ORIGINAL PAPER



Tameness and Rosenthal type locally convex spaces

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Received: 16 April 2022 / Accepted: 15 March 2023 © The Author(s) under exclusive licence to The Royal Academy of Sciences, Madrid 2023

Abstract

Motivated by Rosenthal's famous l^1 -dichotomy in Banach spaces, Haydon's theorem, and additionally by recent works on tame dynamical systems, we introduce the class of *tame* locally convex spaces. This is a natural locally convex analogue of *Rosenthal* Banach spaces (for which any bounded sequence contains a weak Cauchy subsequence). Our approach is based on a bornology of *tame* subsets which in turn is closely related to eventual fragmentability. This leads, among others, to the following results:

- extending Haydon's characterization of Rosenthal Banach spaces, by showing that a lcs E is tame iff every weak-star compact, equicontinuous convex subset of E^* is the strong closed convex hull of its extreme points iff $\overline{\operatorname{co}}^{w^*}(K) = \overline{\operatorname{co}}(K)$ for every weak-star compact equicontinuous subset K of E^* ;
- E is tame iff there is no bounded sequence equivalent to the generalized l^1 -sequence;
- strengthening some results of W.M. Ruess about Rosenthal's dichotomy;
- applying the Davis-Figiel-Johnson-Pelczyński (DFJP) technique one may show that every *tame operator* $T: E \rightarrow F$ between a lcs E and a Banach space F can be factored through a tame (i.e., Rosenthal) Banach space.

Keywords Asplund space \cdot Bornologies \cdot Double limit property \cdot Haydon theorem \cdot Reflexive space \cdot Rosenthal dichotomy \cdot Rosenthal space \cdot Tame locally convex \cdot Tame system

Mathematics Subject Classification 46A03 · 46A17 · 46B22 · 37Bxx · 54Hxx

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1 Introduction

In the present work, we introduce and study a locally convex analogue of Rosenthal Banach spaces. As in [15, 16, 18], we say that a Banach space V is *Rosenthal* if any bounded sequence contains a weak Cauchy subsequence, or equivalently, if V does not contain an isomorphic copy of l^1 . Such Banach spaces appear in many publications (especially, after Rosenthal's classical work [48]), usually without any special name.

In order to better understand our approach and related classes, we present our definition in the framework of the smallness hieararchy for bounded subsets in lcs. In this way, we also provide natural locally convex analogues of Asplund and reflexive Banach spaces.

Smallness hierarchy of bounded subsets. The relationship between a space *E* and its topological dual E^* , via various classical bornologies on *E*, is one of the central themes in the theory of locally convex spaces. For every bounded subset *B* of *E* and an equicontinuous weak-star compact subset *K* of E^* (notation: $K \in \text{eqc}(E^*)$), we can think of *B* as a bounded family of real-valued functions over *K* (via the canonical bilinear map $E \times E^* \to \mathbb{R}$). This "tango" between *B* and *K* is a source of many interesting properties of the entire space. Namely, we want to study whether the family $\tilde{B} := {\tilde{b} : K \to \mathbb{R}}_{b\in B}$ is small (in some sense), and then study the locally convex spaces whose all bounded subsets are small in the same way.

This is related to the general topological question: what might be a hierarchy of *smallness* for a bounded family $B \subset \mathbb{R}^K$ of real functions on an abstract compact space K? We present a framework for this kind of comparisons using the concept of bornological classes (Sect. 3).

We suggest three cases which seem to be very natural. They are very important in the theory of dynamical systems and their representations on widely known classes of Banach spaces. See Sect. 11 and joint works of the second author with Eli Glasner [15, 18]. Consider the following three conditions on B:

- (1) *B* is *tame* on *K* (does not contain any sequence which is combinatorially independent in the sense of Rosenthal, Definition 2.21);
- (2) *B* is a fragmented family (Definition 2.12) of functions on K;
- (3) *B* has the Grothendieck's double limit property (DLP) on *K* (Definition 2.28).

Remark 1.1 These three conditions do not seem immediately comparable. However, (3) \Rightarrow (2) \Rightarrow (1). As it follows from results of [17, 39], every tame (fragmented, DLP) bounded family *B* of continuous functions on a compact space *X* can be *represented* on a Rosenthal (Asplund, reflexive) Banach space.

These results are based on the Davis–Figiel–Johnson–Pelczyński factorization technique [6]. See also Lemma 9.7 and Theorem 9.8.

The tameness can be expressed also in terms of *eventual fragmentability* (Definition 2.12 and Lemma 2.24). This gives an alternative explanation of $(2) \Rightarrow (1)$. As to $(3) \Rightarrow (2)$, note

that B has DLP on K iff the natural image of B into the Banach space C(K) is relatively weakly compact.

Recall that a *representation* of a bounded map $B \times K \to \mathbb{R}$ on a Banach space V is a pair (v, α) of bounded maps $v: B \to V$, $\alpha: K \to V^*$, where α is weak-star continuous and $f(x) = \langle v(f), \alpha(x) \rangle$ for all $f \in B, x \in K$.



For the converse direction (justifying these representations above), note that a Banach space V is:

- (1) Rosenthal (not containing a copy of l^1) iff the closed unit ball B_V of V is a tame family of functions on the weak-star compact unit ball B_{V^*} of V^* ;
- (2) Asplund iff B_V is a a fragmented family of functions on B_{V^*} ;
- (3) reflexive iff B_V has DLP on B_{V^*} .

These three characterizations and Remark 1.1 suggest corresponding locally convex analogues via three bornologies of tame, Asplund and DLP subsets, as defined below.

Definition 1.2 Let E be a lcs.

- (1) We say that a bounded subset $B \subset E$ is:
 - (a) tame if B is a tame family on every weak-star compact equicontinuous subset K ∈ eqc (E*);
 - (b) Asplund if B is a fragmented family on every $K \in \text{eqc}(E^*)$;
 - (c) DLP if B is DLP on every $K \in eqc(E^*)$.
- (2) We say that a lcs E is:
 - (a) *tame* $(E \in (\mathbf{T}))$ if every bounded subset in *E* is tame, Definition 5.3;
 - (b) Namioka-Phelps ($E \in (\mathbf{NP})$) if every bounded subset in E is Asplund, Definition 4.10;
 - (c) DLP ($E \in (DLP)$) if every bounded subset in E is DLP, Definition 4.2.

Asplund subsets play a major role in Banach space theory (sometimes under different names); see [5, 9]. The class (**NP**) was first defined in [33] using a different but equivalent approach.

Properties and examples. The class (T) is quite large. First of all, note that

$$(\mathbf{DLP}) \subset (\mathbf{NP}) \subset (\mathbf{T}).$$

This can be derived from Remark 1.1. Using results of Diestel–Morris–Saxon [7], we show in Proposition 5.7 that (**T**) is *properly* larger than the variety generated by all Banach Rosenthal spaces. Furthermore, (**T**) has nice stability properties (Theorem 5.5). Among other results, we show that (**T**) is closed under: subspaces, arbitrary products, locally convex direct sums and bound covering continuous linear images.

These properties are verified using the concept of fragmentability (which originally comes from Namioka–Phelps [41], Jayne–Rogers [27]) and its natural generalization for families (borrowed from recent study of tame dynamical systems).

Fragmented families (Definition 2.12) are closely related to tameness, providing an important sufficient condition. Beyond representation theory (Remark 1.1), a more direct reason is that *B* is tame on *K* iff *B* is *eventually fragmented* in the sense of [15] (i.e., every sequence in *B* contains a subsequence which is fragmented on *K*). We apply here some useful results of Rosenthal [48] and Talagrand [53], synthesized in Lemma 2.24.

One of the challenges is to find when standard constructions lead to NP or tame lcs. For lcs of the type $C_k(X)$ we have a concrete (and somewhat expected) criterion, Proposition 5.9, which (up to some reformulations) is quite close to a known result by Gabriyelyan–Kakol–Kubiś–Marciszewski [12, Lemma 6.3].

Free locally convex spaces. Another important construction producing lcs is the classical free locally convex space L(X), defined for every Tychonoff space X.

For every compact space K, its free lcs L(K) is *multi-reflexive* (i.e., embedded into a product of reflexive Banach spaces), as it was proved in a very recent paper by Leiderman and Uspenskij [32]. Since multi-reflexive lcs (by Theorem 4.6) is (**DLP**), we obtain that L(K) is (**DLP**).

More generally, in Theorem 8.4 we show that L(X) is (**DLP**) (hence, (**NP**) and (**T**)) for every Tychonoff space X. In particular, we get that $L(\mathbb{N}^{\mathbb{N}})$ is (**DLP**) for the Polish space $\mathbb{N}^{\mathbb{N}}$ of all irrationals. In contrast, another result from [32] shows that $L(\mathbb{N}^{\mathbb{N}})$ is not multi-reflexive. Moreover, while every semi-reflexive lcs is (**DLP**), the spaces L(X), which are (**DLP**), are very rarely semi-reflexive (Theorem 8.8).

Rosenthal type properties. Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a lcs *E* is *weak Cauchy* if the scalar sequence $u(x_n)$ is convergent for every $u \in E^*$. Rosenthal's celebrated dichotomy theorem (see [48]) asserts that every bounded sequence in a Banach space either has a weak Cauchy subsequence or a subsequence equivalent to the unit vector basis of l_1 (an l_1 -sequence).

Definition 1.3 We say (as in [15, 16, 18]) that a Banach space V is *Rosenthal* if every bounded sequence in A has a weak Cauchy subsequence.

Definition 1.4 Let *E* be a lcs. Define the following properties of *E*:

- (Ros) Every bounded sequence in E has a subsequence which is weak Cauchy.
- (**R**₁) There is no bounded sequence in *E* which is equivalent to the l^1 -basis (in the sense of Definition 6.1).
- $(\overline{\mathbf{R_1}})$ The Banach space l^1 cannot be embedded into *E*.

All these three properties are equivalent in Banach spaces by Rosenthal's classical results, [48].

Note that [13] uses some similar notation $((R_1) \text{ and } (R_2))$ to represent similar concepts $((\mathbf{Ros}) \text{ and } (\mathbf{R}_1), \text{ respectively})$. Some authors (e.g., [13] and [11]) say that a lcs *E* has the *Rosenthal property* if it satisfies the Rosenthal dichotomy (every bounded sequence has a subsequence that is either weak Cauchy or equivalent to the l^1 -basis). In this paper, we always refer to Definition 1.4.

In Sect. 7 we prove the following theorems:

Theorem 1.5 (7.5) *For any lcs we have* (**Ros**) \implies (**T**) = (**R**₁) \implies (**R**₁).

Note that $(\mathbf{Ros}) \neq (\mathbf{T})$ (Theorem 1.9) and $(\mathbf{R}_1) \neq (\overline{\mathbf{R}_1})$ (Example 6.4). For every locally complete lcs we have $(\mathbf{R}_1) = (\overline{\mathbf{R}_1})$ (Lemma 6.7).

Theorem 1.6 (7.2) [Tame dichotomy in lcs] Let *E* be a locally convex space. Then every bounded subset in *E* is either tame, or has a subsequence equivalent to the l^1 -sequence.

Theorem 1.7 (7.7) If all bounded sets of a lcs E are metrizable, then (**Ros**) = (**T**) = (**R**₁), and the following generalized Rosenthal's dichotomy holds: any bounded sequence in E either has a weak Cauchy subsequence or an l^1 -subsequence.

The latter result gives, as a corollary, a well-known result of Ruess which extends Rosenthal's non-containtment of l^1 -criteria to a quite large class of lcs.

Fact 1.8 (Ruess [49, Thm 2.1 and Prop. 3.3]) Let *E* be a locally complete lcs with metrizable bounded sets. Then (**Ros**) = ($\overline{\mathbf{R}_1}$) and the following dichotomy holds: any bounded sequence in *X* either has a weak Cauchy subsequence or a subsequence which spans an isomorphic copy of l^1 .

The following result shows the limitations in general lcs for the existence of a Rosenthal type dichotomy.

Theorem 1.9 (7.8) *There exists a tame, complete (even reflexive) lcs which:*

- (i) is not a Rosenthal lcs;
- (ii) does not contain any l^1 -subsequence;
- (iii) contains a dense, Rosenthal subspace.

As a corollary: Rosenthal's dichotomy does not hold for such locally convex spaces.

This also shows that (\mathbf{Ros}) is not closed under the completion. The same is unclear for (\mathbf{T}) .

For every lcs E, there exists the strongest topology between all locally convex tame topologies which are weaker than the original topology. This is proved in Theorem 7.13 using the bipolar theorem. In fact, it is proven for every polarly compatible bornological class (Definition 3.21).

In Theorem 9.8 we apply the DFJP technique [6] and show that every tame (NP, DLP) operator $T: E \rightarrow F$ between a lcs E and a Banach space F can be factored through a tame (Asplund, reflexive) Banach space.

Haydon's theorem for tame locally convex spaces. Recall that, according to Mazur's theorem, weak and norm closures are the same for convex subsets in Banach (or, even in locally convex) spaces. This property for weak-star closure in the dual is not true in general. Haydon's theorem comes as an important compromise. It generalizes an earlier result for separable Banach spaces which was proved by Odell and Rosenthal in [43].

In Sect. 10, we prove a generalized version of Haydon's theorem for locally convex spaces.

Theorem 1.10 (10.12) For a locally convex space *E*, the following are equivalent:

- (1) E is tame.
- (2) Every weak-star compact, equicontinuous convex subset of E^* is the strong closed convex hull of its extreme points.
- (3) For every weak-star compact, equicontinuous subset K of E^* we have:

$$\overline{\operatorname{co}}^{w^*}(K) = \overline{\operatorname{co}}(K).$$

Open questions. See 5.6, 7.14, 8.7.

2 Definitions: fragmentability, independence and tameness

Topological concepts. All topological spaces below are assumed to be completely regular and Hausdorff (that is, Tychonoff). Recall that a function $f: X \to Y$ between topological spaces is said to be a *Baire class 1 function* [28] if the inverse image of every open set is F_{σ} . f has the *point of continuity property* (in short: PCP) if for every closed nonempty $A \subset X$ the restriction $f_{|A|}: A \to Y$ has a continuity point.

Locally convex spaces. We include the following standard definitions. In this work *E* will usually denote a real locally convex space. A subset $B \subset E$ is said to be *bounded* if for every neighborhood *O* of the zero in *E* there exists $c \in \mathbb{R}$ such that $B \subset cO$. For every linear continuous operator $u: E_1 \to E_2$ and every bounded subset $B \subset E_1$ its image u(B) is also bounded in E_2 . Also, *B* is bounded if and only if it is weakly bounded (see, [25, Thm. 8.3.4]). The boundedness is *countably determined*. That is, *B* is bounded iff all its countable subsets are bounded.

Definition 2.1

- (1) A subset $S \subset E$ is said to be
 - (a) convex if when x, y ∈ S and 0 ≤ α ≤ 1 then αx + (1 − α)y ∈ S. The convex hull co S of S is defined as the smallest convex set containing S. Explicitly:

$$\operatorname{co}(S) := \left\{ \sum_{n=1}^{N} \alpha_n x_n \mid \forall 1 \le n \le N : x_n \in S, \, \alpha_n \in [0, 1], \, \sum_{n=1}^{N} \alpha_n = 1 \right\}.$$

(b) balanced if αS ⊆ S for every α ∈ ℝ satisfying |α| ≤ 1. The balanced hull bal Sof A is defined as the smallest balanced set containing A. Explicitly:

bal
$$S := \{ \alpha x \mid x \in S, \alpha \in [-1, 1] \}.$$

(c) a disk (or absolutely convex) if it is both balanced and convex. The absolutely convex hull acx S of S is defined as the smallest disked set containing S. Explicitly:

$$\operatorname{acx} S = \operatorname{co} (\operatorname{bal} S).$$

- (2) A *barrel* is a closed disk S which is absorbing, meaning that $E = \bigcup_{n \in \mathbb{N}} nS$.
- (3) The gauge $q_S: E \to \mathbb{R}$ of S is defined as: $q_S(x) := \inf\{r > 0 \mid x \in rS\}$.
- (4) *E* is said to be *locally complete* if for every closed bounded disk S ⊆ E, the linear span Span(S) ⊆ E of S is complete with respect to q_S.
- (5) As in [25], we denote the *polar* of a subset S of a locally convex space E as:

 $S^{\circ} := \{ \varphi \in E^* \mid \forall x \in S : |\varphi(x)| \le 1 \}.$

Similarly, for every $S \subset E^*$ its polar is:

$$S^{\circ} := \{ x \in E \mid \forall \varphi \in S : |\varphi(x)| \le 1 \}.$$

Now, for every $S \subset E$ its *bipolar* $S^{\circ\circ}$ is defined as $(S^{\circ})^{\circ}$.

(6) A subset M ⊆ E* is said to be *equicontinuous*, if there exists some neighborhood 0 ∈ U ⊆ E such that M ⊆ U°.

Fact 2.2 (Bipolar Theorem) [25, p. 149] For every $S \subseteq E$ the bipolar $S^{\circ\circ} \subset E$ is equal to the weak closed absolutely convex hull $\overline{acx}^w S$.

Fact 2.3 (Mazur's Theorem [52, p. 65, Cor. 2]) For every convex subset of a lcs E, its closure is identical with its weak closure. Hence, $\overline{co(S)} = \overline{co(S)}^w$ for every $S \subset E$.

Denote by eqc (E^*) the system of all equicontinuous weak-star compact subsets in E^* (*equicontinuous compactology* in terms of [25]). It is a basis of the system of all equicontinuous subsets in E^* as it follows from Alaouglu–Bourbaki's theorem.

Fact 2.4 (Alaouglu–Bourbaki) For every equicontinuous subset $A \subset E^*$, its weak-star closure is equicontinuous and weak-star compact.

Let $f: E_1 \to E_2$ be a continuous linear operator between lcs. Write

$$f^* \colon E_2 \to E_1, \langle v, f^*(\varphi) \rangle = \langle f(v), \varphi \rangle$$

for its adjoint, where

$$E \times E^* \to \mathbb{R}, (u, \varphi) \mapsto \langle v, \varphi \rangle = \varphi(v)$$

is the canonical bilinear form.

Definition 2.5 Recall that the *strong topology on the dual* E^* of a lcs E is the topology of bounded convergence. The standard uniformity of E^* has the uniform basis $\{U[B, \varepsilon]\}$, where $\varepsilon > 0$ and B runs over all bounded subsets of E. Here

$$U[B,\varepsilon] := \{ (\varphi_1, \varphi_2) \in E^* \times E^* : |\varphi_1(b) - \varphi_2(b)| < \varepsilon \ \forall b \in B \},\$$

Fact 2.6 [42, Thm. 8.11.3] Suppose that $T: E \to F$ is a linear continuous operator between lcs. Then it is also weakly continuous. Moreover, T^* is both weak-star and strongly continuous.

A (dense) subspace F of a lcs E is said to be *large* in E (see [45, p. 254]) if every bounded set in E is contained in the closure of a bounded set in F. Every dense subspace in a normed space V is large. Also, the same is true for every separable metrizable lcs V.

Lemma 2.7 Let *F* be a large dense subspace of *E* and *i* : $F \hookrightarrow E$ be the inclusion map. Then $i^* : E^* \to F^*$ is a topological isomorphism with respect to the strong topology.

Proof It is easy to see that i^* is a bijection since F is dense in E. Applying Fact 2.6, all that is left is to show that i^* is strongly open. For every bounded subset $B \subseteq E$ and $\varepsilon > 0$ define:

$$W_E(B,\varepsilon) := \{ \varphi \in E^* \mid \forall x \in B : |\varphi(x)| < \varepsilon \}.$$

We define $W_F(B, \varepsilon)$ analogously. We will show that $i^*(W_E(B, \varepsilon))$ is always an open neighborhood of zero in F^* . By definition, for every bounded $B \subseteq E$ there exists a bounded $B' \subseteq F$ such that $B \subseteq \overline{B'}$. It is easy to see that:

$$W_F\left(B', \frac{1}{2}\varepsilon\right) \subseteq i^*\left(W_E\left(B, \varepsilon\right)\right).$$

Lemma 2.8 Let $F \subseteq E$ be a subspace and $i: F \hookrightarrow E$ is the inclusion map. If $M \subseteq F^*$ is a weak-star compact equicontinuous subset, then there exists a weak-star compact, equicontinuous subset $N \subseteq E^*$ such that $M = i^*(N)$.

