Every topological group is a group retract of a minimal group

Michael Megrelishvili

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

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Abstract

We show that every Hausdorff topological group is a group retract of a minimal topological group. This first was conjectured by Pestov in 1983. Our main result leads to a solution of some problems of Arhangel'skii. One of them is the problem about representability of a group as a quotient of a minimal group (Problem 519 in the first edition of 'Open Problems in Topology'). Our approach is based on *generalized Heisenberg groups* and on groups arising from group representations on Banach spaces.
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Contents

1. Introduction

A Hausdorff topological group G is *minimal* (introduced by Stephenson [34] and Doïtchinov [9]) if G does not admit a strictly coarser Hausdorff group topology. *Totally minimal* groups are defined by Dikranjan and Prodanov [7]

as those Hausdorff groups G for which all Hausdorff quotients are minimal. Later these groups were studied also by Schwanengel [32] under the name *q-minimal groups*. First we recall some facts about minimality. For more comprehensive information about minimal groups theory we refer to the books [8,15], review papers [2] and [5] and also a recent article by Dikranjan and the present author [6].

Unless stated otherwise, all spaces in this paper are at least Hausdorff. Let Y be a topological subspace of X. As usual, a continuous map $f : X \to Y$ is a retraction if $f(y) = y$ for every $y \in Y$. Then Y is said to be a retract of X. If X is Hausdorff then a retraction $f : X \to Y$ is a quotient map and Y is closed in X. If X and Y are topological groups then *group retract* will mean that the corresponding retraction is a group homomorphism.

Most obvious examples of minimal groups are compact groups. Stephenson showed [34] that every Abelian locally compact minimal group must be compact. Prodanov and Stoyanov established one of the most fundamental results in the theory proving that every Abelian minimal group is precompact [8, Section 2.7]. Dierolf and Schwanengel [4] using semidirect products found some interesting examples of non-precompact (hence, non-Abelian) minimal groups. These results imply for instance that an arbitrary discrete group is a group retract of a locally compact minimal group. Also the semidirect product $\mathbb{R} \rtimes \mathbb{R}_+$ of the group of all reals $\mathbb{R}$ with the multiplicative group $\mathbb{R}_+$ of positive reals is minimal. Now it is known [17] that $\mathbb{R}^n \rtimes \mathbb{R}_+$ is minimal for every $n \in \mathbb{N}$. It follows that many minimal groups may have non-minimal quotients (in other words, quite often minimal groups fail to be totally minimal) and non-minimal closed subgroups. Note that all *closed* subgroups of an Abelian minimal group are again minimal (see [8, Proposition 2.5.7]). Motivated by these results Arhangel'skii posed the following two natural questions.

Question A. *(See [1, Problem VI.6].) Is every topological group a quotient of a minimal group*?

Question B. *(See [2, Section 3.3F, Question 3.3.1(a)] and [5, page 57].) Is every topological group G a closed subgroup of a minimal group M*?

Question A appears also in the volume of 'Open Problems in Topology' [24, Problem 519] and in two review papers [2, Question 3.3.1] and [5, Question 2.9].

The following was conjectured by Pestov.

Conjecture *(Pestov 1983). Every topological group is a group retract of a minimal topological group.*

Uspenskij in his first version of [38] (April, 2000) announced a positive answer which was reflected in the survey [2, Theorem 3.3F.2]. However, Uspenskij later withdrew his announcement after he found a gap in the proof.

Remus and Stoyanov [30] proved that every compactly generated locally compact Abelian group is a group retract of a minimal locally compact group. In [16] we show that *Heisenberg type groups* frequently are minimal (see Section 3). For instance, if G is locally compact Abelian with the canonical duality mapping $\omega : G^* \times G \to \mathbb{T}$ then the corresponding generalized Heisenberg group $H(\omega) := (\mathbb{T} \times G^*) \rtimes G$ is minimal. It follows that every Abelian locally compact group is a group retract of a minimal locally compact group as G is obviously a natural retract of $H(\omega)$ (see Section 2).

By [16, Theorem 4.13] every Abelian topological group is a quotient of a minimal group. Note also that by [18, Theorem 6.12] every topological subgroup G of the group $Iso(V)$ of all linear isometries of a reflexive (even Asplund) Banach space V is a group retract of a minimal group. This includes all locally compact groups because they are subgroups of $Iso(H)$ where H is a Hilbert space.

In the present paper we obtain the following main result (see Theorem 7.2).

Main theorem. *Every topological group is a group retract of a minimal group.*

It shows that Pestov's Conjecture is true. At the same time it solves simultaneously Questions A and B in a strong form. One of the conclusions is that the preservation of minimality under quotients fails as strongly as possible when passing from totally minimal to minimal groups. Note also that if we do not require that G is *closed* in M then this weaker form of Question B (namely: every topological group G is a subgroup of a minimal group M) follows by a result of Uspenskij [38, Theorem 1.3]. On the other hand in Uspenskij's result the minimal group M in addition is: (a) *Raikov-complete* (that is complete with respect to the two sided uniformity); (b) topologically simple and hence

totally minimal; (c) *Roelcke precompact* (that is the infimum $\mathcal{U}_{\mathcal{L}} \wedge \mathcal{U}_{\mathcal{R}}$ (see Section 2) of right and left uniformities is precompact); and (d) preserves the weight of G.

Our construction preserves some basic topological properties like the weight, character, and the pseudocharacter. More precisely: in the main theorem we prove that every topological group G can be represented as a group retract of a minimal group M such that simultaneously $w(M) = w(G)$, $\chi(M) = \chi(G)$ and $\psi(M) = \psi(G)$ hold. In particular, if G is metrizable (or second countable) then the same is true for M. Moreover if G is Raikov-complete or *Weil-complete* (the latter means that G is complete with respect to the right uniformity) then in addition we can assume that M also has the same property. This gives an immediate negative answer to the following

Question C. *(Arhangel'skii, see also [2, Section 3.3D].) Let G be a minimal group which is Raikov-complete. Must $\chi(G) = \psi(G)$? What if G is Weil-complete?*

Note that minimal topological groups with different $\chi(G)$ and $\psi(G)$ (but without completeness assumptions) were constructed independently by Pestov [27], Shakhmatov [33] and Guran [14] (see also [8, Notes 7.7]). This was one of the motivations of Question C.

From our main theorem we derive also that in fact every compact homogeneous Hausdorff space admits a transitive continuous action of a minimal group (see Corollary 7.3 below) which means that minimality makes no obstacle in this setting. This fact negatively answers the following.

Question D. *(Arhangel'skii [1, Problem VI.4], see also [2, Section 3.3G].) Suppose that a minimal group acts continuously and transitively on a compact Hausdorff space. Must X be a dyadic space? Must X be a Dugundji space?*

In the proof of our main result we essentially use the methods of [16]. The main idea of [16] was to introduce a systematic method for constructing minimal groups using group representations and generalized Heisenberg groups. For instance we already proved (see [16, Theorem 4.8] or Theorems 4.4 and A.11 below) that if a group G is *birepresantable*, that is if it admits sufficiently many representations into continuous bilinear mappings (in short: *BR-group*; see Definition 4.3(3)) then G is a group retract of a minimal group. In the present paper we explore this reduction by showing that in fact every topological group is a BR-group. In the proof we use some new results about representations into bilinear mappings. We show (see Theorem 5.10) for instance that a bounded function $f : G \to \mathbb{R}$ on G is left and right uniformly continuous if and only if f is a *matrix coefficient* $f = m_{v,\psi}$ (and hence $f(g) = \psi(vg)$ for every $g \in G$) of a continuous Banach *co-birepresentation* (see the definition in Section 4) $h : G \to Iso(V)$ by linear isometries such that $v \in V$ and $\psi \in V^*$ is a G-continuous vector. This result was inspired by a recent joint paper with E. Glasner [13] characterizing *strongly uniformly continuous* functions on a topological group G in terms of suitable matrix coefficients. The technique in the latter result, as in some related results of [19,21,13], is based on a dynamical modification of a celebrated factorization theorem in Banach space theory discovered by Davis, Figiel, Johnson and Pelczyński [3].

For the readers convenience, in Appendix A we include some proofs of [16].

2. Preliminaries: Actions and semidirect products

Let X be a topological space. As usual denote by $w(X)$, $\chi(X)$, $\psi(X)$, $d(X)$ the weight, character, pseudocharacter and the density of X respectively. All cardinals are assumed to be infinite.

A (*left*) *action* of a topological group G on a space X, as usual, is a function $\pi : G \times X \to X$, $\pi(g, x) := gx$ such that always $g_1(g_2 x) = (g_1 g_2)x$ and $ex = x$ hold, where $e = e_G$ is the neutral element of G. Every $x \in X$ defines an *orbit map* $\tilde{x} : G \to X$, $g \mapsto gx$. Also every $g \in G$ induces a *g-translation* $\pi^g : X \to X$, $x \mapsto gx$. If the action π is continuous then we say that X is a *G-space*. Sometimes we write it as a pair (G, X).

Let G act on X_1 and on X_2. A map $f : X_1 \to X_2$ is said to be a *G-map* if $f(gx) = gf(x)$ for every $(g, x) \in G \times X_1$. A *G-compactification* of a G-space X is a continuous G-map $\alpha : X \to Y$ with a dense range into a compact G-space Y.

A *right action* $X \times G \to X$ can be defined analogously. If G^{op} is the *opposite group* of G with the same topology then the right G-space (X, G) can be treated as a left G^{op}-space (G^{op}, X) (and vice versa). A map $h : G_1 \to G_2$ between two groups is a *co-homomorphism* (or, an *anti-homomorphism*) if $h(g_1 g_2) = h(g_2)h(g_1)$. This happens iff $h : G_1^{op} \to G_2$ (the same assignment) is a homomorphism.

For a real normed space V denote by B_V its closed unit ball $\{v \in V \colon \|v\| \leqslant 1\}$. Denote by $Iso(V)$ the topological group of all linear (onto) isometries $V \to V$ endowed with the *strong operator topology*. This is the topology of pointwise convergence inherited from V^V. Let V^* be the dual Banach space of V and let

$$\langle\,,\rangle \colon V \times V^* \to \mathbb{R}, \qquad (v, \psi) \mapsto \langle v, \psi \rangle = \psi(v)$$

be the canonical (always continuous) bilinear mapping. Let $\pi \colon G \times V \to V$ be a continuous left action of G on V by linear isometries. This is equivalent to saying that the natural homomorphism $h \colon G \to Iso(V)$, $g \mapsto \pi^g$ is continuous. The *adjoint action* $G \times V^* \to V^*$ is defined by $g\psi(v) := \psi(g^{-1}v)$. Then the corresponding canonical form is G-invariant. That is

$$\langle gv, g\psi \rangle = \langle v, \psi \rangle \quad \forall (g, v, \psi) \in G \times V \times V^*.$$

Similarly, let $V \times G \to V$ be a continuous *right* action of G on V by linear isometries. Then the corresponding *adjoint action* (from the left) $G \times V^* \to V^*$ is defined by $g\psi(v) := \psi(vg)$. Then we have the following equality

$$\langle vg, \psi \rangle = \langle v, g\psi \rangle \quad \forall (g, v, \psi) \in G \times V \times V^*.$$

Adjoint actions of G on V^* do not remain continuous in general (see for example [18]).

The Banach algebra (under the supremum norm) of all continuous real valued bounded functions on a topological space X will be denoted by $C(X)$. Let (G, X) be a left G-space. It induces the right action $C(X) \times G \to C(X)$, where $(fg)(x) = f(gx)$, and the corresponding co-homomorphism $h \colon G \to Iso(C(X))$. While the g-translations $C(X) \to C(X)$ are continuous (being isometric), the orbit maps $\tilde{f} \colon G \to C(X)$, $g \mapsto fg$ are not necessarily continuous. However if X is a compact G-space then every $\tilde{f}$ is continuous and equivalently the action $C(X) \times G \to C(X)$ is continuous.

For every topological group G denote by $RUC(G)$ the Banach subalgebra of $C(G)$ of *right uniformly continuous* (some authors call these functions *left uniformly continuous*) bounded real valued functions on G. These are the functions which are uniformly continuous with respect to the *right uniform structure* $\mathcal{U}_{\mathcal{R}}$ on G. Thus, $f \in RUC(G)$ iff for every $\varepsilon > 0$ there exists a neighborhood V of the identity element $e \in G$ such that $\sup_{g \in G} |f(vg) - f(g)| < \varepsilon$ for every $v \in V$. It is equivalent to say that the orbit map $G \to C(G)$, $g \mapsto fg$ is norm continuous where fg is the *left translation* of f defined by $(fg)(x) := f(gx)$.

Analogously can be defined the algebra $LUC(G)$ of *left uniformly continuous* functions (and the *right translations*). These are the functions which are uniformly continuous with respect to the *left uniform structure* $\mathcal{U}_{\mathcal{L}}$ on G.

