separable) enveloping semigroup E(X,T) is a Rosenthal compact space. More precisely we have:

Theorem 0.5 ([37] (A dynamical version of BFT dichotomy)). Let (X,T) be a compact metric dynamical system and let E be its enveloping semigroup. Either

(1) E is a separable Rosenthal compact space (hence E is a Fréchet-Urysohn space and card $E \leq 2^{\aleph_0}$); or

(2) E contains a homeomorphic copy of $\beta \mathbb{N}$ (hence card $E = 2^{2^{\aleph_0}}$).

The first possibility holds if and only if (X,T) is a tame system.

The metrizable tame systems are exactly those systems which admit a representation on a separable Rosenthal Banach space [38] (a Banach space is called *Rosenthal* if it does not contain an isomorphic copy of l_1). An important subclass of the class of Rosenthal Banach spaces is the class of Asplund Banach spaces. (A Banach space V is Asplund iff its dual has the Radon-Nikodým property iff the dual of every separable linear subspace of V is separable. Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund, and every Asplund space is Rosenthal.) We denote by RN the class of *Radon-Nikodým dynamical systems* (see [37]). A (metrizable) dynamical system is said to be RN if it is representable on a (resp., separable) Asplund Banach space.

Any RN system X is hereditarily non-sensitive (HNS) and any HNS system is tame. A metric system is RN iff it is HNS. It was shown in [45] that a metric system (X, T) is RN iff its enveloping semigroup E(X, T) is metrizable.

Any continuous topological group action on a dendrite is tame, [43]. Circularly (in particular, linearly) ordered dynamical *T*-systems are RN, hence tame, [41].

Yet another characterization of tameness, of combinatorial nature via the notion of independence tuples, is due to Kerr and Li [52]. For more information and references about tame systems see [40, 42].

For the definitions of WAP and AP see Remark 1.3. For the definition and basic properties of HNS we refer to [37, 45]. The following inclusions hold in general

 $\mathrm{AP} \subset \mathrm{WAP} \subset \mathrm{HNS} \subset \mathrm{Tame}\,.$

Proposition 0.6. For metrizable systems we have the following inclusions

 $RN = HNS \subset Tame_2 \subset Tame_1 \subset Tame$.

Proof. The dynamical BFT dichotomy (Theorem 0.5) implies that tame₁ and tame₂ systems are tame (hence, E(X,T) is a Rosenthal compactum in both cases). In fact, a tame₁ system is tame because a first countable space is Fréchet-Urysohn. Finally, the above mentioned result of R. Pol (Theorem 0.4) implies that Tame₂ \subset Tame₁.

When a metric system is tame₂ we have the following detailed information. This is a version of a theorem from [37, Proposition 15.1], which in turn is based on results of S. Todorčević ([61, Theorem 3]):

Theorem 0.7. Let X be a metric tame, point transitive T-system, where T is an arbitrary topological group.

Then

- (1) Either E(X,T) contains an uncountable discrete subspace (i.e. it is not $tame_2$) or,
- (2) There exists a metric tame system (Z,T) and a factor map $Z \to X$ such that E(X,T) = E(Z,T) and such that, with $\pi \colon E(X,T) \to Z$, the evaluation map $p \mapsto pz_0$, where $z_0 \in Z$ is any transitive point, we have

$$|\pi^{-1}(z)| \le 2, \quad \forall z \in Z.$$

Proof. Suppose that E(X,T) does not contain an uncountable discrete subspace. Then there exists, by [37, Theorem 15.1] a factor map $\phi: E(X,T) \to Y$ for some metric system (Y,T) such that $|\phi^{-1}(y)| \leq 2$ for every $y \in Y$. Let $x_0 \in X$ and $y_0 \in Y$ be transitive points and set $z_0 = (x_0, y_0)$ and $Z = \overline{T(x_0, y_0)}$. Then the system (Z,T) is metric and tame with $E(X,T) \to Z \to X$, such that E(Z,T) = E(X,T), and the evaluation map $\pi: p \mapsto p(x_0, y_0)$ from $E(X,T) \to Z$ to is at most 2 to 1.

Remark 0.8. Dynamical systems with the group \mathbb{Z} of integers as the acting group are sometimes called *cascades*. Often when dealing with cascades we change our notation and denote our system as (X, T), where here the letter T denotes the homeomorphism of X which corresponds to $1 \in \mathbb{Z}$.

Remark 0.9.

- (1) Note that every linearly ordered compact separable space X is homeomorphic to a special linearly ordered space K_A which can be obtained using a splitting points construction. More precisely, by a result of Ostaszewski (see [55] and its reformulation [54, Result 1.1]) for X there exist: a closed subset $K \subset [0, 1]$ and a subset $A \subset K$ such that $K_A = (K \times \{0\} \cup (A \times \{1\}))$ is endowed with the corresponding lexicographic order inherited from $K \times \{0, 1\}$. Every splitting point is *singular* (see definition in Section 8). Hence, Lemma 8.6 implies that K_A is metrizable if and only A is countable.
- (2) The splitting point construction can be generalized in several directions. Among others for *circularly ordered* compact spaces. In [41, 44] we use such circularly ordered versions of Ostaszewski type spaces K_A (which, as in [41], we denote by Split(K; A)) the space which we get after splitting points of $A \subset K$ in the circularly ordered space K. We use such examples also in the present paper. In particular, in Proposition 4.1 we have the space \mathbb{T}_A , with $A := \{n\alpha : n \in \mathbb{Z}\}$ and the double circle $\mathbb{T}_{\mathbb{T}}$ (see also Corollary 8.3).

Some highlights of the present work.

- In Theorem 2.3 we give a sufficient condition for the tame₁ property for almost one-to-one extensions X → Y of a tame₁ system Y. This implies, in particular, that X is tame₁ for the following symbolic systems: (a) Tribonacci 3-letter substitution system; (b) Arnoux-Rouzy substitutions; (c) Brun substitution; (d) Jacobi-Perron substitution.
- (2) By Theorem 2.5 the Floyd minimal set is $tame_1$.
- (3) By Corollary 6.6 the action of a hyperbolic group Γ on its *Gromov boundary* $\partial \Gamma$ is tame but not tame₁.
- (4) Proposition 4.1 shows that there exists a tame₁ almost automorphic Zsystem which is not tame₂.
- (5) By Theorem 7.1 every linear action $\operatorname{GL}(n,\mathbb{R})\times\mathbb{R}^n\to\mathbb{R}^n$ is tame but, for $n\geq 2$ it is not tame₁.
- (6) As we have shown in [41] circularly ordered *T*-dynamical systems are tame. This implies that many Sturmian like symbolic dynamical systems are tame. Now we can say more: Here we show that Sturmian like systems are tame₂, Corollary 8.3. And every linearly ordered metric system is tame₁, Theorem 8.8.

- (7) By Proposition 8.10, the circularly ordered system $(\mathbb{T}, H_+(\mathbb{T}))$ is tame but not tame₁, where $H_+(\mathbb{T})$ is the Polish topological group of all c-order preserving homeomorphisms of the circle \mathbb{T} . Note also that the "circular analog of Helly's space" $M_+(\mathbb{T}, \mathbb{T})$ (which is a separable Rosenthal compactum) is not first countable (see Remark 8.11).
 - 1. G_{δ} -points in enveloping semigroups E(X,T)

We recall the following well known result.

Lemma 1.1 ([23, 3.1.F]). Let X be a compact space. A point $x_0 \in X$ is a G_{δ} point if and only if there is a countable basis for the topology at x_0 .

The following useful result is a generalization of [11, Proposition 2].

Proposition 1.2. Let X be a set, (Y,d) a metric space, and $E \subset Y^X$ a compact subspace in the pointwise topology. The following conditions are equivalent:

- (1) a point $p \in E$ admits a countable local basis in E;
- (2) there is a countable set $C \subset X$ which determines p in E, meaning that for any given $q \in E$, the condition q(c) = p(c) for all $c \in C$ implies that q(x) = p(x) for every $x \in X$.

Proof. We note that the following collection of subsets forms a basis for the topology at a point $p \in E$:

(1.1)
$$V(x_1, x_2, \dots, x_n; \epsilon) = \{ q \in E : d(px_i, qx_i) < \epsilon; \ 1 \le i \le n \},\$$

with $\{x_1, x_2, \ldots, x_n\}$ a finite subset of X and $\epsilon > 0$.

Assuming that p admits a countable basis for its topology, there is then also a basis for this topology whose elements of the form (1.1), and the union of all those finite sets that appear in this representation clearly forms a countable set $C \subset X$ which determines p.

Conversely, when p is determined by its values on a countable subset $C \subset X$, it is clearly a G_{δ} point of E and thus, by Lemma 1.1 it admits a countable base for its topology.

Remark 1.3.

- (1) Recall that a T-system X is WAP (weakly almost periodic) if and only if every $p \in E(X,T)$, as a function $p: X \to X$, is continuous [22]. Clearly then for a metric WAP system each countable dense subset $D \subset X$ determines every $p \in E(X,T)$.
- (2) A system Y is AP (= equicontinuous) if and only every p ∈ E(X) is a homeomorphism of X. In this case E(X,T) is a compact topological group. If, in addition, the action of T on X is point transitive (i.e., has a dense orbit) then X is a coset space E(X,T)/H for some closed subgroup H of E(X,T). When T is abelian, H is trivial and X is identified with the topological compact group E(X,T). In this case each element p ∈ E(X,T) is completely determined by its value at any single point x of X.

Remark 1.4. Note that if one can choose a countable subset $C \subset X$, as in Proposition 1.2, which is independent of p, then the map $p \mapsto (px)_{x \in C} \in X^C$, is a topological embedding, and it follows that E(X) is metrizable. Thus, in view of [45], the existence of such a set is a sufficient condition for a metric (X, T) to be

a HNS system. (It is also necessary, because cylindrical sets in $E(X) \subset X^X$ form a basis for the topology, so that when E(X) is a metric space, this basis admits a countable subfamily which is also a basis.)

By a theorem of Bourgain [11], in every Rosenthal compactum K the set of G_{δ} -points is dense. As remarked by Debs, since any nonempty G_{δ} subset of K contains a nonempty compact G_{δ} subset, it follows from Bourgain's result that the set of all G_{δ} -points of a Rosenthal compactum is actually non-meager. (In fact, let $D \subset X$ be the collection of G_{δ} -points in X. If D is meager it is contained in a union $\bigcup_{n \in \mathbb{N}} K_n$ where each K_n is closed with empty interior. Now

$$X \setminus \bigcup_{n \in \mathbb{N}} K_n = \bigcap_{n \in \mathbb{N}} K_n^c$$

is a G_{δ} subset of X, hence contains a compact G_{δ} set which, in turn, contains a G_{δ} point by the proof of Bourgain's theorem).

Debs remarks in [15] however that the question whether the set of all G_{δ} -points of a Rosenthal compactum K is comeager in K is still open.

Question 1.5. For a metric tame system (X,T) is the set of all G_{δ} -points of E(X,T) comeager ?

In order to formulate a related question we need some notations. Let I be the unit interval, $\Omega = I^I, \Sigma = \{0, 1\}^I$. Let

$$\Sigma_c = \{ \sigma \in \Sigma : | \operatorname{supp}(\sigma) | \le \aleph_0 \}.$$

For $p \in \Omega$ and $C \subset I$ let

$$\Omega_C(p) = \{ q \in \Omega : q \upharpoonright C = p \upharpoonright C \}.$$

Now, suppose $A \subset \Omega$ is a closed subset. Consider the sets:

$$\mathcal{G} = \{(p, C) : \Omega_C(p) \cap A = \{p\}\} \subset A \times \Sigma_c,$$
$$G = \{p \in A : \exists C \in \Sigma_c, \ \Omega_C(p) \cap A = \{p\}\} = \pi_1(\mathcal{G}),$$

where elements of Σ_c are identified with their supports and π_1 is the projection on Ω . So G is a kind of an 'analytic' set.

Question. Is G a Baire set; i.e. of the form $U \triangle M$ for $U \subset \Omega$ open and $M \subset \Omega$ meager ?

In a private communication Jan van Mill has shown that in general the answer to this question may be negative. However, in our case we have further information concerning the set A that might be relevant. The compact set $A \subset \Omega$ is a separable Rosenthal compactum (so in particular a Fréchet-Urysohn space). Thus by Bourgain's theorem it has a dense (and non-meager) subset of G_{δ} points.

Note that if we know that the set of G_{δ} points in a separable compactum K is a Baire set, then it is comeager.

Question 1.6. Among the class of separable Rosenthal compacta, which members are homeomorphic to E(X,T) (or Adh (X,T), see Section 5) for some system (X,T) (with (X,T) metric, minimal, T abelian, $T = \mathbb{Z}$ etc.) ?

2. When is an almost automorphic system tame?

Almost automorphic dynamical system were first studied by Veech in [62]. Since then they play a central role in various aspects of the theory of dynamical systems.

Definition 2.1. Let $\pi: (X,T) \to (Y,T)$ be a factor map of metric minimal dynamical systems.