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Proof A consequence of [25, Cor. 8.7.2] and the Alaouglu–Bourbaki's Theorem (Fact 2.4).

It is well-known that if B is a bounded disk, then its gauge q_B is a norm.

Fact 2.9 [45, Proposition 3.2.2] Let $B \subseteq E$ be a bounded disc in E. Then (E_B, q_B) is a normed space and its topology is finer than that induced by E.

The following is a consequence of Fact 2.9 and [25, p. 105 Prop. 1].

Lemma 2.10 Let *E* be a locally convex space. If $A \subseteq E$ is bounded, closed and absolutely convex, then $A = B_{E_A}$ (the unit ball of the semi-normed space (E_A, q_A)).

Fragmentability. The following definition is a generalized version of the *fragmentability* concept.

Definition 2.11 [26, 33] Let (X, τ) be a topological space and (Y, μ) a uniform space. *X* is (τ, μ) -*fragmented* by a (not necessarily continuous) function $f: X \to Y$ if for every nonempty subset *A* of *X* and every $\varepsilon \in \mu$ there exists an open subset *O* of *X* such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε -small in *Y*. We also say in that case that the function *f* is *fragmented* and write $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$ with its natural uniformity, then we write simply $\mathcal{F}(X)$.

When Y = X, $f = id_X$ and μ is a metric uniformity, we retrieve the usual definition of fragmentability (more precisely, (τ, μ) -fragmentability) in the sense of Jayne and Rogers [27]. Implicitly, it already appears in a paper of Namioka and Phelps [41].

If $f: (X, \tau) \to (Y, \mu)$ has PCP then it is fragmented. If (X, τ) is hereditarily Baire (e.g., compact, or Polish) and (Y, μ) is a pseudometrizable uniform space, then f is fragmented iff f has PCP. If X is Polish and Y is a separable metric space, then $f: X \to Y$ is fragmented iff f is a Baire class 1 function. See [14, 15].

Definition 2.12

- [14] We say that a *family of functions* F = {f: (X, τ) → (Y, μ)} is *fragmented* if the condition of Definition 2.11.1 holds simultaneously for all f ∈ F. That is, f(O ∩ A) is ε-small for every f ∈ F.
- (2) [15] *F* is an *eventually fragmented family* if every sequence in *F* has a subsequence which is a fragmented family on *X*.

Definition 2.12.1 was introduced in arxiv preprints of [14] and also independently (under the name: *equi-fragmented*) in the Ph.D. Thesis of M.M. Guillermo [22].

Lemma 2.13 Let $F = \{f : (X, \tau) \to (Y, \mu)\}$ be a family of functions. Then F is a fragmented family iff the mapping $\pi_F : X \to Y^F$, $\pi_F(x)(f) = f(x)$ is (τ, μ_U) -fragmented, where μ_U is the uniform structure of uniform convergence on the set Y^F of all mappings from F into (Y, μ) .

Proof Straighforward.

Lemma 2.14 Let $\alpha : X \to X'$ be a continuous map between compact spaces, (Y, μ) be a uniform space and $F \subseteq Y^{X'}$ be a family of functions. If F is fragmented, then so is $F \circ \alpha \subseteq Y^X$. If α is surjective, then the converse is also true.

Proof Combination of Lemma 2.13 and [15, Lemma 2.3.5].

Lemma 2.15

- (1) Let F be a fragmented family of real-valued functions on a topological space X. Then acx (F) is also fragmented.
- (2) [15, Prop. 4.15] Let F be an eventually fragmented family of real-valued functions on a compact space X. Then co(F) is also eventually fragmented.

Proof (1) If $f_i(D)$ is ε -small for every i = 1, ..., n and $\sum_{i=1}^n |c_i| \le 1, c_i \in \mathbb{R}$, then $\sum_{i=1}^n c_i f_i(D)$ is ε -small.

Lemma 2.16 Let $F \subset X$ be a fragmented family of functions from a topological space X into a uniform space Y. Then the pointwise closure \overline{F}^p is also a fragmented family.

Proof Let $A \subset X$ be a nonempty subset and $\varepsilon \in \mu$. Choose $\delta \in \mu$ such that $\delta^3 \subset \varepsilon$. There exist an open subset $O \subset X$ such that $O \cap A \neq \emptyset$ and $f(O \cap A)$ is δ -small for every $f \in F$. Let $h \in \overline{F}^p$. For this *h* and a given pair $x, y \in O \cap A$ (by definition of the pointwise topology), there exists $f_0 \in F$ such that

$$(h(x), f_0(x)) \in \delta, (f_0(y), h(y)) \in \delta.$$

Since $f_0(O \cap A)$ is δ -small, we have $(f_0(x), f_0(y)) \in \delta$. So, we obtain $(h(x), h(y)) \in \varepsilon$. Therefore, $h(O \cap A)$ is ε -small, as desired.

The following lemma is inspired by results of Namioka and it can be deduced after some reformulations from [40, Theorems 3.4 and 3.6].

Lemma 2.17 [19, Theorem 2.6] Let F be a bounded family of real-valued continuous functions on a compact space X. The following conditions are equivalent:

- (1) *F* is a fragmented family of functions on *X*.
- (2) Every countable subfamily C of F is fragmented on X.
- (3) For every countable subfamily C of F the pseudometric space (X, ρ_C) is separable, where

$$\rho_C(x_1, x_2) := \sup_{f \in C} |f(x_1) - f(x_2)|.$$

Lemma 2.18 Let X be a topological space. If $F \subseteq C(X)$ is a (eventually) fragmented family, then so is its closure \overline{F} in the uniform topology of C(X).

Proof The case of fragmented families is a consequence of Lemma 2.16. We are left with the case of eventually fragmented families.

Suppose that $\{f_n\}_{n \in \mathbb{N}} \subseteq \overline{F} \subseteq C(X)$. By definition, we can find $\{g_n\}_{n \in \mathbb{N}} \subseteq F$ such that

$$\forall x \in X : |f_n(x) - g_n(x)| < \frac{1}{n}.$$

Since *F* is eventually fragmented, we can find a subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\{g_{n_k}\}_{k \in \mathbb{N}}$ is fragmented. We claim that $\{f_{n_k}\}_{k \in \mathbb{N}}$ is also fragmented.

Let $A \subseteq X$ be non-empty and $\varepsilon > 0$. By definition, there exists some open $O \subseteq X$ such that $A \cap O \neq \emptyset$ and $g_{n_k}(A \cap O)$ is $\frac{1}{3}\varepsilon$ -small for every $k \in \mathbb{N}$. Choose $x \in A \cap O$ and $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \leq \frac{1}{3}\varepsilon$. Since $\{f_n\}_{n \in \mathbb{N}} \subseteq C(X)$, we can find a neighborhood $x \subseteq U \subseteq O$ such that for every $1 \leq m \leq n_0$, $f_m(U)$ is ε -small.

- (1) If $n_k \leq n_0$, then $f_{n_k}(U \cap A) \subseteq f_{n_k}(U)$ is ε -small by our construction.
- (2) Otherwise, $||f_{n_k} g_{n_k}|| \le \frac{1}{3}\varepsilon$. Moreover, $g_{n_k}(U \cap A) \subseteq g_{n_k}(O \cap A)$ is $\frac{1}{3}\varepsilon$ -small. Therefore, we conclude that $f_{n_k}(U \cap A)$ is ε -small.

In either case, $f_{n_k}(U \cap A)$ is ε -small, as required. Also, note that $x \in U \cap A \neq \emptyset$.

Corollary 2.19 Let X be a compact space. If $F \subseteq C(X)$ is a (eventually) fragmented family, then so is $\overline{\operatorname{acx}}^w F$, where the closure is taken with respect to the weak topology induced by the supremum norm.

Proof By Mazur's Theorem (Fact 2.3), $\overline{acx}^w F = \overline{acx} F$. Now, we can apply Lemma 2.15 and Lemma 2.18 to get the desired result.

Remark 2.20 An important example for the use of fragmented families (Definition 2.12) is in the case of bounded sets in Banach spaces. If $B \subseteq V$ is a bounded subset of a Banach space and $K \subseteq V^*$ is a weak-star compact subset, then we can view B as a family of functions over K. In this case, B is fragmented iff for every non-empty subset $A \subseteq K$ and $\varepsilon > 0$ there exists a weak-star open subset $O \subseteq V^*$ such that $O \cap A$ is not empty and diam $\{\langle v, x \rangle : x \in O \cap A, v \in B\} < \varepsilon$.

For some other properties of fragmented maps and fragmented families, we refer to [14, 15, 17, 26, 33, 34, 40]. Basic properties and more applications of fragmentability in topological dynamics can be found in [15, 17, 18, 33, 34].

Independent and tame families of functions. A sequence of real functions $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ on a set *X* is said to be (combinatorially) *independent* (see [48, 53]) if there exist real numbers a < b (bounds of independence) such that

$$\bigcap_{n\in P} f_n^{-1}(-\infty,a) \cap \bigcap_{n\in M} f_n^{-1}(b,\infty) \neq \emptyset$$

for all finite disjoint subsets P, M of \mathbb{N} .

Definition 2.21 [17, 19] A bounded family F of real-valued (not necessarily, continuous) functions on a set X is a *tame family* if F does not contain an independent sequence.

Lemma 2.22 [19, Lemma 6.4] Suppose that $\pi : X \to Y$ is a map and $F \subseteq \mathbb{R}^{Y}$ is a family of bounded functions. If F is tame then $F \circ \pi$ is tame. Moreover, if π is onto, the converse is also true.

The following fact from [30] can easily be derived using the finite intersection property characterization of the compactness.

Fact 2.23 Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is an independent family of continuous functions over a compact X. Then there are $a < b \in \mathbb{R}$ such that for every disjoint, possibly infinite P, $M \subseteq \mathbb{N}$:

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset.$$

By [37], every bounded family of (not necessarily continuous) functions $[0, 1] \rightarrow \mathbb{R}$ with total bounded variation (e.g., *Haar systems*) is tame. This remains true replacing the set [0, 1] by any circularly (e.g., linearly) ordered set.

As to the negative examples. The sequence of projections on the Cantor cube and the sequence of Rademacher functions on the unit interval both are independent (hence, nontame).

A critically important example of a nontame sequence is the standard basis sequence $\{e_n : n \in \mathbb{N}\}$ in l^1 as a family of functions on the unit ball of $(l^1)^* = l^\infty$.

The following useful lemma synthesizes some known results. It is based mainly on results of Rosenthal and Talagrand. The equivalence of (1), (3) and (4) is a part of [53, Thm. 14.1.7]. For the case (1) \Leftrightarrow (2), note that every bounded independent sequence $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ is an l^1 -sequence (in the sup-norm), [48, Prop. 4]. On the other hand, as the proof of [48, Thm. 1] shows, if $\{f_n\}_{n \in \mathbb{N}}$ has no independent subsequence then it has a pointwise convergent subsequence. Bounded pointwise-Cauchy sequences in C(X) (for compact X) are weak-Cauchy as it follows by Lebesgue's theorem. Now Rosenthal's dichotomy theorem [48] asserts that $\{f_n\}$ has no l^1 -sequence. In [15, Sect. 4] we show why eventual fragmentability of F can be included in this list (item (5)).

Lemma 2.24 Let K be a compact space and F is a bounded subset in the Banach space C(K). The following conditions are equivalent:

- (1) *F* does not contain an l^1 -sequence.
- (2) F is a tame family on K.
- (3) Each sequence in F has a pointwise convergent subsequence in \mathbb{R}^{K} .
- (4) The pointwise closure cl (F) of F in \mathbb{R}^K consists of fragmented maps.
- (5) F is an eventually fragmented family on K.

Rosenthal's dichotomy and Rosenthal's Banach spaces. For every topological space X denote by C(X) the vector space of all continuous real functions. When X is compact, as usual, we suppose that C(X) is endowed with the supremum norm. So, it will be a Banach subspace of $l^{\infty}(X)$.

Let $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a bounded sequence of functions on a *set* X. Following Rosenthal [48], we say that this sequence is an l^1 -sequence if there exists a constant $\delta > 0$ such that for all $n \in \mathbb{N}$ and choices of real scalars c_1, \ldots, c_n , we have

$$\delta \cdot \sum_{i=1}^{n} |c_i| \le \left\| \sum_{i=1}^{n} c_i f_i \right\|_{\infty}.$$

Then the closed linear span of $\{f_n\}_{n\in\mathbb{N}}$ in $l^{\infty}(X)$ is linearly homeomorphic to the Banach space l^1 . In fact, in this case the map $l^1 \to l^{\infty}(X)$, $(c_n)_{n\in\mathbb{N}} \to \sum_{n\in\mathbb{N}} c_n f_n$ is a linear homeomorphic embedding.

A sequence of vectors in a Banach space can be defined to be equivalent to an l^1 -sequence analogously. According to Rosenthal's dichotomy, every bounded sequence in a Banach space either has a weak Cauchy subsequence or admits an l^1 -sequence. Thus, a Banach space V does not contain an l^1 -sequence (equivalently, does not contain an isomorphic copy of l^1) iff every bounded sequence in V has a weak Cauchy subsequence, [48]. As in [15, 18], we call a Banach space satisfying these equivalent conditions a *Rosenthal Banach space*.

Definition 2.25 Let V be a normed space and $M \subset V^*$ be a subset in the dual space V^* . A bounded subset F of V is said to be *tame for M* if F, as a family of functions on M, is a tame family. If F is tame for the unit ball B_{V^*} of V^* (equivalently, for every bounded subset), then we simply say that F is a *tame subset in* V.

Lemma 2.26 Let V be a normed space, $A \subseteq V$ and $M \subseteq V^*$ be bounded subsets. If A is not tame over M, then A contains an l^1 -sequence in V.

Proof It is a known consequence of the Hahn–Banach theorem that V is isometrically embedded into $C(B_{V^*})$. Applying Lemma 2.24, we get the desired result.

The following characterization of Rosenthal Banach spaces is a reformulation of some known results (see, in particular, [51] and Lemma 2.24).

Lemma 2.27 Let V be a Banach space. The following conditions are equivalent:

- (1) V is a Rosenthal Banach space;
- (2) each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B_{V^*} of V^* . Equivalently, $B_{V^{**}} \subset \mathcal{F}(B_{V^*})$;
- (3) the unit ball B_V is a tame subset of V;
- (4) any bounded subset of V is tame for any bounded subset of V^* .

Proof (1) \Rightarrow (4) A consequence of Lemma 2.26. (4) \Rightarrow (3) Trivial. (3) \Rightarrow (2) Suppose that B_V is a tame family over B_{V^*} . Using Lemma 2.24, we can conclude that $cl_p(B_V) \subseteq \mathcal{F}(B_{V^*})$. On the other hand, $B_{V^{**}} = cl_p(B_V)$ by Goldstein's theorem. Hence, $B_{V^{**}} \subset \mathcal{F}(B_{V^*})$. (2) \Rightarrow (1) Use [51, Thm. 3].

The Double Limit Property (DLP). Recall Grothendieck's double limit property.

Definition 2.28 Let $F \subset \mathbb{R}^K$ be a family of real functions on a set K. Then F is said to have the *double limit property* (DLP) if for every sequence $\{f_n\}_{n\in\mathbb{N}}$ in F and every sequence $\{x_n\}_{n\in\mathbb{N}}$ in K, the limits

$$\lim_{n} \lim_{m} f_n(x_m) \text{ and } \lim_{m} \lim_{n} f_n(x_m)$$

are equal whenever they both exist.

We will often write that a subset is DLP rather than the more correct "has the DLP".

Lemma 2.29

(1) If $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence of functions on K and $\{x_n\}_{n\in\mathbb{N}} \subseteq K$, then there exist subsequences $\{n_k\}_{k\in\mathbb{N}}, \{m_t\}_{t\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{k\in\mathbb{N}}\lim_{t\in\mathbb{N}}f_{n_k}(x_{m_t}) \text{ and } \lim_{t\in\mathbb{N}}\lim_{k\in\mathbb{N}}f_{n_k}(x_{m_t})$$

exist.

- (2) If $A \subset l^{\infty}(K)$ is a bounded family of functions over K satisfying the DLP, then so does the balanced hull bal A (see Definition 2.1).
- (3) If A_1 , A_2 are bounded sets of functions over K satisfying the DLP, then so does A + B.
- (4) Suppose that $\varphi: K_1 \to K_2$ is a continuous map and $F \subseteq C(K_2)$ is DLP. Then $F \circ \varphi \subseteq C(K_1)$ is DLP. Moreover, if φ is surjective, then the converse is also true. Namely, if $F \circ \varphi \subseteq C(K_1)$ is DLP then so is F.

Proof

We will show a more general fact. Suppose that A and B are sets and (·, ·): A × B → ℝ is a map. Also, let {a_n}_{n∈ℕ} ⊆ A and {b_m}_{m∈ℕ} ⊆ B be sequences such that (·, ·) is bounded over them. Then there exist subsequences {n_k}_{k∈ℕ}, {m_t}_{t∈ℕ} ⊆ ℕ such that the limit

$$\lim_{k\in\mathbb{N}}\lim_{t\in\mathbb{N}}\langle a_{n_k}, b_{m_t}\rangle$$

exist. By applying this fact twice we will get the desired result. First, note that the sequence $\{\langle a_n, b_m \rangle\}_{m \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded for every $n \in \mathbb{N}$. We can thus use induction and a diagonal argument to construct a subsequence $\{m_t\}_{t\in\mathbb{N}}$ such that $\{\langle a_n, b_{m_t} \rangle\}_{t\in\mathbb{N}}$ converges for every $n \in \mathbb{N}$. Write $\alpha_n := \lim_{t\in\mathbb{N}} \langle a_n, b_{m_t} \rangle$. Note that $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a bounded sequence so there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ converges. However:

$$\lim_{k\in\mathbb{N}}\alpha_{n_k}=\lim_{k\in\mathbb{N}}\lim_{t\in\mathbb{N}}\langle a_{n_k},b_{m_t}\rangle,$$

as required.

(2) Suppose that $\{\alpha_n f_n\}_{n \in \mathbb{N}} \subseteq \text{bal } A$ and $\{x_m\}_{m \in \mathbb{N}} \subseteq K$ such that

$$\lim_{n\in\mathbb{N}}\lim_{m\in\mathbb{N}}\alpha_n f_n(x_m) \text{ and } \lim_{m\in\mathbb{N}}\lim_{n\in\mathbb{N}}\alpha_n f_n(x_m)$$

exist. By definition, $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [-1, 1]$ and is therefore bounded. Thus, we can apply Lemma 2.29 and the Bolzano-Weierstrass theorem to find subsequences $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $\{m_t\}_{t \in \mathbb{N}} \subseteq \mathbb{N}$ such that:

$$\lim_{k\in\mathbb{N}}\alpha_{n_k}, \lim_{k\in\mathbb{N}}\lim_{t\in\mathbb{N}}f_{n_k}(x_{m_t}) \text{ and } \lim_{t\in\mathbb{N}}\lim_{k\in\mathbb{N}}f_{n_k}(x_{m_t})$$

exist. Moreover, since A is DLP, we know that

$$\lim_{k\in\mathbb{N}}\lim_{t\in\mathbb{N}}f_{n_k}(x_{m_t})=\lim_{t\in\mathbb{N}}\lim_{k\in\mathbb{N}}f_{n_k}(x_{m_t}).$$

Together we get:

$$\lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} \alpha_n f_n(x_m) = \lim_{k \in \mathbb{N}} \lim_{t \in \mathbb{N}} \alpha_{n_k} f_{n_k}(x_{m_t}) = \left(\lim_{k \in \mathbb{N}} \alpha_{n_k}\right) \lim_{k \in \mathbb{N}} \lim_{t \in \mathbb{N}} f_{n_k}(x_{m_t})$$
$$= \left(\lim_{k \in \mathbb{N}} \alpha_{n_k}\right) \lim_{t \in \mathbb{N}} \lim_{k \in \mathbb{N}} f_{n_k}(x_{m_t}) = \lim_{t \in \mathbb{N}} \lim_{k \in \mathbb{N}} \alpha_{n_k} f_{n_k}(x_{m_t})$$
$$= \lim_{m \in \mathbb{N}} \lim_{n \in \mathbb{N}} \alpha_n f_n(x_m)$$

as required.

