SOME EXERCISES IN TOPOLOGICAL GROUPS 04.07.22

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Contents

1. Homework 1

Exercise 1.1.

(1) Show that every normed (in fact, any seminormed) space $(E, +, ||\cdot||)$ is a topological group and the family $\{B_n(0)\}_{n\in\mathbb{N}}$, with

$$B_n(0) := \{x \in E : ||x|| < \frac{1}{n}\}$$

is a local base at 0.

(2) Show that $(\mathbb{Z}, +, d_5)$, where d_5 is the 5-adic metric [1], is a Hausdorff topological ring and give an example of its local base at 0 every member of which is a subgroup of $\mathbb{Z}$.

(3) Every group G with the trivial topology $\tau_{tr} := \{\emptyset, G\}$ is a topological group. The same is true with respect to the discrete topology $\tau_{discr} := P(G)$.

(4) Give an example of a topological group with nontrivial topology which is not Hausdorff.

Proof.

(1) We show that addition is continuous. Indeed, let $\varepsilon > 0$ and define $\delta := \frac{1}{2}\varepsilon$. We claim that if $\|x_1 - x_2\| < \delta$ and $\|y_1 - y_2\| < \delta$ then

$$\|(x_1 + y_1) - (x_2 + y_2)\| < \varepsilon.$$

Indeed:

$$\begin{aligned}\|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\leq \|(x_1 - x_2)\| + \|(y_1 - y_2)\| \\ &< \delta + \delta = \varepsilon.\end{aligned}$$

[1] Recall that for every distinct $x, y \in \mathbb{Z}$ we have
$d_5(x, y) = \frac{1}{5^k}$, where $k = k(x, y) := \max\{i : 5^i|(x - y)\}$)

Another quick proof comes using the sequences. Let $\lim a_n = a. \lim b_n = b$. Equivalently, $\lim \|a - a_n\| = 0$, $\lim \|b - b_n\| = 0$. Then

$$\lim \|(a+b) - (a_n + b_n)\| = 0$$

because $0 \leq \|(a+b) - (a_n + b_n)\| \leq \|a - a_n\| + \|b - b_n\|$.

It is also easy to see that the inversion operation

$$E \to E, x \mapsto -x$$

is continuous. Indeed, let $\lim a_n = a$. Equivalently, $\lim \|a - a_n\| = 0$. Then also,

$$\lim \|a - a_n\| = \lim \|a_n - a\| = \lim \|(-a) - (-a_n)\| = 0.$$

So, $\lim(-a_n) = -a$.

To see that $\{B_n(0)\}$ is a local base, recall that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and therefore $B_n(0) \subseteq B_\varepsilon(0)$.

(2) To see that addition and inversion are continuous we can use similar arguments as in the previous item.

We now show that multiplication is continuous. In fact, we will show that the multiplication map $\cdot\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (x, y) \mapsto x \cdot y$ is a Lipschitz map and therefore continuous. Note that

$$\max\{n \in \mathbb{N} \mid 5^n \mid ab\} = \max\{n \in \mathbb{N} \mid 5^n \mid a\} \cdot \max\{m \in \mathbb{N} \mid 5^m \mid b\}.$$

Thus, for every $0 \neq a, b \in \mathbb{Z}$ we have:

$$\begin{aligned} \|ab\|_5 &:= (\max\{n \in \mathbb{N} \mid 5^n \mid ab\})^{-1} \\ &= (\max\{n \in \mathbb{N} \mid 5^n \mid a\})^{-1} \cdot (\max\{m \in \mathbb{N} \mid 5^m \mid b\})^{-1} \\ &= \|a\|_5 \|b\|_b. \end{aligned}$$

We can also easily verify this identity when $a = 0$ or $b = 0$.

A basis of neighborhoods is given by $B_n(0) = 5^n\mathbb{Z}$, $n \in \mathbb{N}$.

(3) If G is trivial, then the multiplication and inversion are continuous as maps into a trivial topology. If G is discrete, they are continuous as maps from a discrete space.

(4) Consider the topological group $(\mathbb{R}, \tau_{tr}) \times \mathbb{R}$. It is a product of topological groups and therefore also a topological group. However, it is clearly not Hausdorff nor trivial.

□

Exercise 1.2. Let $G \in \mathrm{TGr}$. Prove that for every $U \in N(e)$ of the identity $e \in G$ and every given $n \in \mathbb{N}$ there exists $V \in N(e)$ such that $V = V^{-1}$ and $V^n := \underbrace{VV \cdots V}_{n \ \ times} \subset U$.

Proof. First of all check by induction that $G \to G, x \mapsto x^k$ is continuous for every given $k \in \mathbb{Z}$ (consider separately two cases of natural k and $k < 0$). Then use again induction for $n \in \mathbb{N}$ to complete the proof in general.

Now, by the continuity of $G \to G, x \mapsto x^n$ at the identity point $e \in G$, for every nbd $U \in N(e)$ and every given natural $n \in \mathbb{N}$ there exists $W \in N(e)$ such that $W^n \subset U$. Now take the symmetric nbd $V := W \cap W^{-1} \subset W$. □

Exercise 1.3. Find Hausdorff topological groups (G_1, τ_1) and (G_2, τ_2) and a continuous onto injective homomorphism $f : (G_1, \tau_1) \to (G_2, \tau_2)$ such that f is not a homeomorphism (that is, f^{-1} is not continuous). This shows that not every continuous algebraic isomorphism is an isomorphism in the category TGr of topological groups.

Proof. $id : (\mathbb{R}, \tau_{discr}) \to \mathbb{R}$. □

Exercise 1.4. Prove that

(1) G is *homogeneous* [2] as a topological space.
(2) * Moreover, G is *bi-homogeneous* in the following sense: for every pair $(x, y) \in G \times G$ there exists $f \in \text{Homeo}\,(G, G)$ such that $f(x) = y$ and $f(y) = x$.
(3) Which of the following topological spaces are of the group type: [3]
   (a) *Cantor set.*
   (b) $X := \{x \in \mathbb{R}^2 : \ ||x|| = 5\}$.
   (c) $X := \{x \in \mathbb{R}^3 : \ ||x|| < 5\}$.
   (d) The integers $\mathbb{Z}$ with the cofinite topology.

Proof. (1) (Sketch) $l_a : G \to G$ is a homeomorphism and $l_a(x) = y$ for $a := yx^{-1}$ (follows also directly from (2)).