Denote by $\mathcal{U}_{\mathcal{L}} \wedge \mathcal{U}_{\mathcal{R}}$ the *lower uniformity* of G. It is the infimum (greatest lower bound) of left and right uniformities on the set G. The intersection $UC(G) := RUC(G) \cap LUC(G)$ is a left and right G-invariant closed subalgebra of $RUC(G)$. Clearly, for every bounded function $f \colon G \to \mathbb{R}$ we have $f \in UC(G)$ iff $f \colon (G, \mathcal{U}_{\mathcal{L}} \wedge \mathcal{U}_{\mathcal{R}}) \to \mathbb{R}$ is uniformly continuous. We need the following important fact.

Lemma 2.1. *(See Roelcke–Dierolf [31].)*

(1) *For every topological group G the lower uniformity $\mathcal{U}_{\mathcal{L}} \wedge \mathcal{U}_{\mathcal{R}}$ generates the given topology of G.*
(2) *For every topological group G the algebra $UC(G)$ separates points from closed subsets in G.*

Proof. (1) See Roelcke–Dierolf [31, Proposition 2.5]. For a direct argument we can use the fact that a typical neighborhood of $x \in G$ with respect to the lower uniformity $\mathcal{U}_{\mathcal{L}} \wedge \mathcal{U}_{\mathcal{R}}$ has the form UxU where U is a neighborhood of e.

(2) Follows from (1). $\square$

Note that in general the infimum $\mu_1 \wedge \mu_2$ of two compatible uniform structures on a topological space X need not be compatible with the topology of X (see for example [36, Remark 2.2]).

Let (X, τ) and (G, σ) be topological groups and

$$\alpha \colon G \times X \to X, \qquad \alpha(g, x) = gx = g(x)$$

be a given (left) action. We say that X is a *G-group* if α is continuous and every g-translation $\alpha^g \colon X \to X$ is a group automorphism of X. For every G-group X denote by $X \rtimes_\alpha G$ the corresponding *topological semidirect product* (see

for example [31, Section 6] or [8, Ch. 7]). As a topological space this is the product $X \times G$. The standard multiplicative group operation is defined by the following rule. For a pair $(x_1, g_1), (x_2, g_2)$ in $X \rtimes_\alpha G$ let

$$(x_1, g_1) \cdot (x_2, g_2) := (x_1 \cdot g_1(x_2), g_1 \cdot g_2).$$

Sometimes the closed normal subgroup $X \times \{e_G\}$ of $X \rtimes_\alpha G$ will be identified with X and similarly, the closed subgroup $\{e_X\} \times G$ will be identified with G. The projection $p : X \rtimes_\alpha G \to G$, $p(x, g) = g$ is a group homomorphism and also a retraction. In particular, G is a quotient of $X \rtimes_\alpha G$. The kernel of this retraction $\ker(p)$ is just $X \times \{e_G\}$.

Definition 2.2. Let (X, τ) and (G, σ) be Hausdorff topological groups and

$$\alpha : G \times X \to X, \qquad \alpha(g, x) = gx = g(x)$$

be a continuous action by group automorphisms.

(1) The action α is *topologically exact* (*t-exact*, for short) if there is no strictly coarser, *not necessarily Hausdorff*, group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ', τ, τ)-continuous (see [17]).
(2) More generally, let $\{\alpha_i : G \times Y_i \to Y_i\}_{i \in I}$ be a system of continuous G-actions on the groups Y_i. We say that this system is *t-exact* if there is no strictly coarser, not necessarily Hausdorff, group topology $\sigma' \subsetneq \sigma$ on G such that all given actions remain continuous.
(3) X is a *G-minimal group* if there is no strictly coarser Hausdorff group topology $\tau' \subsetneq \tau$ on X such that α remains continuous with respect to the triple (σ, τ', τ') of topologies (see [30]).

Lemma 2.3. *Let $(X \rtimes_\alpha G, \gamma)$ be a topological semidirect product. Suppose that X is G-minimal with respect to α. Then for every coarser Hausdorff group topology $\gamma_1 \subset \gamma$ we have $\gamma_1|_X = \gamma|_X$.*

Proof. Since $P := (X \rtimes_\alpha G, \gamma_1)$ is a topological group the conjugation map

$$(P, \gamma_1) \times (P, \gamma_1) \to (P, \gamma_1), \qquad (a, b) \to aba^{-1}$$

is continuous. Then its restriction

$$(G, \gamma_1|_G) \times (X, \gamma_1|_X) \to (X, \gamma_1|_X), \qquad (g, x) \to g(x) = gxg^{-1}$$

is also continuous. Since $\gamma_1|_G \subset \gamma|_G$ it follows that the action of the given group $(G, \gamma|_G)$ on the Hausdorff group $(X, \gamma_1|_X)$ is continuous, too. Since $\gamma_1|_X \subset \gamma|_X$ and X is G-minimal we obtain $\gamma_1|_X = \gamma|_X$. $\quad \square$

Theorem 2.4. *Let G be a Hausdorff topological group and let X be an Abelian G-minimal group. Then if the given action $\alpha : G \times X \to X$ is t-exact then $X \rtimes_\alpha G$ is minimal.*

Proof. See Theorem 1.4 of [17] (or combine Lemma 2.3 and Corollary A.6). $\quad \square$

Lemma 2.5. *(See [31, Proposition 12.5].) Let $P := X \rtimes_\alpha G$ be a topological semidirect product. If X and G are Raikov-complete (Weil-complete) then P is a Raikov-complete (respectively, Weil-complete) group.*

Remark 2.6.

(1) Note that in [16] the original definition of t-exactness contains a superfluous condition of *algebraic exactness*. The latter means that the kernel of the action $\ker(\alpha) := \{g \in G : gx = x \ \forall x \in X\}$ is trivial. The reason is that since G is Hausdorff every t-exact action is algebraically exact. Indeed assuming the contrary let $H := \ker(\alpha)$ be the *non-trivial* kernel of the action α. Consider the quotient group G/H with the coset topology τ/H and the map $q : G \to G/H$, $g \mapsto q(g) = gH$. Then the induced action of G/H on X is continuous. It follows that the preimage topology $\tau' := q^{-1}(\tau/H)$ on G is a group topology such that α remains continuous. Since $q^{-1}(\tau/H)$, being not Hausdorff, is strictly coarser than the original (Hausdorff) topology of G we obtain that α is not t-exact.

(2) Suppose that X is a G-group under the action $\pi : G \times X \to X$ such that the semidirect product $X \rtimes_\pi G$ is minimal. Then if π is algebraically exact then π necessarily is t-exact. Indeed, otherwise there exists a strictly coarser group topology τ' on G such that $\alpha : (G, \tau') \times X \to X$ remains continuous. Since X is Hausdorff for every $x \in X$ and every g from the τ'-closure $cl_{\tau'}(\{e\})$ of the singleton $\{e\}$ we have $gx = x$. Since the action is algebraically exact we get $cl_{\tau'}(\{e\}) = \{e\}$. Thus, τ' is Hausdorff. Then the semidirect product $X \rtimes_\alpha (G, \tau')$ is a Hausdorff topological group and its topology is strictly coarser than the original topology on $X \rtimes_\alpha (G, \tau)$. This contradicts the minimality of the latter group.

(3) The direct product $X \times G$ of two minimal Abelian (even cyclic) groups X and G may not be minimal. Take for example $X = G = (\mathbb{Z}, \tau_p)$ with the p-adic topology τ_p (see Doïchinov [9]). Since X is minimal it also can be treated as a G-minimal group with respect to the trivial action of G on X. Then the direct product is just the semidirect product in our setting. It follows (as expected, of course) that the t-exactness is essential in Theorem 2.4. This example also demonstrates that the words 'not necessarily Hausdorff' in Definition 2.2(2) of the t-exactness cannot be omitted.

(4) If X is a locally compact Hausdorff group and G is a subgroup of $Aut(X)$ endowed with the standard *Birkhoff topology* (see [8,16]) then the corresponding action is t-exact.

(5) For every normed space V and a topological subgroup G of $Iso(V)$ the action of G on V is t-exact.

(6) According to [10, Example 10] there exists a totally minimal precompact group X such that a certain semidirect product $X \rtimes \mathbb{Z}_2$ with the two-element cyclic group $\mathbb{Z}_2$ is not minimal. The given action of $\mathbb{Z}_2$ on X is t-exact. Indeed, by the construction the action is not trivial. On the other hand every strictly coarser group topology on the (discrete) group $\mathbb{Z}_2$ is the trivial topology. This example demonstrates that Theorem 2.4 is not true in general for non-Abelian X.

3. Generalized Heisenberg groups

We need a natural generalization of the classical three-dimensional Heisenberg group. This generalization is based on semidirect products defined by biadditive mappings. See, for example, [29,25,16]. For additional properties and applications of this construction we refer also to [17,22,6].

Let E, F, A be Abelian groups. A map $w : E \times F \to A$ is said to be *biadditive* if the induced mappings

$$\omega_x : F \to A, \qquad w_f : E \to A, \qquad \omega_x(f) := \omega(x, f) =: \omega_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$. Sometimes we look at the elements f of F as functions defined on E, the value $f(x)$, for an element x of E is defined as $\omega(x, f)$. We say that ω is *separated* if the induced homomorphisms separate points, that is, for every $x_0 \in E$, $f_0 \in F$ there exist $f \in F$, $x \in E$ such that $f(x_0) \neq 0_A$, $f_0(x) \neq 0_A$, where 0_A is the zero element of A.

Definition 3.1. Let E, F and A be Abelian topological groups and $\omega : E \times F \to A$ be a continuous biadditive mapping. Denote[1] by

$$H(\omega) = (A \times E) \rtimes_{\omega^\nabla} F$$

the semidirect product (say, *generalized Heisenberg group* induced by ω) of F and the group $A \times E$ with respect to the action of F on $A \times E$ defined as follows

$$\omega^\nabla : F \times A \times E \to A \times E, \qquad \omega^\nabla\big(f, (a, x)\big) = \big(a + f(x), x\big),$$

where $f(x) = \omega_f(x)$. The resulting group, as a topological space, is the product $A \times E \times F$. The group operation is defined by the following rule. For a pair

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2)$$

define

$$u_1 \cdot u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big).$$

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[1] The notation of the present paper about generalized Heisenberg groups and some related objects do not always agree with the notation of [16].

Then $H(\omega)$ becomes a two-step nilpotent (in particular, metabelian) topological group. Elementary computations for the commutator $[u_1, u_2]$ give

$$[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1} = \big(f_1(x_2) - f_2(x_1), 0_E, 0_F\big).$$

If ω is separated then the center of the group $H(\omega)$ is the subgroup A.

The very particular case of the canonical bilinear form $\langle\,,\,\rangle: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defines the classical $2n+1$-dimensional Heisenberg group. If A, E and F are complete groups then by Lemma 2.5 the corresponding Heisenberg group $H(\omega)$ is Weil-complete.

In [16] we show that generalized Heisenberg groups are useful in the theory of minimal topological groups. One of the results obtained there is as follows. For every locally compact Abelian group G the Heisenberg group $H(\Delta) = \mathbb{T} \times G^* \rtimes_\Delta G$ of the canonical biadditive mapping $\Delta: G^* \times G \to \mathbb{T}$, $\Delta(\chi, g) = \chi(g)$, where $\mathbb{T}$ denotes the circle group, is minimal. It follows that every locally compact Abelian group is a group retract of a locally compact minimal group.

For more examples of minimal groups that come from biadditive mappings see [6].

The following definition generalizes slightly [16, Definition 3.1].

Definition 3.2. Let E and F be (semi)normed spaces and $\omega: E \times F \to \mathbb{R}$ be a bilinear map. We say that

(1) ω is a *left strong duality* if for every norm-unbounded sequence $x_n \in E$ the subset $\{f(x_n): n \in \mathbb{N}, \ f \in B_F\}$ is unbounded in $\mathbb{R}$. This is equivalent to saying that $\{f(x_n): n \in \mathbb{N}, \ f \in O\} = \mathbb{R}$ for every neighborhood O of the zero 0_F (or, even, for every nonempty open subset O in F).
(2) ω is a *right strong duality* if for every norm-unbounded sequence $f_n \in F$ the subset $\{f_n(x): n \in \mathbb{N}, \ x \in B_E\}$ is unbounded in $\mathbb{R}$.
(3) We call simply a *strong duality* if ω is left and right strong.

If $\omega: E \times F \to \mathbb{R}$ is a strong duality with normed spaces E and F then ω necessarily is separated. Indeed, let $v \in V$, $v \neq 0_E$ and $f(v) = 0$ for every $f \in F$. Since E is a normed space and $v \neq 0_E$ we have $\|v\| > 0$. Then the sequence $x_n := nv_n$ is unbounded in V. On the other hand $\{f(x_n): n \in \mathbb{N}, \ f \in B_F\} = \{0\}$. This means that ω is not left strong. Similar proof in the case of $f \in F$, $f \neq 0_F$ with $f(v) = 0$ for every $v \in V$.

Example 3.3. (See [16].)