(3) and (4) are easy to check.

Fact 2.30 (N.J. Young [57, Thm. 2]) Let *E* and *F* be topological vector spaces and $A \subseteq E, B \subseteq F$. Furthermore, let $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$ be a bilinear map bounded on $A \times B$. If *A* has the DLP as a family of functions over *B*, then so does the bipolar $A^{\circ\circ}$.

If E is locally convex, we can apply the Bipolar Theorem (Fact 2.2) to conclude that $\overline{\operatorname{acx}}^w A$ is DLP over B.

Lemma 2.31 Suppose that K is compact and $A \subseteq C(K)$ is a bounded family of functions that satisfies the DLP. Then so does $\overline{\operatorname{acx}}^w A$.

Proof Suppose that $A \subseteq C(K)$ is a bounded family with the DLP over K. Now, consider K as a subspace of the free topological vector space L(K). By definition, every $f \in A$ can be extended uniquely to a continuous linear operator on L(K). This gives a bilinear pairing $\langle \cdot, \cdot \rangle \colon A \times K$ defined by $\langle f, x \rangle \coloneqq f(x)$.

Also, since A is bounded, so is the image of $A \times K$ under this bilinear map. By applying Fact 2.30, we conclude that $\overline{acx}^w A$ is DLP over K.

Additional preliminaries. Suppose that *E* is a vector space and $A \subseteq E$ is a subset. The open segment between two distinct points $x, y \in E$ is defined as

$$(x, y) := \{ \alpha x + (1 - \alpha)y \mid 0 < \alpha < 1 \}.$$

A point $x \in A$ is said to be an *extreme point of* A if it does not belong to any open segment contained in A, [25]. If A is convex, $x \in A$ is extreme in A if and only if $x = \frac{1}{2}(a + b)$ for $a, b \in A$ implies x = a = b. We write ext A for the set of all extreme points of A.

Lemma 2.32 Let $T: E \to F$ be a continuous and linear map between locally convex spaces and $A \subseteq E$ be compact. Then:

(1) ext $T(A) \subseteq T(\text{ext } A)$;

(2) if T, in addition, is injective then $\operatorname{ext} T(A) = T(\operatorname{ext} A)$;

(3) suppose that $M \subseteq F^*$. Then

$$T(T^*(M)^\circ) = (\operatorname{Im} T) \cap M^\circ.$$

Proof (1) Write $K := \ker T$. Let $y \in \operatorname{ext} T(A)$. Since T is continuous, $K_y := A \cap T^{-1}(\{y\})$ is a closed, non-empty subset of A. It is also compact because A is compact. We claim that $\operatorname{ext} K_y \subseteq \operatorname{ext} A$. Indeed, suppose that $x \in \operatorname{ext} K_y$ and $x = \frac{1}{2}(a + b)$ for $a, b \in A$. As a consequence,

$$y = T(x) = \frac{1}{2}(T(a) + T(b)).$$

However, since $y \in \text{ext } T(A)$, we conclude that y = T(a) = T(b). By definition, $a, b \in K_y$. Finally, since $x \in \text{ext } K_y$, we conclude that x = a = b, proving that $x \in \text{ext } A$.

Applying the Krein-Milman Theorem, ext $K_y \neq \emptyset$. Choose $x_0 \in \text{ext } K_y \subseteq \text{ext } A$. By definition, $y = T(x_0) \in T(\text{ext } A)$, as required.

(2) [42, Thm. 9.2.3]. (3)

$$y \in T(T^*(M)^\circ) \iff \exists x \in T^*(M)^\circ : y = T(x)$$

$$\iff \exists x \in E : y = T(x) \text{ and } \forall \varphi \in M : |(T^*(\varphi))(x)| \le 1$$

$$\iff \exists x \in E : y = T(x) \text{ and } \forall \varphi \in M : |\varphi(T(x))| \le 1$$

$$\iff y \in \operatorname{Im} T \text{ and } \forall \varphi \in M : |\varphi(y)| \le 1$$

$$\iff y \in (\operatorname{Im} T) \cap M^\circ.$$

Definition 2.33 Let $M \in \text{eqc}(E^*)$ and ρ_M be the continuous seminorm on E defined by

$$\rho_M(x) := \sup_{\varphi \in M} |\varphi(x)|.$$

We say that M is (\mathbf{R}_{1}^{*}) if there is no bounded l^{1} -sequence in E with respect to ρ_{M} .

Lemma 2.34 Let $M \subseteq E^*$ be an equicontinuous compact subset. There exist: a continuous, onto and open linear map $\pi : E \to V$ to a normed space V and a weak-star continuous linear operator Δ : Span $M \to V^*$ with dense image, such that $B_{V^*} = \overline{\operatorname{acx}}^*(\Delta(M))$ and $\operatorname{id}_M = \pi^* \circ \Delta$.

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Proof Write $N := \overline{acx}^{w^*}M$ and consider the space $W := \operatorname{Span} N$ with the gauge q_N as norm. For every $x \in E$, consider the evaluation $e_x : N \to \mathbb{R}$ defined by $e_x(\varphi) := \varphi(x)$.

By Lemma 2.10, $B_W = N$. Also, N is weak-star compact, so $e_x(N) = N(x)$ is a bounded subset of \mathbb{R} for every $x \in E$. As a consequence, $e_x \in W^*$ is a bounded functional.

Write $V_1 := W^*$ and define the linear map $\pi_1 : E \to V_1$ by $\pi_1(x) := e_x$. Consider the continuous seminorm ρ_N (Definition 2.33). We claim that π_1 is a seminorm preserving map from (E, ρ_N) to $(W^*, \|\|)$.

Indeed, for every $x \in E$ we have:

$$\|\pi_1(x)\| := \sup_{\varphi \in N} |(\pi_1(x))(\varphi)| := \sup_{\varphi \in N} |\varphi(x)| := \rho_N(x).$$

Thus, π_1 is necessarily continuous and open onto its image.

Let $\Delta_1 : W \to W^{**} = V_1^*$ be the canonical map. Note that it is not necessarily weak-star continuous. We claim that $id_M = \pi_1^* \circ \Delta_1$. Suppose that $\varphi \in M$ and $x \in E$:

$$((\pi_1^* \circ \Delta_1)(\varphi))(x) := (\pi_1^*(\Delta_1(\varphi)))(x)$$
$$= (\Delta_1(\varphi))(\pi_1(x))$$
$$= (\pi_1(x))(\varphi)$$
$$= \varphi(x) = (\mathrm{id}_M(\varphi))(x).$$

Also, by Goldstine's theorem [4, p. IV.17 Proposition 5]:

$$B_{V_1^*} = B_{W^{**}} = \overline{(\Delta_1(B_W))}^{w^*} = \overline{\Delta_1(N)}^{w^*}.$$

Finally, define $V := \text{Im } \pi_1$ and the onto operator $\pi : E \to V$ induced by π_1 . It is easy to see that π remains continuous and open onto its image. Also, let $i : V \to V_1$ be the inclusion map. Define $\Delta : \text{Span } N \to V^*$ by $\Delta = i^* \circ \Delta_1$.

We claim that Δ is weak-star continuous. For the purpose of this proof, write:

$$U_X(a_1,\ldots,a_n;\varepsilon) := \{ f \in X \mid \forall 1 \le i \le n : |f(a_i)| < \varepsilon \}.$$

Suppose that $y_1, \ldots, y_n \in V$ and $\varepsilon > 0$. By definition, we can find $x_1, \ldots, x_n \in E$ such that $\pi_1(x_i) = y_i$ for every $1 \le i \le n$. We will now show that:

$$\Delta\left(U_{\operatorname{Span} N}\left(x_{1},\ldots,x_{n};\varepsilon\right)\right)\subseteq U_{V^{*}}(y_{1},\ldots,y_{n};\varepsilon).$$

Indeed, for every $\varphi \in \Delta (U_{\text{Span }N}(x_1, \dots, x_n; \varepsilon))$ and $1 \le i \le n$ we have:

$$|(\Delta(\varphi))(y_i)| = |(\Delta(\varphi))(\pi_1(x_i))| = |((\pi_1^* \circ \Delta)(\varphi))(x_i)| = |\varphi(x_i)| < \varepsilon.$$

Now:

$$\pi^* \circ \Delta = \pi^* \circ i^* \circ \Delta_1 = (i \circ \pi)^* \circ \Delta_1 = \pi_1^* \circ \Delta_1 = \mathrm{id}_M.$$

In virtue of the Hahn–Banach theorem, $B_{V^*} = i^*(B_{V_1^*})$. Moreover, it is easy to see that

$$\Delta(M) = i^*(\Delta_1(M)) \subseteq i^*(\Delta_1(B_W)) \subseteq i^*(B_{V_1^*}) = B_{V^*},$$

and therefore

$$B_{V^*} = i^* (B_{V_1^*}) = i^* \left(\overline{\Delta_1(N)}^{w^*}\right) \subseteq \overline{(i^* \circ \Delta_1)(N)}^{w^*} = \overline{\Delta(N)}^{w^*}$$
$$= \overline{\Delta(\overline{\operatorname{acx}}^{w^*}M)}^{w^*} \subseteq \overline{\operatorname{acx}}^{w^*} \Delta(M) \subseteq B_{V^*}.$$

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Note that we used the continuity of i^* and Δ . In other words: $B_{V^*} = \overline{\operatorname{acx}}^{w^*} \Delta(M)$. Finally, acx M is dense in N and therefore $\Delta(\operatorname{acx} M)$ is dense in $\Delta(N)$. Since $\Delta(N)$ is absorbing in V^* , $\Delta(\operatorname{Span} M)$ is dense in V^* .

Lemma 2.35 (Equicontinuous Factor Lemma) Let $M \subseteq E^*$ be an equicontinuous compact subset. There exist: a continuous, dense and open onto its image linear map $\pi : E \to V$ to a Banach space V and a linear operator Δ : Span $M \to V^*$ with dense image, weak-star continuous over M, such that $B_{V^*} = \overline{\operatorname{acx}}^*(\Delta(M))$ and $\operatorname{id}_M = \pi^* \circ \Delta$.

Proof Let Δ : Span $M \to V^*$ and $\pi : E \to V$ be the maps described in Lemma 2.34. Consider the completion \widehat{V} and the inclusion map $i : V \to \widehat{V}$. Note that $\Delta(M)$ is weak-star compact as a continuous image of a compact set.

Also, V is normed so we can use the Banach-Steinhaus Theorem to conclude that $\Delta(M)$ is equicontinuous. Using Lemma 2.8, we can find a weak-star compact, equicontinuous subset $\widehat{M} \subseteq V^*$ such that $\Delta(M) = i^*(\widehat{M})$. As a consequence, i^* is a closed map on \widehat{M} . Also, since V is dense in \widehat{V} , i^* is injective. Therefore, i^* is a weak-star homeomorphism on \widehat{M} . We define $\widehat{\pi} := i \circ \pi$ and $\widehat{\Delta} := (i^*)^{-1} \circ \Delta$.

Forgetting the earlier notations of V, π and Δ , we can define $V := \hat{V}, \pi := \hat{\pi}$ and $\Delta := \hat{\Delta}$ to get the desire result.

Remark 2.36 Note that unlike in Lemma 2.34, we can't guarantee that Δ is weak-star continuous over Span *M*, but only over *M*.

Lemma 2.37 Let $D \subseteq E$ be a disk, $\varphi \in E^*$ and $\varepsilon := \varphi^{-1}((-1, 1))$. Suppose that for some disk neighborhood $\delta \subseteq E$ we have $\delta \cap D \subseteq \varepsilon \cap D$. Then there exists $\widehat{\varphi} \in \delta^\circ$ such that $\varphi|_D = \widehat{\varphi}|_D$.

Proof It is easy to see that for every $x \in \text{Span}(D)$ we have $|\varphi(x)| \le q_{\delta}(x)$, where

$$q_{\delta}(x) := \inf\{r > 0 \mid x \in r\delta\}$$

is the gauge of δ . By the Hahn–Banach Theorem, we can find a continuous functional $\widehat{\varphi}$ on *E* which agrees with φ on Span(*D*) and for every $x \in E : |\widehat{\varphi}(x)| \le q_{\delta}(x)$.

Clearly, this also implies that $\widehat{\varphi}$ is continuous and therefore belongs to E^* . Moreover, for every $x \in \delta$ we have $|\widehat{\varphi}(x)| \le q_{\delta}(x) \le 1$. By definition of the polar, $\widehat{\varphi} \in \delta^{\circ}$.

Fact 2.38 [25, p. 131, Corollary 5] Suppose that $A, B \subseteq E$ are non-empty subsets. If A is closed and absolutely convex, B is compact and $A \cap B = \emptyset$, then there exists $\varphi \in E^*$ such that:

$$\sup_{x\in A} |\varphi(x)| < \inf_{y\in B} |\varphi(y)|.$$

The following is a probably well known consequence of the Banach-Grothendieck theorem [25, p. 147 Thm. 2] and Fact 2.38.

Lemma 2.39 Let K be a compact space, and let $i: K \hookrightarrow C(K)^*$ be the standard weak-star embedding. Then $\overline{\operatorname{acx}}^{w^*}i(K) = B_{C(K)^*}$.

3 Bornological classes

A *bornology* \mathcal{B} on a lcs E is a family of subsets in E which covers E, and is hereditary under inclusion (i.e. if $A \in \mathfrak{B}$ and $B \subseteq A$ then $B \in \mathfrak{B}$) and finite unions. So, every bornology contains all finite subsets.

A vector bornology [24, Definition 1:1'2] is a bornology \mathcal{B} on E such that whenever $A, B \in \mathcal{B}$ and $\alpha \in \mathbb{R}$ we have:

(1) $A + B \in \mathcal{B};$

(2) bal $A \in \mathcal{B}$.

If \mathcal{B} is closed under taking convex hulls, it is said to be a *convex bornology*. It is said to be *saturated* if the closure of sets in \mathcal{B} remains in \mathcal{B} . Moreover, it is *separated* if it has no non-trivial bounded subspaces.

Definition 3.1 A bornological class \mathfrak{B} is an assignment

 $Comp \rightarrow \{Bornologies\}, K \mapsto \mathfrak{B}_K$

from the class of all compact spaces *Comp* to the class of vector bornologies such that \mathfrak{B}_K is a separated convex vector bornology on the Banach space *C*(*K*) satisfying the following properties:

- (1) **boundedness**: \mathfrak{B}_K consists of bounded subsets in C(K).
- (2) **consistency**: Suppose that $\varphi \colon K_1 \to K_2$ is a continuous map.
 - (a) If $A \in \mathfrak{B}_{K_2}$, then $A \circ \varphi \in \mathfrak{B}_{K_1}$.
 - (b) If φ is surjective, then the converse is also true, namely that A ∘ φ ∈ 𝔅_{K1} implies A ∈ 𝔅_{K2}.
- (3) **Bipolarity**: If $A \in \mathfrak{B}_K$, then $A^{\circ\circ} = \overline{\operatorname{acx}}^w A \in \mathfrak{B}_K$ where the polar is taken with respect to the dual $C(K)^*$ (note that we use the Bipolar Theorem).

We write [**T**], [**NP**] and [**DLP**] for the classes of tame, fragmented and DLP function families, respectively. Recall that by Lemma 2.24, [**T**] coincides with the class of eventually fragmented families.

Proposition 3.2 [T], [NP] and [DLP] are bornological classes.

Proof First, it is obvious that all three of these classes consist of vector bornologies. We organize the rest of the proof in the following table:

Property	[T]	[NP]	[DLP]	
Boundedness	By definition	By definition	By definition	
Consistency	Lemma 2.22	Lemma 2.14	Lemma 2.29.4	
Bipolarity	Corollary 2.19	Corollary 2.19	Lemma 2.31	

The following is easy to see.

Lemma 3.3 The only separated, convex, vector bornology on \mathbb{R} is the Euclidean bornology B_e (i.e. the bornology of bounded subsets with respect to the Euclidean norm).

Definition 3.4 Let \mathfrak{B} be a bornological class. A bounded subset $B \subseteq E$ is said to be \mathfrak{B} -small if $r_M(B) \in \mathfrak{B}_M$ for every $M \in \text{eqc}(E^*)$, where $r_M \colon E \to C(M)$ is the *restriction operator*. A locally convex space is said to be \mathfrak{B} -small if every bounded subset is \mathfrak{B} -small.

Lemma 3.5 Let E be a locally convex space and \mathfrak{B} a bornological class. The family of \mathfrak{B} -small subsets in E is a saturated, convex vector bornology, denoted by small (\mathfrak{B}, E) .

Proof The only non-trivial assertion is that small (\mathfrak{B}, E) is saturated. Suppose that $A \in \text{small}(\mathfrak{B}, E)$. We show that $\overline{A} \in \text{small}(\mathfrak{B}, E)$. Let $M \in \text{eqc}(E^*)$ and $r: E \to C(M)$ be the restriction operator. By definition, $r(A) \in \mathfrak{B}_M$. Also, since \mathfrak{B} satisfies bipolarity, $\overline{r(A)} \subseteq \overline{r(A)}^w \subseteq (r(A))^{\circ\circ} \in \mathfrak{B}_M$. By continuity, $r(\overline{A}) \subseteq \overline{r(A)}$. Hence, $r(\overline{A}) \in \mathfrak{B}_M$. Therefore, $\overline{A} \in \text{small}(\mathfrak{B}, E)$. п

Lemma 3.6 Suppose that $T: E \to F$ is a continuous linear map between locally convex spaces, $B \subseteq E$ is bounded and $M \in eqc(F^*)$.

Let $r_{T^*(M)}: E \to C(T^*(M))$ and $r_M: F \to C(M)$ be the restriction maps. Then $r_{T^*(M)}(B) \in \mathfrak{B}_{T^*(M)}$ if and only if $r_M(T(B)) \in \mathfrak{B}_M$.

Proof We will show that:

$$r_M(T(B)) = r_{T^*(M)}(B) \circ T^*.$$

The claim is then obvious from the consistency

property (which we apply for the map $M \to T^*(M)$). Indeed, for every $x \in B$ and $\varphi \in M$, we have:

$$(r_M(T(x)))(\varphi) := \varphi(T(x)) = (T^*(\varphi))(x) = (r_{T^*(M)}(x))(T^*(\varphi)) = (r_{T^*(M)}(x) \circ T^*)(\varphi).$$

Lemma 3.7 Suppose that $T: E \to F$ is a continuous linear map and $A \in \text{small}(\mathfrak{B}, E)$. Then $T(A) \in \text{small}(\mathfrak{B}, F)$.

Proof Let $M \in \text{eqc}(F^*)$. By definition, $r_E(A) \in \mathfrak{B}_E$. Applying Lemma 3.6, we conclude that $r_F(T(A)) \in \mathfrak{B}_M$. This is true for every $M \in \text{eqc}(F^*)$, proving the desired result.

Recall that a continuous linear (onto) map $f: E_1 \to E_2$ is said to be *bound covering* if for every bounded $B_2 \subset E_2$ there exists a bounded subset $B_1 \subset E_1$ such that $f(B_1) = B_2$.

Corollary 3.8 The class of \mathfrak{B} -small locally convex spaces is closed under bound covering maps.

In Proposition 5.7, we show that "bound covering" is really essential.