(2) Consider the function $f : G \to G, f(g) = xg^{-1}y$. This is a homeomorphism as a composition of three homeomorphisms $f = l_x \circ l_y \circ i$ (where $i \colon G \to G, i(g) = g^{-1}$).

(3)

(a) Yes. $C \simeq \{0, 1\}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the topological group $\{-1, 1\}^{\mathbb{N}}$.

(b) Yes. The circle $X := \{x \in \mathbb{R}^2 : \ ||x|| = 5\}$ is homeomorphic to $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$.

(c) Yes. The open ball $X := \{x \in \mathbb{R}^3 : \ ||x|| < 5\}$ is homeomorphic to $\mathbb{R}^3$. Indeed, the desired homeomorphism is

$$f : \mathbb{R}^3 \to X, f(x) = \frac{5x}{1 + ||x||}$$

Similar arguments work in every normed space for every open ball.

(d) No. Indeed, it is T_1 but not T_2.

It is also easy to prove directly that the multiplication is not continuous. Observe that for every cofinite subset $V \subset \mathbb{Z}$ we have $V + V = \mathbb{Z}$. □

Exercise 1.5. Let A and B are subsets of $G \in \text{TGr}$ and $g \in G$. Prove that:

(1) If A and B are compact then AB is also compact.
(2) If A and B are connected then AB is also connected.
(3) If A and B are closed then AB need not be closed.
(4) * If A is closed and B is compact then AB is closed.

Proof.

(1) $A \times B$ is compact. Thus, its continuous image $m\,(A \times B) = AB$ is also compact.
(2) $A \times B$ is connected. Thus, its continuous image $m\,(A \times B) = AB$ is also connected.

[2]For every pair $(x, y) \in G \times G$ there exists $f \in Homeo(G, G)$ such that $f(x) = y$.

[3]**Definition:** Let us say that a topological space (X, τ) is of *group type* if the set X admits a group operation $w : X \times X \to X$ such that the triple (X, τ, w) is a topological group.

(3) In the group $\mathbb{R}$ take the closd subsets $A = \mathbb{Z}, B = \left\{n + \frac{1}{2^n}\right\}_{n\in\mathbb{N}}$. Then $A+B = \left\{m + \frac{1}{2^n}\right\}_{m\in\mathbb{Z}, n\in\mathbb{N}}$ and we have $0 = \lim \frac{1}{2^n} \in \overline{(A+B)}$ but $0 \notin A+B$.

(4) Let $c \notin AB$ then $cB^{-1} \cap A = \varnothing$. Thus $\{c\} \times B^{-1} \subseteq m^{-1}(A^c)$. Clearly, $\{c\} \times B^{-1}$ is compact and $m^{-1}(A^c)$ is its open nbd in $G \times G$. By the tube lemma there exists $U \in N(c)$ with $U \times B^{-1} \subseteq m^{-1}(A^c)$. Hence, $UB^{-1} \cap A = \varnothing$. So, $U \cap AB = \varnothing$. Therefore, $c \notin cl(AB)$.

□

Exercise 1.6. Let $G \in$ TGr. Prove that:

(1) $cl(A^{-1}) = cl(A)^{-1}$ and $cl(A)cl(B) \subset cl(AB)$ for every subsets A, B of G.
(2) If $H \leq G$ is a subgroup then $cl(H) \leq G$ is also a subgroup.
(3) If $H \trianglelefteq G$ is a normal subgroup then $cl(H) \trianglelefteq G$ is also a normal subgroup.
(4) If G, in addition, is Hausdorff and $H \leq G$ is abelian then $cl(H) \leq G$ is also an abelian subgroup. Give a counterexample if G is not Hausdorff.

Proof. (1) The inversion is a homeomorphism. Every homeomorphism preserves the closure operator. This explains why $cl(A^{-1}) = cl(A)^{-1}$.

Let $x \in cl(A), y \in cl(B)$. We have to show that $xy \in cl(AB)$. Let $U \in N(xy)$. By the continuity of the multiplication there exist $V \in N(x), W \in N(y)$ such that $VW \subset U$. By our assumptions, $V \cap A \neq \emptyset, W \cap B \neq \emptyset$. Then $VW \cap AB \neq \emptyset$. Since $VW \subset U$ we obtain $U \cap AB \neq \emptyset$.

Remark: In fact, $cl(A)cl(B) \subset cl(AB)$ for every *semitopological semigroup* [4] for **every** subsets A, B. Indeed, it is easy to see that $cl(A)B \subset cl(AB)$ for every *right topological semigroup* [5]. By the continuity of the right translation r_b we obtain

$$cl(A) \cdot b = r_b(cl(A)) \subset cl(r_b(A)) = cl(A \cdot b)$$

for every $b \in B$).

Similarly, $Acl(B) \subset cl(AB)$ for every left topological semigroup. We obtain

$$cl(A)cl(B) \subset cl(Acl(B)) \subset cl(cl(AB)) = cl(AB).$$

(2) $cl(H)cl(H) \subset cl(HH) = cl(H)$ and $cl(H)^{-1} = cl(H^{-1}) = cl(H)$.

(3) Let $H \trianglelefteq G$. Then $f_a(H) = H$ for every conjugation $f_a : G \to G, g \mapsto aga^{-1}$. Now use the fact that every conjugation is a homeomorphism for every topological group. So, $f_a(cl(H)) = cl(f_a(H)) = cl(H)$.

(4) The function $f : G \times G \to G, (x, y) \mapsto [x, y] := xyx^{-1}y^{-1}$ is continuous. Since H is abelian, the restriction $f_H : H \times H \to H$ is the constant function. Namely, $f(h_1, h_2) = e$ for every $h_1, h_2 \in H$. Consider the *subgroup* $cl(H)$ and the restriction $f_{cl(H)} : cl(H) \times cl(H) \to cl(H)$ of f. Since H is dense in $cl(H)$ and $cl(H)$ is *Hausdorff* $f_{cl(H)}$ should also be the constant function.

Hausdorff property is essential. Indeed, take a noncommutative group G with the trivial topology and choose $H := \{e\}$. Then H is abelian but not $cl(H) = G$. □

[4] semigroup with separately continuous multiplication

[5] right translations are continuous

2. Homework 2

Exercise 2.1. Let G_1, G_2 be topological groups and $f\colon G_1 \to G_2$ be a homomorphisms which is continuous at the point $e \in G_1$. Show that f is continuous.