(1) For every normed space V the canonical bilinear form $\langle\,,\,\rangle: V \times V^* \to \mathbb{R}$ is a strong duality.
(2) For every locally compact group G the natural bilinear form

$$\omega: L^1(G) \times \mathcal{K}(G) \to \mathbb{R}$$

is a strong duality.

Here $\mathcal{K}(G)$ is the normed space of all continuous real valued functions with compact supports endowed with the sup-norm. It can be treated as a *proper* subspace of $L^1(G)^* := L^\infty(G)$ such that ω is a restriction of the canonical form $L^1(G) \times L^1(G)^* \to \mathbb{R}$. Hence the second example is not a particular case of (1).

Let $\omega: E \times F \to \mathbb{R}$ be a separated bilinear mapping. Then the generalized Heisenberg group $H(\omega) = \mathbb{R} \times E \rtimes_{\omega^\nabla} F^*$ is not minimal. Indeed the center of a minimal group must be minimal (see for example [8, Proposition 7.2.5]) and the center of $H(\omega)$ is the subgroup $\mathbb{R}$ which is not minimal. Note however that the subgroups V and V^* are *relatively minimal* in $H(\omega)$ (see [6,22]) for the canonical duality $\omega = \langle\,,\,\rangle: V \times V^* \to \mathbb{R}$ for every normed space V.

Now we define as in [16] the semidirect product

$$H_+(\omega) := H(\omega) \rtimes_\alpha R_+ = (\mathbb{R} \times E \rtimes_{\omega^\nabla} F^*) \rtimes_\alpha \mathbb{R}_+$$

where $\mathbb{R}_+$ is the multiplicative group of all positive reals and α is the natural action

$$\alpha: \mathbb{R}_+ \times H(\omega) \to H(\omega), \qquad \alpha\big(t, (a, c, f)\big) = (ta, tx, f).$$

Observe that the third coordinate after the t-translation is just 'f' and not 'tf'.

It turns out that $H_+(\omega)$ is minimal under natural restrictions providing a lot of examples of minimal groups.

Theorem 3.4. *(See [16].) For every strong duality $\omega : E \times F \to \mathbb{R}$ with normed spaces E and F the corresponding group $H_+(\omega) = H(\omega) \lambda_\alpha \mathbb{R}_+$ is minimal.*

Proof. See [16, Proposition 3.6] or Theorem A.10. □

Furthermore, using (twice) Lemma 2.5 we get

Lemma 3.5. *For every continuous bilinear mapping $\omega : E \times F \to \mathbb{R}$ with Banach spaces E and F the corresponding group $H_+(\omega) = H(\omega) \lambda_\alpha R_+ = (\mathbb{R} \times E \lambda_{\omega^\triangledown} F^*) \lambda_\alpha \mathbb{R}_+$ is Weil-complete.*

4. Group representations in bilinear forms

Let G be a topological group. A *representation* (*co-representation*) of G on a normed space V is a homomorphism (respectively co-homomorphism) $h : G \to Iso(V)$. Sometimes we give the representation (co-representation) by the corresponding linear isometric left (respectively, right) action $G \times V \to V$, $(g, v) \mapsto gv = h(g)(v)$ (respectively, $V \times G \to V$, $(v, g) \mapsto vg = h(g)(v)$). A (co)representation $h : G \to Iso(V)$ is *continuous* if h is continuous where $Iso(V)$, as usual, is endowed with the strong operator topology. It is equivalent to say that the corresponding action is continuous.

Definition 4.1. Let V, W be normed spaces and $\omega : V \times W \to \mathbb{R}$ be a continuous bilinear map.

(1) Let $h_1 : G \to Iso(V)$ and $h_2 : G \to Iso(W)$ be two homomorphisms. We say that the pair (h_1, h_2) is a *birepresentation* of G in ω if ω is G-invariant, that is,

$$\omega(gv, g\psi) = \omega(v, \psi) \quad \forall (g, v, \psi) \in G \times V \times W.$$

(2) Let $h_1 : G \to Iso(V)$ be a co-homomorphism and $h_2 : G \to Iso(W)$ be a homomorphism. We say that the pair (h_1, h_2) is a *co-birepresentation* of G in ω if

$$\omega(vg, \psi) = \omega(v, g\psi) \quad \forall (g, v, \psi) \in G \times V \times W.$$

(3) A (co)birepresentation (h_1, h_2) is *continuous* if the functions h_1 and h_2 both are continuous.

The following definition is one of the key ideas of [16], as well as of the present paper. Assume that the pair $\alpha_1 : G \times E \to E$, $\alpha_2 : G \times F \to F$ of actions determines a birepresentation Ψ of G in $\omega : E \times F \to \mathbb{R}$. By the *induced group* $M_+(\Psi)$ of the given birepresentation Ψ we mean the topological semidirect product $H_+(\omega) \lambda_\pi G$, where the action

$$\pi : G \times H_+(\omega) \to H_+(\omega)$$

is defined by $\pi(g, (a, x, f, t)) := (a, gx, gf, t)$, where $gx = \alpha_1(g, x)$ and $gf = \alpha_2(g, f)$.

More generally, let

$$\Phi := \{\Phi_i\}_{i \in I} = \{\omega_i : E_i \times F_i \to \mathbb{R}, \ \alpha_{1i} : G \times E_i \to E_i, \ \alpha_{2i} : G \times F_i \to F_i\}_{i \in I}$$

be a system of continuous G-birepresentations. By the *induced group* $M_+(\Phi)$ of the system Φ we mean the semidirect product

$$\prod_{i \in I} M_+(\omega_i) \lambda_\pi G$$

where the action

$$\pi : G \times \prod_{i \in I} M_+(\omega_i) \to \prod_{i \in I} M_+(\omega_i)$$

is defined coordinatwise by means of the following system $\{\pi_i\}_{i \in I}$ of actions (as defined above):

$$\pi_i : G \times H_+(\omega_i) \to H_+(\omega_i), \qquad \pi_i\big(g, (a, x, f, t)\big) := (a, gx, gf, t).$$

Remark 4.2. Let $h : G \to Iso(V)$ be a co-homomorphism (homomorphism). Denote by h^{op} the *associated ho-momorphism* (respectively co-homomorphism) $h^{op} : G \to Iso(V)$, $g \mapsto h(g^{-1})$. Then the pair (h_1, h_2) is a co-birepresentation of G iff (h_1^{op}, h_2) is a birepresentation of G in ω.

Analogously for every system Φ of co-birepresentations we can define the naturally associated system Φ^{op} of birepresentations.

Definition 4.3.

(1) Let $\Phi := \{ \omega_i : E_i \times F_i \to \mathbb{R}, \ \alpha_{1i} : G \times E_i \to E_i, \ \alpha_{2i} : G \times F_i \to F_i \}_{i \in I}$ be a system of continuous G-birepresentations. We say that this system is *topologically exact* (*t-exact*) if (G, τ) is a Hausdorff topological group and for every strictly coarser (not necessarily Hausdorff) group topology $\tau' \subset \tau$ on G there exists an index $i \in I$ such that one of the actions $\alpha_{1i} : (G, \tau') \times E_i \to E_i$ or $\alpha_{1i} : (G, \tau') \times F_i \to F_i$ is not continuous.
Similarly can be defined *t*-exact systems of *co-birepresentations*. Obviously the system of co-birepresentations is *t*-exact iff the associated system (see Remark 4.2) of birepresentations is *t*-exact.
(2) (See [16, Definition 4.7].) We say that a topological group G is *birepresentable* (shortly: *BR-group*) if there exists a *t*-exact system Φ of linear birepresentations (equivalently, co-birepresentations (see Remark 4.2)) of G in *strong dualities* ω_i.

The following theorem from [16] (see also Theorem A.11) is one of the crucial results in our setting.

Theorem 4.4. *(See [16, Theorem 4.8].)*

(1) *Let Φ be a t-exact system of G-birepresentations into strong dualities $\omega_i : E_i \times F_i \to \mathbb{R}$ with normed spaces E_i and F_i. Then the corresponding induced group $M_+(\Phi)$ is minimal.*
(2) *For every BR-group G there exists a continuous group retraction $p : M \to G$ such that M and also the kernel $\ker(p)$ are minimal.*

Some results of [16,18] show that many important groups (such as additive subgroups of locally convex spaces and locally compact groups) are BR-groups. One of the main results of the present paper (see Theorem 7.1) shows that in fact every topological group is a BR-group. Furthermore, in Definition 4.3(2) we can always choose a system with $|I| = 1$; that is a system Φ with a single birepresentation.

5. Matrix coefficients of group representations

We generalize the usual notion of matrix coefficients to the case of arbitrary (not necessarily canonical) bilinear mappings.

Definition 5.1. Let V, W be (semi)normed spaces and let $\omega : V \times W \to \mathbb{R}$ be a continuous bilinear map. Let $h_1 : G \to Iso(V)$ be a co-representation and $h_2 : G \to Iso(W)$ be a representation such that the pair (h_1, h_2) is a co-birepresentation of G in ω.

(1) For every pair of vectors $v \in V$ and $\psi \in W$ define the *matrix coefficient* $m_{v,\psi}$ as the following function

$$m_{v,\psi} : G \to \mathbb{R}, \qquad g \mapsto \psi(vg)$$

(where $\psi(vg) = \omega(vg, \psi) = \langle vg, \psi \rangle = \langle v, g\psi \rangle$).
(2) We say that a vector $v \in V$ is *G-continuous* if the corresponding orbit map $\tilde{v} : G \to V$, $\tilde{v}(g) = vg$, defined through $h_1 : G \to Iso(V)$, is norm continuous. Similarly one can define a *G-continuous* vector $\psi \in W$.
(3) We say that a matrix coefficient $m_{v,\psi} : G \to \mathbb{R}$ is *bicontinuous* if $v \in V$ and $\psi \in W$ are *G*-continuous vectors. If (h_1, h_2) is a continuous co-birepresentation (Definition 4.1(3)) then every corresponding matrix coefficient is bicontinuous.

First we need the following

Lemma 5.2. *Let $\omega : V \times W \to \mathbb{R}$, $(v, f) \mapsto \omega(v, f) = \langle v, f \rangle = f(v)$ be a bilinear mapping defined for (semi)normed spaces V and W. The following are equivalent.*

(1) *ω is continuous.*
(2) *For some constant $c > 0$ the inequality $|\langle v, f \rangle| \leqslant c \cdot \|v\| \cdot \|f\|$ holds for every $(v, f) \in V \times W$.*

Proof. (1) $\Longrightarrow$ (2): By the continuity of ω at $(0_V, 0_W)$ there exists a constant $\varepsilon > 0$ such that the inequalities $\|v\| \leqslant \varepsilon$, $\|f\| \leqslant \varepsilon$ imply $|f(v)| \leqslant 1$. Then $|(\varepsilon \cdot \frac{f}{\|f\|})(\varepsilon \cdot \frac{v}{\|v\|})| \leqslant 1$ holds for every $(v, f) \in V \times W$. It follows that $|f(v)| \leqslant \frac{1}{\varepsilon^2} \cdot \|v\| \cdot \|f\|$ for every $(v, f) \in V \times W$.
  (2) $\Longrightarrow$ (1): Is trivial. $\square$

Definition 5.3. We say that a bilinear mapping $\omega : V \times W \to \mathbb{R}$ is *regular* if

$$\big|\langle v, f \rangle\big| \leqslant \|v\| \cdot \|f\|$$

holds for every $(v, f) \in V \times W$. If ω in addition is a strong duality then we call it a *regular strong duality*.

For example, the canonical bilinear mapping $\langle\,,\,\rangle : V \times V^* \to \mathbb{R}$ (and hence its any restriction) is regular for every normed space V. Every regular bilinear mapping is continuous by Lemma 5.2.
  The following observation is a modification of [21, Fact 3.5.2].

Lemma 5.4. *Let $\omega : V \times W \to \mathbb{R}$ be a continuous bilinear mapping. Assume that the pair $h_1 : G \to Iso(V)$, $h_2 : G \to Iso(W)$ is a (not necessarily continuous) co-birepresentation of G in ω. Then every bicontinuous matrix coefficient $f = m_{v,\psi} : G \to \mathbb{R}$ is left and right uniformly continuous on G (that is, $f \in UC(G)$).*

Proof. Since ω is continuous by Lemma 5.2 there exists a constant $c > 0$ such that

$$\big|\langle x, y \rangle\big| \leqslant c \cdot \|x\| \cdot \|y\| \quad \forall (x, y) \in V \times W.$$

Since $h_1(G) \subset Iso(V)$ and $h_2(G) \subset Iso(W)$, we have $\|xg\| = \|x\|$ and $\|gy\| = \|y\|$ for every $(g, x, y) \in G \times V \times W$. Clearly $|m_{v,\psi}(g)| = \langle vg, \psi \rangle \leqslant c \cdot \|v\| \cdot \|\psi\|$. Hence $m_{v,\psi}$ is a bounded function. The matrix coefficient $m_{v,\psi}$ is bicontinuous. By Definition 5.1 this means that v and ψ are G-continuous vectors. In order to establish that $f = m_{v,\psi} \in LUC(G)$, observe that

$$\begin{aligned}
\big|f(gu) - f(g)\big| = \big|m_{v,\psi}(gu) - m_{v,\psi}(g)\big| &= \big|\langle vgu, \psi \rangle - \langle vg, \psi \rangle\big| \\
&= \big|\langle vg, u\psi \rangle - \langle vg, \psi \rangle\big| \leqslant c \cdot \|vg\| \cdot \|u\psi - \psi\| = c \cdot \|v\| \cdot \|u\psi - \psi\|.
\end{aligned}$$

Now using the G-continuity of the vector ψ in W, we get that $f \in LUC(G)$.
  Similar verification is valid for the second case $f \in RUC(G)$. $\square$

Principal Theorem 5.10 shows that the representability of a function as a bicontinuous matrix coefficient in fact characterizes functions from $UC(G) = LUC(G) \cap RUC(G)$.