(1) π is said to be an almost one-to-one extension if the set

$$X_0 := \{ x \in X : \pi^{-1}(\pi(x)) = \{ x \} \}$$

is a dense G_{δ} subset of X. When π is an almost one-to-one extension we let

$$Y_0 = \pi(X_0) = \{ y \in Y : |\pi^{-1}(y)| = 1 \}.$$

When the set $Y \setminus Y_0$ is countable we say that π is of *type c*, and we say that π is of *type cc* when $X \setminus X_0$ is countable.¹

- (2) (X,T) is almost automorphic if there exists an almost one-to-one extension $\pi: (X,T) \to (Y,T)$ with an equicontinuous system (Y,T); we denote by AA the class of almost automorphic systems.
- (3) We say that an almost automorphic system (X, T) is:
 - (a) of type AA_c when the corresponding set $Y \setminus Y_0$ is countable;
 - (b) of type AA_{cc} when $X \setminus X_0$ is countable.

Thus, we have

$$AA_{cc} \subset AA_c \subset AA$$
.

As was shown in [33, Corollary 5.4.(2)], a minimal metric tame system (X, T) which admits an invariant measure is almost automorphic and moreover, the invariant measure is unique and the almost one-to-one map $\pi: X \to Y$, from X onto its maximal equicontinuous factor, is an *isomorphic extension*, i.e. π is a measure theoretical isomorphism when Y is equipped with its Haar measure. It was shown recently in [26, Corollary 3.7] that a tame AA system is necessarily regular (i.e. the unique invariant measure is supported on the set X_0 of singleton fibers). Of course there are many examples of AA systems which are not tame. For example there are almost automorphic systems having more than one invariant measure, or having positive entropy, and in [28, Theorem 3.11, Corollary 3.13] there are examples of regular AA systems which are not tame.

Proposition 2.2. Let (X,T) be an AA_{cc} system with T abelian. Then E(X,T) is first countable (hence (X,T) is tame₁).

Proof. Note first that for a minimal equicontinuous system (Y,T) with T abelian, an element $p \in E(Y,T)$ is completely determined by its value at any single point yof Y (Remark 1.3 (2)). Now let $\pi: X \to Y$ be the largest equicontinuous factor of (X,T) and let $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$. By assumption the set $C = X \setminus X_0$ is countable. Given $p \in E(X,T)$, consider the set $C_p = \{x \in X : px \in C\}$. This is at most countable, and taking into account the above remark, if we know the restriction of $p \in E(X,T)$ to C_p , we know p. Now apply Proposition 1.2.

¹In [42] we used the term 'strongly almost one-to-one' for the type cc property.

A more general statement is as follows:

Theorem 2.3. Let $\pi: (X,T) \to (Y,T)$ be an almost one-to-one extension of type cc, and let $\pi_*: E(X) \to E(Y)$ be the induced surjective homomorphism. Suppose $p \in E(X)$ satisfies:

(a) E(Y) is first countable at $\pi_*(p)$;

(b) $\pi_*(p)^{-1}(y)$ is at most countable for every $y \in Y$.

Then E(X,T) is first countable at p. Furthermore, if the conditions above (a), (b) are true for every $p \in E(X)$ then X is tame₁.

Proof. In order to prove that E(X) is first countable at p, according to Proposition 1.2, we have to show that there exists a countable subset C(p) which "determines" p. That is, p'(x) = p(x) for every $x \in C(p)$ and $p' \in E(X, T)$ implies p = p'.

For our $p \in E(X)$ choose first a countable subset $C(\pi_*(p))$ of Y which "determines" $\pi_*(p)$ in E(Y) (such a set exists by Proposition 1.2 because E(Y) is first countable at $\pi_*(p)$). Fix a countable subset A of X such that $\pi(A) = C(\pi_*(p))$.

By Definition 2.1 the set $X \setminus X_0$ is countable, where $X_0 := \{x \in X | \pi^{-1}(\pi(x)) = \{x\}\}$. Set $Y_0 := \pi(X_0)$. Define now

$$B := \pi^{-1}(\pi_*(p)^{-1}(Y \setminus Y_0))).$$

Since $Y \setminus Y_0$ is countable then $\pi_*(p)^{-1}(Y \setminus Y_0)$ is countable (use (b)). Since π is of type cc (Definition 2.1(1)) it follows that the preimage $\pi^{-1}(y)$ is countable for every $y \in Y$. Hence, B is also continuous. Now we claim that the countable set

 $C(p) := A \cup B$

determines p. It is enough to verify the following

Claim. Let $p', p \in E(X)$.

(i) If $p'(a) = p(a) \ \forall a \in A$. Then $\pi_*(p') = \pi_*(p)$.

(ii) Assume that $\pi_*(p') = \pi_*(p)$. If $p'(x) = p(x) \ \forall x \in B$ then p' = p.

Proof of (i). Note that $\pi p'(a) = \pi p(a) \quad \forall a \in A$. Therefore, $\pi_*(p')(\pi(a)) = \pi_*(p)(\pi(a)) \quad \forall a \in A$. So, $\pi_*(p')(y) = \pi_*(p)(y) \quad \forall y \in C(\pi_*(p))$. This means that $\pi_*(p') = \pi_*(p)$.

Proof of (ii). it is enough to show that for every $x \notin B$ we have p'(x) = p(x).

So let $x \notin B$. Then $\pi_*(p)(\pi(x)) \in Y_0$. By definition of X_0 there exists unique element $z \in X$ such that $\pi(z) = \pi_*(p)(\pi(x))$. Since $\pi p(x) = \pi_*(p)\pi(x)$ we have p(x) = z. Moreover, since $\pi_*(p') = \pi_*(p)$ we have also p'(x) = z. So, p' = p.

Remark 2.4.

- (1) The dynamical system (X,T) from Theorem 2.3, in particular, is tame (being tame₁, see Proposition 0.6). This strengthens [42, Theorem 6.16]. Similarly, Proposition 2.2, in turn, strengthens a result of Huang [48].
- (2) Proposition 2.2 together with results of Jolivet [49, Theorem 3.1.1], imply that E(X) is first countable (i.e., X is tame₁) for the following symbolic systems: (a) Tribonacci 3-letter substitution system; (b) Arnoux-Rouzy substitutions; (c) Brun substitution; (d) Jacobi-Perron substitution.

A similar argument will show that the Floyd minimal set [24], which is AA_c but not AA_{cc} , is tame₁. For more details on this dynamical system see [7] and [47].

Theorem 2.5. The Floyd minimal set, which is AA_c but not AA_{cc} , is tame₁.

Proof. Let (X,T) be the Floyd minimal system, with $\pi : X \to Y$ the almost one-toone factor map onto its largest equicontinuous factor Y, which is the 3-adic adding machine. In Figure 1 we recall an idea constructing the homeomorphism which generates the Floyd's cascade. As in [7] we consider X as a subset of the square



FIGURE 1. Floyd system. Figure from Richard Gjerde and Orjan Johansen, Mathematica Scandinavica Vol. 85, No. 1 (1999), pp. 87-104; used with permission.

 $[0,1] \times [0,1]$ in \mathbb{R}^2 , Y as a subset of the unit interval [0,1], and the map π as the projection on the first coordinate. We let

$$X_0 = \{ x \in X : \pi^{-1}(\pi(x)) = \{ x \} \}, \quad Y_0 = \pi(X_0).$$

We know that the set $Y_1 = Y \setminus Y_0$ is countable and that for each $y \in Y_1$ the set $A_y = \pi^{-1}(y)$ is a non-degenerate vertical line segment over the point y. It is not hard to see that for each $p \in E(X, T)$, and such line segment A_y , the restriction of p to A_y is an affine map of A_y onto a closed subinterval of A_{py} , if $py \in Y_1$, and into the singleton $\pi^{-1}(py)$ if $py \in Y_0$. As a consequence we see that for $y \in Y_1$, the map $p \upharpoonright A_y$ is determined by the values of p at any two distinct points of A_y . Choose a set $C \subset X_1 = X \setminus X_0$ such that for each $y \in Y_1$ we have that $C \cap A_y$ is a countable set which is dense in A_y . Of course C is a countable set.

Now given $p \in E(X,T)$, let $C_p = \{x \in X : px \in C\}$. We observe that, because p restricted to a non-degenerate vertical segment which is mapped by p onto a non-degenerate vertical segment, is an affine map (hence one-to-one) C_p is a countable set. We will show that p is determined by its values on C_p ; i.e. if $q \in E(X,T)$ is such that $p \upharpoonright C_p = q \upharpoonright C_p$ then p = q. We first note that E(Y,T) = Y is an abelian group and by Remark 1.3.2 any single element of E(Y) is completely determined by its value at any single point y of Y. So, p and q have the same image in E(Y,T). It then follows that px = qx for all $x \in X_0$. So we now assume that $x \in X_1 = X \setminus X_0$.

Case 1. $px \notin C$. Then $\pi^{-1}(\pi(px)) = \{px\}$ and, since $\pi(px) = \pi(qx)$, we have qx = px.

Case 2. $px \in C$; i.e. $x \in C_p$, hence $y = \pi(px) \in Y_1$ and A_y is a non-degenerate interval. Let $z = \pi(x)$, which by assumption is in Y_1 . Thus A_z is a non-degenerate interval containing x.

Case 2a. $pA_z = \{px\} \subset A_y$ is a singleton. It then follows that also $qA_z = \{px\}$, hence qx = px.

Case 2b. $pA_z \subset A_y$ is a non-degenerate interval. In this case $|pA_z \cap C| = \infty$ and there are distinct points $x_1, x_2 \in A_z$ so that px_1 and px_2 are distinct points of $pA_z \cap C$. In particular $x_1, x_2 \in C_p$ and by assumption $qx_1 = px_1$ and $qx_2 = px_2$. As both $p \upharpoonright A_z$ and $q \upharpoonright A_z$ are affine maps, it follows that $p \upharpoonright A_z = q \upharpoonright A_z$ and, in particular qx = px.

Thus we have shown that indeed p = q, and the countable set C_p determines p. Applying Proposition 1.2 we deduce that E(X,T) is first countable, and hence also that (X,T) is tame₁.

Example 2.6. In a beautiful work [5], Aujogue gives a complete description of the enveloping semigroup of a large class of cut and project \mathbb{R}^d systems called *almost canonical model sets* (which play a major role in mathematical models of quasicristals). These are almost automorphic \mathbb{R}^d actions obtained by the cut and project scheme, from a lattice Σ in $\mathbb{R}^n \times \mathbb{R}^d$ with a window $W \subset \mathbb{R}^n$, which is a polytope satisfying certain conditions. In particular Aujogue shows that these systems are tame with a first countable enveloping semigroup. See also [6].

The interested reader will find many more examples of tame and non-tame almost automorphic systems in the recent works [28], [27].

3. Asymptotic extensions

Recall that the *adherence semigroup* Adh (X, T) of a dynamical system (X, T) is the set of accumulation points of T in E(X, T). It is the semigroup $E(X, T) \setminus T$ iff T forms an open dense subset of E(X, T), iff (X, T) is not weakly rigid (see [8] and Section 5 below). An extension $\pi: (X, T) \to (Y, T)$ of minimal systems is called *asymptotic* if $R_{\pi} \subset ASYM$, where

$$R_{\pi} = \{ (x, x') \in X \times X : \pi(x) = \pi(x') \},\$$

and

$$ASYM = \{(x, x') \in X \times X : px = px', \forall p \in Adh(X, T)\}.$$

By [3, Theorem 6.29] an asymptotic extension is irreducible and, when X is metrizable, it is an almost one-to-one extension. If in addition the system (Y, T) is equicontinuous, then (X, T) is almost automorphic. Note that the Floyd system is not an asymptotic extension of its maximal equicontinuous factor (it has, for example, infinitely many fibers of length 1/2).

Theorem 3.1. Let (X,T) be a minimal metric system, with T an abelian group. Suppose further that it is AA_c , and that the extension $\pi: X \to Y$, where Y is the largest equicontinuous factor of X, is asymptotic. Then X is tame₁.