Proof. Let $g \in G_1$, we will show that f is continuous at g. Consider the translations $T_{g^{-1}}\colon G_1 \to G_1$, $T_{f(g)}\colon G_2 \to G_2$ defined by:

$$T_{g^{-1}}(h) := g^{-1}h,\ T_{f(g)}(x) := f(g)x.$$

Note that since f is a group homomorphism, $f = T_{f(g)} \circ f \circ T_{g^{-1}}$. Indeed

$$(T_{f(g)} \circ f \circ T_{g^{-1}})(h) := f(g)f(g^{-1}h) = f(h).$$

Since both G_1 and G_2 are topological groups, both $T_{g^{-1}}$ and $T_{f(g)}$ are continuous. Furthermore, $T_{g^{-1}}(g) = e$ and f is continuous at e so $f = T_{f(g)} \circ f \circ T_{g^{-1}}$ is continuous at g.

This is true for every $g \in G_1$, proving that f is continuous. $\square$

Exercise 2.2. Let G be a topological group. Prove that:

(1) $\forall U \in N(e)$ $\forall$ compact subset $K \subset G$ $\exists V \in N(e)$:
$xVx^{-1} \subset U$ $\forall x \in K$;

(2) for every compact subset $K \subset G$ and a closed subset $A \subset G$ with $K \cap A = \emptyset$ there exists $U \in N(e)$ s.t. $UK \cap A = \emptyset$.

Proof. (1) Consider the conjugation map $C\colon G \times G \to G$ defined by $C(g,h) := ghg^{-1}$. This map is continuous. Also, for every $x \in G$, we have $C(x,e) = e$. Therefore, we can find open nbds $V_x \in N(e), W_x \in N(x)$ such that $C(W_x, V_x) \subseteq U$. Note that $\{W_x\}_{x \in K}$ is an open cover of K in G. Since K is compact, it has a finite subcover $\{W_x\}_{x\in F}$, where F is a finite subset of K. Then the finite intersection $V := \bigcap_{y \in F} V_y \in N(e)$ is a neighorhood of e in G. For every $x \in K$, we can find $y \in F$ such that $x \in W_y$. Then

$$xVx^{-1} = C(x,V) \subseteq C(W_y, V_y) \subseteq U.$$

(2) Every Hausdorff topological group is T_3, so for every $x \in K$ we can find an open neighborhood $V_x \in N(e)$ such that $V_x x \cap A = \emptyset$. Choose $U_x \in N(e)$ such that $U_x^2 \subset V_x$.

Note that $\{U_x x\}_{x\in K}$ is an open cover of K and K is compact, so there is a finite subcover $\{U_x x\}_{x \in F}$, where F is a finite subset of K. Define $U := \bigcap_{y\in F} U_y$. For every $x \in K$, we can find $y \in F$ such that $x \in U_y y$. We obtain:

$$Ux \subseteq UU_y x \subseteq U_y U_y x = U_y^2 y \subseteq V_y y \subset G \setminus A.$$

Now, it is clear that $UK = \cup_{x\in K} Ux \subset G \setminus A$.

$\square$

Exercise 2.3. Prove or disprove: topological groups $\mathbb{T}^2$ and $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ are topologically isomorphic.

Proof. $\mathbb{T}^2$ is compact and $\mathbb{C}^*$ is not.

* I meant to write in the formulation: $\mathbb{T}^2$ is a quotient group of $\mathbb{C}^*$. $\square$

Exercise 2.4. Let G be a Hausdorff locally compact topological group and H is a closed subgroup of G. Prove that the coset G-space G/H is Hausdorff and locally compact.

Proof. First we will show that G/H is Hausdorff. Suppose that

$$aH \neq bH \in G/H.$$

Since H is closed, so is bH. Every locally compact Hausdorff space is Tychonoff (also true in general for Hausdorff topological groups), so we can find a symmetric $U \in N(e)$ such that $Ua \cap bH = \emptyset$. Find a symmetric $W \in N(e)$ such that $WW \subseteq U$. We claim that $WaH \cap WbH = \emptyset$. Since the quotient map $G \to G/H$ onto the coset space G/H is open, this would imply that we separated aH and bH.

By contradiction, suppose that $x \in WaH \cap WbH$. Thus, there exists $h_1, h_2 \in H, \varepsilon_1, \varepsilon_2 \in W$ such that $x = \varepsilon_1 a h_1 = \varepsilon_2 b h_2$. As a consequence

$$\varepsilon_2^{-1}\varepsilon_1 a = bh_2h_1^{-1}.$$

However, $\varepsilon_2^{-1}\varepsilon_1 \in W^{-1}W = WW \subseteq U$ and $bh_2h_1^{-1} \in bH$ so $Ua \cap bH \neq \emptyset$, a contradiction.

Recall that the image of a locally compact space by an open, continuous map is also locally compact (this is evident by taking the images of compact neighborhoods). Also, the quotient map $G \to G/H$ is open. Moreover, it is clearly continuous. Together, we have shown that the quotient space G/H is locally compact. □

Exercise 2.5. Let $f\colon G \to Y$ be a continuous homomorphism onto of topological groups. Prove that f is open if and only if f is a quotient map.

Proof. Clearly, every open map is quotient. Conversely, suppose that f is quotient and we will show that it is open. Using a similar argument to that of Exercise **??**, it is enough to show that f is open at e. Indeed, suppose that $U \in N(e_G)$. Since f is quotient, to show that $f(U)$ is a neighborhood of e_Y, it is enough to show that $f^{-1}(f(U))$ is a neighborhood of e_G. Indeed, it is easy to see that:

$$f^{-1}(f(U)) = (\ker f)U(\ker f) = \bigcup_{x,y \in \ker f} xUy,$$

which is a neighborhood of e_G as a union of transitions of neighborhoods of e_G.

□

Exercise 2.6. Prove or disprove: $GL_n(\mathbb{R})/D$ is a locally compact Hausdorff topological group, where D denotes the set of all invertible scalar matrices in $GL_n(\mathbb{R})$ for every $n \in \mathbb{N}$.

Proof. This is true. First, note that D is in the center (in fact, is the center) of $GL_n(\mathbb{R})$ and therefore normal, making $GL_n(\mathbb{R})/D$ a group. Indeed, $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n\times n}$ is a locally compact topological group. Also, it is easy to see that D is a closed subgroup of $GL_n(\mathbb{R})$. By our Exercise G/D is a locally compact Hausdorff topological group. □

Exercise 2.7. Prove that the topological groups $\mathbb{T}^2$ and $\mathbb{R}^2/\mathbb{Z}^2$ are topologically isomorphic.