Definition 5.5. We say that a family S of continuous functions on a topological group G is a *local separating family* if S separates the identity $e \in G$ from the closed subsets of G that do not contain the identity. That is, for every neighborhood U of e in G there exist $f \in S$, $\varepsilon > 0$ and a real number $r \in \mathbb{R}$ such that $f(e) = r$ and $f^{-1}(r - \varepsilon, r + \varepsilon) \subset U$.

Lemma 5.6. *Let (G, τ) be a topological group and S be a local separating family of functions. Suppose that $\tau' \subset \tau$ is a coarser group topology on G such that every $f \in S$ is continuous on a topological group (G, τ'). Then $\tau' = \tau$.*

Proof. Observe that by our assumption the homomorphism of groups $1_G : (G, \tau') \to (G, \tau)$, $g \mapsto g$ is continuous at the identity e. Hence this homomorphism is continuous. This implies that $\tau \subset \tau'$. Hence, $\tau' = \tau$, as required. $\square$

Lemma 5.7. *Let (G, τ) be a Hausdorff topological group and let*

$$\Phi := \{\omega_i : E_i \times F_i \to \mathbb{R}, \ \alpha_{1i} : E_i \times G \to E_i, \ \alpha_{2i} : G \times F_i \to F_i\}_{i \in I}$$

be a system of continuous co-birepresentations of G into bilinear mappings ω_i. Let

$$\mathcal{M}_\Phi := \left\{ m_{v,\psi} : G \to \mathbb{R} : (v, \psi) \in E_i \times F_i \right\}_{i \in I}$$

be the family of corresponding matrix coefficients. Suppose that $\mathcal{M}_\Phi$ is a local separating family on G. Then Φ is t-exact.

Proof. Assume that $\tau_1 \subset \tau$ is a coarser group topology on G such that all given co-birepresentations are still continuous. Then by Lemma 5.4, every matrix coefficient $m_{v,\psi} : (G, \tau_1) \to \mathbb{R}$ is (uniformly) continuous for each $m_{v,\psi} \in \mathcal{M}_\Phi$. By our assumption $\mathcal{M}_\Phi$ is a local separating family. By Lemma 5.6 we get $\tau_1 = \tau$. This means that the system Φ is t-exact. $\quad\square$

The following definition was inspired by [13]. Namely by the concept of *Strong Uniform Continuity* (SUC).

Definition 5.8. Let $h : G \to Iso(V)$ be a continuous co-representation on a normed space V and $x_0 \in V^*$. We say that a subset $M \subset V$ is *SUC-small at x_0* if for every $\varepsilon > 0$ there exists a neighborhood U of e such that

$$\sup_{v \in M} \left| \langle v, u x_0 \rangle - \langle v, x_0 \rangle \right| \leqslant \varepsilon \quad \forall u \in U.$$

We collect here some useful properties of SUC-smallness.

Lemma 5.9. *Let $h : G \to Iso(V)$ be a continuous co-representation and $x_0 \in V^*$.*

(a) *The family of SUC-small sets at x_0 is closed under taking: subsets, norm closures, finite linear combinations and convex hulls.*

(b) *If $M_n \subset V$ is SUC-small at $x_0 \in V^*$ for every $n \in \mathbb{N}$ then so is $\bigcap_{n \in \mathbb{N}}(M_n + \delta_n B_V)$ for every positive sequence δ_n such that $\lim \delta_n = 0$.*

(c) *For every $\psi \in V^*$ the following are equivalent*
  (i) *The orbit map $\widetilde{\psi} : G \to V^*$ is norm continuous.*
  (ii) *$\mathcal{B}$ is SUC-small at ψ, where $\mathcal{B} := \{\check{v} : V^* \to \mathbb{R}, \ x \mapsto \check{v}(x) := \langle v, x \rangle\}_{v \in B_V}$.*

(d) *Let $h_2 : G \to Iso(E)$ be a continuous co-representation and let $\gamma : V \to E$ be a linear continuous G-map (of right G-spaces). Assume that $M \subset E$ is an SUC-small set at $\psi \in E^*$. Then $\gamma^{-1}(M) \subset V$ is SUC-small at $\gamma^*(\psi) \in V^*$, where $\gamma^* : E^* \to V^*$ is the adjoint of γ.*

Proof. Assertion (a) is straightforward.

(b) We have to show that the set $\bigcap_{n \in \mathbb{N}}(M_n + \delta_n B_V)$ is SUC-small at x_0. Let $\varepsilon > 0$ be fixed. Since $G x_0$ is a bounded subset of V^* one can choose $n_0 \in \mathbb{N}$ such that $|v(g x_0)| < \frac{\varepsilon}{4}$ for every $g \in G$ and every $v \in \delta_{n_0} B_V$. Since M_{n_0} is SUC-small at x_0 we can choose a neighborhood $U(e)$ such that $|m(u x_0) - m(x_0)| < \frac{\varepsilon}{2}$ for every $u \in U$ and every $m \in M_{n_0}$. Now every element $w \in \bigcap_{n \in \mathbb{N}}(M_n + \delta_n B_V)$ has a form $w = m + v$ for some $m \in M_{n_0}$ and $v \in \delta_{n_0} B_V$. Then for every $u \in U$ we have

$$\left| w(u x_0) - w(x_0) \right| \leqslant \left| m(u x_0) - m(x_0) \right| + \left| v(u x_0) \right| + \left| v(x_0) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

(c) Use that $\|u \psi - \psi\| = \sup_{v \in B_V} |\langle v, u \psi \rangle - \langle v, \psi \rangle|$ and B_V is G-invariant.

(d) Take into account that $\gamma^* : E^* \to V^*$ is also a G-map with respect to the adjoint actions of G on E^* and V^*. By the definition of the adjoint map γ^* for every $(v, u) \in V \times G$ we have

$$\langle v, u \gamma^*(\psi) \rangle - \langle v, \gamma^*(\psi) \rangle = \langle v, \gamma^*(u\psi) \rangle - \langle v, \gamma^*(\psi) \rangle = \langle \gamma(v), u\psi \rangle - \langle \gamma(v), \psi \rangle.$$

This equality implies that $\gamma^{-1}(M) \subset V$ is SUC-small at $\gamma^*(\psi) \in V^*$ (using the assumption that $M \subset E$ is a SUC-small set at $\psi \in E^*$). $\quad\square$

For every $f \in RUC(G)$ denote by $\mathcal{A}_f$ the smallest closed *unital* (that is, containing the constants) G-invariant subalgebra of $RUC(G)$ which contains f. Denote by X_f the Gelfand space of the algebra $\mathcal{A}_f$. We call $\mathcal{A}_f$ and X_f the *cyclic G-algebra* and *cyclic G-system* of f, respectively (see for example, [39, Ch. IV, Section 5] or [12, Section 2]). The corresponding compactification $\alpha_f : G \to X_f$ is a G-compactification. That is, the compact space X_f is a left G-space such that α_f is a G-map and the G-orbit of the point $\alpha_f(e)$ (where e is the identity of G) is dense in X_f. Since $f \in \mathcal{A}_f$ there exists a continuous function $F : X_f \to \mathbb{R}$ such that the following diagram commutes

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha_f} & X_f \\
& \searrow{\scriptstyle f} & \downarrow{\scriptstyle F} \\
& & \mathbb{R}
\end{array}
$$

The following theorem is one of the main results of the present paper having in our opinion its own interest.

Theorem 5.10. *For a topological group G and a function $f : G \to \mathbb{R}$ the following conditions are equivalent.*

(1) $f \in UC(G)$.
(2) *The function $f : G \to \mathbb{R}$ is a (bicontinuous) matrix coefficient for some continuous co-birepresentation $h_1 : G \to Iso(V)$, $h_2 : G \to Iso(W)$ in a regular bilinear mapping $\omega : V \times W \to \mathbb{R}$ with Banach spaces V and W.*

Moreover in the second claim we can always assume without restriction of generality that $d(V) \leqslant d(G)$ and $d(W) \leqslant d(G)$.

Proof. (2) $\Longrightarrow$ (1): Apply Lemma 5.4.

(1) $\Longrightarrow$ (2): Let $f \in UC(G)$. In particular, $f \in RUC(G)$. Consider the corresponding cyclic algebra $\mathcal{A}_f$ and the G-compactification $\alpha_f : G \to X_f$. As we already mentioned, $F \circ \alpha_f = f$ for some continuous function $F : X_f \to \mathbb{R}$. Denote by z the image of the identity $e \in G$ in X_f under the map α_f, that is, $z := \alpha_f(e)$. The compact space X_f is naturally embedded into $C(X_f)^*$ by assigning to $y \in X_f$ the corresponding point measure $\delta_y \in C(X_f)^*$, where $\delta_y(\varphi) := \varphi(y)$ for every $\varphi \in C(X_f)$. In the sequel we will identify X_f with its natural image in $C(X_f)^*$.

Now we show that the G-orbit FG of the vector F in $C(X_f)$, as a family of functions, is SUC-small (see Definition 5.8) at $z \in X_f \subset C(X_f)^*$. Indeed, let $\varepsilon > 0$. By our assumption, $f \in UC(G)$. In particular, $f \in LUC(G)$. Therefore there exists a neighborhood U of the identity e in G such that

$$
\bigl| f(gu) - f(g) \bigr| < \varepsilon \quad \forall (u, g) \in U \times G.
$$

On the other hand, since α_f is a G-map the equality $F \circ \alpha_f = f$ implies that

$$
\bigl| F(guz) - F(gz) \bigr| = \bigl| F(gu\alpha_f(e)) - F(g\alpha_f(e)) \bigr| = \bigl| F(\alpha_f(gu)) - F(\alpha_f(g)) \bigr| = \bigl| f(gu) - f(g) \bigr|.
$$

Therefore we get

$$
\bigl| F(guz) - F(gz) \bigr| = \bigl| \langle Fg, uz \rangle - \langle Fg, z \rangle \bigr| < \varepsilon \quad \forall (u, g) \in U \times G.
$$

This means that FG is SUC-small at z.

Let $Y := co(-FG \cup FG)$ be the convex hull of the symmetric bounded set $-FG \cup FG$. Then Y is a convex symmetric G-invariant subset in $C(X_f)$. Denote by E the Banach subspace of $C(X_f)$ generated by Y, that is E is the norm closure of the linear span $sp(Y)$ of Y in $C(X_f)$. Since X_f is a compact G-space the natural right action of G on $C(X_f)$ (by linear isometries) is continuous. By our construction E is a G-invariant subspace. Hence the restricted action of G on E is well defined and also continuous.

Since Y is bounded convex and symmetric, we can apply the construction of Davis, Figiel, Johnson, and Pełczyński [3]. We mostly use the presentation and the development given by Fabian in the book [11]. Consider the sequence $K_n := 2^n Y + 2^{-n} B_E$, $n \in \mathbb{N}$, of subsets in E. Let $\| \cdot \|_n$ be the Minkowski's functional of the set K_n, that is,

$$
\| v \|_n = \inf\{ \lambda > 0 \mid v \in \lambda K_n \}.
$$

Then $\| \cdot \|_n$ is a G-invariant norm on E equivalent to the given norm of E for every $n \in \mathbb{N}$. For $v \in E$, let

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and} \quad V := \{ v \in E \mid N(v) < \infty \}.$$

Denote by $j : V \hookrightarrow E$ the inclusion map. Then (V, N) is a Banach space and $j : V \to E$ is a continuous linear injection. Furthermore we have

$$F \in Y \subset j(B_V).$$

Indeed, if $y \in Y$ then $2^n y \in K_n$. So, $\|y\|_n \leqslant 2^{-n}$. Thus, $N(y)^2 \leqslant \sum_{n \in \mathbb{N}} 2^{-2n} < 1$.

By our construction Y and B_E are G-invariant. This implies that the natural right action $V \times G \to V$, $(v, g) \mapsto vg$ is isometric, that is $N(vg) = N(v)$. Moreover, by the definition of the norm N on V (use the fact that the G-invariant norm $\| \cdot \|_n$ on E is equivalent to the given norm of E for each $n \in \mathbb{N}$) we can show that this action is norm continuous. Therefore, the continuous co-representation $h_1 : G \to Iso(V)$, $h_1(g)(v) := vg$ on the Banach space (V, N) is well defined.