Proof. Let $\pi: (X,T) \to (Y,T)$ be the largest equicontinuous factor of (X,T), so that π is an almost one-to-one extension. By assumption the set

$$Y_m = \{ y \in Y : |\pi^{-1}(y)| > 1 \}$$

is at most countable. Let

$$X_m = \pi^{-1}(Y_m),$$

$$X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\} = X \setminus X_m, \text{ and}$$

$$Y_0 = \pi(X_0) = \{y \in Y : |\pi^{-1}(y)| = 1\} = Y \setminus Y_m.$$

If $X_0 = X$, the system (X, T) is equicontinuous and $E(X, T) \cong (X, T)$ is metrizable. So we now assume that $X_0 \subsetneq X$, in which case it is a dense G_{δ} subset of X. We recall that the map π induces a surjective (dynamical and semigroup) homomorphism $\pi_* \colon E((X,T) \to E(Y,T))$. In our case, as (Y,T) is minimal, metric, equicontinuous and T is abelian, $E(Y,T) \cong (Y,T)$ is a second countable compact abelian topological group. We have $\pi(px) = \pi_*(p)\pi(x)$ ($x \in X, p \in E(X,T)$). For simplicity we will dispense with the symbol π_* and write this as $\pi(px) = p\pi(x)$. Note that then, considering p as an element of the group E(Y,T), it has an inverse and the notation $p^{-1}y$ for $y \in Y$ makes sense. Now given $p \in E(X,T)$ choose $C(p) \subset \pi^{-1}(p^{-1}Y_m)$ such that

(3.1)
$$\begin{aligned} |C(p) \cap \pi^{-1}(y)| &= 1, \quad \text{if } y \in p^{-1}Y_m \cap Y_0 \\ |C(p) \cap \pi^{-1}(y)| &= 2, \quad \text{if } y \in p^{-1}Y_m \setminus Y_0. \end{aligned}$$

We write $C(p) = C_1 \cup C_2$ where the sets C_1, C_2 are determined by the conditions in (3.1).

Suppose now that $q \in E(X,T)$ is such that $q \upharpoonright C(p) = p \upharpoonright C(p)$; we claim that p = q. Let x be any point in X and let $y = \pi(x)$. As C(p) is nonempty our assumption implies that p and q define the same element of the group E(Y,T). Thus if $px \in X_0$ then clearly px = qx. So we now assume that $px \in X_m$, and it then follows that $x \in \pi^{-1}(p^{-1}Y_m)$. If $y \in p^{-1}Y_m \cap Y_0$ then $\pi^{-1}(y) = \{x\}$ and we have qx = px. Otherwise, $y \in p^{-1}Y_m \setminus Y_0$ and $\pi^{-1}(y)$ is not a singleton. We have to consider two cases.

In case $p \in Adh(X,T) = E(X,T) \setminus T$, then the assumption that π is an asymptotic extension implies that $p\pi^{-1}(y) = p(\pi^{-1}x)$ is a singleton, say $\{z\}$, and then $px = z = px_1 = px_2$, where $\{x_1, x_2\} = C_2 \cap \pi^{-1}(y)$, hence $z = px_1 = qx_1 = px_2 = qx_2$. It follows that also $q \in Adh(X,T) = E(X,T) \setminus T$, hence qx = px = z. Finally if $p \in T$ then $qx_1 = px_1 \neq px_2 = qx_2$, therefore also $q \in T$, whence p = q. In view of Proposition 1.2 our proof is complete.

Now the class of dynamical systems satisfying the conditions of Theorem 3.1 is very large. It contains e.g. all the Sturmian like systems and many other semicocycle extensions (see Section 3.1 below). We demonstrate this with the following construction.

Example 3.2. Here is a simple construction of a minimal cascade (X,T) on a Cantor set with a factor map $\pi: X \to Y$ such that $N(y) := \operatorname{card} \pi^{-1}(y) < \omega$ for all $y \in Y$ but $N(\cdot)$ is not bounded.

Start with an adding machine (or any other minimal system on a Cantor set), say (Y,S). Choose a convergent sequence of distinct points $\{y_n\}_{n=1}^{\infty}$ in Y, with $\lim_{n\to\infty} y_n = y_{\infty} \notin \{y_n\}_{n\in\mathbb{N}}$, such that the orbits $\mathcal{O}_S(y_i)$, $i = 1, 2, \ldots, \infty$, are all distinct. Construct by induction a sequence of disjoint clopen neighborhoods $V_n \ni y_n$, $n < \infty$. Choose for each n a nesting sequence of clopen sets $B_{n,k}$, neighborhoods of y_n , with $B_{n,1} \subset V_n$ and $\lim_{k\to\infty} \text{diam } B_{n,k} = 0$. Next define a function $f: Y \to [0, 1]$ as follows. For a fixed $n < \infty$ let $f = \frac{j}{n^2}$ on $B_{n,k} \setminus B_{n,k+1}$ where $0 \le j \le n-1$ and $j \equiv k \pmod{n}$. On the rest of the space put f = 0. Clearly f is continuous on the complement of $\{y_1, y_2, \ldots\}$. In particular f is continuous at y_{∞} .

Let $y_0 \in Y$ be a point of continuity of f outside of $\{y_n\}_{n=1}^{\infty} \cup \{y_\infty\}$, and let X be the closure of the orbit of $(y_0, \{f(S^jy_0)\}_{-\infty < j < \infty})$ in $Y \times [0, 1]^{\mathbb{Z}}$, where on $[0, 1]^{\mathbb{Z}}$ the action is the shift σ , and T is defined as the restriction to X of the product $S \times \sigma$.

It is not hard to verify that $\pi: X \to Y$, the projection on the first coordinate, has the property $\operatorname{card} \pi^{-1}(S^j y_k) = k$, for $k = 1, 2, \ldots, j \in \mathbb{Z}$, and that $\operatorname{card} \pi^{-1}(y) = 1$ for any other point $y \in Y$. As a point transitive almost 1-1 extension of Y, the system (X, T) is minimal.

When we start with (Y, S) which is an adding machine, it is easy to check, that the system (X,T) is tame and that the extension π is asymptotic. We thus obtain a metric tame system (X,T) with E(X,T) first countable.

Similar constructions will yield many other examples of this type with, say, $|\pi^{-1}(y)| = \aleph_0$ or $\pi^{-1}(y) = [0,1]$, for some $y \in Y$; or in fact, given any compact separable space Ω , we can build such an example with $\pi^{-1}(y) = \Omega$ for some $y \in Y$. See [63, Example 5.7] for such constructions.

Remark 3.3. In Section 9, Theorem 9.1, we will show that, e.g. when T is abelian, the condition AA_c suffices to ensure that if (X,T) is a minimal metric tame system, then the unique minimal left ideal M in E(X,T) is a first countable Rosenthal compactum.

3.1. Semi-cocycles. Let T be a discrete countable group. Let (Y,T) be a metric minimal infinite dynamical system and let K be a compact space. Let $C \subset Y$ be a nonempty finite or countable set. Let $F: X \setminus C \to K$ be a continuous map. Such a function is called *a semi-cocycle* in [17], [16]. We refer to [18] and the recent [28] for the theory of semi-cocycles, whose roots can be traced to [29].

We further assume that for each $c \in C$ the function F cannot be extended continuously to c, and that for every $c \in C$ the set $Tc \cap C$ is finite.

We fix a point $y_0 \in X \setminus C$ and set $f(t) = F(ty_0)$, a function in $\Omega = K^T$. Let $x_0 \in Y \times K^T$ be defined by

$$x_0(t) = (y_0, F(ty_0)) = (y_0, f(t)), \quad t \in T.$$

For $t \in T$ let $T_t: Y \times \Omega \to Y \times \Omega$ be defined by

$$T_t(y,\omega) = (ty, S_t\omega),$$

where for $\omega \in \Omega$ the shift homeomorphism $S_t \colon \Omega \to \Omega$ is defined by $S_t \omega(s) = \omega(st)$. Let

$$X = \operatorname{cls} \mathfrak{O}_T(x_0) = \overline{\{T_t x_0 : t \in T\}} \subset Y \times \Omega.$$

We let $\pi: X \to Y$ denote the projection on the Y coordinate. Clearly $\pi: (X, T) \to (Y, T)$ is a homomorphism of dynamical systems. One can also easily see that π is an almost one-to-one extension, with $\pi^{-1}(\pi(x)) = \{x\}$ for every x with $\pi(x) \notin TC$. In fact, for such x and for a sequence t_i such that $\lim t_i y_0 = \pi(x) = y \notin TC$, we have:

$$\lim_{i \to \infty} T_{t_i} x_0 = x, \quad \text{with} \quad x(t) = (y, F(ty)), \quad (t \in T).$$

It then follows that the system (X, T) is minimal and almost automorphic of type AA_c .

Next fix a point $c \in C$ and let t_i be a sequence in T such that $x = \lim T_{t_i} x_0$ with $\pi(x) = c$. We then have

$$\begin{aligned} x(t) &= \lim_{i \to \infty} (T_{t_i} x_0)(t) = \lim_{i \to \infty} (t_i y_0, f(tt_i)) = \lim_{i \to \infty} (t_i y_0, F(tt_i y_0)) = (c, k(t)), \quad (t \in T) \\ \text{with } k(t) &= \lim_{i \to \infty} F(t_i y_0) \in K. \end{aligned}$$

The element k(t) may depend on t but, by our assumption that $Tc \cap C$ is finite, we now see that any two points in $\pi^{-1}(c)$ may differ in at most finitely many coordinates. The same of course holds for $\pi^{-1}(tc)$ for every $t \in T$. Thus the map π is asymptotic, so that the conditions in Theorem 3.1 are fulfilled. We have proved the following:

Proposition 3.4. Let F be a semi-cocycle as above. Then the map π is an asymptotic extension, whence (by Theorem 3.1) the system (X,T) is tame₁.

4. A TAME₁ ALMOST AUTOMORPHIC SYSTEM WHICH IS NOT TAME₂

Proposition 4.1.

- (1) There exists a tame₁ almost automorphic \mathbb{Z} -system which is not tame₂.
- (2) There are two disjoint minimal tame₂ systems whose product (which is minimal and tame₁) is not tame₂.

Proof. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus, and let $\alpha \in \mathbb{R}$ be a fixed irrational number and $T_{\alpha} \colon \mathbb{T} \to \mathbb{T}$ is the rotation by $\alpha, T_{\alpha}\theta = \theta + \alpha \pmod{1}$. We define a topological space $X = \mathbb{T}_A$ and a continuous map $\pi \colon X \to \mathbb{T}$ as follows. Intuitively, \mathbb{T}_A is a circularly ordered space which we get by splitting on the circle \mathbb{T} the points of the orbit $A := \{n\alpha : n \in \mathbb{Z}\}$. Then the map $\pi \colon X \to \mathbb{T}$ is just the natural projection (gluing back the splitted points). For $\theta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$ the preimage $\pi^{-1}(\theta)$ is a singleton x_{θ} . On the other hand for each $n \in \mathbb{Z}$, $\pi^{-1}(n\alpha)$ consists of exactly two points $x_{n\alpha}^-$ and $x_{n\alpha}^+$. For convenience we will use the notation θ^{\pm} , $(\theta \in \mathbb{T})$ for points of X, where $(n\alpha)^- = x_{n\alpha}^-$, $(n\alpha)^+ = x_{n\alpha}^+$ and $\theta^- = \theta^+ = x_\theta$ for $\theta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$. A basis for the topology at a point of the form $x_{\theta}, \ \theta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$, is the collection of sets $\pi^{-1}(\theta - \epsilon, \theta + \epsilon)$, $\epsilon > 0$. For $(n\alpha)^-$ a basis will be the collection of sets of the form $\{(n\alpha)^{-}\} \cup \pi^{-1}(n\alpha - \epsilon, n\alpha)$, where $\epsilon > 0$. Finally for $(n\alpha)^{+}$ a basis will be the collection of sets of the form $\{(n\alpha)^+\} \cup \pi^{-1}(n\alpha, n\alpha + \epsilon)$. It is not hard to check that this defines a compact metrizable zero dimensional topology on X (in fact X is homeomorphic to the Cantor set) with respect to which π is continuous. Next define $T: X \to X$ by the formula $T\theta^{\pm} = (\theta + \alpha)^{\pm}$. It is easy to see that the system (X,T) is minimal and that the natural factor map $\pi: (T,X) \to (R_{\alpha},\mathbb{T})$ is asymptotic; so that the AA system (X, T) is tame.

We now define for each $\gamma \in \mathbb{T}$ two distinct maps $p_{\gamma}^{\pm} \colon X \to X$ by the formulas

$$p_{\gamma}^+(\theta^{\pm}) = (\theta + \gamma)^+, \qquad p_{\gamma}^-(\theta^{\pm}) = (\theta + \gamma)^-.$$

We leave the easy verification of the following claims to the reader.

(1) For every $\gamma \in \mathbb{T}$ and every sequence, $n_i \nearrow \infty$ with

$$\lim_{i \to \infty} n_i \alpha = \gamma, \text{ and } \forall i, \ n_i \alpha < \gamma,$$

we have $\lim_{i\to\infty} T^{n_i} = p_{\gamma}^-$ in E(X,T). An analogous statement holds for p_{γ}^+ .

- (2) $E(X,T) = \{T^n : n \in \mathbb{Z}\} \cup \{p_{\gamma}^{\pm} : \gamma \in \mathbb{T}\}.$
- (3) The subspace $\{T^n : n \in \mathbb{Z}\}$ inherits from E the discrete topology.
- (4) The adherence semigroup

$$Adh(X,T) = E(X,T) \setminus \{T^n : n \in \mathbb{Z}\} = \{p_{\gamma}^{\pm} : \gamma \in \mathbb{T}\}\$$

is homeomorphic to the "two arrows" space of Alexandroff and Urysohn (see [23, page 212], and also to Ellis' "two circles" example [20, Example 5.29]).