Proof. We will identify $\mathbb{T}^2$ with $(\mathbb{R}/\mathbb{Z})^2$. The map $q\colon \mathbb{R} \to \mathbb{T}, q(x) = cis(2\pi x)$ is a continuous open quotient. Then the same is true about the product map

$$f := q \times q\colon \mathbb{R}^2 \to \mathbb{T}^2.$$

Here $ker f = \mathbb{Z}^2$. Hence, we have a caninically defined continuous group isomorphism $\gamma\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$ such that $f = \gamma \circ p$ (where $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is the natural projection). Since f is a quotient, the group isomorphism $\gamma\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$ necessarily is a homeomorphism. □

Lemma 2.8. *The only closed subgroups of* $\mathbb{R}$ *are of the form:* $\mathbb{R}, \{0\}$ *or* $a\mathbb{Z}$ *for some* $a \in \mathbb{R}$.

Exercise 2.9. Prove or disprove: the subset $\{m\sqrt{3} + n\pi : \ m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Proof. By contradiction, assume that G is not dense in $\mathbb{R}$. Clearly, $G \leq \mathbb{R}$ is a non-trivial subgroup. By Lemma 2.8, we can find $a \in \mathbb{R}$ such that $\overline{G} = a\mathbb{Z}$. In particular, there are $n, m \in \mathbb{Z}$ such that $\pi = na, \sqrt{3} = ma$. As a consequence:

$$\pi = \frac{n}{m}\sqrt{3}.$$

However, this would imply that π is algebraic, a contradiction. □

Exercise 2.10. Prove that every Hausdorff topological quotient group of $\mathbb{R}$ is topologically isomorphic to one of the following groups: $\mathbb{R}$, $\mathbb{T}$, $\{1\}$.

Proof. All (op to the isomorphisms in TGr) Hausdorff topological group quotients of $\mathbb{R}$ are the quotient groups $\mathbb{R}/H$, where H is a closed subgroup. By Lemma 2.8 the closed subgroups:

(1) $H = \mathbb{R}$;
(2) $H = \{0\}$;
(3) $a\mathbb{Z}$ for some $a \in \mathbb{R}$.

Accordingly we have:

(1) $\mathbb{R}/H \simeq \{0\}$;
(2) $\mathbb{R}/\{0\} \simeq \mathbb{R}$;
(3) We claim that $\mathbb{R}/a\mathbb{Z} \simeq \mathbb{T}$ for every $0 \neq a \in \mathbb{R}$ (in the case of $a = 0$ we obtain (2)).

For every $0 \neq a \in \mathbb{R}$ the map $M_a\colon \mathbb{R} \to \mathbb{R}, x \mapsto ax$ is a topological group automorphism. Consider the function

$$f = q \circ M_{\frac{1}{a}}\colon \mathbb{R} \to \mathbb{T}, \ f(x) = cis(\frac{2\pi}{a}x).$$

This is a continuous onto open group homomorphism (as a composition of such homomorphisms). On the other hand, $ler f = a\mathbb{Z}$. Therefore, $\mathbb{R}/ker f = \mathbb{R}/a\mathbb{Z} \simeq \mathbb{T}$. □

3. Homework 3

Exercise 3.1. Prove or disprove: there exist closed subgroups $H_1, H_2 \subseteq \mathbb{T}$ such that the subgroup H_1H_2 is not closed in $\mathbb{T}$.

Proof. Disproving. Note that H_1 and H_2 are compact since $\mathbb{T}$ is compact. By an Exercise, the product H_1H_2 is also compact. Also, $\mathbb{T}$ is Hausdorff so H_1H_2 is closed. □

Exercise 3.2. Prove or disprove: on the group $(\mathbb{R}, \tau)$ there exists a Hausdorff group topology σ on $\mathbb{R}$ such that σ is strictly coarser than τ.

Proof. Proof.

Assuming the contrary, let $\mathbb{R}$ be a minimal topological group. That is, every injective continuous homomorphism $f : \mathbb{R} \to G$ into a Hausdorff topological group G is an embedding of topological groups.

We proved in lecture notes that there exists an injective continuous homomorphism $f : \mathbb{R} \to \mathbb{T} \times \mathbb{T}$. Then f is an embedding of topological groups. Then $f(\mathbb{R})$ (like, $\mathbb{R}$) is a locally compact but not compact subgroup of G. According to a result we proved in lecture notes, $f(\mathbb{R})$ must be a closed subgroup of the compact group $\mathbb{T} \times \mathbb{T}$. This implies that $f(\mathbb{R})$ is compact. Contradiction ! □

Exercise 3.3. Prove or disprove: topological group $\mathbb{T}^2$ is a quotient group of the group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Proof. Proof.

It is equivalent to show that there exists an open continuous onto homomorphism $\mathbb{C}^* \to \mathbb{T} \times \mathbb{T}$.

First of all observe that $\mathbb{C}^*$ is topologically isomorphic to $\mathbb{R}_+ \times \mathbb{T}$. Indeed, the functions

$$f : \mathbb{C}^* \to \mathbb{R}_+ \times \mathbb{T},\ z \mapsto (|z|, \frac{z}{|z|})$$

$$f^{-1} : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{C}^*,\ (r, cis\alpha) \to rcis\alpha$$

are well defined continuous homomorphisms.

Second observation is easy: the groups $\mathbb{R}_+$ and $\mathbb{R}$ are topologically isomorphic. Indeed, consider for example the exponential function $f : \mathbb{R} \to \mathbb{R}_+, f(x) = 2^x$.

The third observation. As we already know the natural homomorphism $q : \mathbb{R} \to \mathbb{T}$ is an open map. Then the induced onto homomorphism $q \times id : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is also open (the product of open maps is open).

Summing up all three observations we obtain an open continuous onto homomorphism $\mathbb{C}^* \to \mathbb{T} \times \mathbb{T}$, as desired. □

Exercise 3.4. Let $G \in LCA$. Prove that

(1) $f_1, f_2 \in G^* \Rightarrow f_1 + f_2 \in G^*$.
(2) $(G^*, +)$ is an abelian group.

Proof.