Let $j^* : E^* \to V^*$ be the adjoint map of $j : V \to E$ and $i^* : C(X_f)^* \to E^*$ be the adjoint of the inclusion $i : E \hookrightarrow C(X_f)$. The composition $\gamma^* = j^* \circ i^* : C(X_f)^* \to V^*$ is the adjoint of $\gamma := i \circ j : V \to C(X_f)$. Denote by ψ the vector $\gamma^*(z) \in \gamma^*(X_f) \subset V^*$. Now our aim is to show the G-continuity of the vector $\psi \in V^*$, that is the continuity of the orbit map $\widetilde{\psi} : G \to V^*$.

Claim. $j(B_V) \subset \bigcap_{n \in \mathbb{N}} K_n = \bigcap_{n \in \mathbb{N}} (2^n Y + 2^{-n} B_E)$.

Proof. The norms $\| \cdot \|_n$ on E are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, $v \in \lambda_n K_n$ for some $0 < \lambda_n < 1$ and $n \in \mathbb{N}$. By the construction K_n is a convex subset containing the origin. This implies that $\lambda_n K_n \subset K_n$. Hence $v \in K_n$ for every $n \in \mathbb{N}$. $\quad \square$

Recall now that FG is SUC-small at $z \in C(X_f)^*$. By Lemma 5.9(a) we know that then $Y := co(-FG \cup FG)$ is also SUC-small at $z \in C(X_f)^*$. Moreover by Lemma 5.9(b) we obtain that $M := \bigcap_{n \in \mathbb{N}} (2^n Y + 2^{-n} B_E) \subset C(X_f)$ is SUC-small at $z \in C(X_f)^*$. The linear continuous operator $\gamma : V \to C(X_f)$ is a G-map. Then by Lemma 5.9(d) it follows that $\gamma^{-1}(M) \subset V$ is SUC-small at $\psi := \gamma^*(z) \in V^*$. The same is true for B_V because by the claim we have $\gamma(B_V) = j(B_V) \subset M$ (and hence, $B_V \subset \gamma^{-1}(M)$). Now Lemma 5.9(c) means that the orbit map $\widetilde{\psi} : G \to V^*$ is G-continuous.

Define W as the Banach subspace of V^* generated by the orbit $G\psi$ in V^*. More precisely, W is the norm closure $cl(sp(G\psi))$ of the linear span $sp(G\psi)$ of $G\psi$ in V^*. Clearly, W is a G-invariant subset of V^* under the adjoint action of G on V^*. The left action of G on W by linear isometries defines the representation $h_2 : G \to Iso(W)$. Moreover, since ψ is G-continuous, it is easy to see that in fact every vector $w \in W$ is G-continuous. This means that h_2 is continuous. Define the bilinear mapping $\omega : V \times W \to \mathbb{R}$ as a restriction of the canonical form $V \times V^* \to \mathbb{R}$. Clearly, ω is regular (hence, continuous) and the pair (h_1, h_2) is a continuous co-birepresentation of G in ω.

By our construction $F \in j(V)$ (because $F \in Y \subset j(B_V)$). Since j is injective the element $v := j^{-1}(F)$ is uniquely determined in V. We already proved that $\psi = \gamma^*(z) \in V^*$ is a G-continuous vector. In order to complete the proof it suffices to show that $f = m_{v, \psi}$. Using the equality $F \circ \alpha_f = f$ and the fact that α_f is a G-map we get

$$\langle Fg, z \rangle = F\big(g\alpha_f(e)\big) = (F \circ \alpha_f)(g) = f(g).$$

On the other hand,

$$m_{v, \psi}(g) = \langle vg, \psi \rangle = \big\langle j^{-1}(F)g, \gamma^*(z) \big\rangle = \big\langle \gamma\big(j^{-1}(F)g\big), z \big\rangle = \langle Fg, z \rangle.$$

Hence, $f = m_{v, \psi}$, as required.

This proves the equivalence (1) $\Longleftrightarrow$ (2).

By the G-continuity of ψ in $W = cl(sp(G\psi))$ we get that $d(W) = d(G\psi) \leqslant d(G)$. Now we check that $d(V) \leqslant d(G)$. First of all E is a subspace of $C(X_f)$ generated by $Y := co(-FG \cup FG)$. Since F is a G-continuous vector in E we have $d(FG) \leqslant d(G)$. Therefore we get that

$$d(E) = d(Y) = d(FG) \leqslant d(G).$$

Now it suffices to show that $d(V) = d(E)$. That is we have to show that the canonical construction of [3] in fact always preserves the density. Indeed, by the construction V is a (diagonal) subspace into the l_2-sum $Z := \sum_{n=1}^{\infty}(E, \|\cdot\|_n)_{l_2}$. So, $d(V) \leqslant d(Z)$. On the other hand we know that every norm $\|\cdot\|_n$ is equivalent to the original norm on V. Hence, $d(E, \|\cdot\|_n) \leqslant d(E)$. Therefore, Z is an l_2-sum of countably many Banach spaces each of them having the density $d(E)$. It follows that

$$d(V) \leqslant d(Z) \leqslant d(E).$$

So we obtain $d(V) \leqslant d(G)$, as required. $\quad\square$

6. Additional properties of strong dualities

In this section we give some additional auxiliary technical results about strong dualities (which did not appear in [16]). The meaning of Theorem 6.3 is that every continuous co-birepresentation in a bilinear mapping naturally leads to a continuous co-birepresentation into strong duality preserving the matrix coefficients.

Definition 6.1. Let $\omega : V \times W \to \mathbb{R}$, and $\omega' : V' \times W' \to \mathbb{R}$ be two continuous bilinear mappings defined for (semi)normed spaces.

(1) We say that ω' *refines* ω (notation: $\omega \succcurlyeq \omega'$) if there exist continuous linear operators of normed spaces $p : V \to V'$, $q : W \to W'$ such that $\omega'(p(v), q(f)) = \omega(v, f)$ for every $(v, f) \in V \times W$. Hence the following diagram commutes

$$
\begin{array}{ccc}
V \times W & \xrightarrow{\;\omega\;} & \mathbb{R} \\
{\scriptstyle p}\big\downarrow \quad \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle 1_{\mathbb{R}}} \\
V' \times W' & \xrightarrow{\;\omega'\;} & \mathbb{R}
\end{array}
$$

(2) Let $\Psi = (h_1, h_2)$ and $\Psi' = (h_1', h_2')$ be two co-birepresentations of G into ω and ω' respectively. We say that Ψ' *refines* Ψ (notation: $\Psi \succcurlyeq \Psi'$) if $\omega \succcurlyeq \omega'$ and one can find p and q satisfying the assumption of the first definition such that p and q are G-maps.
(3) If p and q are onto then we say that ω' is an *onto refinement* of ω. *Dense refinement* will mean that $p(V)$ and $q(W)$ are dense in V' and W' respectively. Similarly, we define *onto refinement* and *dense refinement* of co-birepresentations.

Lemma 6.2.

(1) *If $\Psi \succcurlyeq \Psi'$ then $\mathcal{M}_\Psi \subset \mathcal{M}_{\Psi'}$.*
(2) *If $\omega \succcurlyeq \omega'$ is a dense refinement and ω is left (right) strong duality then ω' is also left (respectively, right) strong duality.*

Proof. (1): For every pair $(v, \psi) \in V \times W$ and $g \in G$ we have

$$m_{v,\psi}(g) = \omega(vg, \psi) = \omega'\big(p(vg), q(\psi)\big) = \omega'\big(p(v)g, q(\psi)\big) = m_{p(v), q(\psi)}(g).$$

Thus, $m_{v,\psi} = m_{p(v), q(\psi)}$. This proves the inclusion $\mathcal{M}_\Psi \subset \mathcal{M}_{\Psi'}$.

(2): Assume that ω is left strong. We show that then ω' is also left strong (we omit the similar details for the 'right strong case'). Let v_n be an unbounded sequence in V'. Since $p(V)$ is dense in V' there exists a sequence x_n in V such that $\|p(x_n) - v_n\| \leqslant 1$. Clearly x_n is also unbounded (otherwise v_n is bounded) because p is a bounded operator. By the continuity of $q : W \to W'$ we can choose $\varepsilon > 0$ such that $\|q(f)\| \leqslant 1$ whenever $\|f\| \leqslant \varepsilon$. Since ω is left strong then $\{f(x_n) = \omega(x_n, f) : n \in \mathbb{N}, \|f\| \leqslant \varepsilon\}$ is unbounded in $\mathbb{R}$. By the inclusion

$$\big\{f(x_n) : n \in \mathbb{N}, \|f\| \leqslant \varepsilon\big\} = \big\{\omega'\big(p(x_n), q(f)\big) : n \in \mathbb{N}, \|f\| \leqslant \varepsilon\big\} \subset \big\{\omega'\big(p(x_n), \phi\big) : n \in \mathbb{N}, \phi \in B_{W'}\big\}$$

the set $\{\phi(p(x_n)) : n \in \mathbb{N}, \phi \in B_{W'}\}$ is unbounded, too. By the continuity of ω' and Lemma 5.2 there exists a constant $c > 0$ such that

$$\big\|\phi(v_n) - \phi\big(p(x_n)\big)\big\| \leqslant c \cdot \|\phi\| \cdot \big\|v_n - p(x_n)\big\| \leqslant c$$

holds for every $\phi \in B_{W'}$ and $n \in \mathbb{N}$. It follows that the set $\{\phi(v_n): n \in \mathbb{N}, \phi \in B_{W'}\}$ is also unbounded in $\mathbb{R}$. This means that ω' is also left strong. $\quad\square$

Theorem 6.3. *For every continuous co-birepresentation Ψ of G into a continuous bilinear mapping*

$$\omega: V \times W \to \mathbb{R}, \qquad (v, f) \mapsto \omega(v, f) = \langle v, f \rangle = f(v)$$

such that V and W are normed spaces there exists a regular strong duality $\omega_0: V_0 \times W_0 \to \mathbb{R}$ with normed spaces V_0 and W_0 and a continuous co-birepresentation Ψ_0 of G into ω_0 such that $\Psi \succcurlyeq \Psi_0$ is an onto refinement.

Proof. Define a seminorm $\| \cdot \|_*$ on V by

$$\|v\|_* := \sup\{\langle v, f \rangle: f \in B_W\}.$$

Note that the seminorm $\| \cdot \|_*$ on V in fact is the *strong polar topology $\beta(V, W)$* (see for example, [26, Section 9.4]) induced on V by the form $\omega: V \times W \to \mathbb{R}$.

Assertion 1. *There exists a constant $c > 0$ such that $\|v\|_* \leqslant c \cdot \|v\|$ for every $v \in V$.*

Proof. By Lemma 5.2 we have the inequality $|\langle v, f \rangle| \leqslant c \cdot \|v\| \cdot \|f\|$ for some constant $c > 0$. Then $\langle v, \frac{f}{\|f\|} \rangle \leqslant c \cdot \|v\|$ for every $f \in W$. Hence,

$$\|v\|_* := \sup\{\langle v, \varphi \rangle: \|\varphi\| = 1\} \leqslant c \cdot \|v\| \quad \forall v \in V. \qquad \square$$

Assertion 2. $\omega_*: (V, \| \cdot \|_*) \times W \to \mathbb{R}, (v, f) \mapsto \omega(v, f)$ *is a regular* (*continuous*) *bilinear map.*

Proof. Clearly, $\langle v, \frac{f}{\|f\|} \rangle \leqslant \|v\|_*$ for every $f \in W$ and $v \in V$. So, $\langle v, f \rangle \leqslant \|v\|_* \cdot \|f\|$. Thus, $\omega_*: (V, \| \cdot \|_*) \times W \to \mathbb{R}$ is regular (hence, continuous, by Lemma 5.2). $\quad\square$

Assertion 3. $\omega_*: (V, \| \cdot \|_*) \times W \to \mathbb{R}$ *is a left strong duality and the pair of natural identity maps $(1_V)_*: V \to (V, \| \cdot \|_*)$, $1_W: W \to W$ defines the natural onto refinement $\omega \succcurlyeq \omega_*$.*

Proof. Let v_n be a norm unbounded sequence in $(V, \| \cdot \|_*)$. Then by the definition of the seminorm $\| \cdot \|_*$ for every $n \in \mathbb{N}$ there exists f_n in the unit ball B_W such that the sequence $f_n(v_n)$ is unbounded in $\mathbb{R}$. This proves that ω_* is a left strong duality.