It thus follows that E is a separable Rosenthal compactum of cardinality 2^{\aleph_0} .

(5) For each $\gamma \in \mathbb{T}$ the complement of the set $C(p_{\gamma}^{\pm})$ of continuity points of p_{γ}^{\pm} is the countable set $\{\theta^{\pm} : \theta + \gamma = n\alpha, \text{ for some } n \in \mathbb{Z}\}$. In particular, each element of E is of Baire class 1.

Algebraically

$$\operatorname{Adh}(X,T) = \{p_{\gamma}^{\pm} : \gamma \in \mathbb{T}\} = \mathbb{T} \times \{1,-1\}$$

and it can be checked that its topology is separable and first countable. Each of the two components $\mathbb{T} \times \{1\}$ and $\mathbb{T} \times \{-1\}$ is dense, and with the induced topology, it is the Sorgenfrey circle. These components though are highly non-measurable sets. Dynamically $M = \operatorname{Adh}(X, T)$ is the unique minimal left ideal of E(X, T) and thus a minimal system (see Section 9 below).

Next let $\beta \in \mathbb{R}$ be an irrational number such that the pair (α, β) is independent over the rational numbers. We consider the dynamical system (Y,T) which is obtained as above from (\mathbb{T}, R_{β}) . The two systems (X, T) and (Y, T) are then *disjoint*, which means that the product system $(X \times Y, T \times T)$ is minimal. It then follows that also the two minimal systems Adh (X, T) and Adh (Y, T) are disjoint, so that the system

$$\Omega = \operatorname{Adh}(X, T) \times \operatorname{Adh}(X, T) \cong (\mathbb{T} \times \{1, -1\}) \times (\mathbb{T} \times \{1, -1\})$$

is a minimal system.

Given $r \in E(X \times Y, T \times T)$ we let $(p,q) = (p_r, q_r) \in E(X, T) \times E(Y, T)$ be its canonical image in $E(X,T) \times E(Y,T)$. This then defines a homomorphism of Adh $(X \times Y, T \times T)$ into $\Omega = \text{Adh}(X,T) \times \text{Adh}(X,T)$ which is clearly injective. Conversely, given a pair $(p,q) \in \text{Adh}(X,T) \times \text{Adh}(X,T)$ there is, by minimality, an $r \in \text{Adh}(X \times Y, T \times T)$ with $r(\text{Id}_X, \text{Id}_Y) = (p,q)$. Thus our homomorphism is also surjective, i.e. an isomorphism of Adh $(X \times Y, T \times T)$ and $\Omega = \text{Adh}(X,T) \times$ Adh (X,T).

It is now easy to check that the subset

$$\Omega_0 = (\mathbb{T} \times \{1\}) \times (\mathbb{T} \times \{-1\}) \subset \Omega$$

is a copy of the Sorgenfrey torus and that its subset

$$\Delta = \{ (p_{\gamma}^+, p_{-\gamma}^+) : \gamma \in \mathbb{T} \}$$

inherits the discrete topology from Ω .

Thus our system $(X \times Y, T \times T)$ is

- a metric minimal Z-system,
- almost automorphic (over the 2-torus $(\mathbb{T} \times \mathbb{T}, R_{\alpha} \times R_{\beta}))$,
- tame $(E(X \times Y, T \times T)$ is a Rosenthal compactum),
- $E(X \times Y, T \times T)$ is first countable (e.g. by Theorem 3.1),
- and $E(X \times Y, T \times T)$ is not hereditarily separable.

This completes our proof.

Remark 4.2. As we have shown $\operatorname{Adh}(X \times Y)$ is the topological square of the two arrows space. Note that it contains the Alexandroff duplicate $D(\{0,1\}^{\mathbb{N}})$ of the Cantor set. It is equivalent to showing that the square S^2 of the split interval S := S([0,1]) contains $D(\{0,1\}^{\mathbb{N}})$ (here $S := \{p^{\pm} : p \in [0,1]\}$ and $p^- < p^+$). In

order to see this consider the Cantor set $C \subset [0,1]$ and define in S^2 the following subset

$$Y := \{ (p^+, (1-p)^+) : p \in C \} \cup \{ (p^-, (1-p)^+) : p \in C \}.$$

Then Y is homeomorphic to $D(\{0,1\}^{\mathbb{N}})$. The "positive half" $\{(p^+, (1-p)^+) : p \in C\}$ is a discrete subset of Y.

5. Weak rigidity

A dynamical system (X, T) (where we assume that the *T*-action is effective) is said to be *weakly rigid* when Adh (X, T) = E(X, T) or equivalently when there is a **net** $\{t_i\} \subset T \setminus \{e\}$ with $t_i \to \text{Id}$ (pointwise on X). It is called *rigid* when there is a **sequence** $\{t_i\} \subset T \setminus \{e\}$ with $t_i \to \text{Id}$ (pointwise on X). Finally (X, T) is *uniformly rigid* when the convergence of the sequence $\{t_i\}$ to Id is uniform. Note that a system (X, T) is not weakly rigid when and only when T is an open dense subset of E(X, T).

The following claim follows directly from the fact that the enveloping semigroup of a metric tame system is a Fréchet-Urysohn topological space (Theorem 0.5).

Proposition 5.1. A tame system is weakly rigid if and only if it is rigid.

Proposition 5.2. For any dynamical system (X,T), weak rigidity implies that

$$\operatorname{Asym}(X,T) = \Delta.$$

Proof. Suppose (X,T) is weakly rigid and let (x,x') be an asymptotic pair. Let $\{t_i\} \subset T \setminus \{e\}$ be a net such that $t_i \to \text{Id}$. Then

$$x = \lim t_i x, \quad x' = \lim t_i x'$$

and $\lim t_i x = \lim t_i x'$, whence x = x'.

Example 5.3 (See [36], [46] and [2]).

- (1) Every distal system is weakly rigid.
- (2) Every WAP cascade is uniformly rigid.
- (3) There are minimal weakly mixing uniformly rigid cascades.

Example 5.4. The Glasner-Weiss examples (see [37, theorem 11.1]) are uniformly rigid cascades which are HNS hence tame. These systems are metric, not equicontinuous and not minimal. Being HNS their enveloping semigroups are actually metrizable (see [45]).

Example 5.5. In [47] the authors show that their 'Auslander systems' (a family of almost automorphic systems which generalize the Floyd system) is never weakly rigid.

Question 5.6.

- (1) Is there an example of a tame rigid (metric, cascade) system (X, T) which is not HNS (i.e. with non-metrizable enveloping semigroup)?
- (2) Is there a minimal, tame, rigid system (X, T) which is not equicontinuous ?

Example 5.7. In [21] the author shows that the action of $G = GL(d, \mathbb{R})$ on the projective space \mathbb{P}^{d-1} , $d \geq 2$, is tame and that the corresponding enveloping semigroup $E(\mathbb{P}^{d-1}, G)$ is not first countable. He also shows that the group G embeds as an open subset of $E(\mathbb{P}^{d-1}, G)$. Thus this action is not weakly rigid.

The system in the following example is a (two-to-one) extension of the system from Example 5.7. This demonstrates that the class Tame₁ is not closed under factors.

Example 5.8 ([1]). The sphere \mathbb{S}^{d-1} with the projective action of the group $G = \operatorname{GL}(d,\mathbb{R})$ (or, of the projective group $\operatorname{PGL}(d,\mathbb{R})$) is tame₁ but not tame₂.

Example 5.9. Let $G = GL(d, \mathbb{R})$ and $X_1 = \mathbb{S}^{d-1}$, $X_2 = \mathbb{P}^{d-1}$. Set $X = X_1 \cup X_2$, the disjoint sum of G-systems. The dynamical system (X, G) is tame and, as the system (X_2, G) is a factor of (X_1, G) , we have $E(X, G) = E(X_1, G)$. We see that the tame₁ system (X, G) admits a subsystem, namely (X_2, G) , which is not tame₁. Thus the class Tame₁ is not closed under subsystems.

6. On tame dynamical systems which are not tame₁

Let (X,T) be a dynamical system with enveloping semigroup E = E(X,T). Let us call an element $p \in E$ a parabolic idempotent with target x_0 if there is a point $x_0 \in X$ such that $px = x_0, \forall x \in X$, and a loxodromic idempotent with target (x_0, x_1) if there are distinct points $x_0, x_1 \in X$ with $px = x_0, \forall x \in X \setminus \{x_1\}$ and $px_1 = x_1$. We say that x_0 and x_1 are the attracting and repulsing points of prespectively. Clearly, if (X, T) admits a parabolic idempotent, it is necessarily a proximal system and therefore contains a unique minimal set $Z \subseteq X$. If (X, T) is a proximal system then every minimal idempotent is parabolic with target at the minimal set (and of course conversely, every parabolic idempotent whose target is in the minimal subset of X is a minimal idempotent).

Proposition 6.1. Let (X,T) be a proximal dynamical system. Let $Z \subset X$ be its (necessarily unique) minimal subset.

- (1) Suppose that there is an uncountable set $B \subset X$ such that for each $b \in B$ there is a loxodromic idempotent p_b with target (a_b, b) , with b as a repulsing point and $a_b \in Z$ the attracting point such that $b \neq a_b$. Then E(X,T)contains the uncountable discrete subset $\{p_b : b \in B\}$, hence it is not hereditarily separable.
- (2) Suppose there is a point $a \in Z$ and an uncountable set of points $B = \{b_{\nu}\} \subset X \setminus \{a\}$ such that each pair (a, b_{ν}) is the target pair of a loxodromic idempotent $p_{(a,b_{\nu})}$ with attracting point a and a repulsing point b_{ν} . Then the parabolic idempotent p_a defined by $p_a x = a$, $\forall x \in X$, does not admit a countable basis for its topology, hence E(X,T) is not first countable.

Proof. (1) Straightforward.

(2) We repeat Ellis' argument in [21] as follows: Assuming otherwise, in view of Lemma 1.2, there is a countable set $C \subset X$ such that for any $q \in E(X,T)$, if $qc = p_ac$ for every $c \in C$ then $q = p_a$. Now the set B is uncountable and we can choose an element $b_{\nu} \in B \setminus C$. It then follows that for every $c \in C$ we have

$$p_{(a,b_{\nu})}c = p_a c = a_s$$

but nonetheless $p_{(a,b_{\nu})}b_{\nu} = b_{\nu} \neq a = p_a b_{\nu}$. Thus $p_{(a,b_{\nu})} \neq p_a$, a contradiction.

Remark 6.2. The subset $\{p_a\} \cup B$ of X is a topological copy of the Alexandroff compactification $A(\mathbf{c})$, of a discrete space of size continuum. Note that, in view of Theorem 0.3, the existence of a copy of $A(\mathbf{c})$ in E(X) is another way of seeing

that (X, T) is not tame₁ (see additional cases where Proposition 6.1(2) applies, in Theorem 6.5, Theorem 7.1 and Proposition 8.10).

Let Γ be an infinite countable uniform convergence group acting on a compact metric space X with infinite limit set $\Lambda \subset X$. Then the system (Λ, Γ) is minimal and every point of Λ is conical (see [13, Proposition 3.3]).

Definition 6.3. A point $b \in \Lambda$ is a *conical limit point* if there is a wandering net, $\gamma_n \in \Gamma$, such that for all $y \in \Lambda \setminus \{b\}$, the ordered pairs $(\gamma_n b, \gamma_n y)$ lie in a compact subset of the space of distinct pairs of Λ .

Claim 6.4. The following conditions are equivalent for a point $b \in \Lambda$:

- (1) It is a conical point.
- (2) There is a collapsing sequence γ_n and a point $a \in \Lambda$ with $\gamma_n y \to a$ for all $y \in \Lambda \setminus \{b\}$ and $\gamma_n b \to b$.
- (3) There is a point $y \in X, y \neq b$ and there are nets x_n in $X \setminus \{b, y\}$ with $x_n \to b$, and $\gamma_n \in \Gamma$, such that the sequence $\gamma_n(b, y, x_n)$ remains in a compact subset of $\Theta^0(X)$.
- (4) The point b is the repulsing point for a loxodromic projection $p \in E(X, \Gamma)$.

Theorem 6.5. Let Γ be an infinite countable uniform convergence group acting on a compact metric space X with infinite limit set $\Lambda \subset X$. Then the dynamical system (X, Γ) is tame but $E(X, \Gamma)$

- (i) is not hereditarily separable; in fact it contains a subset of cardinality 2^{ℵ₀} which is discrete in the relative topology,
- (ii) is not first countable.

Proof. The fact that (X, Γ) is tame follows directly from the definition of being a divergence group. In [52] the authors show, with a direct simple argument, a stronger statement, namely that it is null.

(i) As every point of X is conical, for each $b \in X$ let p_b be a loxodromic idempotent, say with repulsing point b. It is easy to check that every limit point of the collection $B = \{p_b : b \in X\}$ is parabolic. Thus the topology induced on B from $E(X,\Gamma)$ is discrete (by Proposition 6.1.1).