Proposition 3.9 The class of \mathfrak{B} -small locally convex spaces is closed under taking linear subspaces. In fact, if E is a locally convex space, $F \subseteq E$ is a subspace and $B \subseteq F$ is a *bounded subset, then* $B \in \text{small}(\mathfrak{B}, E)$ *if and only if* $B \in \text{small}(\mathfrak{B}, F)$ *.*

Proof First, if $B \in \text{small}(\mathfrak{B}, F)$ then clearly $B \in \text{small}(\mathfrak{B}, E)$ in virtue of Lemma 3.7. Conversely, suppose that $B \in \text{small}(\mathfrak{B}, E)$. We will show that $B \in \text{small}(\mathfrak{B}, F)$. Let $M \in eqc(F^*)$. By Lemma 2.8, we can find $N \in eqc(E^*)$. such that $i^*(N) = M$ where $i: F \to E$ is the inclusion map. Let $r_E: E \to C(N)$ and $r_F: F \to C(M)$ be the restriction maps. $B = i(B) \in \text{small}(\mathfrak{B}, E)$ by definition, and therefore $r_E(i(B)) \in \mathfrak{B}_N$. Applying Lemma 3.6, we conclude that $r_F(B) \in \mathfrak{B}_{i^*(N)} = \mathfrak{B}_M$. This is true for every $M \in \text{eqc}(F^*)$ and therefore $B \in \text{small}(\mathfrak{B}, F)$.

Suppose that $E := \prod_{i=1}^{n} E_i$ is the product of topological vector spaces $\{E_i\}_{i=1}^{n}$. Write $\pi_i : E \to E_i$ for the projection map. Also, let $\Delta_i : E_i \to E$ be the *dissection map* sending an element $x \in E_i$ to an element of *E* whose *i*'th entry is *x* and the rest are 0. Finally, consider the adjoint map $\Delta_i^* : E^* \to E_i^*$ defined by $\Delta_i^*(\varphi) := \varphi \circ \Delta_i$.

Lemma 3.10 Let E_1, \ldots, E_n be locally convex spaces, $A_i \subseteq E_i$ for $i \in \{1, \ldots, n\}$ and $M \subseteq E^*$ be subsets. Consider $E = \prod_{i=1}^n E_i$ and $A = \prod_{i=1}^n A_i$. For every $1 \le i \le n$, let $r_i : E_i \to C(\Delta_i^*(M))$ and $r : E \to C(M)$ be the restriction maps. Then

$$r(A) \subseteq \sum_{i=1}^{n} r_i(A_i) \circ \Delta_i^*.$$

Proof Indeed, suppose that $a = (a_1, ..., a_n) \in A$, meaning that $a_i \in A_i$ for every $1 \le i \le n$. We claim that

$$r(a) = \sum_{i=1}^{n} r_i(a_i) \circ \Delta_i^* \in \sum_{i=1}^{n} r_i(A_i) \circ \Delta_i^*.$$

Let $\varphi \in M$. For every $x \in E$ we can write

$$x = \sum_{i=1}^{n} \Delta_i(\pi_i(x)),$$

and therefore

$$\varphi(x) = \varphi\left(\sum_{i=1}^{n} \Delta_i(\pi_i(x))\right) = \sum_{i=1}^{n} (\varphi \circ \Delta_i)(\pi_i(x)) = \sum_{i=1}^{n} (\Delta_i^*(\varphi))(\pi_i(x)).$$

In particular,

$$(r(a))(\varphi) := \varphi(a) = \sum_{i=1}^{n} (\Delta_i^*(\varphi))(a_i) = \sum_{i=1}^{n} (r_i(a_i) \circ \Delta_i^*)(\varphi).$$

Remark 3.11

(1) [52, IV 4.3, Theorem] Let $E = \prod_{i \in I} E_i$ be a product of lcs E_i . Then its dual E^* is **algebraically** the locally convex direct sum $\bigoplus_{i \in I} E_i^*$ with the corresponding duality

$$E \times E^* \to \mathbb{R}, \ (v, u) = \left((v_i)_{i \in I}, \sum_{i \in I} u_i \right) \mapsto \sum_{i \in I} \langle v_i, u_i \rangle.$$

[25, Section 8.8, Proposition 1] A basis of the equicontinuous compactology eqc (E^*) on E^* is obtained by taking all sets of the form $\sum_{j \in J} H_j$, where J is finite and $H_j \in$ eqc (E_j).

(2) [52, IV 4.3, Corollary 1] Similarly, if $E = \bigoplus_{i \in I} E_i$ is a lc sum then its dual E^* is **algebraically** the locally convex product $\prod_{i \in I} E_i^*$ with the corresponding duality

$$E \times E^* \to \mathbb{R}, \ (v, u) = \left(\sum_{i \in I} v_i, (u_i)_{i \in I}\right) \mapsto \sum_{i \in I} \langle v_i, u_i \rangle.$$

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A basis of the equicontinuous compactology eqc (E^*) on E^* is obtained by taking all sets of the form $\prod_{i \in I} H_i$, where $H_i \in \text{eqc}(E_i)$.

By [52, II, 6.3], for every bounded subset *B* of a locally convex direct sum $\bigoplus_{i \in I} E_i$, there exists a finite set $J \subset I$ such that $pr_i(B)$ is zero for every $i \notin J$.

Lemma 3.12 Let E_1, \ldots, E_n be locally convex spaces and let $A_i \subseteq E_i$ for $i \in \{1, \ldots, n\}$ be \mathfrak{B} -small subsets. Then $A = \prod_{i=1}^n A_i$ is \mathfrak{B} -small in $E = \prod_{i=1}^n E_i$.

Proof Let $M \in eqc(E^*)$. Then $M_i := \Delta_i^*(M) \in eqc(E_i^*)$, where $\Delta_i^* : E^* \to E_i^*$ are defined as in Lemma 3.10. As before, let $r_i : E_i \to C(M_i)$ be the restriction map. By definition, $r_i(A_i) \in \mathfrak{B}_{M_i}$. Since \mathfrak{B} is a bornological class, this implies that $r_i(A_i) \circ \Delta_i^* \in \mathfrak{B}_M$ for every $1 \le i \le n$. Also, \mathfrak{B}_M is a linear bornology and therefore $A' := \sum_{i=1}^n r_i(A_i) \circ \Delta_i^* \in \mathfrak{B}_M$. By Lemma 3.10, $r(A) \subseteq A'$, proving the desired result.

The following remark is a consequence of the previous lemma and Lemma 3.3.

Remark 3.13 If \mathfrak{B} is a bornological class and F is finite, then \mathfrak{B}_F is simply the Euclidean bornology.

Corollary 3.14 Arbitrary products and direct sums of \mathfrak{B} -small spaces are \mathfrak{B} -small.

Proof First consider the case of products. Let $\{E_i\}_{i \in I}$ be a family of \mathfrak{B} -small spaces. Let $B \subseteq E := \prod_{i \in I} E_i$ be bounded, $M \in \text{eqc}(E^*)$ and $r : B \to C(M)$ be the restriction map. We show that $r(B) \in \mathfrak{B}_M$. Indeed, using Remark 3.11.1, we know that there is a finite $J \subseteq I$ and $H_j \in \text{eqc}(E_j^*)$ such that $M \subseteq \sum_{j \in J} H_j$. Thus, the system (B, M) where B is considered as a family of functions over M can be isomorphically embedded in $(\prod_{j \in J} E_j, \prod_{j \in J} E_j^*)$. However, this family is \mathfrak{B} -small as a consequence of Lemma 3.12.

For the case of direct sums, we use a very similar technique, this time leveraging Remark 3.11.2 by factoring the bounded set to finite components rather than the equicontinuous family.

Corollary 3.15 If \mathfrak{B} is a bornological class, then every locally convex E with the weak topology is \mathfrak{B} -small.

Proof First, recall that E_w can be embedded in \mathbb{R}^{E^*} . In virtue of Theorem 3.17, E_w is \mathfrak{B} -small as a subspace of the product of \mathfrak{B} -small spaces.

Lemma 3.16 Suppose that \mathfrak{B} is a bornological class and F is a dense large subspace of a lcs E. Then F is \mathfrak{B} -small if and only if E is \mathfrak{B} -small.

Proof If *E* is \mathfrak{B} -small then so is *F* in virtue of Proposition 3.9. Conversely, assume that *F* is \mathfrak{B} -small. Suppose that $B \subseteq E$ is bounded. By definition, there is a bounded $C \subseteq F$ such that $B \subseteq \overline{C}$. Since *F* is \mathfrak{B} -small, $C \in \text{small}(\mathfrak{B}, F)$. Applying Proposition 3.9, we conclude that $C \in \text{small}(\mathfrak{B}, E)$. Finally, using Lemma 3.5, we conclude that $B \subseteq \overline{C} \in \text{small}(\mathfrak{B}, E)$. \Box

Theorem 3.17 The class of \mathfrak{B} -small locally convex spaces is closed under:

- (1) subspaces
- (2) bound covering maps
- (3) products
- (4) direct sums
- (5) inverse limits.

Moreover, if F is a large, dense subspace of the locally convex space E, and F is \mathfrak{B} -small, then so is E. In particular, if V is a normed \mathfrak{B} -small space, then so is its completion.

Proof Apply Corollary 3.8, Proposition 3.9, Corollary 3.14 and Lemma 3.16.

3.1 Relation to the Mackey topology

Definition 3.18 Let \mathfrak{B} be a bornological class. A bounded subset $B \subseteq E$ is said to be *Mackey* \mathfrak{B} -*small* if for every absolutely convex weak-star compact, (not necessarily equicontinuous) subset $M \subset E^*$, B viewed as a bounded family of functions on M belongs to \mathfrak{B}_M . A locally convex space is said to be *Mackey* \mathfrak{B} -*small* if every bounded subset is \mathfrak{B} -small.

Recall that the *Mackey topology* on a lcs (E, τ) is the strongest topology compatible with its dual E^* . We will often denote it as τ_{μ} . It is exactly the polar topology induced by all weak-star compact, absolutely convex subsets of E^* [52, p. 131]. *E* is said to be a *Mackey space* if $\tau = \tau_{\mu}$.

Proposition 3.19 A bounded subset B in E is Mackey \mathfrak{B} -small if and only if it is \mathfrak{B} -small with respect to the Mackey topology. The same can be said for the entire space E.

Proof Recall ([25, p. 158, Thm. 5]) that the Mackey topology is compatible with the dual E^* . As a consequence, it has the same absolutely convex, weak-star compact subsets as the usual topology. Therefore, if *B* is Mackey \mathfrak{B} -small, then *B* is \mathfrak{B} -small with respect to the Mackey topology.

Conversely, suppose that B is \mathfrak{B} -small with respect to the Mackey topology, and let $M \subseteq E^*$ be an absolutely convex weak-star compact subset.

By definition, the polar M° is an open neighborhood of zero in the Mackey topology. Thus, $M^{\circ\circ}$ is equicontinuous. As a consequence, B is \mathfrak{B} -small over $M^{\circ\circ}$. However, $M \subseteq M^{\circ\circ}$, and therefore B is \mathfrak{B} -small over M, as required.

A lcs E is said to be *barreled* if every barrel (Definition 2.1) is a neighborhood of zero. This class includes all reflexive and complete metrizable (i.e., Frechet) spaces. By [52, p. 132, Lemma 3.4], every barreled or metrizable space is a Mackey space.

Corollary 3.20 If E is a Mackey space (e.g., barreled or metrizable), then it is \mathfrak{B} -small if and only if it is Mackey \mathfrak{B} -small.

For basic information about Mackey topologies and related topics, we refer to [25, 52]. For some generalizations see [1].

3.2 The co-bornology and strongest topologies

Definition 3.21 A bornological class \mathfrak{B} is said to be *polarly compatible* if whenever $A \in \mathfrak{B}_K$ for compact K, then $r_{B_{C(K)^*}}(A) \in \mathfrak{B}_{B_{C(K)^*}}$ where $r_{B_{C(K)^*}}: C(K) \to C(B_{C(K)^*})$ is the canonical map defined by:

$$(r_{B_{C(K)^*}}(f))(\varphi) := \varphi(f).$$

For every lcs (e.g., Banach space) (E, τ) and a bornological class \mathfrak{B} , one may define the *strongest locally convex* \mathfrak{B} -*small topology* $\tau_{\mathfrak{B}}$ on E. We mean the supremum of all \mathfrak{B} -small locally convex topologies on E which are coarser than τ . Since the class of \mathfrak{B} -small spaces is closed under products and subspaces (by Theorem 3.17), we obtain that $\tau_{\mathfrak{B}}$ is well-defined and it is a \mathfrak{B} -small locally convex topology on E (such that $\tau_{\mathfrak{B}} \subseteq \tau$). The weak topology τ_w on E is always \mathfrak{B} -small (by Corollary 3.15), and therefore

$$\tau_w \subseteq \tau_{\mathfrak{B}} \subseteq \tau$$

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As a consequence, $\tau_{\mathfrak{B}}$ is always a Hausdorff locally convex topology on *E*. Below, in Theorem 3.36, we give a description of this topology for polarly compatible classes as a naturally defined polar topology.

Lemma 3.22 [15, Prop. 4.19] Let K be a compact space and $F \subset C(K)$ is bounded. Then F is a tame family for K if and only if F is a tame family for the weak-star compact unit ball $B_{C(K)^*}$. Equivalently, F is a tame subset (in terms of Definition 5.1) of the Banach space C(K).

Corollary 3.23 The class [T] is polarly compatible.

Remark 3.24 [NP] is also polarly compatible. In fact, an analogous statement to Lemma 3.22 holds about fragmented maps [39].

Lemma 3.25 $\overline{\operatorname{acx}}^{w^*}(M) \in \operatorname{eqc}(E^*)$ for every $M \in \operatorname{eqc}(E^*)$.

Proof By definition, there is a neighborhood $\varepsilon \subseteq E$ of zero such that $M \subseteq \varepsilon^{\circ}$. Note that ε° is convex and weak-star compact by Alaouglu–Bourbaki's theorem.

Finally note that $\overline{acx}^{w^*}(M)$ is a weak-star closed subspace of ε° .

Lemma 3.26 Let $M \subseteq E^*$ be an equicontinuous, weak-star compact subset. Write $T := \overline{\operatorname{acx}} M$. Then there exists a surjective $j \colon B_{C(M)^*} \to T$ such that $r_T(x) \circ j = r_{B_{C(M)^*}}(r_M(x))$ for every $x \in E$.

$$\begin{array}{ccc} B_{C(M)^*} \xrightarrow{j} T \\ & \swarrow \\ \gamma \searrow & \downarrow r_T(x) \\ & \mathbb{R} \end{array}$$

where $\gamma := r_{B_{C(M)}*}(r_{M}(x)).$

Proof Let $i: M \to B_{C(M)^*}$ be the natural embedding. Define $M' := \operatorname{acx} i(M)$ and $j': M' \to \operatorname{acx} M$ via:

$$j'\left(\sum_{m=1}^n \alpha_m i(\varphi_m)\right) := \sum_{m=1}^n \alpha_m \varphi_m.$$

It is easy to see that this function is well-defined and linear. We will now show that it is uniformly continuous with respect to the standard uniformities of the weak-star topologies. Let us write

$$U_X(p_1,\ldots,p_t;\varepsilon) := \{(f,g) \in X \times X \mid \forall 1 \le k \le t : |f(p_k) - g(p_k)| < \varepsilon\},\$$

where $X \in \{M', acx M\}$.

Suppose that $x_1, \ldots, x_t \in E$ and $\varepsilon > 0$. It is easy to see that:

$$j'(U_{M'}(r_M(x_1),\ldots,r_M(x_t);\varepsilon)) \subseteq U_M(x_1,\ldots,x_t;\varepsilon).$$

By definition, j' is uniformly continuous.

Now, by Lemma 3.25, $T := \overline{acx}^{w^*} M$ is compact, and in particular complete.

We can thus extend j' to a continuous function $j: \overline{M'}^{w^*} \to T$. By Lemma 2.39, $\overline{M'}^{w^*} = B_{C(M)^*}$.

Since $B_{C(M)^*}$ is compact, $j(B_{C(M)^*})$ is closed. Moreover, acx $M = j(M') \subseteq j(B_{C(M)^*})$ so it is also dense. As a consequence, *j* is surjective.

Now suppose that $x \in E$ and $\Phi = \sum_{m=1}^{n} \alpha_m i(\varphi_m) \in M' \subseteq B_{C(M)^*}$. We thus have:

$$(r_T(x) \circ j)(\Phi) = r_T(x)(j(\Phi)) = r_T(x)(j'(\Phi))$$

= $r_T(x) \left(\sum_{m=1}^n \alpha_m \varphi_m\right) = \left(\sum_{m=1}^n \alpha_m \varphi_m\right)(x)$
= $\sum_{m=1}^n \alpha_m \varphi_m(x) = \sum_{m=1}^n \alpha_m (r_M(x))(\varphi_m)$
= $\sum_{m=1}^n \alpha_m (i(\varphi_m))(r_M(x)) = \left(\sum_{m=1}^n \alpha_m i(\varphi_m)\right)(r_M(x))$
= $\Phi(r_M(x)) = (r_{B_{C(M)^*}}(r_M(x)))(\Phi).$

As a consequence, $(r_T(x) \circ j)|_{M'} = (r_{B_{C(M)^*}}(r_M(x)))|_{M'}$. Since $r_T(x) \circ j$ and $r_{B_{C(M)^*}}(r_M(x))$ are continuous, and M' is dense in $B_{C(M)^*}$, we have $r_T(x) \circ j = r_{B_{C(M)^*}}(r_M(x))$.

Lemma 3.27 Let V be a vector space and let $\mathcal{B} \subseteq \mathcal{P}(V)$ be a family of subsets of V. Then the following requirements are enough to conclude that \mathcal{B} is a convex bornology:

- (1) Singletons belong to \mathcal{B} .
- (2) If $M \in \mathcal{B}$ and $N \subseteq M$ then $N \in \mathcal{B}$.
- (3) Union of two elements of \mathcal{B} remains in \mathcal{B} .
- (4) If $M \in \mathcal{B}$ and $r \in \mathbb{R}$ then $rM \in \mathcal{B}$.
- (5) For every $M \in \mathcal{B}$, the absolutely convex hull acx $M \in \mathcal{B}$.

Proof First, it is easy to see that

$$\bigcup_{A\in\mathcal{B}}A\supseteq\bigcup_{x\in V}\{x\}\supseteq V,$$

so it only remains to show that sums of elements in \mathcal{B} remains in \mathcal{B} . Indeed, if $A_1, A_2 \in \mathcal{B}$, then so are $A_1 \cup A_2$ and 2acx $(A_1 \cup A_2)$. Finally, note that:

$$A_1 + A_2 \subseteq 2acx (A_1 \cup A_2).$$

Definition 3.28 Let \mathfrak{B} be a bornological class and $A \subseteq E$ is a bounded subset. An equicontinuous, $M \subseteq E^*$ is said to be *co*- \mathfrak{B} -*small with respect to* A if $r(A) \in \mathfrak{B}_{\overline{M}^{w^*}}$, where $r: E \to C(\overline{M}^{w^*})$ is the restriction map. If this is true for every bounded subset of E, then we will simply say that M is *co*- \mathfrak{B} -*small*.

Lemma 3.29 Suppose that \mathfrak{B} is a bornological class, E a locally convex space, $M \in eqc(E^*)$ and $A \subseteq E$ is bounded. If M is co- \mathfrak{B} -small with respect to A, then so is $M' := M \cup \{0\}$.

Proof If $0 \in M$ then we are done. Otherwise, recall that M is compact and therefore closed. As a consequence, 0 is isolated in M'. Choose $\varphi_0 \in M$ and write $N := \{0, \varphi_0\}$. We now define $j: M' \to M$ and $s: M' \to N$ by

$$j(\varphi) := \begin{cases} \varphi & \varphi \neq 0\\ \varphi_0 & \varphi = 0 \end{cases}, \ s(\varphi) := \begin{cases} 0 & \varphi \neq 0\\ \varphi_0 & \varphi = 0 \end{cases}.$$

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Note that both of these maps are continuous. In virtue of Remark 3.3, \mathfrak{B}_N is the Euclidean bornology. Namely, every bounded subset belongs to \mathfrak{B}_N . In particular, so does $r_N(A) \in \mathfrak{B}_N$. Thus, $r_N(A) \circ s \in \mathfrak{B}_{M'}$. Moreover, it is easy to see that $r_M(A) \circ j \in \mathfrak{B}_{M'}$. We claim that

$$r_{M'}(A) \subseteq r_M(A) \circ j - r_N(A) \circ s.$$

More specifically, we claim that for every $x \in A$:

$$r_{M'}(x) = r_M(x) \circ j - r_N(x) \circ s.$$

Indeed, suppose that $x \in A$, then for every $\varphi \in M$ we have:

$$(r_{M'}(x))(\varphi) = \varphi(x) = (j(\varphi))(x) = (j(\varphi))(x) - (s(\varphi))(x) = (r_M(x) \circ j - r_N(x) \circ s)(\varphi).$$

Also:

$$(r_{M'}(x))(0) = 0(x) = 0 = \varphi_0(x) - \varphi_0(x) = (j(0))(x) - (s(0))(x)$$
$$= (r_M(x) \circ j - r_N(x) \circ s)(0).$$

As a consequence, we have verified the identity for every $\varphi \in M'$. Since $\mathfrak{B}_{M'}$ is a vector bornology we conclude that $r_{M'}(A) \in \mathfrak{B}_{M'}$ as required.