By Assertion 1 the new $\| \cdot \|_*$-seminorm topology on V is coarser than the original norm topology. It follows that the pair $(1_V)_*: V \to (V, \| \cdot \|_*)$, $1_W: W \to W$ defines the natural onto refinement $\omega \succcurlyeq \omega_*$. $\quad\square$

For the seminormed space $(V, \| \cdot \|_*)$ denote by $(V_0, \| \cdot \|_0)$ the corresponding universal normed space. The elements of V_0 can be treated as the subsets $[v] := \{x \in V: \|x - v\|_* = 0\}$ of V, where $v \in V$. The canonical norm on V_0 is defined by $\|[v]\|_0 := \|v\|_*$. Denote by $\lambda_*: (V, \| \cdot \|_*) \to V_0, v \mapsto [v]$ and $\lambda: V \to V_0, v \mapsto [v]$ the corresponding natural linear continuous onto operators.

Assertion 4. *The bilinear mapping*

$$\omega_L: (V_0, \| \cdot \|_0) \times W \to \mathbb{R}, \qquad ([v], f) \mapsto \omega(v, f)$$

is a well defined left strong regular duality and the pair $\lambda: V \to (V_0, \| \cdot \|)_0$, $1_W: W \to W$ defines the natural onto refinement $\omega \succcurlyeq \omega_L$. Furthermore, if ω is a right strong duality then ω_L is a strong duality.

Proof. Since $\mathbb{R}$ is Hausdorff the continuity of ω_* implies that if $\|v\|_* = 0$ then $f(v) = 0$ for every $f \in W$. Hence, $f(v_1) = f(v_2)$ for every $v_1, v_2 \in [v]$ and $f \in W$. This proves that ω_L is well defined. Since $\omega_L([v], f) = \omega(v, f) = \omega_*(v, f)$, we easily get by Assertion 2 that ω_L is regular. Moreover the pair $(\lambda_*, 1_W)$ defines the (onto, of course) refinement $\omega_* \succcurlyeq \omega_L$. The latter fact implies that ω_L also is left strong by Lemma 6.2(2) and Assertion 3. Since $\omega \succcurlyeq \omega_*$, $\omega_* \succcurlyeq \omega_L$ and $\lambda = \lambda_* \circ (1_V)_*$ we get the natural onto refinement $\omega \succcurlyeq \omega_L$ with respect to the pair $(\lambda, 1_W)$.

Now if ω is a right strict duality then ω_L remains right strong duality by Lemma 6.2(2). $\quad\square$

Now assume that the pair $h_1 : G \to Iso(V, \| \cdot \|)$, $h_2 : G \to Iso(W)$ is a given continuous co-birepresentation Ψ of G in ω. Since $\|v\|_* \leqslant c \cdot \|v\|$ we get that every vector $v \in V$ is G-continuous with respect to $\| \cdot \|_*$. On the other hand, since B_W is G-invariant we get that $\|vg\|_* = \|v\|_*$ for every $g \in G$. Therefore, $(V, \| \cdot \|_*) \times G \to (V, \| \cdot \|_*)$, $(v, g) := vg = h_1(g)(v)$ is a well defined continuous right action.

Define the co-representation $(h_1)_0 : G \to Iso(V_0, \| \cdot \|_0)$ by the natural right action $([v], g) \mapsto [v]g = [vg]$. Then it is a well defined continuous co-representation. The proof is straightforward taking into account the trivial equalities $\|[v]\|_0 = \|v\|_*$, $[v]g = [vg]$ and $\|vg\|_* = \|v\|_*$ for every $(v, g) \in V \times G$. It is also easy to see that the equality $\omega_L([v]g, \psi) = \omega_L([v], g\psi)$ holds for every $([v], g, \psi) \in V_0 \times G \times W$. Hence, the pair $((h_1)_0, h_2)$ defines a continuous co-birepresentation (denote it by Ψ_L) of G in $\omega_L : (V_0, \| \cdot \|_0) \times W \to \mathbb{R}$. Furthermore, each of the maps λ and 1_W from Assertion 4 is a G-map. So in fact we found a co-birepresentation Ψ_L into the left strong regular duality ω_L such that $\Psi \succcurlyeq \Psi_L$.

Similarly, starting now from ω_L and switching left and right, we can construct: a seminormed space $(W, \| \cdot \|_*)$, its universal normed space W_0, a regular *right strong* duality $(\omega_L)_R : V_0 \times W_0 \to \mathbb{R}$ which is an onto refinement of ω_L and a continuous co-birepresentation $(\Psi_L)_R$ of G into $(\omega_L)_R$ such that $\Psi_L \succcurlyeq (\Psi_L)_R$. Denote by ω_0 the duality $(\omega_L)_R$ and by Ψ_0 the co-birepresentation $(\Psi_L)_R$. Then Ψ_0 is the desired co-birepresentation because $\Psi \succcurlyeq \Psi_L \succcurlyeq \Psi_0$ and ω_0 in fact is left and right strong (take into account the analogue of Assertion 4). $\quad\square$

Lemma 6.4.

(1) *Let $\omega : V \times W \to \mathbb{R}$ be a strong duality such that V and W are normed spaces. Denote by $\varrho_V : V \to \widehat{V}$ and $\varrho_W : W \to \widehat{W}$ the corresponding completions. Then the uniquely defined continuous extension*

$$\hat{\omega} : \widehat{V} \times \widehat{W} \to \mathbb{R}$$

is a strong duality and the pair (ϱ_V, ϱ_W) defines the canonical refinement $\omega \succcurlyeq \hat{\omega}$. If ω is regular then the same is true for $\hat{\omega}$.

(2) *Assume that the pair $h_1 : G \to Iso(V)$, $h_2 : G \to Iso(W)$ defines a continuous co-birepresentation Φ. Then there exists a uniquely defined extension to a continuous co-birepresentation $\widehat{\Phi}$ of G into the form $\hat{\omega}$ defined by the pair $\widehat{h_1} : G \to Iso(\widehat{V})$, $\widehat{h_2} : G \to Iso(\widehat{W})$. In fact, $\widehat{\Phi}$ is a dense refinement of Φ.*

Proof. (1) It can be derived by Lemma 6.2(2) because $\hat{\omega}$ is a dense refinement of ω under the natural dense inclusions $\varrho_V : V \to \widehat{V}$ and $\varrho_W : W \to \widehat{W}$. The regularity of $\hat{\omega}$ for regular ω is clear.

(2) Note that for every continuous (not necessarily isometric) linear (left or right) action of a topological group G on a normed space V there exists a uniquely defined canonical linear extension on the Banach space $\widehat{V}$ which is also continuous. This is easy to verify directly or it can be derived also by [18, Proposition 2.6.4]. Straightforward arguments show also that: ϱ_V and ϱ_W are continuous G-maps, $\hat{\omega}(xg, y) = \hat{\omega}(x, gy)$ for every $(x, g, y) \in \widehat{V} \times G \times \widehat{W}$ and the corresponding g-translations $\widehat{V} \to \widehat{V}$ and $\widehat{W} \to \widehat{W}$ are linear isometries. $\quad\square$

Proposition 6.5. *Let G be a Hausdorff topological group and let*

$$\Phi := \{\Phi_i\}_{i \in I} = \{\omega_i : E_i \times F_i \to \mathbb{R}, \ \alpha_{1i} : E_i \times G \to E_i, \ \alpha_{2i} : G \times F_i \to F_i\}_{i \in I}$$

be a system of continuous co-birepresentations of G into regular bilinear mappings ω_i with Banach spaces E_i, F_i. Let

$$\mathcal{M}_\Phi := \big\{m_{v,\psi} : G \to \mathbb{R} \colon (v, \psi) \in E_i \times F_i, \ i \in I\big\}$$

be the family of all corresponding matrix coefficients. Suppose that $\mathcal{M}_\Phi$ is a local separating family of functions on G. Then there exists a continuous t-exact co-birepresentation Ψ of G into a regular strong duality $\omega : E \times F \to \mathbb{R}$. Furthermore we can assume that E and F are Banach spaces and their densities are not greater than $\sup\{d(E_i) \cdot d(F_i) \cdot |I| \colon i \in I\}$.

Proof. Consider the l_2-sum of the given system Φ of co-birepresentations. That is, define naturally the Banach spaces $V := (\sum_{i \in I} E_i)_{l_2}$ and $W := (\sum_{i \in I} F_i)_{l_2}$, the continuous co-representation $h_1 : G \to Iso(V)$ and the continuous representation $h_2 : G \to Iso(W)$. Clearly, $\langle vg, f \rangle = \langle v, gf \rangle$ for the natural bilinear mapping

$$\omega_{l_2} : V \times W \to \mathbb{R}, \qquad \left\langle \sum_i v_i, \sum_i f_i \right\rangle := \sum_i f_i(v_i).$$

Since $|\omega_i(v, f)| \leqslant \|v\| \cdot \|f\|$ for every $i \in I$ it follows by the Schwartz inequality that the form ω_{l_2} is well defined and continuous. Moreover the co-birepresentation $\Psi_{l_2} := (h_1, h_2)$ of G in ω_{l_2} is also well defined and continuous.

Now in order to get a regular strong duality we apply Theorem 6.3 to Ψ_{l_2}. Then we obtain the regular strong duality $(\omega_{l_2})_0 : V_0 \times W_0 \to \mathbb{R}$ and a continuous co-birepresentation $(\Psi_{l_2})_0$ of G in $(\omega_{l_2})_0$ such that $\omega_{l_2} \succcurlyeq (\omega_{l_2})_0$ and $\Psi_{l_2} \succcurlyeq (\Psi_{l_2})_0$. Applying Lemma 6.4 we get the continuous co-birepresentation $\widehat{(\Psi_{l_2})_0}$ of G into a regular strong duality $\widehat{(\omega_{l_2})_0} : \widehat{V_0} \times \widehat{W_0} \to \mathbb{R}$ such that $(\omega_{l_2})_0 \succcurlyeq \widehat{(\omega_{l_2})_0}$ and $(\Psi_{l_2})_0 \succcurlyeq \widehat{(\Psi_{l_2})_0}$. We claim that $\Psi := \widehat{(\Psi_{l_2})_0}$ is the desired co-birepresentation into $\omega := \widehat{(\omega_{l_2})_0} : \widehat{V_0} \times \widehat{W_0} \to \mathbb{R}$. Indeed, first of all observe that for every $i \in I$ the co-representation Φ_i of G in ω_i can be treated as 'a part' of the global co-birepresentation Ψ_{l_2}. Therefore the set $\mathcal{M}_{\Psi_{l_2}}$ of all matrix coefficients defined by Ψ_{l_2} contains the set $\mathcal{M}_\Phi$ which is a local separating family on G. Hence $\mathcal{M}_{\Psi_{l_2}}$ is also local separating. Now by Lemma 6.2(1) the same is true for the families $\mathcal{M}_{(\Psi_{l_2})_0}$ and $\mathcal{M}_\Psi = \mathcal{M}_{\widehat{(\Psi_{l_2})_0}}$ because $\Psi_{l_2} \succcurlyeq (\Psi_{l_2})_0 \succcurlyeq \widehat{(\Psi_{l_2})_0} = \Psi$. It follows by Lemma 5.7 that the co-birepresentation Ψ of G in $\omega = \widehat{(\omega_{l_2})_0} : E \times F \to \mathbb{R}$ is t-exact, where $E := \widehat{V_0}$ and $F := \widehat{W_0}$ are certainly Banach spaces. The completion of normed spaces does not increase the density. So by our construction (using some obvious properties of l_2-sums) one can assume in addition that the densities of E and F are not greater than $\sup\{d(E_i) \cdot d(F_i) \cdot |I| : i \in I\}$. □

7. Proof of the main theorem and some consequences

First we prove the following crucial result.

Theorem 7.1.

(1) *Every Hausdorff topological group G is a BR-group.*
(2) *Moreover, there exists a t-exact birepresentation*

$$\Psi := \big\{ \omega : E \times F \to \mathbb{R}, \ h_1 : G \to Iso(E), \ h_2 : G \to Iso(F) \big\}$$

of G such that: ω *is a regular strong duality*; E *and F are Banach spaces with the density not greater than $w(G)$.*

Proof. (1) directly follows from (2) (and Definition 4.3). Hence it suffices to show (2). By Lemma 2.1 the algebra $UC(G)$ separates points and closed subsets of G. Choose a subfamily $S \subset UC(G)$ with cardinality $|S| \leqslant \chi(G)$ such that S is a local separating family (see Definition 5.5) for G. By Theorem 5.10 every $f \in S \subset UC(G)$ can be represented as a bicontinuous matrix coefficient by some *regular* bilinear mapping. More precisely, there exists a continuous co-birepresentation Φ_f defined by the pair $h_f : G \to Iso(V_f)$, $h'_f : G \to Iso(W_f)$ in a regular bilinear mapping $\omega_f : V_f \times W_f \to \mathbb{R}$ with Banach spaces V_f and W_f such that $f = m_{v,\psi}$ for some pair $(v, \psi) \in V_f \times W_f$. Moreover we can assume that $d(V_f) \leqslant d(G)$ and $d(W_f) \leqslant d(G)$.