(ii) By [30, Corollary II, 2.2] every minimal proximal system is weakly mixing; i.e. the product system $X \times X$ is topologically transitive, and X being a metric space, the set of transitive points

$$A = \{(x, y) \in X \times X : \overline{\Gamma(x, y)} = X \times X\}$$

is a dense G_{δ} subset of $X \times X$. We then conclude, by the theorem of Kuratowski and Ulam, that there is a residual subset D of X such that for every $x \in D$ the set $A_x = \{y \in X : (x, y) \in A\}$ is residual in X. In particular, A_x is of cardinality 2^{\aleph_0} .

Fix a point $(x_0, y_0) \in A$ and let (z, w) be an arbitrary point in $X \times X$ with $z \neq w$. There is the a sequence $\gamma_n \in \Gamma$ with $\gamma_n(x_0, y_0) \to (z, w)$. We can assume that γ_n is a collapsing sequence and then we have a pair $(a, b) \in X \times X$ such that $\gamma_n x \to a$ for every $x \in X \setminus \{b\}$ and $\gamma_n b \to b$. Let $p_{(a,b)}$ denote the corresponding idempotent in $E(X, \Gamma)$. Thus $p_{(a,b)}x = a$ for every $x \in X \setminus \{b\}$ and $p_{(a,b)}b = b$.

It then follows that either $z = \lim \gamma_n x_0 = a$ and $w = \lim \gamma_n = b$, or $z = \lim \gamma_n x_0 = b$ and $w = \lim \gamma_n = a$. Of course, by replacing γ_n by γ_n^{-1} , we can assume with no loss of generality that the first case occurs.

As a consequence of this discussion we conclude that:

For every $a \in D$ there is an a residual set of points $B_a = \{b_\nu\} \subset X \setminus \{a\}$ such that each pair (a, b_ν) is the target pair of a loxodromic idempotent $p_{(a,b_\nu)}$ with attracting point a and a repulsing point b_ν .

Now apply Proposition 6.1 (2).

A Gromov hyperbolic group acting on its boundary is an example of a group satisfying the conditions of Theorem 6.5. In fact, it was shown by Bowditch [14, Theorem 0.1] that essentially these are the only examples: If an infinite countable group Γ is acting as a uniform convergence group on a compact perfect metric space X, then Γ is hyperbolic and the dynamical system (X, Γ) is isomorphic to the action of Γ on its Gromov boundary.

Corollary 6.6. The action of a hyperbolic group Γ on its Gromov boundary $\partial\Gamma$ is tame but not tame₁.

Example 6.7 (Dynkin and Maljutov [19]). The free group F_2 on two generators, say a and b, is hyperbolic and its boundary can be identified with the compact metric space Ω (a Cantor set) of all the one-sided infinite reduced words w on the symbols a, b, a^{-1}, b^{-1} . The group action is

$$F_2 \times \Omega \to \Omega, \quad (\gamma, w) = \gamma \cdot w,$$

where $\gamma \cdot w$ is obtained by concatenation of γ (written in its reduced form) and w and then performing the needed cancelations (see [19]). The resulting dynamical system is tame (in fact, null), and the enveloping semigroup $E(\Omega, F_2)$ is Fréchet-Urysohn but not first countable.

Question 6.8. Suppose T is abelian (or even more specifically that $T = \mathbb{Z}$). Is there a tame metric (minimal) system (X, T) such that E(X, T) is not first countable ?

7. LINEAR ACTIONS ON EUCLIDEAN SPACES

For many important actions $T \times X \to X$ the phase space is not compact. One may still study the tameness of this action via *T*-compactifications of *X*. It is especially natural whenever *X* is locally compact. Since *T*-factors preserve the class Tame (but not Tame₁) the question can be reduced to the case of the onepoint *T*-compactification of *X*.

Theorem 7.1. Every linear action $GL(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ is tame but, for $n \ge 2$ it is not tame₁.

Proof. Let $X_n := \mathbb{R}^n \cup \{\infty\}$ be the one point compactification of \mathbb{R}^n and $G \times X_n \to X_n$, with $G = \operatorname{GL}(n, \mathbb{R})$, be the canonical extension. We have to show that X_n is a tame G-system. For $p \in E(X, G)$ let

$$V_p := p^{-1}(\mathbb{R}^n) = \{ v \in \mathbb{R}^n : p(v) \in \mathbb{R}^n \} = \{ v \in X : p(v) \neq \infty \}.$$

Claim. The subset V_p is a linear subspace (hence, closed) in \mathbb{R}^n and the restriction $p \upharpoonright V_p \colon V_p \to \mathbb{R}^n$ is a (continuous) linear map.

Indeed, let g_i be a net in G such that $\lim g_i = p$ in E. If $u, v \in V_p$ then $\lim g_i u = p(u) \in \mathbb{R}^n$ and $\lim g_i v = p(v) \in \mathbb{R}^n$. Then by the linearity of maps g_i we obtain $\lim g_i(c_1u + c_2v) = c_1p(u) + c_2p(v) \in \mathbb{R}^n$. Since V_p is finite dimensional, the linear map $p \upharpoonright V_p \colon V_p \to \mathbb{R}^n$ is necessarily continuous. \Box

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 \Box

Using this claim we obtain that the cardinality of $E(X_n)$ is not greater than 2^{\aleph_0} . Now the tameness follows by the dynamical BFT dichotomy (Theorem 0.5).

To see that (X_n, G) is not tame₁ we first observe that the following lemma holds. The proof is by a verification of the conditions in Proposition 6.1 (2).

Lemma 7.2. Let $S < GL(2, \mathbb{R})$ be the subgroup

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, \ b \in \mathbb{R} \right\}.$$

Then the action of S on the space

$$Y_1 = \left\{ \left(\begin{array}{c} t \\ 1 \end{array} \right) : \ t \in \mathbb{R} \right\} \cup \{\infty\} \subset \mathbb{R}^2 \cup \{\infty\}$$

is tame but not $tame_1$.

Denoting, for each $s \in \mathbb{R}$

$$Y_s = \left\{ \left(\begin{array}{c} t \\ s \end{array} \right) : \ t \in \mathbb{R} \right\} \cup \{\infty\} \subset \mathbb{R}^2 \cup \{\infty\},$$

we have $X_2 = \bigsqcup_{s \in \mathbb{R}} Y_s$ and it follows that $E(X_2, S) = E(Y_1, S)$.

Next observe that for any $n \ge 2$ the dynamical system (X_2, S) is a subsystem of the system (X_n, S) and we now consider S as a subgroup of $GL(n, \mathbb{R})$, embedded at the left top corner:

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} A & \mathbf{0}_{2,n-2} \\ \mathbf{0}_{n-2,2} & \mathbf{I}_{n-2,n-2} \end{pmatrix}.$$

Moreover, as under this embedding the action of S on X_n has the property that it acts as the identity on the last n-2 coordinates of a vector in \mathbb{R}^n , it follows that $E(X_n, S) = E(X_2, S) = E(Y, S)$. Now, since $E(X_n, S) \subset E(X_n, \operatorname{GL}(n, \mathbb{R}))$ we conclude that $E(X_n, \operatorname{GL}(n, \mathbb{R}))$ is also not tame₁.

The following lemma presents another way of seeing that the action of $GL(2,\mathbb{R})$ on $X_2 = \mathbb{R}^2 \cup \{\infty\}$ is not tame₁.

Lemma 7.3. The action $GL(2, \mathbb{R}) \times X_2 \to X_2$ is not tame₁.

Proof. By Proposition 1.2 it is enough to show that there exists $p \in E(X_2)$ such that countable subsets C(p) of X_2 cannot determine p. Take the following ∞ -idempotent

$$p = p_{\infty} \colon X_2 \to X_2, \ p(x) = \infty \ \forall x \neq 0_2 \ \text{and} \ p(0_2) = 0_2.$$

It really belongs to $E(X_2)$ because if $t_n \in \operatorname{GL}(2, \mathbb{R})$ is the scalar matrix with $t_n(x) = nx$ for every $x \in \mathbb{R}^2$, then $\lim t_n = p$ in $E(X_2)$. Let, as before, $V_p := p^{-1}(\mathbb{R}^2)$. Then for such p we have $V_p = \{0_2\}$. Assume in contrary that C(p) is a countable subset of X_2 which determines p. We can suppose that C(p) does not contain 0_2 (and also ∞). There exists a one-dimensional subspace L in \mathbb{R}^2 which does not meet C(p). Moreover, there exists $q_L \in E(X_2)$ such that $q_L^{-1}(\mathbb{R}^2) = L$ and $q_L(x) = x$ for every $x \in L$. Up to a conjugation, we can suppose that

$$L = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}, \quad g_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, \quad q_L = \lim g_n.$$

Then $p(x) = q_L(x) = \infty$ for every $x \in C(p)$ (but clearly, $p \neq q_L$).

Remark 7.4. The enveloping semigroup $E(X_n)$ of the action from Theorem 7.1 can be identified with the semigroup of all partial linear endomorphisms of \mathbb{R}^n . To see this observe that the claim from the proof of Theorem 7.1 can be reversed. Namely, every partial linear endomorphism $f: V \to W$ of \mathbb{R}^n defines an element $p \in E(X_n)$ such that

$$V = p^{-1}(\mathbb{R}^n), \ p(V) = W, \ p(x) = f(x) \ \forall x \in V, \ p(y) = \infty \ \forall y \notin V.$$

Moreover, that assignment is a semigroup isomorphism: (partial) composition corresponds to the product of suitable elements from the enveloping semigroup.

Remark 7.5. Theorem 0.3 predicts that $E(X_2)$ in Lemma 7.3 should contain the Alexandroff compactification $A(\mathbf{c})$, of a discrete space of size continuum. The following subset of $E(X_2)$ is indeed a topological copy of $A(\mathbf{c})$

 $\{p_{\infty}\} \cup \{q_L : L \text{ is a one-dimensional linear subspace in } \mathbb{R}^2\}.$

Lemma 7.6. The action of S on the space

$$Y_1^* = \left\{ \left(\begin{array}{c} t\\ 1 \end{array} \right) : \ t \in \mathbb{R} \right\} \cup \{\pm \infty\}$$

is tame₁.

Proof. Clearly every element $p \in \operatorname{Adh}(Y_1^*, S)$ fixes the points $\pm \infty$. Suppose now that $p \in \operatorname{Adh}(Y_1^*, S)$ satisfies $p \begin{pmatrix} s \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ 1 \end{pmatrix}$ for some $s \in \mathbb{R}$. It is then easy to check that it has either the form p_{0s}^o or p_{0s}^i , where for $s \in \mathbb{R}$,

$$p_{0s}^{o} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{cases} \infty & t > s \\ \begin{pmatrix} s \\ 1 \end{pmatrix} & t = s \\ -\infty & t < s, \end{cases} \qquad p_{s}^{i} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ 1 \end{pmatrix} \text{ for all } t \in \mathbb{R}.$$

Applying $g \in S$ with $g \begin{pmatrix} s \\ 1 \end{pmatrix} = \begin{pmatrix} r \\ 1 \end{pmatrix}$ we get the elements $gp_{0s}^o = p_{rs}^o$, where

$$p_{rs}^{o} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{cases} \infty & t > s \\ \begin{pmatrix} r \\ 1 \end{pmatrix} & t = s \\ -\infty & t < s, \end{cases}$$

and $gp_s^i = p_r^i$.

To complete the list of all the elements of $Adh(Y_1^*, S)$ we have to add the limits:

$$p_{\infty} = \lim_{s \to -\infty} p_{0s}^{o}, \qquad p_{\infty} \begin{pmatrix} t \\ 1 \end{pmatrix} = \infty \quad \text{for all } t \in \mathbb{R},$$

and

$$p_{-\infty} = \lim_{s \to \infty} p_{0s}^o, \qquad p_{-\infty} \begin{pmatrix} t \\ 1 \end{pmatrix} = -\infty \quad \text{for all } t \in \mathbb{R},$$

1

and finally

$$p_{\pm\infty,s} = \lim_{r \to \pm\infty} p_{rs}^o, \qquad p_{\pm\infty} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{cases} \infty & t > s \\ \pm \infty & t = s \\ -\infty & t < s. \end{cases}$$



FIGURE 2. A map of $E(Y_1^*, S)$.

Now clearly each p_{rs}^o , p_s^i , $r, s \in \mathbb{R}$, is determined by its value at three points, while the elements $p_{\pm\infty}$ and $p_{\pm\infty,\infty}$ are determined by their values on the set

$$\left\{ \binom{n}{1} : n \in \mathbb{Z} \right\}.$$

Now apply Proposition 1.2.

As in Examples 5.7 and 5.8 we observe here a situation where a tame₁ system (Y_1^*, S) admits the system (Y_1, S) as a factor, and the latter has an enveloping semigroup which is not first countable.