Lemma 3.30 Let \mathfrak{B} be a polarly compatible bornological class and let $A \subseteq E$ be bounded. The family of co- \mathfrak{B} -small subsets with respect to A of E^* is a weak-star saturated, convex bornology.

Denote this bornology as small^{*} (\mathfrak{B}, E, A) . We also write

small^{*} (
$$\mathfrak{B}, E$$
) := $\bigcap_{A \subseteq E}$ small^{*} (\mathfrak{B}, E, A)

where A runs over bounded subsets. Clearly, small* (\mathfrak{B}, E) is also a weak-star saturated, locally convex bornology.

Proof We will prove the requirements of Lemma 3.27:

- Saturated: by definition.
 Without loss of generality, we will assume that *M* is weak-star closed for the rest of the proof.
- (2) Singletons belong to small* (\mathfrak{B}, E, A) : Consequence of Remark 3.13.
- (3) If $M \in \text{small}^*(\mathfrak{B}, E, A)$ and $N \subseteq M$, then $N \in \text{small}^*(\mathfrak{B}, E, A)$: Consider the inclusion map $i : N \to M$. Let $r_M : E \to C(M)$ and $r_N : E \to C(N)$ be the restriction maps. By definition, $r_M(A) \in \mathfrak{B}_M$. In virtue of the consistency property, $r_M(A) \circ i \in \mathfrak{B}_N$. Also, note that $r_N(A) = r_M(A) \circ i$. By definition, $N \in \text{small}^*(\mathfrak{B}, E, A)$.
- (4) Union of finite elements of small* (𝔅, E, A) remains in small* (𝔅, E, A): Let M and N be members of small* (𝔅, E, A). We show that M ∪ N ∈ small* (𝔅, E, A). Consider the space P := M × N × {1, 2} and the continuous function T : P → (M ∪ N) defined by:

$$T(\varphi, \psi, n) := \begin{cases} \varphi & n = 1 \\ \psi & n = 2 \end{cases}$$

By definition, $r_M(A) \in \mathfrak{B}_M$, $r_N(A) \in \mathfrak{B}_N$; we will check that $r_{M \cup N}(A) \in \mathfrak{B}_{M \cup N}$. Because *T* is surjective, and in virtue of the consistency property, we can equivalently show that $r_{M \cup N}(A) \circ T \in \mathfrak{B}_P$. In virtue of Lemma 3.29, we can assume without loss of generality that $0 \in M$ and $0 \in N$. Now, consider the maps $\theta_M : P \to M$ and $\theta_N : P \to N$ defined as

$$\theta_M(\varphi, \psi, n) := \begin{cases} \varphi & n = 1 \\ 0 & n = 2 \end{cases}$$
$$\theta_N(\varphi, \psi, n) := \begin{cases} 0 & n = 1 \\ \psi & n = 2 \end{cases}$$

Using the consistency property, we know that $(r_M(A) \circ \theta_M) + (r_N(A) \circ \theta_N) \in \mathfrak{B}_P$. We claim that

$$r_{M\cup N}(A) \circ T \subseteq (r_M(A) \circ \theta_M) + (r_N(A) \circ \theta_N),$$

completing this part of the proof. Indeed, for every $x \in A$, we have:

$$r_{M\cup N}(x) \circ T = (r_M(x) \circ \theta_M) + (r_N(x) \circ \theta_N).$$

- (5) Suppose that $M \in \text{small}^*(\mathfrak{B}, E, A)$ and $\alpha \in \mathbb{R}$, we show that $\alpha M \in \text{small}^*(\mathfrak{B}, E, A)$. Consider the scalar map $S_{\alpha} \colon M \to \alpha M$ defined by $S_{\alpha}(\varphi) := \alpha \varphi$. By definition, $r_M(A) \in \mathfrak{B}_M$. Note that $r_M(A) = r_{\alpha M}(A) \circ S_{\alpha}$. In virtue of consistency, $r_{\alpha M}(A) \in \mathfrak{B}_{\alpha M}$.
- (6) If $M \in \text{small}^*(\mathfrak{B}, E, A)$, then so is its closed absolutely convex hull

$$T := \overline{\operatorname{acx}}^{w^*} M \in \operatorname{small}^* (\mathfrak{B}, E, A).$$

Let $i: M \hookrightarrow B_{C(M)^*}$ be the inclusion map. By definition, M is equicontinuous, so using Lemma 3.26, consider a continuous surjection $j: B_{C(M)^*} \to T$ such that for every $x \in E$, $r_T(x) \circ j = r_{B_{C(M)^*}}(r_M(x))$. In virtue of its construction, we have:

$$r_T(A) \circ j = r_{B_C(M)^*}(r_M(A)).$$

Moreover, since \mathfrak{B} is polarly compatible, $r_{B_{C}(M)^*}(r_M(A)) \in \mathfrak{B}_{B_{C}(M)^*}$. Finally, using the consistency property, we know that $r_T(A) \in \mathfrak{B}_T$, as required.

Recall that every convex bornology \mathcal{B} on the dual E^* , induces a locally convex topology on E [24, Thm. 5:1'1(a)]. This is the *polar* topology defined by the basis:

$$\{M^{\circ} \mid M \in \mathcal{B}\}.$$

Also, if \mathcal{B} consists of equicontinuous subsets only, then its polar topology is weaker than the original topology [24, Thm. 5:1'3]

Definition 3.31 Let \mathfrak{B} be a polarly compatible bornological class, and (E, τ) be a locally convex space. Recall that Lemma 3.30 applies in this case so small^{*} (\mathfrak{B}, E) is a convex bornology. We define $\tau_{\mathfrak{B}}$ to be the polar topology generated by small^{*} (\mathfrak{B}, E). Since small^{*} (\mathfrak{B}, E) consists of equicontinuous subsets, $\tau_{\mathfrak{B}} \subseteq \tau$.

Note that the requirements of Definition 3.21 are necessary to obtain Lemma 3.30, as can be seen in the following lemma.

Lemma 3.32 Suppose that \mathfrak{B} is a bornological class such that for every E and every bounded $A \subseteq E$, small^{*} (\mathfrak{B}, E, A) is a saturated locally convex bornology (with respect to the weak-star topology). Then \mathfrak{B} is polarly compatible.

Proof Suppose that *K* is compact and $A \in \mathfrak{B}_K$. Write $r: C(K) \to C(B_{C(K)^*})$ for the restriction map and $i: K \to B_{C(K)^*}$ for the evaluation map. Also, define K' := i(K). We need to show that $r(A) \in \mathfrak{B}_{B_{C(K)^*}}$. It is easy to see that $A = r(A) \circ i$. As a consequence, $r(A) \in \mathfrak{B}_{K'}$. In other words, $r(A) \in \text{small}^* (\mathfrak{B}, C(K)^*, K')$ and therefore

$$r(A) \in \operatorname{small}^*\left(\mathfrak{B}, C(K)^*, \overline{\operatorname{acx}}^{w^*}K'\right).$$

As a consequence,

$$r(A) \in \mathfrak{B}_{\overline{\operatorname{acx}}^{w^*}K'}.$$

However, by Lemma 2.39, $\overline{\operatorname{acx}}^{w^*}K' = B_{C(K)^*}$, as required.

Corollary 3.33 The class [DLP] is polarly compatible.

Proof A consequence of Fact 2.30 and Lemma 3.32. Note that in Fact 2.30, E and F are interchangeable.

Lemma 3.34 For every locally convex space (E, τ) , we have:

$$\tau_w \subseteq \tau_{\mathfrak{B}} \subseteq \tau \subseteq \tau_\mu$$

where τ_w is the weak topology and τ_μ is the Mackey topology.

Proof First, let us note that

$$\mathcal{F} \subseteq \operatorname{small}^*(\mathfrak{B}, E) \subseteq \mathcal{E} \subseteq \mathcal{C}$$

where \mathcal{F}, \mathcal{E} and \mathcal{C} are the bornologies of finite subsets, equicontinuous subsets, and weakstar compact absolutely convex subsets, respectively. It is known that their respective polar topologies are τ_w , τ and τ_μ ([52, p. 131, Cor. 1]). As a consequence:

$$\tau_w \subseteq \tau_{\mathfrak{B}} \subseteq \tau \subseteq \tau_{\mu}.$$

Lemma 3.35 Let (E, τ) be a locally convex space. The topology $\tau_{\mathfrak{B}}$ is \mathfrak{B} -small.

Proof Let $A \subseteq E$ be a bounded subset and $M \subseteq E^*$ be a weak-star compact, equicontinuous subset (with respect to $\tau_{\mathfrak{B}}$). We will show that $r(A) \in \mathfrak{B}_M$.

By Lemma 3.34:

 $\tau_w \subseteq \tau_{\mathfrak{B}} \subseteq \tau_{\mu}.$

By [52, p. 132, Corollary 2], we conclude that the bounded subsets of $\tau_{\mathfrak{B}}$ are the same as those of τ . Thus, A is bounded with respect to the original topology τ .

Since *M* is equicontinuous, we can find a neighborhood ε of zero in *E* such that $M \subseteq \varepsilon^{\circ}$. By definition, we can find a subset $N \in \text{small}^*(\mathfrak{B}, E)$ such that $N^{\circ} \subseteq \varepsilon$. We therefore have

$$N^{\circ\circ} \supseteq \varepsilon^{\circ} \supseteq M.$$

By the bipolar theorem (Fact 2.2),

$$N^{\circ\circ} = \overline{\operatorname{acx}}^w N.$$

By Lemma 3.30, small* (\mathfrak{B}, E) is weak-star saturated and locally convex, so

$$N^{\circ\circ} = \overline{\operatorname{acx}}^{w^*} N \in \operatorname{small}^*(\mathfrak{B}, E).$$

In particular, $r_{N^{\circ\circ}}(A) \in \mathfrak{B}_{N^{\circ\circ}}$, hence $r_M(A) \in \mathfrak{B}_M$.

Deringer

Theorem 3.36 For every lcs (E, τ) , $\tau_{\mathfrak{B}}$ is the strongest locally convex, \mathfrak{B} -small topology coarser than τ .

Proof First, by Lemma 3.35, $\tau_{\mathfrak{B}}$ is indeed a \mathfrak{B} -small topology. Now, suppose that $\tau_{\mathfrak{B}} \subseteq \sigma \subseteq \tau$ is a locally convex \mathfrak{B} -small topology. By Lemma 3.34, we can write

$$\tau_w \subseteq \tau_{\mathfrak{B}} \subseteq \sigma \subseteq \tau \subseteq \tau_{\mu}.$$

In virtue of [52, p. 132, Cor. 2], we know that $\tau_{\mathfrak{B}}$ and σ have the same bounded sets. Now, suppose that $\varepsilon \in \sigma$ is a neighborhood of zero. We can find a $\delta \in \sigma$ such that $\overline{\operatorname{acx}} \delta \subseteq \varepsilon$. Note that because both $\tau_{\mathfrak{B}}$ and σ are compatible with the dual E^* , the closure of convex sets (like acx δ) is equal to the weak closure ([25, p. 131, Cor. 6]). We will show that $\overline{\operatorname{acx}} \delta \in \tau_{\mathfrak{B}}$, proving that $\sigma \subseteq \tau_{\mathfrak{B}}$.

Since σ is \mathfrak{B} -small, we know that $r(A) \in \mathfrak{B}_{\delta^{\circ}}$ for every bounded $A \subseteq E$. Since we already established that the bounded subsets of $\tau_{\mathfrak{B}}$ and σ agree, it means that $\delta^{\circ} \in \text{small}^*(\mathfrak{B}, E_{\tau})$. Again, by definition, $\delta^{\circ\circ} \in \tau_{\mathfrak{B}}$. Using the Bipolar Theorem (Fact 2.2), we know that $\overline{\operatorname{acx} \delta} = \delta^{\circ\circ}$, as required.

4 Locally convex analogues of reflexive and Asplund Banach spaces

DLP locally convex spaces. The following well-known observation can be derived by results of [3, Appendix A].

Fact 4.1 Let $F \times K \to \mathbb{R}$ be a bounded separately continuous map where F and K are compact. Then F, as a family of maps on K, is DLP.

Following the lines of Proposition 3.2 and Definition 3.4, we give

Definition 4.2 Let *E* be a locally convex space.

- We say that a bounded subset B of E is DLP if $B \in \text{small}([DLP], E)$. Explicitly, B is DLP if it is DLP as a family of functions over every $M \in \text{eqc}(E^*)$.
- *E* is said to be DLP, and write $E \in (DLP)$ if *E* is [DLP]-small. In other words, $E \in (DLP)$ if and only if every bounded subset of *E* is DLP.

The following is a direct consequence of Theorem 3.17.

Theorem 4.3 *The class* (**DLP**) *is closed under taking:*

- (1) *subspaces*
- (2) bound covering maps
- (3) products
- (4) *direct sums*
- (5) inverse limits.

Moreover, if F is a large, dense subspace of the locally convex space E, and $F \in (DLP)$, then $E \in (DLP)$. In particular, if V is a normed DLP space, then so is its completion.

Proposition 4.4 *Every relatively weakly compact subset B in a lcs E is DLP.*

Proof Let *B* be weakly compact in a lcs *E* and $M \in \text{eqc}(E^*)$. Then the natural map $w: B \times M \to \mathbb{R}$ is separately continuous. Observe that *w* is a bounded map. Indeed, there exists a neighborhood *O* of zero in *E* such that |u(x)| < 1 for every $x \in O$ and $u \in M$. Since *B* is bounded, there exists $c \in \mathbb{R}$ such that $B \subset cO$. Then |u(x)| < c for every $x \in B$ and $u \in M$. Now, by Fact 4.1, we obtain that *B* is DLP on *M*.

The converse is true for Banach spaces. Namely, a bounded subset B of a Banach space V is DLP iff B is relatively weakly compact (see [3, Thm. A5]). As a consequence, a Banach space V is reflexive iff it is DLP.

Fact 4.5 [52, Ch. IV, 5.5] A locally convex space is semi-reflexive if and only if every bounded subset is relatively weakly compact.

Recall that a lcs E is said to be *boundedly-complete* (or, *quasi-complete*, [25]) if every closed bounded subset in E is complete. An equivalent condition is that every bounded Cauchy net converges. Every boundedly-complete lcs is sequentially complete and every sequentially complete lcs is locally complete (Definition 2.1). Note that every weakly compact subset in a lcs is complete, [21, p. 90]. This implies (by Fact 4.5) that every semi-reflexive space is boundedly-complete. In Theorem 4.6 we make this more precise.

Theorem 4.6 *E* is semi-reflexive if and only if *E* is boundedly-complete and *DLP*.

Proof First suppose that E is semi-reflexive. Then our claim is a consequence of Proposition 4.4 and Fact 4.5. The converse is a conclusion of Proposition 4.7 below.

Proposition 4.7 (version of Grothendieck's result [29, Thm. 17.12])

Let E be a boundedly-complete lcs. Then the following are equivalent for a subset $B \subseteq E$:

(1) *B* is bounded and *DLP*.

(2) *B* is relatively weakly compact.

(3) The closed convex hull C := co(B) is weakly compact.

Proof (3) \Rightarrow (2) Obvious.

 $(2) \Rightarrow (1)$ Let *B* be a relatively weakly compact subset in *E*. Then it is well-known that *B* is bounded. Also *B* is DLP by Proposition 4.4.

(1) \Rightarrow (3) Let $B \subseteq E$ be a bounded DLP subset. We have to show that $C := \overline{co(B)}$ is weakly compact. First, C is complete being bounded and closed in a boundedly-complete space E.

Also, by Lemma 3.5, *C* is DLP as well. By [52, II, 5.4, Corollary 2], there exists an embedding $T: E \hookrightarrow V$ where $V := \prod_{\lambda \in \Lambda} V_{\lambda}$ and $\{V_{\lambda}\}_{\lambda \in \Lambda}$ are Banach spaces.

By Fact 2.6, it is enough to show that T(C) is weakly compact.

The subset T(C) is complete in T(E) and also in V, because T is a uniform embedding (as a linear embedding). Therefore, T(C) is closed in V. Moreover, T(C) is convex. So, it is even weakly closed in V ([25, p. 131, Corollary 6]).

As a consequence, it is enough to show that T(C) is relatively weakly compact.

By Theorem 4.3, the projection $C_{\lambda} := \pi_{\lambda}(T(C)) \subseteq V_{\lambda}$ is DLP for every $\lambda \in \Lambda$. Since $C_{\lambda} \subseteq V_{\lambda}$ is a bounded DLP subset in a Banach space V_{λ} , we can apply [3, Thm. A.5] to conclude that it is weakly relatively compact.

Now observe that $T(C) \subseteq \widehat{C} := \prod_{\lambda \in \Lambda} C_{\lambda}$. By [52, p. 137, Thm 4.3], the weak topology of the product is the product of the weak topologies. Using Tychonoff's Theorem, we conclude that \widehat{C} is also weakly compact in $V = \prod_{\lambda \in \Lambda} V_{\lambda}$. Finally, $T(C) \subseteq \widehat{C}$ is relatively weakly compact.

Proposition 4.7 implies that in the boundedly-complete space E, the closed convex hull of every weakly compact subset is weakly compact (generalized Krein-Smulian theorem). This need not be true in arbitrary lcs if E is not boundedly-complete.

For example, let *E* be the normed subspace of l^2 which consists of all sequences of finite support. Then the set $\{\frac{1}{n}e_n\}_{n\in\mathbb{N}} \cup \{0_{l^2}\}$ is compact but the closure of its convex hull is not

compact (and even not complete). Note that E is DLP because it is a subspace of l^2 . By Theorem 4.3, the DLP is hereditary. Since E is not semi-reflexive, this example shows that boundedly-completeness is essential also in Theorem 4.6.

Moreover, in this space every bounded neighborhood of the origin is DLP but not relatively weakly compact.

Remark 4.8 We mention some interesting subclasses in DLP. Among others:

- (1) Semi-reflexive lcs (Theorem 4.6);
- (2) Quasi-Montel lcs (every bounded subset is uniformly precompact). Schwartz and nuclear lcs are quasi-Montel.
 Important examples in Analysis: the spaces C[∞](Ω) and D(Ω) (for an open subset Ω in Rⁿ). Also, the space of analytic functions H(Ω) over a domain;
- (3) For every locally convex space E, the lcs (E, w) with its weak topology is (DLP). Indeed, (E, w) is a subspace of ℝ^E;
- (4) Every space C_p(X), in its pointwise topology (for every topological space X), is (**DLP**). Indeed, C_p(X) is a subspace of ℝ^X.

Namioka–Phelps (NP) locally convex spaces. Recall the following well-known characterization of Asplund Banach spaces.

Fact 4.9 (Namioka–Phelps [41]) A Banach space $(V, || \cdot ||)$ is Asplund (the dual of every separable (Banach) subspace is separable) iff very bounded weakly-star compact subset $K \subset E^*$ is (weak*,norm)-fragmented.

The second assertion can be reformulated as follows: the unit ball in E is a fragmented family of functions on the unit ball of the dual space.

The following locally convex version of Asplund spaces was introduced in [33].

Definition 4.10 [33] A locally convex space *E* is said to be a Namioka–Phelps (**NP**) space if every $K \in eqc(E^*)$ is (weak*, strong)-fragmented.