(1) First, $f_1 + f_2$ is clearly a homomorphism (as the sum of two homomorphisms). To see that $f_1 + f_2$ remains continuous, it is enough to check the continuity at $e \in G$. Let $\varepsilon > 0$ and d be the metric on $\mathbb{T}$. Since f_1 and f_2 are continuous,

there exists some $U \in N_G(e)$ such that $\rho(f_i(x), 0) < \frac{1}{2}\varepsilon$ for every $x, \in U, i \in \{1, 2\}$. As a consequence,

$$\rho((f_1 + f_2)(x), 0) = \rho(f_1(x) + f_2(x), 0) \leq \rho(f_1(x), 0) + \rho(f_2(x), 0) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

(2) We already established that $(G^*, +)$ is closed wrt the addition. The associativity is trivial (by the same property in $\mathbb{T}$). Next, note that for every $f \in G^*$ we have $-f \in G^*$ defined as $(-f)(x) = -(f(x))$. Clearly, $-f \in G^*$ and $f + (-f) = 0$. Therefore, we have shown that $(G^*, +)$ is a group.

Finally, $\mathbb{T}$ is abelian so for every $f_1, f_2 \in G^*$ and $x \in G$ we have:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x).$$

As a consequence, $f_1 + f_2 = f_2 + f_1$, proving that $(G^*, +)$ is abelian.

□

Exercise 3.5. Show that every $G \in LCA$ is a closed subgroup of some $P \in LCA$ such that P is autodual (meaning that $P \simeq P^*$).

Proof. Consider the group $H := G \times G^*$. As we mentioned in class notes, the dual group preserves preserved the finite products, so

$$H^* = (G \times G^*)^* \simeq G^* \times G^{**} \simeq G^* \times G \simeq G \times G^* = H.$$

Also, $G \equiv G \times \{0\} \subseteq H$ is a closed subgroup. □

Exercise 3.6. Show that the group G^* is a topological group (with respect to the compact open topology) for every $G \in LCA$.

Proof. After Exercise 3.4, we only need to show that addition and inversion are continuous.

We have to prove that

$$G^* \times G^* \to G^*, (s, t) \mapsto s - t$$

is continuous at every given point $(s_0, t_0) \in G^*$. Recall that the family

$$[K, O] := \{f : G \to \mathbb{T} : \; f(K) \subset O\}$$

with compact $K \subset G$ and open $O \subset \mathbb{T}$ is a subbase of the compact open topology on G^*. We use the following simple well known

Fact. *Let $f : X \to Y$ be a function between topological spaces and γ be a prebase for the topology of Y. Then for the continuity of f it is sufficient (and of course also necessary) that $f^{-1}(U)$ is open for every $U \in \gamma$.*

Let $s_0 - t_0 \in [K, O]$. That is, $(s_0 - t_0)(K) \subset O$. By the continuity of $s_0 - t_0 : G \to \mathbb{T}$ the set $(s_0 - t_0)(K)$ is compact in $\mathbb{T}$. Then by Exercise 2.2.2 there exists $V \in N(0)$ in $(\mathbb{R}/\mathbb{Z}, +) =: \mathbb{T}$ such that $V + (s_0 - t_0)(K) \subset O$. Take a symmetric nbd $W \in N(0)$ such that $W + W \subset V$. Define

$$O_1 := s_0 + [K, W], \; O_2 := t_0 + [K, W].$$

Then if $s \in O_1, t \in O_2$ then $s - t \in O$. □

Exercise 3.7. Let (M,d) be a metric space. Denote by G the group of all isometries, with respect to the natural composition. Define on G the pointwise topology τ_p. Prove that (G,τ_p) is a Hausdorff topological group.

Proof. We will show that the operation $P\colon G\times G\to G$ defined by $P(f,g):=f\circ g^{-1}$ is continuous. Let $F\subseteq M$ be finite and $\varepsilon>0$. Write:

$$[F,\varepsilon]:=\{f\in G\mid d(x,f(x))<\varepsilon\}$$

Note that $[F,\varepsilon]$ is a neighborhood of the identity $e=id_M$ of G.

We claim that $P\left(\left[F,\frac{1}{2}\varepsilon\right]\times\left[F,\frac{1}{2}\varepsilon\right]\right)\subseteq[F,\varepsilon]$. Let $f,g\in\left[F,\frac{1}{2}\varepsilon\right]$ and $x\in F$. Because f,g are isometries, we know that:

$$\begin{aligned} d((P(f,g))(x),x) &= d(f(g^{-1}(x)),x)\\ &= d(f(g^{-1}(x)),f(f^{-1}(x)))\\ &= d(g^{-1}(x),f^{-1}(x))\\ &\le d(g^{-1}(x),x)+d(f^{-1}(x),x)\\ &= d(x,g(x))+d(x,f(x))\\ &< \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon. \end{aligned}$$

In other words, $P(f,g)\in[F,\varepsilon]$, as required.

(G,τ_p) is *Hausdorff*. Indeed, let $g\neq e$ in $G=\operatorname{Iso}(M,d)$. Then there exists $x_0\in M$ such that $gx_0\neq x_0$. Hence, $\varepsilon:=d(gx_0,x_0)>0$. Then $g\notin[F,\varepsilon]\in N_e(G)$. □

4. Homework 4

Exercise 4.1.

(1) Prove that G^* is closed in T^G for every discrete abelian G.
(2) Prove that the compact group G^* is metrizable for every countable discrete abelian group G.

Proof.

(1) Remark (indirect proof) If G is discrete, then in virtue of Pontryagin duality, G^* is compact (with respect to the compact open topology). As a consequence, it is closed in every Hausdorff space that contains it.

(direct proof) Let us show that G^* is closed in $\mathbb{T}^G$ with respect to the Tychonoff topology. Since every function is continuous from a discrete space, G^* contains all characters (all possible homomorphisms of G to $\mathbb{T}$). In other words:

$$G^*=\bigcap_{a,b\in G}\left\{f\in\mathbb{T}^G\mid f(e)=1_G\text{ and }f(ab)=f(a)f(b)\right\},$$

where 1_G is the constant function mapping every $g\in G$ to $1\in\mathbb{T}$. It is clearly closed as the intersection of closed subsets.

Every

$$\{f\in\mathbb{T}^G\mid f(e)=1_G\text{ and }f(ab)=f(a)f(b)\}$$

is closed because if the convergence of generalized sequences $\lim f_i = f$ in the product space $\mathbb{T}^G$ means that $\forall x \in G \quad \lim f_i(x) = f(x)$.