8. Order preserving actions

A standard example of a circularly ordered space is the circle \mathbb{T} . An abstract circular (some authors prefer the name, *cyclical*) order R on a set X can be defined as a certain ternary relation. Intuitively, it is a linear (i.e., total) order which has been bent into a "circle". For formal classical definitions and related new dynamical topics see [41, 44].

Let (X, T) be a compact dynamical system. If X is a circularly ordered (cordered) space, compact in its interval topology, and every t-translation $X \to X$ is circular ordered preserving (COP) then we say that (X, T) is a circularly ordered system.

The class of *linearly ordered dynamical systems* (LOP) is defined similarly. Note that every linearly ordered **compact** T-system is a circularly ordered T-system. A prototypic example of a circularly ordered system is the circle \mathbb{T} equipped with the action of the group $H_+(\mathbb{T})$ (or, some of its subgroup) on \mathbb{T} . Several important Sturmian like symbolic dynamical systems are circularly ordered dynamical systems, hence tame. This is so because every circularly ordered system is tame. See the details in [41].

Let X, Y be linearly (circularly) ordered sets. Below we denote by $M_+(X, Y)$ the set of linear (circular) order preserving maps from X into Y. Similarly, by $H_+(X)$ we denote the group of all LOP (resp., COP) homeomorphisms $X \to X$. When X is compact, $H_+(X)$ is a topological group carrying the usual compact open topology.

Our results show that there are naturally defined circularly ordered tame systems of all three types. Moreover, the (pointwise compact) space $M_+(X, Y)$ is first countable for every compact metrizable linearly ordered spaces X, Y (Theorem 8.7). This result is a generalized version of the Helly theorem (for X = Y = [0, 1]).

This fact implies that every linearly ordered metric dynamical T-system is tame₁ (Theorem 8.8). In contrast, the circular version of the Helly space $M_+(\mathbb{T},\mathbb{T})$ is not first countable (Remark 8.11). This result demonstrates once more the relative complexity of circular orders when compared with linear orders.

Orderable enveloping semigroups and the Tame₂ class. By [53] in linearly ordered topological spaces (LOTS) the separability is hereditary. That is, in a separable LOTS every topological subspace is separable. This can be easily extended to the case of c-ordered spaces, using the *covering space*, [41] (which is linearly ordered).

Lemma 8.1. Let R be a circular order on X such that X in its interval topology is separable. Then every topological subspace of X is separable.

Proof. By [41, Prop. 2.10] for every $c \in X$ there exists a LOTS $X(c) := ([c^-, c^+], \leq_c)$ such that X(c) we get from X splitting the point z into two new points z^-, z^+ . The natural projection $q: [c^-, c^+] \to X$ (identifying the endpoints of X(c)) is a c-order preserving quotient map. Here, the linear order \leq_c on X(c) is defined by

 $z^- \leq_z a \leq_z z^+$, $a <_z b \Leftrightarrow [z, a, b] \quad \forall a \neq b \neq z \neq a$,

where [z, a, b] indicates the fact that $(z, a, b) \in R$ (the triple is positively oriented).

Moreover, it is easy to see that the restriction of q on every interval $[a, b]_{\leq_z}$ onto $[a, b]_{\circ}$ is an isomorphism of linearly ordered sets, where $c^- \neq a$ or $c^+ \neq b$. The topologies on X and X(c) are both interval topologies. It follows that if D is a topologically dense subset of X then the same set D is dense in $[c^-, c^+]$. Hence, if the c-ordered space X is separable then the LOTS X(c) is also separable. Since X(c) is hereditarily separable and $q: [c^-, c^+] \to X$ is continuous onto, we obtain that X is also hereditarily separable.

In particular, if the enveloping semigroup is c-ordered then its separability is hereditary. In view of our Definition 0.2 this leads to the following sufficient condition.

Theorem 8.2. If the enveloping semigroup E(X), as a compact space, is circularly ordered then the original metrizable dynamical system X is tame₂.

Corollary 8.3. The Sturmian like cascades $\text{Split}(\mathbb{T}, R_{\alpha}; A)$ are tame₂ (but not *HNS*).

Proof. We can apply Theorem 8.2 because for these systems the enveloping semigroup E becomes (by [41, Cor. 6.5]) a circularly ordered cascade, where $E = \mathbb{T}_{\mathbb{T}} \cup \mathbb{Z}$ is a c-ordered compact (non-metrizable) subset of the c-ordered lexicographic order $\mathbb{T} \times \{-, 0, +\}$.

Question 8.4. For which c-ordered systems the enveloping semigroup is c-ordered (at least as a compact space) ?

Generalized Helly theorem and Tame₁ class. It is a well known fact in classical analysis that every order preserving bounded real valued function $f: X \to \mathbb{R}$ on an interval $X \subseteq \mathbb{R}$ has one-sided limits. This can be extended to the case of linear order preserving functions $f: X \to Y$ such that X is first countable and Y is sequentially compact; see [25]. The following lemma is easy to verify.

Lemma 8.5. Let $p: X \to Y$ be LOP where X and Y are separable metrizable LOTS (linearly ordered topological spaces) and Y is compact. Then

- (1) p has one-sided limits $p(a^-), p(a^+)$ at any $a \in X$ and $p(a) \in [p(a^-), p(a^+)]$.
- (2) p is continuous at a if and only if $p(a^-) = p(a^+)$ if and only if $p(a^-) = p(a^+) = p(a)$.
- (3) p has at most countably many discontinuity points.

Let (X, \leq) be a linearly ordered set. Let us say that $u \in X$ is a singular point and write $u \in sing(X)$, if there is a maximal element below u or a minimal element above it. It is equivalent to saying that for every $v \neq u$, either [u, v) or (v, u] is a clopen set.

Lemma 8.6. Let (X, \leq) be a linearly ordered compact metric space. Then $sing(X, \leq)$ is at most countable.

Proof. For every $u \in sing(X)$ choose exactly one clopen (nonempty) set [u, v) or (v, u], where $v \neq u$. This assignment defines a 1-1 map from sing(X) into the set clop(X) of all clopen subsets of X. Now observe that clop(X) is countable for every compact metric space (take a countable basis \mathcal{B} of open subsets; then every clopen subset is a finite union of some members of \mathcal{B}).

Recall that the Helly space $M_+([0, 1], [0, 1])$ is first countable (see for example, [60, page 127]). The following theorem was inspired by this classical fact.

Theorem 8.7 (Generalized Helly space). Let X and Y be linearly ordered sets with their interval topologies such that X and Y are compact metric spaces. Then the (compact) set $M_+(X,Y)$ of all LOP maps is first countable in the pointwise topology.

Proof. Let $p \in M_+(X, Y)$. Lemma 8.5 guarantees that p has at most countably many discontinuities. Denote this set by disc(p). Since X is compact and metrizable, there exists a countable dense subset A in X. By Lemma 8.6 the set sing(X) is countable. Consider the following countable set

$$C := disc(p) \cup sing(X) \cup A.$$

It is enough to show that C satisfies the condition of Lemma 1.2 with $E := M_+(X,Y)$. So, let $q \in M_+(X,Y)$ be such that $q(c) = p(c) \quad \forall x \in C$. We have to show that $q(c) = p(c) \quad \forall x \in X$.

Assuming the contrary, let $q(x_0) \neq p(x_0)$ for some $x_0 \in X$. Then, clearly, $x_0 \in cont(p)$. Choose $y_1, y_2 \in Y$ such that $p(x_0) \in (y_1, y_2)$ and

$$q(x_0) \notin (y_1, y_2)$$

There exists an open neighborhood O of x_0 in X such that $p(O) \subset (y_1, y_2)$. Since open intervals form a topological basis of the topology of a circular order, we can suppose that O is an open interval (x_1, x_2) containing x_0 . By our choice, x_0 is not singular. Therefore, (x_1, x_0) and (x_0, x_2) are nonempty. Since A is dense in X, there exist $a_1, a_2 \in A$ such that $a_1 \in (x_1, x_0)$ and $a_2 \in (x_0, x_2)$. Then $a_1 < x_0 < a_2$. So, $q(a_1) \le q(x_0) \le q(a_2)$. Since $q(a_1) = p(a_1), q(a_2) = p(a_2)$ and $p(a_1), p(a_2) \in (y_1, y_2)$ it follows that $y_1 < q(x_0) < y_2$. We get

$$q(x_0) \in (y_1, y_2).$$

This contradiction completes the proof.

Theorem 8.8. Every linearly ordered compact metric dynamical system is tame₁.

Proof. Let X be a compact metrizable linearly ordered dynamical system. Every element $p \in E(X)$ is a LOP selfmap $X \to X$, because $M_+(X,X)$ is pointwise closed. So, E is a subspace of $M_+(X,X)$ which is first countable by Theorem 8.7.

Example 8.9. Consider the linearly ordered system $([0, 1], H_+([0, 1]))$. The enveloping semigroup of this c-order preserving system is a (compact) subspace of the Helly space which is first countable. So, this system is tame₁. It is not tame₂. In fact, it is (like the Helly space) not hereditarily separable. There exists a discrete uncountable subspace in the enveloping semigroup. For each $z \in [0, 1]$ consider the functions

$$f_z : [0,1] \to [0,1], \quad f_z = \begin{cases} 0 & \text{for } x < z \\ \frac{1}{2} & \text{for } x = z \\ 0 & \text{for } x > z. \end{cases}$$

Then $\{f_z : z \in (0,1)\}$ is an uncountable discrete subset of E(X).

Recall again that every separable linearly

Circularly ordered systems, in general, are not tame₁.

Proposition 8.10. Let $H_+(\mathbb{T})$ be the Polish topological group of all c-order preserving homeomorphisms of the circle \mathbb{T} . The minimal circularly ordered dynamical system $(\mathbb{T}, H_+(\mathbb{T}))$ is tame but not tame₁.

Proof. This system is tame being a circularly ordered system, [41]. Let us show that the enveloping semigroup of this c-order preserving system is not first countable. Choose any point $a \in \mathbb{T}$. For every $b \neq a$ in \mathbb{T} the pair (a, b) is the target pair of a loxodromic idempotent $p = p_{(a,b)}$ with attracting point a and a repulsing point b.



FIGURE 3. loxodromic idempotent p

Then, according to Proposition 6.1.2, the parabolic idempotent p_a defined by $p_a x = a$, $\forall x \in \mathbb{T}$, does not admit a countable basis for its topology. This example, in particular, shows that a *null system* need not be tame₁.

Remark 8.11. Theorems 8.7 and 8.8 cannot be extended, in general, to circular orders. Indeed, the "circular analog of Helly's space" $M_+(\mathbb{T},\mathbb{T})$ (which is a separable Rosenthal compactum) is not first countable. Also its subspace, the enveloping semigroup $E(\mathbb{T})$ of the circularly ordered system $(\mathbb{T}, H_+(\mathbb{T}))$ from Example 8.10, is not first countable.

Remark 8.12. Let $\mathbb{Q}_0 = \mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$ be the circled rationals with the discrete topology and $G = \operatorname{Aut}(\mathbb{Q}_0)$ the automorphism group (in its pointwise topology) of the circularly ordered set \mathbb{Q}_0 . In [44] we have shown that the universal minimal *G*-system of the Polish topological group *G* can be identified with the circularly ordered compact metric space $\operatorname{Split}(\mathbb{T}; \mathbb{Q}_\circ)$, which is constructed by splitting the points of \mathbb{Q}_0 in \mathbb{T} . In a similar way to the proof of Proposition 8.10 one shows that the *G*-system M(G) is tame but not tame₁.

In view of Examples 5.7, 5.8, Lemmas 7.3 and 7.6, and the examples in the present section, we pose the following.

Question 8.13. Let (X,T) be a metric, point transitive, tame system which is not tame₁. Is there always an extension $\pi: (X^*, Y) \to (X,T)$ such that (X^*,T) is metric, point transitive and tame₁? Is there such an extension π which is also at most two-to-one?

It is though worth mentioning that all of these examples are connected to ordered structures, which may be the secrete behind the existence of the tame₁ extension.

9. MINIMAL LEFT IDEALS

According to Ellis' theory every enveloping semigroups E(X,T) of a dynamical system (X,T) contains at least one minimal left ideal. These minimal left ideals are the same as the minimal subsets of the dynamical system (E(X,T),T). A minimal left ideal M has the following form: The set J of idempotents in M is nonempty. We choose an arbitrary idempotent $u \in J$ and set G = uM. Then G is a group, $M = J \cdot G = \bigcup_{v \in J} vG$, and the representation of an element $p \in M$ as $p = v\alpha$, with $v \in J$ and $\alpha \in G$ is unique. For each $\alpha \in G$, the map $R_{\alpha} : p \mapsto p\alpha$ is a homeomorphism of M onto itself which commutes with the (left) action of T on M, so that R_{α} is an automorphism of the dynamical system (M,T). Moreover, the map $\alpha \mapsto R_{\alpha}$ is a group isomorphism between G and Aut(M,T).