We get a system

$$\Phi := \{\Phi_f\}_{f \in S} = \big\{ \omega_f : V_f \times W_f \to \mathbb{R}, \ h_f : G \to Iso(V_f), \ h'_f : G \to Iso(W_f) \big\}_{f \in S}$$

of continuous co-birepresentations of G. By our construction the corresponding set of all matrix coefficients $\mathcal{M}_\Phi$ contains the local separating family S of functions on G. We can apply Proposition 6.5. Then there exists a continuous t-exact co-birepresentation Ψ' of G into a regular *strong* duality $\omega : E \times F \to \mathbb{R}$ with Banach spaces E and F the densities of them are not greater than $\sup\{d(V_f) \cdot d(W_f) \cdot |S| : f \in S\}$. Since $|S| \leqslant \chi(G)$, $w(G) = d(G) \cdot \chi(G)$, and $d(V_f) \leqslant d(G)$, $d(W_f) \leqslant d(G)$, we obtain that $\max\{d(E), d(F)\} \leqslant w(G)$. Finally, in order to get the desired *birepresentation* Ψ from our co-birepresentation Ψ', just define it according to Remark 4.2 as $\Psi := (\Psi')^{op}$. □

Now we obtain our main result:

Theorem 7.2. *For every Hausdorff topological group G there exists a minimal group M and a topological group retraction $p : M \to G$.*

Furthermore we can assume that:

(a) *There exists a regular strong duality $\omega: E \times F \to \mathbb{R}$ with Banach spaces E, F and a t-exact birepresentation Ψ of G into ω such that*: $\max\{d(E), d(F)\} \leqslant w(G)$ *and M can be constructed as the induced group $M_+(\Psi)$ of Ψ. That is*

$$M := M_+(\Psi) = H_+(\omega) \rtimes_\pi G = \big((\mathbb{R} \times E \rtimes_{\omega^\nabla} F) \rtimes_\alpha \mathbb{R}_+\big) \rtimes_\pi G.$$

(b) $p: M \to G$ *is the natural group retraction and the group $H_+(\omega) = \ker(p)$ is minimal, Weil-complete and solvable.*
(c) $w(M) = w(G)$, $\chi(M) = \chi(G)$ *and* $\psi(M) = \psi(G)$.
(d) *If G is Raikov-complete (Weil-complete) then M also has the same property.*
(e) *If G is solvable then M is solvable.*

Proof. (a): Use Theorems 7.1 and 4.4 (see also Theorem A.11).

(b): By Proposition 3.4 the kernel $\ker(p) = H_+(\omega)$ of the retraction p is a minimal group. Furthermore, using Lemma 3.5 we can conclude that the group $H_+(\omega)$ is Weil-complete. The solvability of the group $H_+(\omega) = (\mathbb{R} \times E \rtimes_{\omega^\nabla} F) \rtimes_\alpha \mathbb{R}_+$ is trivial.

(c): Take into account that M, as a topological space, is homeomorphic to the product of G and the metrizable space $H_+(\omega)$ with weight $\leqslant w(G)$.

(d): Use Lemmas 2.5 and 3.5.

(e): Use (b). $\quad\square$

In particular, by Theorem 7.2 we can conclude now that Pestov's conjecture is true: every topological group is a group retract of a minimal topological group.

Corollary 7.3. *Every compact Hausdorff homogeneous space admits a transitive continuous action of a minimal group.*

Proof. Let X be a compact Hausdorff homogeneous space. Then the group $G := Homeo(X)$ of all homeomorphisms of X is a topological group with respect to the usual compact open topology. The natural action $\alpha: G \times X \to X$ is continuous. This action is transitive because X is homogeneous. By Theorem 7.2 there exists a minimal group M and a continuous group retraction $p: M \to G$. Then the action $M \times X \to X$, $mx := \alpha(p(m), x)$ is also continuous and transitive. $\quad\square$

Remark 7.4.

(1) Theorem 7.2 implies that there exist Raikov-complete (Weil-complete) minimal topological groups such that $\chi(M)$ and $\psi(M)$ are different (as far as possible for general groups). This answers negatively Question C (see Introduction).
(2) Applying Corollary 7.3 to a not dyadic compact homogeneous space X we get an immediate negative answer to Question D (see Introduction).

The following application of Theorem 7.2 has been found recently by Uspenskij [38]: every topological group is a quotient of a Weil-complete minimal group. Indeed, by [37] every topological group G is a quotient of a Weil-complete group W. By Theorem 7.2, W is a group retract (hence, a quotient) of a Weil-complete minimal group.

Recall that a topological group K is *perfectly minimal* in the sense of Stoyanov (see for example [8]) if the product $K \times P$ is minimal for every minimal group P. By the test of perfect minimality [16, Theorem 1.14] a minimal group K is perfectly minimal iff its center is perfectly minimal. It is easy to see that the center of the group $M = H_+(\omega) \rtimes_\pi G$ is trivial. Indeed, the center of its subgroup $H_+(\omega)$ is already trivial. Here it is important to note that the bilinear mapping ω (being a strong duality) is separated (see the text after Definition 3.2). Therefore, it follows that in fact, M in Theorem 7.2 is perfectly minimal.

Recall that every locally compact Abelian group G is a group retract of a generalized Heisenberg group $H(\Delta) = \mathbb{T} \times G^* \rtimes_\Delta G$ (see Section 2 or [16]) which certainly is locally compact. For non-Abelian case the following question (recorded also in [2, Question 3.3.5]) seems to be open.

Question 7.5. *(See [16, Question 2.13.1].) Is it true that every locally compact Hausdorff group is a group retract (quotient, or a subgroup) of a locally compact minimal group?*

Theorem 7.1 shows that every topological group G admits sufficiently many continuous representations into continuous bilinear mappings. It turns out that in general we cannot replace general bilinear mappings $E \times F \to \mathbb{R}$ by the *canonical* bilinear mappings $\langle , \rangle : V \times V^* \to \mathbb{R}$. More precisely, let $h : G \to Iso(V)$ be a given continuous representation of G on V. Then we have the adjoint representation $h^* : G \to Iso(V^*)$, $h^*(g)(\psi) = g\psi$, where $(g\psi)(v) = \psi(g^{-1}v)$. One attractive previous idea to prove that every group G is a BR-group was trying to find sufficiently many representations h of G such that h^* is also continuous. If h^* is continuous then (h, h^*) becomes a continuous birepresentation in $\langle , \rangle$. Every topological group can be treated as a topological subgroup of $Iso(V)$ for a suitable Banach space V (see for example, [35,28]); the t-exactness is clear by Remark 2.6(5). Thus, we could derive directly that every topological group G is a BR-group. Although this result is really true (Theorem 7.1) however in its proof we cannot use that direct naive argument. The reason is that in general h^* is not continuous (see for example [18]).

This remark suggests the following definition.

Definition 7.6. We say that a topological group G is *adjoint continuously representable* (in short: *ACR-group*) if there exists a continuous representation $h : G \to Iso(V)$ on a Banach space V such that the adjoint representation $h^* : G \to Iso(V^*)$ is also continuous and the continuous birepresentation (h, h^*) of G is t-exact.

It seems to be interesting to study the class of adjoint continuous representable groups. Note that h^* is continuous for every continuous representation $h : G \to Iso(V)$ on an Asplund (e.g., reflexive) Banach space V (see [18]). It follows that every Asplund representable, e.g., reflexively representable, group G is an ACR-group (where Asplund (respectively, reflexively) representability means that G can be embedded into $Iso(V)$ for some Asplund (respectively, reflexive) space V). For instance every locally compact Hausdorff group G (being Hilbert representable) is an ACR-group. Only recently became clear that this result cannot cover all groups. Indeed the group $Homeo_+[0, 1]$ (the topological group of all orientation preserving homeomorphisms of $[0, 1]$) is not reflexively representable [20] and even not Asplund representable [13]. Moreover the following stronger result [13] is true (this was proved also by V. Uspenskij): if $G := Homeo_+[0, 1]$ and $h : G \to Iso(V)$ is a continuous representation on a Banach space V such that the adjoint representation $h^* : G \to Iso(V^*)$ is also continuous then h is trivial. It follows that $Homeo_+[0, 1]$ is not an ACR-group. For more information and questions about group representations on Banach spaces we refer to [12,13,23].

Acknowledgements

Appendix A

For the readers convenience we include here (sometimes simplified) proofs of some principal results from [16] which we need in the present paper.

For every topological group (P, γ) and its subgroup H denote by γ / H the usual quotient topology on the coset space P/H. More precisely, if $pr : P \to P/H$ is the canonical projection then $\gamma / H := \{OH : O \in \gamma\} = \{pr(O) : O \in \gamma\}$. If $q : P \to G$ is an onto homomorphism. Then on G we can define the quotient (group) topology which in fact is the topology $q(\gamma)$.

The following well known result is very useful.

Lemma A.1 (*Merson's lemma*). *Let (G, γ) be a not necessarily Hausdorff topological group and H be a not necessarily closed subgroup of G. Assume that $\gamma_1 \subset \gamma$ be a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1 / H = \gamma / H$. Then $\gamma_1 = \gamma$.*

Proof. See for example [8, Lemma 7.2.3]. □

Definition A.2. Let $q : X \to Y$ be a (not necessarily group[2]) retraction of a group X on a subgroup Y. We say that q is *central* if

$$q\left(xyx^{-1}\right) = y \quad \forall (x, y) \in X \times Y.$$

Lemma A.3. *Let* $H(\omega) = A \times E \rtimes_{\omega^\nabla} F$ *be the Heisenberg group of the biadditive mapping* $\omega : E \times F \to A$. *Then the natural projections* $q_E : H(\omega) \to E$, $q_F : H(\omega) \to F$ *and* $q_A : H(\omega) \to A$ *are central.*

Proof. If $u := (a, x, f) \in H(\omega)$, $y \in E$, $\varphi \in F$ and $a \in A$ then $uyu^{-1} = (f(y), y, 0_F)$, $u\varphi^{-1}u^{-1} = (-\varphi(x), 0_E, \varphi)$ and $uau^{-1} = a$. $\quad\square$

Proposition A.4. *Let* (M, γ) *be a topological group such that* M *is* algebraically[3] *a semidirect product* $M = X \rtimes_\alpha G$. *If* $q : X \to Y$ *is a continuous central retraction of the topological subgroup* X *on a topological* G*-subgroup* Y *of* X, *then the action*

$$\alpha|_{G \times Y} : (G, \gamma/X) \times (Y, \gamma|_Y) \to (Y, \gamma|_Y) \tag{A.1}$$

is continuous.

Proof. By our assumption M *algebraically* is the semidirect product $M = X \rtimes_\alpha G$. Therefore we have $M/X = \{X \times \{g\}\}_{g \in G}$. We sometimes identify M/X and G. This justifies also the notation $(G, \gamma/X)$. Note also that then the group topologies γ/X and $pr(\gamma)$ are the same on G, where $pr : M \to G = M/X$, $(x, g) \mapsto g$ denotes the canonical projection.

Clearly, each g-transition $(Y, \gamma|_Y) \to (Y, \gamma|_Y)$ is continuous. Hence it suffices to show that action (A.1) is continuous at (e_G, y) for every $y \in Y$, where e_G is the neutral element of G. Fix an arbitrary $y \in Y$ and a neighborhood $O(y)$ of y in $(Y, \gamma|_Y)$. Since the retraction $q : (X, \gamma|_X \to (Y, \gamma|_Y)$ is continuous (at y) there exists a neighborhood U_1 of $y := (y, e_G)$ in (M, γ) such that

$$q(U_1 \cap X) \subset O.$$

The conjugation $M \times M \to M$, $(a, b) \to aba^{-1}$ is continuous (at (e_M, y)). We can choose: a neighborhood U_2 of y in M and a neighborhood V of e_M in M such that

$$vU_2v^{-1} \subset U_1 \quad \forall v \in V.$$

Now, we claim that (for the canonical projection $pr : M \to G = M/X$) we have

$$\alpha(g, z) \in O \quad \forall z \in U_2 \cap Y, \quad g \in pr(V).$$

Indeed, if $v = (x, g) \in V$ and $z \in U_2 \cap Y$ then $vzv^{-1} \in U_1$ because $vU_2v^{-1} \subset U_1$. From the normality of X in M we have $vzv^{-1} \in X$. Thus, $vzv^{-1} \in U_1 \cap X$. Elementary computations show that

$$vzv^{-1} = (x, g)(z, e_G)(x, g)^{-1} = \left(x\alpha(g, z)x^{-1}, e_G\right) = x\alpha(g, z)x^{-1}.$$

Using the inclusion $q(U_1 \cap X) \subset O$ we get $q(x\alpha(g, z)x^{-1}) \in O$. Since q is a central retraction and $\alpha(g, z) \in Y$, we obtain $q(x\alpha(g, z)x^{-1}) = \alpha(g, z)$. Therefore, $\alpha(g, z) \in O$ for every $g \in pr(V)$ and $z \in U_2 \cap Y$. Finally observe that $pr(V)$ is a neighborhood of e_G in $(G, \gamma/X)$ and $U_2 \cap Y$ is a neighborhood of y in $(Y, \gamma|_Y)$. This means that the action A.1 is continuous at (e_G, y). $\quad\square$

Proposition A.5. *Let* $M = (X \rtimes_\alpha G, \gamma)$ *be a topological semidirect product and* $\{Y_i\}_{i \in I}$ *be a system of* G*-subgroups in* X *such that the system of actions*

$$\{\alpha|_{G \times Y_i} : G \times Y_i \to Y_i\}_{i \in I}$$

is t-exact. Suppose that for each $i \in I$ *there exists a continuous central retraction* $q_i : X \to Y_i$. *Then if* $\gamma_1 \subset \gamma$ *is a coarser group topology on* M *such that* $\gamma_1|_X = \gamma|_X$ *then* $\gamma_1 = \gamma$.