(2) If G is countable and discrete, then $\mathbb{T}^G$ is metrizable (as a countable product of metrizable space). Therefore, G^* is also metrizable as a subspace of $\mathbb{T}^G$.

$\square$

Exercise 4.2. Prove that a (Hausdorff) topological group G is minimal if and only if every continuous injective group homomorphism $f : G \to P$ into a Hausdorff topological group P is an embedding.

Proof. First suppose that G is minimal and let $f\colon G \to P$ be a continuous, injective group homomorphism where P is Hausdorff. Write τ for the topology of G and let σ be the weak topology on G generated by f. Since f is injective and P is Hausdorff, then σ is an Hausdorff group topology. In virtue of minimality, $\tau \subseteq \sigma$.

Suppose now that $e_G \in O$ is a neighborhood of the identity in G. By the definition of the weak topology, there exists a neighborhood $e_P \in U \subseteq f(G)$ such that $f^{-1}(U) \subseteq O$. In other words, $U \subseteq f(O)$, meaning that $f(O)$ is also a neighborhood of zero. Note that we used the fact that $U \subseteq f(G)$. This is true for every neighborhood O making f an open mapping (and therefore, an embedding).

Conversely, suppose that $\sigma \subseteq \tau$ is a Hausdorff group topology, weaker than τ. Then the identity $id\colon (G, \tau) \to (G, \sigma)$ is a continuous injective group homomorphism. Therefore, it is an embedding. This implies that $\sigma = \tau$, proving that τ is minimal. $\square$

Exercise 4.3. Which of the following topological groups are minimal:

(1) $\mathbb{Z}$
(2) $\mathbb{R}$
(3) $\mathbb{C}$
(4) $\mathbb{T}^3$
(5) $\mathbb{C}^*$

Proof.

(1) $\mathbb{Z}$ is not minimal, it has the p-adic topology which is weaker than the discrete topology.
(2) $\mathbb{R}$ is not minimal in virtue of Exercise 3.2.
(3) Let τ be a strictly weaker topology on $\mathbb{R}$ described in the previous item. Then $(\mathbb{C}, \tau \times \tau)$ is strictly weaker than $\mathbb{C}$ with the euclidean topology. By definition, $\mathbb{C}$ is not minimal.
(4) $\mathbb{T}^3$ is compact Hausdorff and therefore minimal.
(5) It is easy to see that $\mathbb{C}^*$ is topologically isomorphic to $\mathbb{T} \times \mathbb{R}$. We already established that $\mathbb{R}$ is not minimal, so neither is $\mathbb{C}^*$ (following a similar argument to (3)).

$\square$

Exercise 4.4. Let $G \in LCA$. Prove that G is compact if and only if G is minimal.

Proof. First, if G is compact then it is clearly minimal.

Conversely, suppose that $(G.\tau)$ is locally compact and minimal. Consider the weak topology τ_w on G generated by all possible homomorphisms $f : G \to \mathbb{T}$ (here G is discrete). Such homomorphisms separate the points as we already established for every discrete abelian G. Hence, we have a topological group embedding $i : (G, \tau_w) \hookrightarrow \mathbb{T}^G$. Clearly, $\tau_w \subseteq \tau$. The minimality of (G, τ) guarantees that $\tau_w = \tau$. Since (G, τ) is a locally compact and $\mathbb{T}^G$ is Hausdorff, the subgroup $i(G)$ must be closed in $\mathbb{T}^G$. Therefore, $i(G)$ and, hence, also (G, τ)), are compact.

□

Exercise 4.5. Prove that every compact abelian Hausdorff topological group G can be embedded into some torus group $\mathbb{T}^S$.

Proof. In virtue of Pontryagin duality, G^* is discrete. Using Exercise 4.1, we know that G^{**} is closed in $\mathbb{T}^{G^*}$. Recall that $G \simeq G^{**}$, so by writing $S = G^*$ we get that G can be embedded in $\mathbb{T}^S$. □

Exercise 4.6. Prove that the class NA is closed under topological subgroups, products and quotient groups.
Proof.

(1) *subgroups*: Let G be an NA group, and $H \leq G$ be a subgroup. By definition, there exists a local basis $\gamma = \{O_j \mid j \in J\}$ for G where $\{O_j\}_{j\in J}$ are subgroups. It is easy to see that $\gamma_H := \{O_j \cap H \mid j \in J\}$ is a local basis for H, consisting of subgroups. By definition, H is an NA group.
(2) *products*: Suppose that G and H are NA groups. Write $\gamma_G = \{U_j \mid j \in J\}$ and $\gamma_H = \{V_i \mid i \in I\}$ for the local bases consisting of subgroups. It is easy to see that $\{U_j \times V_i \mid (j, i) \in J \times I\}$ is a local basis for $G \times H$.
(3) *subgroups*: Suppose that G is an NA group and H is a normal subgroup. By definition, there exists a local basis $\gamma = \{O_j \mid j \in J\}$ for G where $\{O_j\}_{j\in J}$ are subgroups. It is easy to see that $\gamma' := \{O_j H \mid j \in J\}$ is a local basis of G/H consisting of subgroups, making G/H an NA group.

□

Exercise 4.7. Let (X, μ) be a uniform space. Prove that $(X, top(\mu))$ (defined in lecture notes) is a topological space. Show that this topology is Hausdorff iff

$$\cap\{\varepsilon : \varepsilon \in \mu\} = \{(x, x) : x \in X\}$$

Proof. We will show the following properties:

(1) $\{\emptyset, X\} \subseteq top(\mu)$: obvious.
(2) If $U, V \in top(\mu)$, then $U \cap V \in top(\mu)$: Let $x \in U \cap V$. By definition, there exists $\varepsilon_1, \varepsilon_2 \in \mu$ such that if $(x, y_1) \in \varepsilon$ and $(x, y_2) \in \varepsilon$ then $y_1 \in U, y_2 \in V$. Define $\delta := \varepsilon_1 \cap \varepsilon_2 \in \mu$. It is easy to see that if $(x, y) \in \delta$, then $(x, y) \in \varepsilon_1$ and $(x, y) \in \varepsilon_2$. As a consequence, $y \in U \cap V$, as required.
(3) Suppose that $\{O_j\}_{j\in J}$ is a family in $top(\mu)$. Write $O := \bigcup_{j\in J} O_j$. Suppose that $x \in O$. By definition, there exist some $j_0 \in J$ and $\varepsilon \in \mu$ such that if $(x, y) \in \varepsilon$ then $y \in O_{j_0}$. In particular, $y \in O$. This is true for every $x \in O$ making it open in $top(\mu)$.