A minimal system (X,T) admits a single left minimal ideal if and only if the proximal relation $P \subset X \times X$ is an equivalence relation, but in general even a minimal metrizable system can admit 2^c minimal left ideals (see [34]). It may happen that the proximal relation P on a minimal system (X,T) is an equivalence relation but is not closed (see [59]). The relation P is closed if and only if E(X,T)has a unique minimal ideal M and J is a closed subset of M.

Recall that a metric minimal tame dynamical system (X,T) which admits an invariant measure is almost automorphic [33], and in particular its proximal relation is a closed equivalence relation. Thus the enveloping semigroup of such a system always admits a unique minimal left ideal M and its subset of idempotents J is a closed subset of M. In fact, in many such systems it may happen that this unique minimal left ideal coincides with the adherence semigroup A(X,T).

Let (X,T) be a metric minimal tame system admitting a *T*-invariant probability measure. Then the system (X,T) is almost automorphic and the extension

 $\pi: X \to Y$, where Y is the largest equicontinuous factor of X, is an almost one-to-one map. Let, as before

$$X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}, \quad Y_0 = \pi(X_0) = \{y \in Y : |\pi^{-1}(y)| = 1\}.$$

Proposition 9.1. Let (X,T) be a metric minimal tame system admitting a *T*-invariant probability measure. Then the enveloping semigroup E(X,T) contains a unique minimal left ideal $M \subset E(X,T)$, and the space M is a separable Rosenthal compactum. If $Y \setminus Y_0$ is countable, i.e. when (X,T) is AA_c , then M is first countable.

Proof. It is shown in [30] that the minimal system (X, T) is almost automorphic; i.e. it is an almost one-to-one extension of its largest equicontinuous factor (Y, T). It then follows that the proximal relation on X is a closed equivalence relation and thus E(X, T) admits a unique minimal left ideal M and its set of idempotents J is closed. Being a closed subset of a Rosenthal compactum M is also a Rosenthal compactum. Each T-orbit in M is dense and thus M is separable.

The dynamical system (Y,T) is minimal and equicontinuous and therefore has the form Y = K/L where K is a compact second countable topological group and $L \leq K$ is a closed subgroup. The enveloping semigroup of the system (Y,T) = (K/L,T) is the system (K,T) and it follows that there is a homomorphism

$$\tilde{\pi} \colon (M,T) \to (K,T),$$

which is a proximal extension. it also follows that the restriction of $\tilde{\pi}$ to G is a group isomorphism

$$\tilde{\pi} \upharpoonright G \colon G \to K.$$

We also have $\tilde{\pi}^{-1}(e) = J$, where *e* is the identity element of *K* and it follows that *J* is a closed G_{δ} subset of *M*.

We observe that it suffices to show that each $v \in J$ is a G_{δ} point of J. In fact, as $\{R_{\alpha} : \alpha \in G\} \cong \operatorname{Aut}(M,T)$ it will follow that every $p = v\alpha \in vG$ is also a G_{δ} point of M.

Now it follows from our assumption the $Y \setminus Y_0$ is countable that in fact J is metrizable. For each $y \in Y \setminus Y_0$ we choose a point $x_y \in \pi^{-1}(y)$ and define the evaluation map

$$eva_u: v \mapsto vx_u, \quad J \to \pi^{-1}(y).$$

Now the countable collection of continuous maps $\{eva_y\}_{y \in Y \setminus Y_0}$ separates points on J and our claim follows. This completes the proof of the proposition. \Box

Remark 9.2. If in the above proposition we replace the condition that $Y \setminus Y_0$ be countable by the assumption that the set J of minimal idempotents is metrizable, we obtain the same result.

Example 9.3.

- (1) In the case of a hyperbolic group acting on its Gromov boundary the unique minimal ideal is isomorphic to the action on the boundary, hence actually metrizable (see Corollary 6.6).
- (2) The Sturmian system is an example where Adh (X, T) is minimal and hence it coincides with the unique minimal left ideal. Topologically it is the split circle (see [37, Example 14.10]). In this example |J| = 2.

- (3) For any system which satisfies the assumptions of Theorem 3.1 we have that Adh(X,T) is minimal and hence coincides with the unique minimal left ideal.
- (4) For the Floyd system (X,T) we have

$$E(X,T) \supseteq \operatorname{Adh}(X,T) \supseteq M$$

where M is the unique minimal left ideal in E(X,T) (see [47]).

(5) In the example [42, Example 8.43] we see that Adh (X, T) = E(X, T) \Z = M is the unique minimal left ideal of E(X, T). Algebraically it is the product set T^d × F, where F is the collection of ordered orthonormal bases for ℝ^d. It is easy to see that J = F and that the topology induced on J is actually the natural compact metric topology on F. Thus, by Remark 9.2 the system (X, T) is tame₁. Note that this AA system is not AA_c.

10. Are the groups vG, $v \in J$ always homeomorphic to each other?

For the Sturmian system (X, T) (or any other system of the form $\text{Split}(\mathbb{T}, R_{\alpha}; A)$), the Adherence semigroup Adh(X, T) is the Ellis' two circles system and it coincides with the unique minimal left ideal M of E(X, T). The set J of minimal idempotents in M consists of two elements

$$J = \{u_+, u_-\}.$$

If we choose $u_+ = u$ then the group G = uM is algebraically isomorphic to the circle group \mathbb{T} , and as a topological space it is homeomorphic to the Sorgenfrey circle. The map $g \mapsto -u_-g$ is a topological isomorphism from G onto u_-G .

We next consider a similar but more complicated example where the same phenomenon holds (compare with Subsection 3.1).

Example 10.1. Let $\alpha \in (0, 1)$ be an irrational number. We start our construction with the dynamical system (\mathbb{T}, R_{α}) defined on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by $R_{\alpha}\theta = \theta + \alpha$ (mod 1). Let

$$F(\theta) = \begin{cases} \cos(\frac{2\pi}{\theta}) & 0 < \theta < 1/2\\ 2\theta & 1/2 \le \theta < 1\\ 0 & \theta = 0 \end{cases}$$

a function on \mathbb{T} with 0 as the unique discontinuity point. Set $f(n) = F(n\alpha)$, a



FIGURE 4. The graph of the function $F(\theta)$

function in $\Omega = [-1,2]^{\mathbb{Z}}$. Let $x_0 \in \mathbb{T} \times [-1,2]^{\mathbb{Z}}$ be defined by $x_0(n) = (0, F(n\alpha)) = (0, f(n)), \quad n \in \mathbb{Z}.$

Let $T: \mathbb{T} \times \Omega \to \mathbb{T} \times \Omega$ be defined by

$$T(\theta, \omega) = (\theta + \alpha, S\omega),$$

where for $\omega \in \Omega$ the shift homeomorphism $S \colon \Omega \to \Omega$ is defined by $S\omega(n) = \omega(n+1)$. Let

$$X = \operatorname{cls} \mathcal{O}_T(x_0) = \overline{\{T^n x_0 : n \in \mathbb{Z}\}} \subset \mathbb{T} \times \Omega.$$

We let $\pi: X \to \mathbb{T}$ denote the projection on the \mathbb{T} coordinate. Clearly $\pi: (X,T) \to (\mathbb{T}, R_{\alpha})$ is a homomorphism of dynamical systems. One can also easily see that π is an almost one-to-one extension, with $\pi^{-1}(\pi(x)) = \{x\}$ for every $x \in X$ such that $\pi(x) \notin \mathcal{O}_{R_{\alpha}}(0)$. In fact for a sequence n_i such that $\lim n_i \alpha = \theta \notin \mathcal{O}_{R_{\alpha}}(0)$, we have:

$$\lim_{i \to \infty} T^{n_i} x_0 = x, \quad with \quad x(n) = (\theta, F(\theta + n\alpha)), \quad (n \in \mathbb{Z}).$$

It then follows that the system (X,T) is minimal and almost automorphic. Let $n_i \in \mathbb{Z}$ be a sequence such that $n_i \alpha \to 0$ with $n_i \alpha \in [1/2, 1)$. Assume that $x_2 := \lim T^{n_i} x_0$ exists. It is then easy to see that x_2 has the form

$$x_2(n) = \begin{cases} (0, f(n)) & n \neq 0\\ (0, 2) & n = 0. \end{cases}$$

On the other hand if $n_i \in \mathbb{Z}$ is a sequence such that $n_i \alpha \to 0$ with $n_i \alpha \in (0, 1/2)$ and such that $x_t := \lim T^{n_i} x_0$ exists, then x_t has the form

$$x_t(n) = \begin{cases} (0, f(n)) & n \neq 0\\ (0, t) & n = 0, \end{cases}$$

for a unique $t \in [-1, 1]$.

This gives us a full description of the fiber over 0:

$$\pi^{-1}(0) = \{x_2\} \cup \{x_t : t \in [-1, 1]\}.$$

Of course this defines also $\pi^{-1}(n\alpha)$ for every $n \in \mathbb{Z}$ and, as $\pi^{-1}(\theta)$ is a singleton for every θ which is not in the R_{α} -orbit of 0, we now have a complete description of all the points of X.

As a base for the topology at x_2 we can take the collection $\{V_k(x_2) : k = 2, 3, ... \}$, where

$$V_k(x_2) = \pi^{-1}(1 - 1/k, 1) \cup \{x_2\}.$$

The collection $\{V_{k,l}(x_t): k, l=2,3,\dots\}$, where

$$V_{k,l}(x_t) = \left(\pi^{-1}(0, 1/k) \cap \{x : x(0) = (\pi(x), s) \& |s - t| < 1/l\}\right) \cup \{x_s : |t - s| < 1/l\}$$

will serve as a basis for the topology at x_t $(0 \le t \le 1)$.

Of course we have $T^n V_k(x_2) = V_k(T^n x_2)$ and $T^n V_k(x_t) = V_k(T^n x_t)$, for $n \in \mathbb{Z}$. Finally for a point $x \notin \pi^{-1}(\mathcal{O}_{R_{\alpha}}(0))$ the collection $\{V_k(x), k = 2, 3, ...\}$, where

$$V_k(x) = \pi^{-1}(\pi(x) - 1/k, \pi(x) + 1/k),$$

is a basis for the topology at x.

Next note that, as the elements of $\pi^{-1}(0)$ differ from each other only at one coordinate, the extension π is asymptotic. It then follows that our system (X,T)

is tame and that E(X,T) is first countable (see Theorem 3.1). Thus we proved the following:

Claim 10.2. The system (X,T) is AA_c (but not AA_{cc}) and it is tame₁.

We will next determine the nature of the elements of the adherence semigroup $Adh(X,T) = E(X,T) \setminus \{T^n : n \in \mathbb{Z}\}.$

We start by determining the idempotents in Adh (X,T). If $v = v^2$ is such an idempotent, then clearly $\pi_*(v) = \operatorname{Id}_{\mathbb{T}} = 0$, where $\pi_* \colon E(X,T) \to E(\mathbb{T},R_\alpha) \cong \mathbb{T}$ is the canonical homomorphism induced by π , and where $E(\mathbb{T},R_\alpha)$ is identified with \mathbb{T} . Now if $T^{n_i} \to v$, then it follows that $n_i \alpha \to 0$, and thus $T^{n_i} x_0 \to v x_0 = x_\epsilon$, with either $\epsilon = 2$ or $\epsilon = t \in [-1,1]$. We now note that this determines v, since then vx = x when $x \notin \mathcal{O}_T(x_0)$ and $vT^j x = T^j v x = T^j x_\epsilon$ for every $x \in \pi^{-1}(0)$ and $j \in \mathbb{Z}$. We then write $v = v_\epsilon$. Conversely if n_i is a sequence in \mathbb{Z} such that $n_i \alpha \to 0$ and $T^{n_i} x_0 \to x_\epsilon$, then we must have $T^{n_i} \to v_\epsilon$ in E(X,T). Note that clearly v_ϵ is a minimal idempotent.

Let now $p = \lim T^{n_j}$ be an element of $\operatorname{Adh}(X,T)$, with $\lim n_j \alpha = \gamma \in \mathbb{T}$. For $x \in \pi^{-1}(-\gamma)$ we have $px = x_{\epsilon}$ for some ϵ and it follows that $p(\pi^{-1}(-\gamma)) = \{x_{\epsilon}\}$. We now check and see that $v_{\epsilon}p = p$. It follows immediately that $\operatorname{Adh}(X,T) = M$, the unique minimal left ideal of E(X,T). Denote by J the set of minimal idempotents in $\operatorname{Adh}(X,T)$; thus

$$J = \{v_2\} \cup \{v_t : t \in [-1, 1]\}.$$

Note that $v_{\epsilon}v_{\eta} = v_{\epsilon}$ for every ϵ and η . For convenience let $u = v_2$ and we set G = uM. With this notation we have that for p as above, $p = v_{\epsilon}p$ and denoting $g_{\gamma} = up \in G$, we get $p = v_{\epsilon}g_{\gamma}$. Thus

$$G = uM = \{g_{\gamma} : \gamma \in \mathbb{T}\} \cong \mathbb{T},$$

and every $p \in M$ has uniquely the form $p = v_{\epsilon}g_{\gamma}$ with $g_{\gamma} \in G$ and $v_{\epsilon} \in J$.