Definition 4.11 We say that a bounded subset A of a lcs E is an Asplund set in E if A is fragmented on every $K \in \text{eqc}(E^*)$.

The first assertion of the following lemma appears also in [17, 39].

Lemma 4.12

- (1) Let K be a compact space and $F \subset C(K)$ be a norm bounded subset. If F is DLP on K then F is a fragmented family on K.
- (2) Every DLP (e.g., relatively weakly compact) subset B in a lcs E is Asplund and hence (DLP) ⊂ (NP).

Proof It is easy to see that (2) follows from (1). To show (1), note that every (relatively) weakly compact subset in a lcs is Asplund being fragmented on every $K \in \text{eqc}(E^*)$. This follows using Namioka' joint continuity theorem (as in [33, Prop. 3.5]). On the other hand, a bounded family of continuous functions on a compact space K has DLP iff its natural image into the Banach space C(K) is relatively weakly compact.

Like in Definition 4.2 (DLP), both of these definitions can be reformulated in terms of a bornological class [**NP**]. We can also formulate the following theorem as a consequence of Theorem 3.17.

Theorem 4.13 The class (NP) is closed under taking:

- (1) subspaces
- (2) bound covering maps
- (3) products
- (4) direct sums
- (5) inverse limits.

Moreover, if F is a large, dense subspace of the locally convex space E, and $F \in (\mathbf{NP})$, then $E \in (\mathbf{NP})$.

Remark 4.14 Recall that a bounded subset A in a Banach space V is said to be an Asplund subset (Fabian [9, p. 22]), or, a Stegall subset in terms of Bourgin [5, p. 121] if the pseudometric space (V^*, ρ_C) is separable for every countable $C \subset A$, where

$$\rho_C(\phi, \psi) := \sup_{x \in C} |\phi(x) - \psi(x)|.$$

Equivalently, (B_{V^*}, ρ_C) is separable.

This definition is compatible with Definition 4.11 as it follows from Lemma 2.17.

Lemma 4.15 The following conditions are equivalent:

(1) *E* is (**NP**);

(2) every (countable) bounded $B \subset E$ subset is an Asplund subset in E.

Proof *E* is (**NP**) means that every subset $K \in eqc(E^*)$ is (weak*, strong)-fragmented. That is, for every subset $A \subseteq K$, $\varepsilon > 0$ and bounded set $B \subseteq V$, there exists a weak-star open set $O \subseteq E^*$ such that $A \cap O$ is nonempty and (ε, B) -small. This is equivalent to $(A \cap O)(x)$ being ε -small for every $x \in B$. This just means that *B* is a fragmented family of functions on *K*.

By Lemma 2.17, it is equivalent that for every countable subset C of B is fragmented on K. \Box

Note that (NP) lcs has several remarkable properties. For the continuity of dual group actions, see [33] and for the fixed point theorems, [16] and [56].

Examples 4.16 [33]

- (1) A Banach space is (NP) iff it is Asplund.
- (2) Every Frechet differentiable lcs is (NP).

(3) If the dual V^* is a linear subspace in a product of separable lcs, then V is (NP).

5 Locally convex analogue of Rosenthal Banach spaces

Definition 5.1 We say that a bounded subset B of a lcs E is *tame* in E if one of the following equivalent conditions (by Lemma 2.24) is satisfied:

(i) *B* is tame (Definition 2.21) over every $K \in eqc(E^*)$. In other words, $B \in small([T], E)$.

(ii) *B* is eventually fragmented over every $K \in \text{eqc}(E^*)$.

Every fragmented family is eventually fragmented. Therefore, every Asplund subset in *E* is tame.

In the spirit of Definition 3.18, we will say that a bounded subset is *Mackey* tame if it is tame over *every* weak-star compact (not necessarily equicontinuous) subset $K \subseteq E^*$.

Remark 5.2 By Lemma 3.5 and Proposition 3.2, the family of tame (Asplund, DLP) subsets in a given lcs is a *convex bornology* in the sense of [24] and *a saturated bornology* in the sense of [25, p. 153].

By Lemma 2.27, a Banach space V is Rosenthal iff every bounded subset $F \subset V$ is tame. Motivated by this reformulation, we introduce here the following locally convex analogue of Rosenthal Banach spaces.

Definition 5.3 A locally convex space *E* is said to be *tame* if every bounded subset *B* of *E* is tame. In other words, *E* is tame if and only if *E* is [**T**]-small. In this case we write $E \in (\mathbf{T})$.

Similarly, a space is *Mackey tame* if every bounded subset is Mackey tame. In this case we will write $E \in (\mathbf{mT})$.

As in Lemma 4.15, we may assume without loss of generality that B is countable. We have the inclusions (note that s. stands for "subsets"):

{weakly compact s.} \subset (DLP s.) \subset {Asplund s.} \subset {tame s.} \subset {bounded s.} {semi-reflexive lcs} \subset (DLP) \subset (NP) \subset (T) \subset {*lcs*}

Proposition 5.4

- (1) For Mackey spaces (e.g., barreled or metrizable), Mackey tameness is equivalent to tameness.
- (2) A Banach space is a tame lcs iff it is a Rosenthal Banach space.

Proof Corollary 3.20 and Lemma 2.27 respectively.

The following is a direct consequence of Theorem 3.17.

Theorem 5.5 The class (**T**) is closed under taking:

- (1) subspaces
- (2) bound covering maps
- (3) products
- (4) direct sums
- (5) inverse limits.

Moreover, if F is a large, dense subspace of the locally convex space E, and $F \in (\mathbf{T})$, then $E \in (\mathbf{T})$. In particular, if V is a normed tame space, then so is its completion.

Question 5.6 Is it true that the completion of a DLP/NP/tame lcs is always DLP/NP/tame?

A nonempty class of lcs is said to be a *variety* [7] if it is closed under the operations of taking subspaces, quotients, arbitrary products and isomorphisms. In particular, it is closed also under the inverse limits. For every subclass K of locally convex spaces, the intersection of all varieties containing K is a variety *generated* by K. Notation: $\mathcal{V}(K)$.

Proposition 5.7 The variety $\mathcal{V}(R)$ generated by the class of all Banach tame (i.e., Rosenthal) spaces is **properly** contained in (**T**). In particular, not every tame lcs can be embedded into a product of tame Banach spaces. Similar assertion is true for (**NP**).

Proof Use results of this section and also the following facts:

(a) The classes (**DLP**), (**NP**) and (**T**) are not closed under quotients. As it was mentioned in [52, IV, Ex. 20], there exists a Frechet Montel (hence, (**DLP**)) space *E* and a closed subspace *M* such that the quotient space *E*/*M* is the Banach space *l*¹ (which, of course, is not tame).

(b) The class *R* of all Banach tame (i.e., Rosenthal) spaces is closed under finite products and quotients, Hence by [7, Thm. 1.4], the variety $\mathcal{V}(R)$ of all lcs generated by R is just the class of all subspaces in products of Rosenthal Banach spaces. This implies that the Montel Frechet space *E* from (a), which is (**DLP**) and hence tame, cannot be embedded into a product of tame Banach spaces because a quotient of *E* is l^1 (and variety is closed under quotients).

Fact 5.8 [44, Main Theorem] Let K be a compact space. The following are equivalent:

- (1) l^1 cannot be embedded in C(K).
- (2) The dual of every separable Banach subspace of C(K) is separable.
- (3) *K* is scattered.

In other words, C(K) as a Banach space is Rosenthal, iff it is Asplund iff K is scattered.

This result motivates a generalization to the locally convex space $C_k(X)$ with the compact open topology when X is not compact. In [12, Lemma 6.3], Gabriyelyan–Kakol–Kubiś– Marciszewski gave a natural generalization showing that the same statement remains valid for the lcs $C_k(X)$ where X is a Tychonoff space X which is not necessarily compact, if we require that every compact subset of X is scattered. We give now another generalization of Fact 5.8 involving tame and NP lcs.

Proposition 5.9 For every Tychonoff topological space X the following are equivalent:

- (1) $C_k(X)$ is a tame lcs.
- (2) $C_k(X)$ is (NP).
- (3) Every compact subset of X is scattered.

Proof For every compact $K \subseteq X$, write $E_K := C(K)$ for the Banach space of continuous functions over K. Let D be the directed family of all compact subsets in X. By [25, p. 70, Proposition 3], $E = C_k(X)$ can be embedded in the inverse limit $\lim_{K \in D} E_K$. To make notation easier, we will assume without loss of generality that $E \subseteq \lim_{K \in D} E_K$.

(3) \Rightarrow (2) : Suppose that every compact $K \subseteq X$ is scattered. We will show that $C_k(X)$ is (**NP**). Applying Fact 5.8, we know that E_K is (**NP**) if and only if K is scattered. Using Theorem 5.5, we conclude that $\lim_{K \in D} E_K$ is also (**NP**), and therefore so is $E = C_k(X) \subseteq \lim_{K \in D} E_K$.

 $(2) \Rightarrow (1)$: Obvious.

 $(1) \Rightarrow (3)$: We will show that if *E* is tame, then every compact subset of *X* is scattered. Let $K \subseteq X$ be a compact subset. Consider the restriction map $r: C_k(X) \to C(K)$. It is easy to see that *r* is continuous. We claim that it is bound covering. Then, applying Theorem 5.5 we will conclude that C(K) is also tame. Finally, we will use Fact 5.8 to show that *K* is indeed scattered.

Now, to see that r is bound covering, consider the Stone-Čech compactification βX .

Suppose that $f \in C(K)$. Since *K* is compact, *f* is bounded. Also, *K* is closed in βX as a compact subset in a compact Hausdorff space. We can therefore apply the Tietze extension theorem to find $\hat{f} \in C(\beta X)$ such that $\hat{f}_{|K} = f$ and

$$\sup_{x \in \beta X} |f(x)| = \sup_{x \in K} |f(x)|.$$

Let us write $e: C(K) \to C_k(X)$ for the map sending every $f \in C(K)$ to $\widehat{f}|_X$. Clearly, r(e(f)) = f and therefore r(e(B)) = B for every $B \subseteq C(K)$.

Moreover, for every $f \in C(K)$ and compact $K' \subseteq X$, we have

$$\sup_{x \in K'} |(e(f))(x)| \le \sup_{x \in \beta X} |\widehat{f}(x)| = ||f||.$$

As a consequence, if $B \subseteq C(K)$ is bounded by M > 0, then so is e(B). By definition, r is bound covering.

Remark 5.10 A Banach space V is Asplund iff the dual of every separable subspace is separable. In contrast, there exists a separable NP space E with nonseparable dual (Remark 5.11 below). However, it is unclear for us if a lcs is NP iff the dual of every separable *Banach* subspace is separable. This question is interesting also in order to compare our Proposition 5.9 and [12, Lemma 6.3]. The "only if part" is clear because the class (**NP**) is closed under subspaces.

Remark 5.11 Let V be a (**NP**) lcs. In contrast to the case of Banach spaces (recall that (**NP**) Banach spaces are exactly Asplund Banach spaces), the dual of a separable linear subspace E of V is not necessarily separable. Indeed, consider the product space $V = \mathbb{R}^{c}$. Then \mathbb{R}^{c} is a reflexive lcs (in particular, (**NP**)). \mathbb{R}^{c} is separable (because of the Pondiczery Theorem [55, Thm. 16.4.c]).

However, its dual is not separable. Indeed, by [52], its dual V^* in its weak-star topology can be identified with the locally convex direct sum $\bigoplus_{i \in I} \mathbb{R}_i$ of continuum many copies of \mathbb{R} . It is easy to see that V^* in its weak-star topology is not separable. Therefore, V^* in its strong topology cannot be separable.

6 Generalized *I*¹-sequences

Definition 6.1 Let *E* be a locally convex space. A bounded sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ is said to be *equivalent to the* l^1 *-basis* (or simply an l^1 *-sequence*) if there exist: a continuous seminorm ρ on *E* and $\delta > 0$, such that for every $c_1, \ldots, c_n \in \mathbb{R}$

$$\delta \sum_{i=1}^{n} |c_i| \le \rho \left(\sum_{i=1}^{n} c_i x_i \right).$$

Generalizing Definition 1.4, we say that a subset A in a locally convex space E satisfies (\mathbf{R}_1) if it has no bounded l^1 -sequences.

The space *E* satisfies (\mathbf{R}_1) iff every bounded subset satisfies (\mathbf{R}_1). If there is no embedding of l^1 into *E*, then we will say that *E* satisfies ($\overline{\mathbf{R}_1}$).

Lemma 6.2 Let $(X, \|\cdot\|)$ be a Banach space and let Y be a locally convex space. Suppose that $\varphi \colon X \to Y$ is a continuous linear map. Then φ is an embedding if and only if there is a continuous seminorm ρ on Y and $\delta > 0$ such that $\rho(\varphi(x)) \ge \delta \|x\|$ for every $x \in X$.

Proof Suppose that this condition is indeed satisfied. Clearly, φ is injective, because for every $0 \neq x \in X$ we have $\rho(\varphi(x)) \ge \delta ||x|| > 0$. Hence, $\varphi(x) \neq 0$.

Since φ is linear, it is enough to show that $\varphi(B_X)$ is a neighborhood of 0 in $\varphi(X)$. We claim that

$$\varphi(X) \cap B_{\rho}(\delta) \subseteq \varphi(B_X).$$

Indeed, if $y \in \varphi(X) \cap B_{\rho}(\delta)$ then there is $x \in X$ such that $\varphi(x) = y$. Note that

$$\delta \ge \rho(y) = \rho(\varphi(x)) \ge \delta \|x\|,$$

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and therefore $||x|| \le 1$. By definition, $x \in B_X$ and $y \in \varphi(B_X)$.

Conversely, suppose that φ is an embedding. By definition, $\varphi(B_X)$ is a neighborhood of 0 in $\varphi(X)$. So there are continuous seminorms ρ_1, \ldots, ρ_n on Y and $\delta' > 0$ such that

$$\varphi(X) \cap B_{\rho_1,\dots,\rho_n}(\delta') \subseteq \varphi(B_X),$$

where

$$B_{\rho_1,\dots,\rho_n}(\delta') := \{ y \in Y \mid \forall 1 \le i \le n \colon \rho_i(y) \le \delta' \}$$

First, define $\rho := \max_{1 \le i \le n} \rho_i$ which is clearly another continuous seminorm on *Y*, and note that

$$B_{\rho_1,\ldots,\rho_n}(\delta') = B_{\rho}(\delta').$$

Next, suppose that $x \in X$. If x = 0 then our condition is clearly satisfied. Otherwise, ||x|| > 0. Write $\tilde{x} := \frac{2}{||x||}x$ and note that $||\tilde{x}|| = 2 > 1$. Thus, $\tilde{x} \notin B_X$.

We claim that $\varphi(\tilde{x}) \notin B_{\rho}(\delta')$ as a consequence. Indeed, suppose by contradiction that $\varphi(\tilde{x}) \in B_{\rho}(\delta')$. Thus,

$$\varphi(\widetilde{x}) \in B_{\rho}(\delta') \cap \varphi(X) \subseteq \varphi(B_X).$$

By definition, there exists some $y \in B_X$ such that $\varphi(y) = \varphi(\tilde{x})$. However, φ is injective so $\tilde{x} = y \in B_X$, which is impossible since $\tilde{x} \notin B_X$. This contradiction shows that $\varphi(\tilde{x}) \notin B_\rho(\delta')$. By definition, $\rho(\varphi(\tilde{x})) \ge \delta'$ and therefore

$$\rho\left(\varphi\left(\frac{2}{\|x\|}x\right)\right) = \frac{2}{\|x\|}\rho(\varphi(x)) \ge \delta'$$
$$\downarrow$$
$$\rho(\varphi(x)) \ge \|x\|\frac{\delta'}{2}.$$

Finally, define $\delta := \frac{\delta'}{2}$.

The following is a direct consequence of Lemma 6.2.

Lemma 6.3 $(\mathbf{R}_1) \Rightarrow (\overline{\mathbf{R}_1})$. More precisely, let $\varphi: l^1 \to X$ be an embedding into a lcs X. Then $\{\varphi(e_n)\}_{n \in \mathbb{N}} \subseteq X$ is an l^1 -sequence.

Example 6.4 The converse to Lemma 6.3 need not be true, $(\overline{\mathbf{R}_1}) \neq (\mathbf{R}_1)$.

Proof Indeed, consider the normed subspace X of l^1 consisting of finitely supported sequences:

$$X := \left\{ \alpha \in l^1 \mid \exists n_0 \in \mathbb{N} \; \forall n \ge n_0 : \alpha_n = 0 \right\}.$$

It is easy to see that the standard basis $\{e_n\} \subseteq X$ remains an l^1 -sequence. However, there is no embedding $\varphi : l^1 \to X$.

By contradiction, suppose that $\varphi: l^1 \to X$ is an embedding. By dim X we mean the Hamel dimension of X. It is easy to see that dim $X = \aleph_0$ while dim $Y = \dim l^1 = \aleph$, a contradiction.

Lemma 6.5 Suppose that X and Y are locally convex spaces and $\varphi \colon X \to Y$ is a continuous linear map. If $A \subseteq X$ is bounded and satisfies (**R**₁), then so does $\varphi(A)$.

Proof By contradiction, suppose that $\varphi(A) \subseteq Y$ does not satisfy (**R**₁). By definition, there exist: a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq \varphi(A) \subseteq Y$, a continuous seminorm ρ on Y and $\delta > 0$ such that for every $c_1, \ldots, c_n \in \mathbb{R}$

$$\delta \sum_{i=1}^{n} |c_i| \leq \rho \left(\sum_{i=1}^{n} c_i y_i \right).$$

Since φ is continuous and linear, there is a continuous seminorm σ on X such that

 $\rho(\varphi(x)) \le \sigma(x)$

for every $x \in X$. Choose $x_n \in A \subseteq X$ such that $\varphi(x_n) = y_n$. The sequence $\{x_n\}$ is bounded because A is bounded. We claim that $\{x_n\} \subseteq A$ is an l^1 -sequence with respect to σ and δ , which contradicts A satisfying (**R**₁). Indeed, for every $c_1, \ldots, c_n \in \mathbb{R}$

$$\sigma\left(\sum_{i=1}^{n} c_{i} x_{i}\right) \geq \rho\left(\varphi\left(\sum_{i=1}^{n} c_{i} x_{i}\right)\right) = \rho\left(\sum_{i=1}^{n} c_{i} \varphi(x_{i})\right)$$
$$= \rho\left(\sum_{i=1}^{n} c_{i} y_{i}\right) \geq \delta \sum_{i=1}^{n} |c_{i}|.$$

The following lemma can be extrapolated from [49, Propositions 3.1 and 3.3].

Lemma 6.6 Let X be a locally complete lcs. If $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is a bounded l^1 -sequence, then there is a linear topological embedding $\varphi \colon l^1 \to X$ such that for every $n \in \mathbb{N}$, $\varphi(e_n) = x_n$. Moreover, if ρ is a continuous seminorm on X and $\delta > 0$ make $\{x_n\}_{n \in \mathbb{N}}$ into an l^1 -sequence, then:

$$\rho(\varphi(\alpha)) \ge \delta \|\alpha\|$$

for every $\alpha \in l^1$.

Proof Let $\{x_n\}_{n\in\mathbb{N}}$ be equivalent to the l^1 -basis. Define $\varphi: l^1 \to X$ by $\varphi(\alpha) := \sum_{n\in\mathbb{N}} \alpha_n x_n$, where $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l^1$. Before we continue, we verify the convergence of this sum. Write *B* for the smallest closed disc containing $\{x_n\}_{n\in\mathbb{N}}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, so is *B*. By our assumption, *X* is locally complete. Hence, (X_B, q_B) is complete. Moreover, $\sum_{n=0}^{N} \alpha_n x_n \in X_B$, so we can use the Cauchy criterion for convergence. Let $\varepsilon > 0$. By the definition of l^1 , there exists some $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |\alpha_n| \le \varepsilon$. Note that for every $n \in \mathbb{N}, q_B(x_n) \le 1$ because $x_n \in 1 \cdot B$. Thus, for every $m \ge n_0$:

$$q_B\left(\sum_{n=n_0}^m \alpha_n x_n\right) \le \sum_{n=n_0}^m |\alpha_n| q_B(x_n) \le \sum_{n=n_0}^m |\alpha_n| \le \sum_{n=n_0}^\infty |\alpha_n| \le \varepsilon.$$

We have therefore shown that $\sum_{n \in \mathbb{N}} \alpha_n x_n \xrightarrow{q_B} x \in X_B$. Now, apply Fact 2.9 to conclude that $\lim_{n \in \mathbb{N}} \sum_{n \in \mathbb{N}} \alpha_n x_n = x \in X$ with respect to the original topology of X.