Now, suppose that $top(\mu)$ is Hausdorff. Thus, for every $x \neq y \in X$ there exists $\varepsilon \in \mu$ such that $(x, y) \neq \varepsilon$. As a consequence, $(x, y) \notin \bigcap \mu$. On other words,

$$\{(x, x) \mid x \in X\} = \{(x, y) \in X^2 \mid x \neq y\}^c \supseteq \bigcap \mu$$

Also, since every $\varepsilon \in \mu$ is reflexive, $(x,x) \in \varepsilon$. Therefore, $(x,x) \in \bigcap \mu$ for every $x \in X$. We have successfully shown that

$$\cap\{\varepsilon \mid \varepsilon \in \mu\} = \{(x,x)\colon x \in X\}.$$

Conversely, assume that $\cap\{\varepsilon \mid \varepsilon \in \mu\} = \{(x,x)\colon x \in X\}$. Suppose that $x \neq y \in X$. Since $(x,y) \notin \bigcap \mu$, there exists some $\varepsilon \in \mu$ such that $(x,y) \notin \varepsilon$. Find $\delta \in \mu$ such that $\delta \circ \delta \subseteq \varepsilon$. Write

$$U := \delta[x] := \{z \in X \mid (x,z) \in \delta\} \text{ and } V := \delta[y] := \{z \in X \mid (y,z) \in \delta\}.$$

We claim that U and V are disjoint. Indeed, if $w \in U \cap V$, then $(x,w),(w,y) \in \delta$. As a consequence, $(x,y) \in \delta \circ \delta \subseteq \varepsilon$, a contradiction. □

Exercise 4.8.

(1) Let G be a (Hausdorff) topological group. Show that (G, μ_r) (defined in lecture notes) indeed is a (Hausdorff) uniform space.
(2) Let G_1, G_2 be topological groups and $f : G_1 \to G_2$ be a group homomorphism. Show that f is continuous iff f is uniformly continuous with respect to the corresponding right uniformities μ_r^1, μ_r^2.

Proof.

(1) We show the properties for the basis elements $\{E_U^r\}_{e \in U \subseteq X}$.
   (a) $\forall \varepsilon \in \mu_r : \Delta \subseteq \varepsilon$: Indeed, for every neighborhood $e \in U \subseteq G$ we have $(x,x) \in E_U^r$. As a consequence, $\Delta = \{(x,x) \mid x \in X\}$ is contained in every entourage.
   (b) $\forall \varepsilon \in \mu_r : \varepsilon^{-1} \in \varepsilon$: Suppose that $e \in U \subseteq G$ is a neighborhood of zero. Define If $(x,y) \in E_U^r$, then $xy^{-1} \in U$ by definition. Therefore, $(y, x^{-1}) \in U^{-1}$. Equivalently, $(y,x) \in E_{U^{-1}}^r$. Thus $(E_U^r)^{-1} = E_{U^{-1}}^r \in \mu_r$.
   (c) $\forall \varepsilon \in \mu_r \exists \delta \in \mu_r : \delta \circ \delta \subseteq \varepsilon$: let $e \in U \subseteq G$ be an open neighborhood. Find a neighborhood V such that $VV \subseteq U$. It is easy to see that:
   $$E_V^r \circ E_V^r \subseteq E_U^r.$$
   (d) $\forall \varepsilon_1, \varepsilon_2 \in \mu_r : \varepsilon_1 \cap \varepsilon_2 \in \mu_r$: let $e \in U, V \subseteq G$ be neighborhoods of G. It is easy to see that:
   $$E_U^r \cap E_V^r = E_{U \cap V}^r \in \mu_r.$$
   (e) $\delta \in \mu_r, \delta \subseteq \varepsilon \Rightarrow \varepsilon \in \mu$: the uniformity μ_r is defined via a basis and so it satisfies this requirement.
   (f) If G is Hausdorff, then so is μ_r: application of Exercise **??** and the fact that $top(\mu_r)$ is the original topology of G.
(2) Clearly, if f is uniformly continuous, then it is continuous. Conversely, let $e_2 \in U \subseteq G_2$ be a neighborhood. We need to find a neighborhood $e_1 \in V \subseteq G_1$ such that if $(x,y) \in E_V^r$ then $(f(x), f(y)) \in E_U^r$. Define $V := f^{-1}(U)$. Indeed, if $(x,y) \in E_V^r$, then $xy^{-1} \in V$ and therefore
   $$f(x)(f(y))^{-1} = f(xy^{-1}) \in f(V) \subseteq U.$$
   By definition, $(f(x), f(y)) \in E_U^r$.

□

5. Homework 5

Exercise 5.1. Let $f: G_1 \to G_2$ be a continuous homomorphism between topological groups. Prove that $f: (G_1, \mu_r) \to (G_2, \mu_r)$ is a uniform map.

Proof. Exercise 4.8.2. $\square$

Exercise 5.2. Let G be a NA topological group. Show that (G, μ_r) is a NA uniform space.

Proof. By definition, there exists a local basis $\{H_\lambda\}_{\lambda\in\Lambda}$ consisting of subgroups for the topology of G. As a consequence, $\{E^r_{H_\lambda}\}_{\lambda\in\Lambda}$ is a basis for the right uniformity of (G, μ_r). We claim that each $E^r_{H_\lambda}$ is an equivalence relation, proving that (G, μ_r) is indeed an NA uniform space.

We only need to show transitivity, namely, let $\lambda \in \Lambda$ and $x, y, z \in G$ such that $(x, y), (y, z) \in E^r_{H_\lambda}$. By definition, $xy^{-1}, yz^{-1} \in H_\lambda$. As a consequence:

$$xz^{-1} = x(y^{-1}y)z^{-1} = (xy^{-1})(yz^{-1}) \in H_\lambda H_\lambda = H_\lambda,$$

as required. $\square$

Exercise 5.3. Prove that there exists a proper representation $h: S_{\mathbb{N}} \hookrightarrow \mathrm{Iso}(l_2)$ of the symmetric topological group $S_{\mathbb{N}}$ on the Hilbert space l_2.