Given $p \in Adh(X,T)$ with $\pi_*(p) = \gamma$, in order to determine the unique ϵ for which $v_{\epsilon}p = p \in v_{\epsilon}G$, we have to evaluate $px(0) = (0,\epsilon)$ for some (any) $x \in \pi^{-1}(-\gamma)$.

The collection G is actually a group which algebraically is isomorphic to \mathbb{T} , with $g_{\gamma} \leftrightarrow \gamma$, and the map $r_{\gamma} \colon p \mapsto pg_{\gamma}$ defines a (continuous) automorphism of the minimal system $(M,T) = (\operatorname{Adh}(X,T),T)$. Thus $G \cong \operatorname{Aut}(\operatorname{Adh}(X,T),T) = \{r_{\gamma} \colon \gamma \in \mathbb{T}\}.$

As the topology of A(X,T) is the topology of pointwise convergence, it is now easy to see that the topology induced on J is such that $u = u_2$ is an isolated point of J and that the collection $\{v_t : t \in [-1,1]\}$ is naturally homeomorphic to the interval [-1,1].

For $x \in X$ let eva_x : Adh $(X,T) \to X$ denote the evaluation map defined by

$$eva_x(p) = px, \quad (p \in Adh(X,T)).$$

Clearly the collection of sets

$$\{\operatorname{eva}_{x}^{-1}(V): x \in X, V \text{ open in } X\}$$

forms a sub-base for the topology of Adh(X,T).

We will next show that G, with the topology it inherits from Adh(X,T), is homeomorphic to the Sorgenfrey circle (i.e. the half open intervals topology on \mathbb{T}).

We now have that when x ranges over X, and we write $\pi(x) = -\theta$, sets of the form:

$$G \cap \operatorname{eva}_x^{-1}(V_k(x_2)) = \{g_{\gamma} \colon \gamma - \theta \in (1 - 1/k, 1)\} \cup \{g_{\theta}\} \cong (\theta - 1/k, \theta],$$

$$G \cap \operatorname{eva}_x^{-1}(V_{k,l}(x_t)) = \{g_{\gamma} \colon \gamma - \theta \in (0, 1/k)\} \cong (\theta, \theta + 1/k),$$

and for $x' \notin \pi^{-1}(\mathcal{O}_{R_{\alpha}})$ with $\pi(x') = \theta'$

$$G \cap \operatorname{eva}_x^{-1}(V_k(x')) = \{g_{\gamma} \colon \gamma - \theta \in (\theta' - 1/k, \theta' + 1/k) \cong (\theta' + \theta - 1/k, \theta' + \theta + 1/k),$$

$$k, l = 2, 3, \dots, t \in [-1, 1],$$

together with all their T^n -translates, form a basis for the topology of G. This completes the proof of the claim that the relative topology on G is the left-half-open interval Sorgenfrey topology on \mathbb{T} .

Similarly we see that for each $t \in [-1,1]$ the topology induced on the group v_tG is the right-half-open interval Sorgenfrey topology on \mathbb{T} .

Thus for $t, s \in [-1, 1]$ the map

 $L_{v_*}: v_s G \to v_t G, \quad v_s g_\gamma \mapsto v_t g_\gamma$

is a topological isomorphism, while for u and t the map

$$L_u \colon v_t G \to G, \quad v_t g_\gamma \mapsto -g_\gamma$$

is a topological isomorphism.

Question 10.3. Is there a minimal dynamical system (X,T) such that in E(X,T)there is a minimal left ideal M and two idempotents $u, v \in M$ such that the groups uM and vM (with the induced topology from M) are not homeomorphic?

11. The β -rank of a tame system

Let (X,T) be a metric (tame) dynamical system, E = E(X,T) its enveloping semigroup. For $p \in E$ define the the oscillation function of p at $x \in X$ as

$$\operatorname{osc}(p, x) = \inf\{\sup_{x_1, x_2 \in V} d(px_1, px_2) : V \subset X \text{ open }, x \in V\}$$

and similarly, for A a subset of X with $x \in A$,

$$\operatorname{osc}(p, x, A) = \operatorname{osc}(p \upharpoonright A, x).$$

0 Consider, for each $\epsilon > 0$, the derivative operation

$$A \mapsto A'_{\epsilon,p} = \{x \in A : \operatorname{osc}(p, x, A) \ge \epsilon\}$$

and by iterating define $A^{\alpha}_{\epsilon,p}$ for $\alpha < \omega_1$. Let

$$\beta(p,\epsilon,A) = \begin{cases} \text{least } \alpha \text{ with } A^{\alpha}_{\epsilon,p} = \emptyset, & \text{if such an } \alpha \text{ exists} \\ \omega_1 & \text{otherwise.} \end{cases}$$

Set $\beta(p,\epsilon) = \beta(p,\epsilon,X)$ and define the oscillation rank $\beta(p)$ of p by

$$\beta(p) = \sup_{\epsilon > 0} \beta(p, \epsilon).$$

Note that p is continuous iff $\beta(p) = 1$.

Finally define the β -rank of the system (X, T) as the ordinal

$$\beta(X,T) = \sup\{\beta(p) : p \in E\}$$

The following claim follows from [12], and [51, Theorem 3, Theorem 8] (where the α, β and γ ranks are shown to be essentially equivalent for bounded Baire class 1 functions).

Claim 11.1. For a metric tame dynamical system (X,T) we have $\beta(X,T) < \omega_1$.

Claim 11.2. For the Sturmian system we have $\beta(X,T) = 2$.

Proof. In order to compute $\beta(u_+)$ we observe that for a fixed $\epsilon > 0$ there are only finitely many points, all of the form $x = T^n x_0^-$, where $\operatorname{osc}(u_+, x) \ge \epsilon$. Thus $\beta(u_+, \epsilon) = 2$ for every $\epsilon > 0$, hence $\beta(u_+) = 2$. Similarly $\beta(u_-) = 2$ and, as $\beta(pg_{\gamma}) = \beta(r_{\gamma}(p)) = \beta(p)$ for every $p \in \operatorname{Adh}(X, T)$ and $\gamma \in G$, we conclude that $\beta(p) = 2$ for every $p \in \operatorname{Adh}(X, T)$. Of course $\beta(T^n) = 1$ and we conclude that $\beta(X, T) = 2$, as claimed. \Box

A similar proof will show that:

Claim 11.3. For the " $\cos(\frac{1}{x})$ -system" (X,T) (Example 10.1) we have $\beta(X,T) < \omega_1$.

Claim 11.4. The Dynkin-Maljutov system (Ω, F_2) (Example 6.7) has β -rank 2.

Proof. We have $\beta(t) = 1$ for every $t \in F_2$, $\beta(p) = 1$ for a parabolic $p \in Adh(\Omega, F_2)$, and $\beta(p) = 2$ for a loxodromic p. Thus $\beta(\Omega, F_2) = 2$ as claimed.

The Sturmian system is obtained by considering a cover of \mathbb{T} by the two closed sets $[0, \alpha]$ and $[\alpha, 1]$. In [56] Paul shows that, in fact, every almost automorphic **symbolic** minimal system arises from a finite separating cover of its maximal equicontinuous factor. Both functions $\mathbf{1}_{[0,\alpha]}$ (in the Sturmian system) and the function F (in the $\cos(\frac{1}{x})$ -system) are semicontinuous, hence with β -rank 2 (as Baire class 1 functions), and, as we have seen, the β -rank of these dynamical systems is 2. We will show next that for some separating cover A_0, A_1 of the circle by two closed sets (so that again the β rank of the function $\mathbf{1}_{A_0}$ is 2), the corresponding almost automorphic system (X, T) is not even tame, so that $\beta(X, T) = \omega_1$.

The following propositions and theorem are proved in [26, Proposition 3.5, Proposition 3.6, Corollary 3.7].

Proposition 11.5. Let (\mathbb{T}, R_{α}) be the rotation system with α irrational and let $B \subset \mathbb{T}$ be a residual subset. Let $A_0, A_1 \subset \mathbb{T}$ be two closed subsets that satisfy

- (1) cls int $(A_0) = A_0$ and cls int $(A_1) = A_1$.
- (2) int $(A_0) \cap \operatorname{int} (A_1) = \emptyset$.
- (3) $C = \partial A_0 = \partial A_1$ is a Cantor set.

Then there exists an infinite subset $I \subset \mathbb{Z}$ such that for every $a \in \{0, 1\}^I$, there is a point $x \in B$ with $T^i x = x + i\alpha \in int(A_{a(i)})$ for every $i \in I$.

Now let $F = \mathbf{1}_{A_1}$ i.e.

$$F(\theta) = \begin{cases} 1 & \theta \in A_1, \\ 0 & \text{otherwise} \end{cases}$$

Choose a point $y_0 \in \text{int } A_1$ and let $x_0 = f \in \{0, 1\}^{\mathbb{Z}}$ be defined by $x_0(n) = f(n) = F(y_0 + n\alpha)$, $(n \in \mathbb{Z})$. We set $X = \text{cls} \{T^n x_0 : n \in \mathbb{Z}\}$, where T is the shift map on $\{0, 1\}^{\mathbb{Z}}$.

The dynamical system (X, T) is almost automorphic with a maximal equicontinuous factor (\mathbb{T}, R_{α}) , where the factor map $\pi: (X, T) \to (\mathbb{T}, R_{\alpha})$ is an almost

one-to-one extension and, in particular, $\pi^{-1}(y_0) = \{x_0\}$. It is also a cut and project system as follows (see [56], [9]):

Proposition 11.6. The almost automorphic subshift (X,T) is isomorphic to the cut and project dynamical system $(\Omega(\lambda(W)),\mathbb{Z})$ obtained from the cut and project scheme $(\mathbb{Z},\mathbb{T},L)$, with lattice $L = \{(n,n\alpha) : n \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{T}$, where

- $0 \in \mathbb{T}$ has a unique preimage under the factor map $\beta : (\Omega(\lambda(W)), \mathbb{Z}) \to (\mathbb{T}, R_{\alpha}).$
- $W = \beta([1]), where [1] = \{\xi \in \{0, 1\}^{\mathbb{Z}} : \xi(0) = 1\}.$

Moreover, the window W is proper, that is, int(W) = W.

Now in this situation the following theorem applies.

Theorem 11.7. Suppose that $(\mathbb{Z}, \mathbb{T}, L)$ is a cut and project scheme. If W is a proper window and ∂W is a Cantor set, then the system $(\Omega(\mathcal{A}(W)), \mathbb{Z})$ admits an infinite free set $I \subset G$ i.e. for every $a \in \{0, 1\}^I$, there is a point $x \in \Omega(\mathcal{A}(W))$ with x(i) = a(i) for every $i \in I$. Thus the system $(\Omega(\mathcal{A}(W)), \mathbb{Z})$ is not tame.

Question 11.8. Is there, for every ordinal $\alpha < \omega_1$, a tame metric system (X, T) with $\beta(X, T) = \alpha$? Or, to be more modest, find a dynamical system with $\beta(X, T) = 3$.

12. More open questions

A separable Banach space V is Rosenthal if and only if its *enveloping semigroup* $\mathcal{E}(V)$ is a (separable) Rosenthal compactum (see the BFT dichotomy for Banach spaces [39, Theorem 5.22]).

Another characterization, due to Odell and Rosenthal, is that a separable Banach space V is Rosenthal if and only if every element $x^{**} \in B^{**}$ is a Baire class 1 function on B^* , iff B^{**} is a separable Rosenthal compactum. One can ask what is the nature of the spaces B^{**} in terms of the Todorčević classes mentioned in Theorem 0.3.

Now, as noted by Bourgain [10] the point 0 is not a G_{δ} subset of B^{**} for any Rosenthal space which is not Asplund. In particular, it follows that B^{**} is not first countable (at 0). This implies that the enveloping semigroup $\mathcal{E}(V)$ of V is not hereditarily separable (because, by [39, Lemma 2.6.2] the weak-star compact ball B^{**} is a continuous image of $\mathcal{E}(V)$).

Question 12.1. Is there a separable Rosenthal Banach space V such that $\mathcal{E}(V)$ is first countable but not metrizable (equivalently, V is not Asplund) ?

Question 12.2. Every continuous topological group action on a dendrite D is tame, [43]. Is it always tame₁?

This is the case for the simplest nontrivial dendrite D = [0, 1]. This can be proved by a minor modification of the arguments in Example 8.9. Note that the circle S^1 , which is a local dendrite, does admit an action of $GL(d, \mathbb{R})$ which is tame but not tame₁ (Example 5.7). See also Proposition 8.10.

For other questions in the present work see: Questions 1.5, 1.6, 5.6, 6.8, 8.4, 8.13, 10.3 and 11.8.

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DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, RAMAT AVIV, ISRAEL Email address: glasner@math.tau.ac.il URL: http://www.math.tau.ac.il/~glasner

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT-GAN, ISRAEL Email address: megereli@math.biu.ac.il URL: http://www.math.biu.ac.il/~megereli