Note that φ is clearly linear. Next we show that φ is continuous. Let ρ be a continuous seminorm on X, $\varepsilon > 0$ and $M \in \mathbb{R}$ be a bound for $\{\rho(x_n)\}_{n \in \mathbb{N}}$. A similar calculation shows that

$$\rho(\varphi(\alpha)) \le M \sum_{n \in \mathbb{N}} |\alpha_n| = M \|\overline{\alpha}\| \le M \frac{\varepsilon}{M} = \varepsilon$$

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for every $\alpha \in \frac{\varepsilon}{M} B_{l^1}$. Note that for every $\alpha \in l^1$ such that $\alpha_n = 0$ for every $n > n_0$, we have

$$\rho(\varphi(\alpha)) = \rho\left(\sum_{i=1}^{n_0} \alpha_i x_i\right) \ge \delta \sum_{i=1}^{n_0} |\alpha_i| = \delta \|\alpha\|.$$

Since the finitely supported elements in l^1 are dense, and by virtue of the continuity of φ , this inequality also applies for every $\alpha \in l^1$. Applying Lemma 6.2, we conclude that φ is indeed an embedding.

The following is a direct consequence of Lemma 6.3 and Lemma 6.6.

Lemma 6.7 For locally complete lcs, the conditions (\mathbf{R}_1) and $(\overline{\mathbf{R}_1})$ are equivalent.

Recall that we gave definitions of ρ_M and (\mathbf{R}_1^*) in Definition 2.33.

Lemma 6.8 Let *E* be a locally convex space and let $M \subseteq E^*$ be an equicontinuous, weak-star compact, disked subset. If *M* does not satisfy (\mathbf{R}_1^*), then there exist an embedding $T: V \to E$ where *V* is a dense normed subspace of l^1 and $\delta > 0$ such that

$$\delta B_{V^*} \subseteq T^*(M).$$

Proof Let \widehat{E} be the completion of E. By definition, there is a bounded sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq E$ which is equivalent to the usual l^1 -basis with respect to ρ_M . By Lemma 6.6, there exists an embedding $T' : l^1 \to \widehat{E}$ with respect to ρ_M such that $T'(e_m) = x_m$. Moreover, there exists $\delta > 0$ such that

$$\forall \alpha \in l^1 : \rho_M(T(\alpha)) \ge \delta \|\alpha\|_1$$

Write $V := (T')^{-1}(E)$ and $T := T'_{|V}$.

By contradiction, suppose that $\delta B_{V^*} \not\subseteq T^*(M)$. In this case, there exists $\varphi \in \delta B_{V^*} \setminus T^*(M)$. Recall that by Banach–Grothendieck theorem the dual of (V^*, w^*) is simply V.

Also, $T^*(M)$ is clearly absolutely convex and closed. Using Fact 2.38, we can find $\alpha \in V$ such that

$$\sup_{\theta \in M} |(T^*(\theta))(\alpha)| < |\varphi(\alpha)|.$$

Thus we get:

$$\delta \|\alpha\|_{1} \leq \rho_{M}(T(\alpha)) = \sup_{\theta \in M} |\theta(T(\alpha))|$$
$$= \sup_{\theta \in M} |(T^{*}(\theta))(\alpha)| < |\varphi(\alpha)|$$
$$\leq \|\varphi\|_{1} \|\alpha\|_{1} \leq \delta \|\alpha\|_{1},$$

a contradiction.

7 Rosenthal type properties

For the definitions of (\mathbf{R}_1) and $(\overline{\mathbf{R}_1})$, see Definitions 1.4 and 6.1.

Theorem 7.1 $(T) = (R_1)$.

More precisely, a bounded subset $B \subseteq X$ *of a lcs is tame* if and only if *B satisfies* (**R**₁).

Proof First suppose that $B \subseteq X$ does not satisfy (\mathbf{R}_1) . We will show that B is not tame. Without loss of generality, we can assume that $B = \{x_n\}_{n \in \mathbb{N}} \subseteq X$ is a bounded sequence equivalent to the l^1 -basis. Write \widehat{X} for the completion of X. It is easy to see that $\{x_n\}_{n \in \mathbb{N}}$ remains an l^1 -sequence as a subset of \widehat{X} (directly applying Definition 6.1 and extending the seminorm to \widehat{X}). By Lemma 6.6, there is an embedding $\varphi : l^1 \to \widehat{X}$ such that $\varphi(e_n) = x_n \in X$. Write $B' := \{e_n\}_{n \in \mathbb{N}} \subseteq l^1$. Clearly, B' is not tame in l^1 . Since φ is an embedding, $B = \varphi(B')$ is also not tame in \widehat{X} . However, by Theorem 5.5, this implies that B is not tame in X, which is a contradiction.

Conversely, suppose that $B \subseteq X$ satisfies (**R**₁), we will show that *B* is tame. Let $M \in$ eqc (X^*) and let $\pi : X \to V$ and $\Delta : M \to V^*$ be as in Lemma 2.35. Since Δ is weak-star continuous, $\Delta(M)$ is weak-star compact and therefore equicontinuous by the Banach–Steinhaus Theorem (for the Banach space *V*). Since *B* satisfies (**R**₁) and considering Lemma 6.5, so does $\pi(B)$. By Lemma 2.26, $\pi(B)$ is tame over $\Delta(M)$. Applying Lemma 3.6, we conclude that *B* is tame over $M = \pi^*(\Delta(M))$.

The following corollary of Theorem 7.1 gives a generalization of Rosenthals's dichotomy to all locally convex spaces in terms of the tameness.

Theorem 7.2 [Tame dichotomy in lcs] Let *E* be a locally convex space. Then every bounded subset in *E* is either tame, or has a subsequence equivalent to the l^1 -sequence.

Definition 7.3 A subset $A \subseteq X$ of a lcs is said to be *Rosenthal* (write $A \in (\mathbf{Ros})$) if every bounded sequence in A has a weak Cauchy subsequence. We say that X is *Rosenthal* ($X \in (\mathbf{Ros})$) if every bounded subset of X is Rosenthal.

Proposition 7.4 $(Ros) \Longrightarrow (mT)$.

More precisely, every bounded Rosenthal subset $B \subseteq X$ (e.g., every bounded weak Cauchy subsequence) in a lcs X is Mackey tame.

Proof Suppose that $B \subseteq X$ is a bounded Rosenthal set and let $M \subseteq X^*$ be weak-star compact. We claim that *B* is tame over *M*. Assuming the contrary, suppose that $\{x_n\}_{n\in\mathbb{N}}\subseteq B$ is independent over *M*. We can assume without loss of generality that $\{x_n\}_{n\in\mathbb{N}}$ is both weak-Cauchy and independent over *M*. Let $a < b \in \mathbb{R}$ be the bounds of independence of $\{x_n\}_{n\in\mathbb{N}}$ over *M*. Since *M* is weak-star compact, we can use Fact 2.23 to find $\varphi \in M$ such that:

$$\varphi(x_n) \in \begin{cases} (b, \infty) & n \in 2\mathbb{N} \\ (-\infty, a) & \text{otherwise} \end{cases}$$

This implies that $\{\varphi(x_n)\}_{n \in \mathbb{N}}$ is not a Cauchy sequence, which is a contradiction.

Theorem 7.5 In every locally convex space X, the following holds:

$$(\mathbf{Ros}) \Longrightarrow (\mathbf{mT}) \Longrightarrow (\mathbf{T}) = (\mathbf{R}_1) \Longrightarrow (\mathbf{R}_1).$$

Proof Combine Theorem 7.1, Proposition 7.4 and Lemma 6.3.

Theorem 7.6 Let X be a locally convex space whose bounded subsets are metrizable. Then (Ros) = (T).

More precisely, If $B \subseteq X$ is a bounded subset then it is Rosenthal if and only if it is tame.

Proof We already established in Proposition 7.4 that every Rosenthal bounded subset is tame. We will see the converse applied in this setting. Let $B \subseteq X$ be tame, we will show that it

is also Rosenthal. Without loss of generality, we can assume that $B = \{x_n\}_{n \in \mathbb{N}} \subseteq X$ is a bounded sequence. Note that acx *B* is also bounded, and therefore metrizable.

Let $\{\varepsilon_m \cap \operatorname{acx} B\}_{m \in \mathbb{N}}$ be a descending basis of disks for the neighborhoods of $0 \in \operatorname{acx} B$, and let $\{\varepsilon_m^\circ\}_{m \in \mathbb{N}}$ be the corresponding polars. Every such polar is equicontinuous by definition and also weak-star compact by the Banach–Alaoglu Theorem. Since *B* is tame, it must be tame over each ε_m° .

Define $n_k^{(0)} := k$. Using induction, we construct subsequences $\left\{n_k^{(m)}\right\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that

 $\begin{array}{l} (1) \ \left\{ n_k^{(m+1)} \right\}_{k \in \mathbb{N}} \subseteq \left\{ n_k^{(m)} \right\}_{k \in \mathbb{N}}. \\ (2) \ \left\{ \varphi \left(x_{n_k^{(m)}} \right) \right\}_{k \in \mathbb{N}} \text{ converges for every } \varphi \in \varepsilon_m^\circ. \end{array}$

The induction step is done by applying Lemma 2.24 on $\{n_k^{(m)}\}_{k\in\mathbb{N}}$ to find a subsequence $\{n_k^{(m+1)}\}_{k\in\mathbb{N}}$ that converges on $\varepsilon_{m+1}^{\circ}$. Now, defining $n_k := n_k^{(k)}$ we get that $\{\varphi(x_{n_k})\}_{k\in\mathbb{N}}$ converges for every $\varphi \in \bigcup_{m\in\mathbb{N}} \varepsilon_m^{\circ}$.

We will now show that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a weak Cauchy sequence. Let $\varphi \in X^*$ and let ε be a neighborhood of zero such that $|\varphi(\varepsilon)| \leq 1$. By the construction of $\{\varepsilon_m\}_{m\in\mathbb{N}}$, there is some $m_0 \in \mathbb{N}$ such that $\varepsilon_{m_0} \cap \operatorname{acx} B \subseteq \varepsilon \cap \operatorname{acx} B$.

Using Lemma 2.37, we can find $\widehat{\varphi} \in \varepsilon_{m_0}^{\circ}$ such that $\varphi_{|acx B} = \widehat{\varphi}_{|acx B}$. However, $\{x_{n_k}\}_{k \in \mathbb{N}} \subseteq acx B$ and therefore $\{\varphi(x_{n_k})\}_{k \in \mathbb{N}} = \{\widehat{\varphi}(x_{n_k})\}_{k \in \mathbb{N}}$ converges. By definition, $\{x_{n_k}\}_{k \in \mathbb{N}}$ is weak Cauchy.

Theorem 7.7 If all bounded sets of a lcs X are metrizable, then

$$(\mathbf{Ros}) = (\mathbf{mT}) = (\mathbf{T}) = (\mathbf{R}_1) \Longrightarrow (\mathbf{R}_1)$$

and the following dichotomy holds: any bounded sequence in X either has a weak Cauchy subsequence or an l^1 -subsequence.

Proof Combine Theorem 7.5 and Theorem 7.6.

Theorem 7.7 and Lemma 6.7 directly imply a well-known result of Ruess (Fact 1.8) which extends Rosenthal's non-containtment of l^1 -criteria to a quite large class of lcs.

Theorem 7.8 There exists a strongly tame complete (even reflexive) lcs which:

- (i) is not a Rosenthal lcs;
- (ii) does not contain any l^1 -subsequence;
- (iii) contains a dense, Rosenthal subspace.

As a corollary: Rosenthal's dichotomy does not hold for such locally convex spaces.

Proof Write $C := \{0, 1\}^{\mathbb{N}}$ and consider the space $X := \mathbb{R}^C$ with the product topology. It is known that X is complete and reflexive. By Theorem 5.5, X is tame. Also, every reflexive space is barreled [25, Proposition 11.4.2] so by Proposition 5.4, X is strongly tame. We will see now that it is not Rosenthal. Consider the *n*-th projection $\pi_n : C \to \{0, 1\}$. We claim that $\{\pi_n\}_{n \in \mathbb{N}} \subseteq X$ has no weak Cauchy subsequence. Indeed, for every subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ define $\alpha \in C$ whose *n*-th entry is

$$\alpha_n := \begin{cases} 1 & n = n_k, k \in 2\mathbb{N} \\ 0 & \text{else} \end{cases}$$

and let $e_{\alpha} \colon X \to \mathbb{R}$ be the evaluation functional at α . It is easy to see that $\{e_{\alpha}(\pi_{n_k})\}_{k \in \mathbb{N}}$ does not converge and therefore $\{\pi_{n_k}\}_{k \in \mathbb{N}}$ is not weak Cauchy.

Finally, consider the dense subspace $Y \subseteq X$ of functions with finite support

$$Y := \{ f \in \mathbb{R}^C \mid |\operatorname{Supp}(f)| < \aleph_0 \} \subseteq \mathbb{R}^C.$$

We will show that Y is a Rosenthal space. Let $\{x_n\}_{n \in \mathbb{N}} \subseteq Y$ be a bounded sequence. By definition,

$$S := \bigcup_{n \in \mathbb{N}} \operatorname{Supp}(x_n) \subseteq C$$

is countable so we can enumerate it as $S = \{\alpha_m\}_{m \in \mathbb{N}}$. Using induction and a diagonal argument, it is easy to find a subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\{x_{n_k}\}_{k \in \mathbb{N}}$ converges pointwise on *S*. Moreover, for every $\alpha \in C \setminus S$, $k \in \mathbb{N}$ we have $x_{n_k}(\alpha) = 0$. Therefore, $\{x_{n_k}\}$ converges pointwise on *C* and is therefore weak-Cauchy.

Remark 7.9 Theorem 7.2 can be considered as a locally convex version of Rosenthal's classical dichotomy (in Banach spaces). As shown in Theorem 7.6, "Tame dichotomy" of Theorem 7.2 implies that Rosenthal's classical dichotomy from Banach spaces can be extended to a much larger class of lcs with metrizable bounded subsets. This strengthens a well-known result of Ruess [49] (Fact 1.8). On the other hand, Rosenthal's dichotomy is not true for general lcs as Theorem 7.8 demonstrates.

Strongest tame image of a lcs. In this subsection, we apply the results of Sect. 3.2 to the bornological class of tame functions. By 3.23, this is indeed a polarly compatible bornology.

Definition 7.10 An equicontinuous subset $M \subseteq E^*$ is said to be *co-tame* if $M \in \text{small}^*([\mathbf{T}], E)$. Explicitly, M is co-tame if and only if every bounded $A \subseteq E$ is tame over M. Similarly can be defined *co-Asplund* and *co-DLP* subsets.

The following remark provides a useful class of co-Asplund (hence, co-tame) subsets.

Remark 7.11 Let *E* be a lcs and $K \in eqc(E^*)$. Suppose that *K* is separable (or, more generally, *uniformly Lindelof* [33]) in the standard strong topology on E^* . Then *K* is (weak^{*}, strong)-fragmented by [33, Prop. 3.10]. It follows that every bounded subset $B \subset E$ is fragmented on *K*. Hence *K* is co-Asplund (so, also co-tame). This explains also assertion (3) in Examples 4.16.

Lemma 7.12 The family of all co-tame subsets small* ([**T**], E) is a locally convex bornology. In particular, the weak-star closed absolutely convex hull of a co-tame subset is co-tame. It is also true relative to a given bounded subset B. This means that if B is tame over M, it is also tame over $\overline{\operatorname{acx}}^{w^*}(M)$.

Proof Application of Lemma 3.30 in light of [**T**] being polarly compatible.

Theorem 7.13 Let (E, τ) be a locally convex space and small^{*} ([**T**], E) be the locally convex bornology of co-tame subsets. Then the polar topology τ_{tame} is the strongest tame locally convex topology, coarser than τ , and

$$\tau_w \subseteq \tau_{tame} \subseteq \tau \subseteq \tau_\mu$$

where τ_{μ} is the Mackey topology.

Proof Theorem 3.36 and Lemma 3.34.

Similar results are valid also for the class (NP).

Problem 7.14 Study τ_{tame} and τ_{NP} for remarkable (e.g., classical) locally convex (or, Banach) spaces (E, τ) .

8 Free locally convex spaces revisited

Given a class P of Banach spaces, a locally convex space E is called *multi-P* (see [32]) if E can be isomorphically embedded into a product of spaces that belong to P.

For every compact space K its free lcs L(K) is *multi-reflexive*, as it was proved in a recent paper by Leiderman and Uspenskij [32]. Since multi-reflexive lcs is DLP (by Theorem 4.6), we obtain that L(K) is DLP. In Theorem 8.4, we give a stronger result using the following two facts.

Fact 8.1 [10, Proposition 2.7] For a subset A of L(X), the following assertions are equivalent:

- (1) A is bounded;
- (2) Supp(A) has compact closure in the Dieudonné completion μX and $C_A := \sup_{\chi \in A \cup \{0\}} \|\chi\|$ is finite;
- (3) Supp(A) is functionally bounded in X and C_A is finite.

Fact 8.2 [54, Thm. 2] If $K \subseteq X$ is a compact subset of a Tychonoff space X, then L(K) naturally can be viewed as a subspace of L(X).

Proof It is easy to see that K is C-imbedded into X, that is, every continuous function on K can be extended to X (a proof can be found in Proposition 5.9). Moreover, K is compact and therefore, by [54, Thm. 2], the inclusion $L(K) \subseteq L(X)$ is an embedding.

Lemma 8.3 The free lcs L(X) is DLP (in particular, is (**NP**)) for every Dieudonné complete space X.

Proof Let $B \subseteq L(X)$ be bounded and let $M \subseteq C(X) = L(X)^*$ be a weak-star compact, equicontinuous subset. We will show that B has the DLP over M.

Since X is Dieudonné complete and by Fact 8.1, $\text{Supp}(B) \subseteq X$ has compact closure. Let

$$K := \operatorname{Supp}(B) \subseteq X.$$

By Fact 8.2, the inclusion $L(K) \subseteq L(X)$ is an embedding. Thus, $B \subseteq L(K) \subseteq L(X)$.

Moreover, M can be viewed as an equicontinuous subset of $L(K)^*$. We already mentioned that L(K) is multi-reflexive by [32]. Then L(K) is DLP because the class is closed under products and subspaces (see Theorem 4.3). Therefore B is DLP on M. This is true for every bounded B and weak-star compact, equicontinuous $M \subseteq L(X)^*$. By definition, L(X) is DLP.

Theorem 8.4 L(X) is DLP for every Tychonoff space X.

Proof Write μX for the Dieudonné completion of X. From the first sentence of the proof of [54, Thm. 5], we know that L(X) is embedded in $L(\mu X)$. By Lemma 8.3, $L(\mu X)$ is DLP. Now, by Theorem 4.3, so is L(X) as its subspace.

Remark 8.5 We are indebted to S. Gabriyelyan for suggesting Theorem 8.4 and its present proof as a consequence of Lemma 8.3.

Remark 8.6

- (1) By Theorem 8.4, L(P) is DLP (in particular, is NP) for the Polish space $P = \mathbb{N}^{\mathbb{N}}$ of all irrationals. In contrast, another result from [32] shows that L(P) is not multi-reflexive.
- (2) Note that every lcs X is a linear topological quotient of some lcs which is DLP. Indeed, the identity map $id: X \to X$ can be canonically extended to the linear onto map $L(X) \to X$, which is a quotient (open) map (because its restriction $id: X \to X$ is an onto factor map).