Proof. Consider the map $h\colon S_{\mathbb{N}} \to Iso(l_2)$ defined as

$$(h(\sigma))\left(\{a_n\}_{n\in\mathbb{N}}\right) := \{a_{\sigma(n)}\}_{n\in\mathbb{N}}.$$

We will show that this map is continuous. Indeed, suppose that $\{\sigma_\lambda\}_{\lambda\in\Lambda} \subseteq S_{\mathbb{N}}$ converges to σ. We will show that $\lim\limits_{\lambda\in\Lambda} h(\sigma_\lambda)$ converges in $Iso(l_2)$. Let $a \in l_2$ and $\varepsilon > 0$. There exists some $n_0 \in \mathbb{N}$ such that $\sum\limits_{n=n_0}^{\infty} a_n < \frac{1}{2}\varepsilon$. Also, we can find $\lambda_0 \in \Lambda$ such that for every $\lambda_0 \leq \lambda \in \Lambda$ we have $\sigma_\lambda(n) = \sigma_{\lambda_0}(n)$ for every $1 \leq n < n_0$. For such λ, we have

$$\begin{aligned}
\|(h(\sigma_\lambda))(a) - (h(\sigma))(a)\| &= |\sum_{n=0}^{\infty} a_{\sigma_\lambda(n)} - a_{\sigma(n)}| \\
&= |\sum_{n=0}^{n_0-1} \left(a_{\sigma_\lambda(n)} - a_{\sigma(n)}\right) + \sum_{n=n_0}^{\infty} \left(a_{\sigma_\lambda(n)} - a_{\sigma(n)}\right)| \\
&= |\sum_{n=n_0}^{\infty} a_{\sigma_\lambda(n)} - a_{\sigma(n)}| \\
&\leq \sum_{n=n_0}^{\infty} a_{\sigma_\lambda(n)} + \sum_{n=n_0}^{\infty} a_{\sigma(n)} \\
&\leq 2\frac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

To see that the inverse of h is continuous, recall that $\mathbb{N}$ is embedded in l_2 via $n \mapsto e_n$. Let $r\colon Im(h) \to \mathbb{N}^{\mathbb{N}}$ defined as the restriction to $\{e_n\}_{n\in\mathbb{N}}$. The SOT is actually the same as the weak topology induced on $Iso(l_2)$ by l_2 and therefore the restriction is continuous. It is easy to see that $r = h^{-1}$, proving our claim. $\square$

Exercise 5.4. Show that every discrete countable group G admits a proper representation $h : G \hookrightarrow \mathrm{Iso}(l_\infty)$ on the Banach space $(l_\infty, || \cdot ||_{sup})$.

Proof. Since G is countable, it is easy to see that l_∞ is isomorphic to $l_\infty(G)$, so we can equivalently find a representation on $Iso(l_\infty(G))$. Consider the map $h\colon G \to Iso(l_\infty(G))$ defined as

$$\forall f \in l_\infty(G),\ x, y \in G : ((h(x))\,(f))(y) := f(x^{-1}y).$$

Since G is discrete, h is continuous.

Now, we will see that h is open, proving that it is indeed an embedding. It is enough to show that $h(\{e\})$ is open. In-fact, consider $1_e \in l_\infty(G)$ defined by:

$$1_e(g) := \begin{cases} 1 & g = e \\ 0 & \text{else} \end{cases}.$$

Also, consider the neighborhood $h(\{e\}) \in U \subseteq Iso(l_\infty(G))$:

$$U := \{T \in Iso(l_\infty(G)) \mid \|1_e - T(1_e)\|_\infty < 1\}.$$

We claim that $U \cap Im(h) = \{h(e)\}$. Indeed, for every $e \neq g \in G$ we have:

$$\begin{aligned} \|1_e - (h(g))(1_e)\|_1 &\geq |1_e(e) - ((h(g))(1_e))(e)| \\ &= |1 - 1_e(g^{-1})| \\ &= |1 - 0| = 1. \end{aligned}$$

□

Exercise 5.5. Let $G \times X \to X$ be a continuous action on a compact Hausdorff space X. Show that $C(X) = \mathrm{RUC}_G(X)$.

Proof. Clearly, $\mathrm{RUC}_G(X) \subseteq C(X)$. To see the converse, suppose that $f \in C(X)$ and $\varepsilon > 0$. Since f is continuous, for every $x \in X$ we can find a neighborhood $x \in U_x \subseteq X$ such that for every $y \in U_x$, $|f(x) - f(y)| < \frac{1}{2}\varepsilon$. Moreover, since the action is continuous, we can find a neighborhood $e \in V_x \subseteq G$ such that $V_x V_x x \subseteq U_x$.

Since X is compact, we can find a finite subcover $x_1, \ldots, x_n \in X$ of $\{V_x x\}_{x \in \mathbb{N}}$. Define $V := \bigcap_{i=1}^{n} V_n$. We claim that for every $v \in V$ and $x \in X$ we have $|f(vx) - f(x)| < \varepsilon$. Indeed, for every $x \in X$ there exists $1 \leq i \leq n$ such that $x \in V_{x_i} x_i$. As a consequence,

$$vx \in Vx \subseteq V_{x_i} x \subseteq V_{x_i} V_{x_i} x_i \subseteq U_{x_i}.$$

Therefore:

$$\begin{aligned} |f(vx) - f(x)| &= |f(vx) - f(x_i) + f(x_i) - f(x)| \\ &\leq |f(vx) - f(x_i)| + |f(x_i) - f(x)| \\ &\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

□

Exercise 5.6. Let $\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the natural action by rotations (of the circle group $\mathbb{T}$ on the plane $X = \mathbb{R}^2$). Construct a continuous bounded function $f \in C_b(X)$ such that $f \notin \mathrm{RUC}_G(X)$.

Proof. For $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, we have the following action (by rotations) :

$$\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad t \cdot (x, y) := (x \cos t + y \sin t, -x \sin t + y \cos t).$$

In particular,

Consider the function $f(x,y) = \begin{cases} 1 & y \geq 1 \\ y & -1 \leq y \leq 1 \\ -1 & y \leq -1 \end{cases}$

Let $\varepsilon = 1$. For every nbd U of identity in $\mathbb{T}$ there exists $t \in U$ and $x \in \mathbb{R}$ such that $-x \sin t + \cos t \leq -1$. Since $t \cdot (x, 1) = (x \cos t + \sin t, -x \sin t + \cos t)$, we have $f(t \cdot (x, y)) = -1$. Therefore we get

$$|f(t \cdot (x, y)) - f((x, y))| = |-1 - 1| = 2 > \epsilon = 1.$$

$\square$

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