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[2] That is, q is not necessarily a homomorphism.
[3] That is the topology on M is not necessarily the product topology of $X \times G$.

Proof. Proposition A.4 shows that each action

$$\alpha|_{G \times Y_i} : (G, \gamma_1/X) \times Y_i \to Y_i$$

is continuous. Clearly, $\gamma_1/X \subset \gamma/X$. By Definition 2.2(2) the group topology γ_1/X coincides with the given topology γ/X of G. Now, Merson's Lemma A.1 implies that $\gamma_1 = \gamma$. □

Corollary A.6. *Let $(X \rtimes_\alpha G, \gamma)$ be a topological semidirect product and let $\alpha : G \times X \to X$ be t-exact. Suppose that X is Abelian and $\gamma_1 \subset \gamma$ is a coarser group topology which agrees with γ on X. Then $\gamma_1 = \gamma$.*

Proof. Since X is Abelian, the identity mapping $X \to X$ is a central retraction. □

The commutativity of X is essential here as we already mentioned in Remarks 2.6(6).

Lemma A.7. *Let $(E, \|\cdot\|)$ be a normed space. Denote by σ the norm topology on E. Suppose that $\sigma' \subset \sigma$ is a strictly coarser, not necessarily Hausdorff, group topology on E. Then every σ'-open nonempty subset U in E is norm-unbounded.*

Proof. Since σ' is strictly coarser than the given norm topology, there exists $\varepsilon_0 > 0$ such that every σ'-neighborhood O of 0_E in E contains an element x with $\|x\| \geq \varepsilon_0$. It suffices to prove our lemma for a σ'-neighborhood U of 0_E. Since σ' is a group topology, for each natural n there exists a σ'-neighborhood V_n such that $nV_n \subset U$. One can choose $x_n \in V_n$ with the property $\|x_n\| \geq \varepsilon_0$. Then $\|n \cdot x_n\| \geq n \cdot \varepsilon_0$. Since $nx_n \in U$ (and n is arbitrary), this means that U is norm-unbounded. □

Lemma A.8. *Let $\omega : (E, \sigma) \times (F, \tau) \to \mathbb{R}$ be a strong duality with normed spaces E and F. Assume that $\sigma' \subset \sigma$ and $\tau' \subset \tau$ are coarser, not necessarily Hausdorff, group topologies on E and F respectively such that $\omega : (E, \sigma') \times (F, \tau') \to \mathbb{R}$ is continuous. Then necessarily $\sigma' = \sigma$ and $\tau' = \tau$.*

Proof. We show that $\sigma' = \sigma$. We omit the similar arguments for $\tau' = \tau$.

By our assumption $\omega : (E, \sigma') \times (F, \tau') \to \mathbb{R}$ is continuous. Then this map remains continuous replacing τ' by the stronger topology τ. That is the map $\omega : (E, \sigma') \times (F, \tau) \to \mathbb{R}$ is continuous, too. Assume that σ' is strictly coarser than σ. By Lemma A.7 every σ'-neighborhood of 0_E is norm-unbounded in $(E, \|\cdot\|)$. By the continuity of $\omega : (E, \sigma') \times (F, \tau) \to \mathbb{R}$ at the point $(0_E, 0_F)$ there exist: an σ'-neighborhood U of 0_E and a τ-neighborhood V of 0_F such that $\{f(u): u \in U, \ f \in V\} \subset (-1, 1)$. Since U is norm-unbounded the set $\{f(u): u \in U, \ f \in V\} \subset (-1, 1)$ is also unbounded in $\mathbb{R}$ by Definition 3.2. This contradiction completes the proof. □

Proposition A.9. *Let $H(\omega) = \mathbb{R} \times E \rtimes_{\omega^\nabla} F$ be the Heisenberg group of the strong duality $\omega : E \times F \to \mathbb{R}$ with normed spaces E and F. Assume that $\gamma_1 \subset \gamma$ is a coarser group topology on $H(\omega)$ such that $\gamma_1|_{\mathbb{R}} = \gamma|_{\mathbb{R}}$. Then $\gamma_1 = \gamma$.*

Proof. Denote by γ the given product topology on $H(w)$. Let $\gamma_1 \subset \gamma$ be a coarser group topology on $H(w)$ such that $\gamma_1|_{\mathbb{R}} = \gamma|_{\mathbb{R}}$. By Merson's lemma it suffices to show that $\gamma_1/\mathbb{R} = \gamma/\mathbb{R}$.

First we establish the continuity of the map

$$w : (E, \gamma_1|_E) \times (F, \gamma_1/\mathbb{R} \times E) \to (\mathbb{R}, \gamma_1|_{\mathbb{R}}) = (\mathbb{R}, \gamma|_{\mathbb{R}}). \tag{A.2}$$

We prove the continuity of the map A.2 at an arbitrary pair $(x_0, f_0) \in E \times F$. Let O be a neighborhood of $f_0(x_0)$ in $(\mathbb{R}, \gamma_1|_{\mathbb{R}})$. Choose a neighborhood O' of $(f_0(x_0), 0_E, 0_F)$ in $(H(w), \gamma_1)$ such that $O' \cap \mathbb{R} = O$. Consider the points $\bar{x}_0 := (0_{\mathbb{R}}, x_0, 0_F)$, $\bar{f}_0 := (0_{\mathbb{R}}, 0_E, f_0) \in H(w)$. Observe that the commutator $[\bar{f}_0, \bar{x}_0]$ is just $(f_0(x_0), 0_E, 0_F)$. Since $(H(w), \gamma_1)$ is a topological group there exist γ_1-neighborhoods U and V of $\bar{x}_0$ and $\bar{f}_0$ respectively such that $[v, u] \in O'$ for every pair $v \in V, u \in U$. In particular, for every $\bar{y} := (0_{\mathbb{R}}, y, 0_F) \in U \cap E$ and $v := (a, x, f) \in V$ we have $[v, \bar{y}] = (f(y), 0_E, 0_F) \in O' \cap \mathbb{R} = O$. We obtain that $f(y) \in O$ for every $f \in q_F(V)$ and $\bar{y} \in U \cap E$. This means that we have the continuity of A.2 at (f_0, x_0) because $q_F(V)$ is a neighborhood of f_0 in the space $(F, \gamma_1/\mathbb{R} \times E)$ and $U \cap E$ is a neighborhood of x_0 in $(E, \gamma_1|E)$. Since the given biadditive mapping is a strong duality it follows by Lemma A.8 that the topology $\gamma_1/\mathbb{R} \times E$ on F coincides with the given topology $\tau = \gamma/\mathbb{R} \times E$.

Quite similarly one can prove that the map

$$w : (E, \gamma_1/\mathbb{R} \times F) \times (F, \gamma_1|_F) \to (\mathbb{R}, \gamma_1|_\mathbb{R})$$

is continuous, which implies that $\gamma_1/\mathbb{R} \times F = \gamma/\mathbb{R} \times F$.

Denote by σ and τ the given norm topologies on E and F, respectively.

By the equalities $\gamma_1/\mathbb{R} \times E = \gamma/\mathbb{R} \times E = \tau$ in F and $\gamma_1/\mathbb{R} \times F = \gamma/\mathbb{R} \times F = \sigma$ in E it follows that the maps

$$q_E : \big(H(w), \gamma_1\big) \to (E, \sigma), \qquad (a, x, f) \mapsto x$$

and

$$q_F : \big(H(w), \gamma_1\big) \to (F, \tau), \qquad (a, x, f) \mapsto f$$

are continuous. Then we obtain that

$$q_{E \times F} : \big(H(w), \gamma_1\big) \to E \times F, \qquad (a, x, f) \mapsto (x, f)$$

is also continuous, where $E \times F$ is endowed with the product topology induced by the pair of topologies (σ, τ). This topology coincides with $\gamma/\mathbb{R}$. Then $\gamma_1/\mathbb{R} \supset \gamma/\mathbb{R}$. Since $\gamma_1 \subset \gamma$ we have $\gamma_1/\mathbb{R} \subset \gamma/\mathbb{R}$. Hence $\gamma_1/\mathbb{R} = \gamma/\mathbb{R}$, as desired. $\square$

Theorem A.10. *(See [16, Theorem 3.9].) For every strong duality $\omega : E \times F \to \mathbb{R}$ with normed spaces E and F the corresponding group $H_+(\omega) = H(\omega) \rtimes_\alpha \mathbb{R}_+$ is minimal.*

Proof. Denote by γ the given topology on $H_+(\omega)$ and suppose that $\gamma_1 \subset \gamma$ is a coarser Hausdorff group topology. The group $\mathbb{R} \rtimes \mathbb{R}_+$ is minimal [4] (see Introduction) and it naturally is embedded in $H_+(\omega)$. Therefore, $\gamma_1|_\mathbb{R} = \gamma|_\mathbb{R}$. From Proposition A.9 immediately follows that $\gamma_1|_{H(\omega)} = \gamma|_{H(\omega)}$.

Now observe that by Lemma A.3 the natural retraction $q : H(\omega) \to \mathbb{R}$ is central and the action of $\mathbb{R}_+$ on $\mathbb{R}$ is t-exact (see Remark 2.6(2)). By Proposition A.5 (in the situation: $G := \mathbb{R}_+$, $X := H(\omega)$, $Y := \mathbb{R}$) we get $\gamma_1 = \gamma$. $\square$

The following result is a particular case of [16, Theorem 4.8]. For simplicity we give the arguments only for the system with a single birepresentation. This particular case is enough for the main result of the present paper.

Theorem A.11. *(Compare [16, Theorem 4.8].) Let Φ be a t-exact birepresentation of a topological group G into a strong duality $\omega : E \times F \to \mathbb{R}$ with normed spaces E and F.*

(1) *Then the corresponding induced group*

$$M := M_+(\Phi) = \big((\mathbb{R} \times E \rtimes_{\omega^\triangledown} F) \rtimes_\alpha \mathbb{R}_+\big) \rtimes_\pi G$$

is minimal.
(2) *The projection $p : M \to G$ is a group retraction such that M and also the kernel* $\ker(p)$ *are minimal groups.*

Proof. (1): By our definitions

$$M := H_+(\omega) \rtimes_\pi G = \big((\mathbb{R} \times E \rtimes_{\omega^\triangledown} F) \rtimes_\alpha \mathbb{R}_+\big) \rtimes_\pi G.$$

Let $\gamma_1 \subset \gamma$ be a coarser Hausdorff group topology on M. Theorem A.10 establishes the minimality of $H_+(\omega) = (\mathbb{R} \times E \rtimes_\omega F) \rtimes_\alpha \mathbb{R}_+$. In particular, γ_1 and γ agree on the subgroup $H := H(\omega) = \mathbb{R} \times E \rtimes_{\omega^\triangledown} F$.

By Lemma A.3 the natural projections $H \to E$ and $H \to F$ are central. Since the birepresentation of G in ω is t-exact we can apply Proposition A.5 (with $Y_1 := E$, $Y_2 := F$) to the group $H \rtimes G$. It follows that γ_1 agrees with γ on the subgroup $H \rtimes G$ of M.

It is important now that (M, γ) is an internal topological semidirect product (see [31, Section 6]) of $H \rtimes G$ with $\mathbb{R}_+$ (observe that $H \rtimes G$ is a normal subgroup of $M = (H \rtimes G) \cdot \mathbb{R}_+$ and $(H \rtimes G) \cap \mathbb{R}_+ = \{e_M\}$). This presentation enables us to apply Proposition A.5, this time in the following situation: $G := \mathbb{R}_+$, $X := H \rtimes G$, $Y := \mathbb{R}$ and $q : X \to Y$ is the natural projection. Clearly $q : H \rtimes G \to \mathbb{R}$ is a continuous central retraction (with respect to the topologies $\gamma|_{H \rtimes G}$ and $\gamma|_\mathbb{R}$). The action of $\mathbb{R}_+$ on $\mathbb{R}$ is t-exact (Remark 2.6(2)). As we mentioned above γ_1 agrees with γ on the subgroup

$X := H \searrow G$ of M. So all requirements of Proposition A.5 are fulfilled for the topological group (M, γ) and the coarser topology $\gamma_1 \subset \gamma$. As a conclusion we get $\gamma_1 = \gamma$.

(2): $\ker(p) = H_+(\omega)$ is minimal by Theorem A.10. $\quad \square$

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