Question 8.7 *Examine if* $L(\mathbb{N}^{\mathbb{N}})$ *is multi-Asplund or, at least, multi-Rosenthal.*

In [54, p. 679, Corollary], Uspenski shows that for a Dieudonné complete space, L(X) is complete if and only if there are no infinite compact subsets. The following is a similar result that shows the "scarcity" of semi-reflexivity in the realm of free locally convex spaces.

Theorem 8.8 Let X be a Dieudonné complete space. Then L(X) is semi-reflexive if and only if X has no infinite compact subset.

Proof By Fact 4.5, semi-reflexivity is equivalent to the Heine–Borel property for the weak topology.

First, suppose that X has no infinite compact subset. We will show that every bounded subset is weakly relatively compact. Let $B \subseteq L(X)$ be bounded. By Fact 8.1, B has compact support $K \subseteq X$. Since every compact subset of X is finite, we conclude that $\text{Supp}(B) \subseteq K$ is also finite. As a consequence, $B \subseteq \text{Span}(\text{Supp}(B))$ is a bounded subset in a finite dimensional topological space. In this case, the weak topology of Span(Supp(B)) coincides with the original topology, and the Heine–Borel property is satisfied.

Conversely, assume that L(X) is semi-reflexive. By contradiction, assume that X has an infinite compact subset $K \subseteq X$. Since K is infinite, we can find a discrete countable subset $\{x_n\}_{n \in \mathbb{N}} \subseteq K$. For every $n \in \mathbb{N}$ define:

$$\chi_n := \sum_{m=1}^n x_m,$$

and consider the set $B := {\chi_n}_{n \in \mathbb{N}}$. Clearly, *B* is bounded in virtue of Fact 8.1. By our assumption, *B* is weakly relatively compact, and therefore has an accumulation point $\chi \in L(X)$.

Write $\chi = \sum_{i=1}^{k} \alpha_i y_i$ and choose some $n_0 \in \mathbb{N}$ such that $x_{n_0} \neq y_i$ for every $1 \le i \le k$. By the choice of $\{x_n\}_{n \in \mathbb{N}}$, there is some neighborhood $x_{n_0} \in U \subseteq X$ such that

$$A := \{x_n\}_{n_0 \neq n \in \mathbb{N}} \cup \{y_i\}_{i=1}^k \subseteq X \setminus U.$$

Since *X* is completely regular, we can find $f \in C(X) = (L(X))^*$ such that

$$f(x_{n_0}) = 1$$
 and $\forall x \in A : f(x) = 0$.

By the definition of accumulation point, there is some $n_0 \leq N \in \mathbb{N}$ such that

$$||f(\chi) - f(\chi_N)|| < \frac{1}{2}.$$

However, $f(\chi) = 0$ and $f(\chi_N) = 1$, which is a contradiction.

We were informed recently by Gabriyelyan that Proposition 8.8 is a partial case of a more general result which was proved in his (accepted) joint work with Banakh [2].

Deringer

9 DFJP factorization for lcs

Recall that a continuous linear operator $T: E \to F$ between two lcs is said to be *(weakly) compact* if there exists a neighborhood O of zero in E such that T(O) is relatively (weakly) compact in F, [21, p. 94]. A natural possibility to generalize these definitions is to require that T(O) is small in some other sense, i.e., T(O) belongs to some bornology of small subsets. In particular, we have the following definitions.

Definition 9.1 We say that a linear continuous map $T: E \to F$ between lcs is *tame (NP, DLP)* if there exists a zero neighborhood $U \subseteq E$ such that $T(U) \subseteq F$ is a tame (NP, DLP) subset in F.

Note that by our definitions T(U) is bounded. Hence, the identity map of a tame not normable lcs is not tame.

The following fundamental result is a part of a classical work about DFJP factorization, [6]. In [15, 17, 36, 39], some simplifications and adaptations were added.

Fact 9.2 [6, Lemma 1] Let X be a Banach space and $W \subseteq X$ be an absolutely convex bounded subset. Write $U_n := 2^n W + 2^{-n} B_X$ and $\|\cdot\|_n$ for the gauge of U_n . Also, consider $[x] := (\sum_{n \in \mathbb{N}} \|x\|_n^2)^{1/2}$ and $Y = \{x \in X \mid [x] < \infty\}$. Finally, let $j: Y \to X$ be the identity map. Then:

(1) $W \subseteq B_Y$.

(2) $(Y, [\cdot])$ is a Banach space and j is continuous.

(3) $j^{**}: Y^{**} \to X^{**}$ is one to one and $(j^{**})^{-1}(X) = Y$.

(4) *Y* is reflexive if and only if *W* is weakly relatively compact.

Remark 9.3 Note that in Fact 9.2, $j^*: X^* \to Y^*$ is dense as a consequence of Lemma 9.4.

Lemma 9.4 Let $f: E_1 \to E_2$ be a continuous linear map between locally convex spaces. Then $f^*: E_2^* \to E_1^*$ is one to one if and only if $f: E_1 \to E_2$ has a dense image.

Proof Observe that $f(E_1)$ is not dense in E_2 if and only if there exists $0 \neq \varphi \in E_2^*$ such that $f^*(\varphi) = 0$.

Lemma 9.5 Let K be a compact topological space and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of sets of bounded functions on K. Write B for the unit ball of C(K) and:

$$A := \bigcap_{n \in \mathbb{N}} \left(A_n + \frac{1}{2^n} B \right).$$

If all $\{A_n\}_{n \in \mathbb{N}}$ are (eventually) fragmented over K, so is A.

Proof We will only prove the case of eventually fragmented maps. The other case is slightly easier. Suppose that $\{x_m\}_{m \in \mathbb{N}} \subseteq A$ is an infinite subset. We need to find a subsequence $\{m_k\} \subseteq \mathbb{N}$ such that $\{x_{m_k}\}_{k \in \mathbb{N}}$ is fragmented over *K*. By its definition, for every $n, m \in \mathbb{N}$, we can find $y(n, m) \in A_n$ such that $x_m \in y(n, m) + \frac{1}{2^n}B_n$ (we use function notation rather than subscript to make it easier to read). Using induction, we build a sequence $\left\{ \left\{ m_k^{(n)} \right\}_{k \in \mathbb{N}} \right\}_{n \in \mathbb{N}}$ of descending infinite subsequences. We define $m_k^{(-1)} := k$ (the trivial subsequence), and at each step choose a subsequence $\left\{ m_k^{(n+1)} \right\}_{k \in \mathbb{N}} \subseteq \left\{ m_k^{(n)} \right\}_{k \in \mathbb{N}}$ such that $\left\{ y \left(n + 1, m_k^{(n+1)} \right) \right\}_{k \in \mathbb{N}}$

is fragmented. Now, define $m_k := m_k^{(k)}$. It is easy to see that $Y_n := \{y (n, m_k)\}_{k \in \mathbb{N}}$ remains fragmented. We will show that so does $A' := \{x_{m_k}\}_{k \in \mathbb{N}} \subseteq A$.

Let $\varepsilon > 0$ and $T \subseteq K$ be non-empty. We need to find an open $O \subseteq K$ such that $T \cap O$ is not empty and $f(T \cap O)$ is ε -small for every $f \in A'$. There exists some $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \frac{1}{3}\varepsilon$. By our construction, $\{y(n_0, m_k)\}_{k \in \mathbb{N}}$ is fragmented so we can find an open $O \subseteq K$ such that $O \cap T$ is not empty and $f(O \cap T)$ is $\frac{1}{3}\varepsilon$ -small for every $f \in Y_n$. Since $A' \subseteq Y_n + \frac{1}{2^{n_0}}B$, it is easy to see that for every $f \in A'$, $f(O \cap T)$ is ε -small, as required. \Box

Fact 9.6 [39] Let $j: V_1 \to V_2$ be a continuous linear operator between Banach spaces such that the adjoint $j^*: V_2^* \to V_1^*$ is norm-dense. Let $F \subset V_1$ be a bounded subset.

(1) If j(F) is a tame subset in V_2 , then F is a tame subset in V_1 .

(2) If j(F) is an Asplund subset in V_2 , then F is an Asplund subset in V_1 .

Proof (1) In order to show that F is a tame subset in V_1 it is equivalent to check that F is eventually fragmented on $B_{V_1^*}$ (Lemma 2.24). We have to prove that for every sequence $\{f_n\}_{n \in \mathbb{N}}$ in F there exists a subsequence $\{f_n\}_{k \in \mathbb{N}}$ which is fragmented on $B_{V_1^*}$. Equivalently, $(B_{V_1^*}, \rho_C)$ is separable (Lemma 2.17), where

$$C:=\{f_{n_k}\} \text{ and } \rho_C(x_1, x_2):=\sup_{f\in C} |f(x_1) - f(x_2)|.$$

By our assumption, j(F) is a tame subset in V_2 . Hence, there exists a subsequence $C := \{f_{n_k}\}$ of $\{f_n\}$ such that $j(C) = \{j(f_{n_k})\}$ is a fragmented family on $B_{V_2^*}$. Equivalently, $(B_{V_2^*}, \rho_{j(C)})$ is separable (Lemma 2.17). Then $(V_2^*, \rho_{j(C)})$ is also separable. By the definition of the adjoint operator, we have $\langle j(x), v^* \rangle = \langle x, j^*(v^*) \rangle$ for every $x \in V_1, v^* \in V_2^*$. This implies that $(j^*(V_2^*), \rho_C)$ is separable. Then its ρ_C -closure $cl_{\rho_C}(j^*(V_2^*))$ is also ρ_C -separable. Clearly, $cl_{\rho_C}(j^*(V_2^*)) \supset cl_{norm}(j^*(V_2^*)) = V_1^*$ (because *C* is a bounded sequence in V_1). Therefore, $(V_1, *, \rho_C)$ and also $(B_{V_1^*}, \rho_C)$ are separable, as desired.

(2) The case of Asplund subsets is similar (and easier).

Lemma 9.7 Let A be a bounded tame (Asplund, relatively weakly compact) subset in a Banach space X. Then there exist a Rosenthal (Asplund, reflexive) Banach space Y, a continuous linear injective map $j: Y \to X$ such that A is a subset of $j(B_Y)$ and the adjoint map $j^*: X^* \to Y^*$ is dense.

Proof Define $W := \operatorname{acx} A$. The case of relatively weakly compact subsets is a consequence of Fact 9.2. Let us consider the cases where A is tame or Asplund. By Lemma 3.5, W remains tame (Asplund). Next, let $j: Y \to X$ and $\{U_n\}_{n \in \mathbb{N}}$ be as described in Fact 9.2. It is well known and easy to see that

$$A \subseteq W \subseteq j(B_Y) = B_Y \subseteq U := \bigcap_{n \in \mathbb{N}} U_n.$$

Then $j^*: X^* \to Y^*$ is dense by Remark 9.3. By Lemma 2.24, the tameness (Asplundness) of bounded families of continuous functions on compact sets is equivalent to eventual fragmentability (fragmentability). By Lemma 9.5, we conclude that U and $j(B_Y)$ also are tame (Asplund) subsets. By Fact 9.6, B_Y is tame (Asplund). Therefore, Y is Rosenthal (Asplund).

Theorem 9.8 Every tame (NP, DLP) operator $T: E \rightarrow X$ between a lcs E and a Banach space X can be factored through a Rosenthal (Asplund, reflexive) Banach space.

Proof By our assumption, there exists a zero neighborhood O in E such that T(O) is tame (Asplund, weakly relatively compact) in X. We apply Lemma 9.7 to the subset $A:=T(O) \subset X$ in order to construct a *Rosenthal (Asplund, reflexive) Banach space* V and a continuous injective linear operator $j: V \to X$ such that A is a subset of $j(B_V)$. Now, consider the linear operator $u := j^{-1} \circ T: E \to V$. Since $u(O) \subset B_V$, we obtain that u is continuous. Then $T = j \circ u$ is the required factorization.

It would be interesting to find some additional natural bornologies which are consistent with DFJP-factorization.

10 Generalization of Haydon's Theorem and tame spaces

In its original statement Haydon's theorem characterizes Rosenthal Banach spaces as follows.

Theorem 10.1 (Haydon [23, Thm. 3.3]) *Let V be a Banach space. The following are equivalent:*

- (1) V contains no l^1 -sequence;
- (2) every weak-star compact convex subset of V^* is the norm closed convex hull of its extreme points;
- (3) for every weak-star compact subset T of V^* ,

$$\overline{\operatorname{co}}^{w^*}(T) = \overline{\operatorname{co}}(T).$$

Our generalized version, in a brief summary, can be expressed as follows.

Proposition 10.2 For a locally convex space E, the following are equivalent:

- (1) *E* is tame (in virtue of Theorem 7.1, equivalent to not containing of an l^1 -sequence);
- (2) every equicontinuous, weak-star compact convex subset of E* is the strong closed convex hull of its extreme points. That is, co^{w*}(ext M) = co(ext M) for every convex M ∈ eqc (E*);
- (3) for every equicontinuous, weak-star compact subset T of E^* ,

$$\overline{\operatorname{co}}^{w^*}(T) = \overline{\operatorname{co}}(T).$$

In fact, Theorem 10.12 will show an even more localized result for a given equicontinuous subset M.

Application of the DFJP construction. By the Krein–Milman theorem, if $M \in \text{eqc}(E^*)$ is convex, then $M = \overline{\text{co}}^{w^*}(\text{ext } M)$. In light of Haydon's theorem, we give the following definition.

Definition 10.3 Let $M \in \text{eqc}(E^*)$ be convex. We say that a bounded set $B \subset E$ is *anti-H* for *M* if there exist a functional $\psi \in M$ and $\varepsilon > 0$ such that

$$U[B, \varepsilon](\psi) \cap \operatorname{co}(\operatorname{ext} M) = \emptyset.$$

The following is a direct consequence of the previous definition.

Lemma 10.4 $\overline{\operatorname{co}}^{w^*}(\operatorname{ext} M) = \overline{\operatorname{co}}(\operatorname{ext} M)$ if and only if there is no bounded $B \subseteq E$ which is anti-H for M.

Lemma 10.5 Let $T: E_1 \to E_2$ be a continuous linear map between lcs, $T^*: E_2^* \to E_1^*$ be *its adjoint, B be a bounded subset in* E_1 *and* $M \in eqc(E_2^*)$ *be convex. Then*

(1) $(\varphi_1, \varphi_2) \in U[T(B), \varepsilon]$ in $E_2^* \iff (T^*(\varphi_1), T^*(\varphi_2)) \in U[B, \varepsilon]$ in E_1^* ;

(2) for every $\varphi \in E_2$ we have

$$U[T(B), \varepsilon](\varphi) = (T^*)^{-1}(U[B, \varepsilon](T^*(\varphi)));$$

- (3) if T(B) is anti-H over M then B is anti-H over $T^*(M)$;
- (4) if the image of T is dense in E₂, then the converse is also true, namely, B being anti-H over T*(M) implies that T(B) is anti-H over M.

Proof

(1) By definition of the adjoint $T^*: E_2^* \to E_1^*$, we have

$$\langle b, T^*(\varphi) \rangle = \langle T(b), \varphi \rangle \ \forall b \in B.$$

Now apply the descriptions of $U[B, \varepsilon]$, $U[T(B), \varepsilon]$ according to Definition 2.5. (2) Using (1) we get:

$$\begin{split} \varphi' \in U[T(B), \varepsilon](\varphi) &\iff (\varphi, \varphi') \in U[T(B), \varepsilon] \\ &\iff (T^*(\varphi), T^*(\varphi')) \in U[B, \varepsilon] \\ &\iff T^*(\varphi') \in U[B, \varepsilon](T^*(\varphi)) \\ &\iff \varphi' \in (T^*)^{-1}(U[B, \varepsilon](T^*(\varphi))). \end{split}$$

(3) By definition, there exists $\varphi \in M$ such that:

$$U[T(B), \varepsilon](\varphi) \cap \operatorname{co}(\operatorname{ext} M) = \emptyset.$$

Write $\psi := T^*(\varphi)$. Suppose by contradiction that *B* is not anti-*H* for $T^*(M)$. Thus, we can find

$$\psi' \in U[B, \varepsilon](\psi) \cap \operatorname{co}(\operatorname{ext} T^*(M)).$$

By Lemma 2.32.1, co (ext $T^*(M)$) $\subseteq T^*(co (ext M))$. We can therefore find $\varphi' \in co (ext M)$ such that $\psi' = T^*(\varphi')$. Also, by (2):

$$\psi' = T^*(\varphi') \in U[B, \varepsilon](T^*(\varphi)) \Rightarrow \varphi' \in (T^*)^{-1}(U[B, \varepsilon](T^*(\varphi))) = U[T(B), \varepsilon](\varphi).$$

Using all the information on φ' so far, we get

$$\varphi' \in U[T(B), \varepsilon](\varphi) \cap \operatorname{co}(\operatorname{ext} M) = \emptyset,$$

a contradiction.

(4) Since T has a dense image, T^* is injective and so Lemma 2.32.2 is applicable:

$$B \text{ is anti-}H \text{ for } T^*(M) \iff \exists \varphi \in M : U[B, \varepsilon](T^*(\varphi)) \cap \text{ co } (\text{ext } T^*(M)) = \emptyset$$
$$\implies \exists \varphi \in M : T^*(U[T(B), \varepsilon](\varphi)) \cap \text{ co } (\text{ext } T^*(M)) = \emptyset$$
$$\iff \exists \varphi \in M : T^*(U[T(B), \varepsilon](\varphi)) \cap T^*(\text{ co } (\text{ext } M)) = \emptyset$$
$$\iff \exists \varphi \in M : U[T(B), \varepsilon](\varphi) \cap \text{ co } (\text{ext } M) = \emptyset$$
$$\iff T(B) \text{ is anti-}H \text{ for } M.$$

Note that the equivalence marked by \star is true in virtue of another application of T^* being injective.

Proposition 10.6 Let V be a Banach space, $A \subseteq V$ be bounded and $M \in eqc(V^*)$ be convex. If A is anti-H with respect to M, then A is not tame in V.

Proof By contradiction, assume that A is tame in V. Let W and $j: W \to V$ be the Rosenthal Banach space and the map described in the factorization Lemma 9.7 (caution: the notation was V and not W in Lemma 9.7). Thus, $A \subseteq j(B_W)$ and therefore $j(B_W)$ is also anti-H over M. By Lemma 10.5.3, B_W is anti-H over $j^*(M)$. However, $j^*(M)$ is a weak-star compact subset of a Rosenthal Banach space - a contradiction to Haydon's Theorem 10.1.

Proposition 10.7 Let E be a lcs.

If $M \in eqc(E^*)$ is co-tame and convex, then $M = \overline{co}(ext M)$.

Proof Clearly, $M \supseteq \overline{co} (\text{ext } M)$. To show the converse, assume by contradiction that $M \notin \overline{co} (\text{ext } M)$. By Lemma 10.4, we can find a bounded $B \subseteq E$ which is anti-H for M. Let $V, \pi : E \to V$ and Δ : Span $M \to V^*$ be the maps described in Lemma 2.35. Write $N := \Delta(M) \in \text{eqc } (V^*)$. Since Δ is weak-star continuous, N is weak-star compact in V^* and therefore equicontinuous by the Banach–Steinhaus theorem (V is a Banach space). Note that $M = \pi^*(N)$ so we can say that B is anti-H over $\pi^*(N)$. The map π is dense, so by Lemma 10.5.4 we conclude that $\pi(B)$ is anti-H over N.

However, *M* is co-tame so *B* is tame over $M = \pi^*(N)$ by definition. By Lemma 3.6, $\pi(B)$ is tame over *N*. Moreover, by Lemma 7.12, $\pi(B)$ is tame over

$$\overline{\operatorname{acx}}^{w^*}N = \overline{\operatorname{acx}}^{w^*}\Delta(M) = B_{V^*}.$$

Thus, $\pi(B)$ is tame in V. Proposition 10.6 gives the contradiction.

The Haydon Property and other results.

Definition 10.8 Let *E* be a locally convex space and let $M \in eqc(E^*)$. We say that *M* has the *Haydon property* if

$$\overline{\operatorname{co}}^{w^*}(N) = \overline{\operatorname{co}}(N)$$

for every weak-star closed $N \subseteq M$. If every $M \in eqc(E^*)$ has the Haydon property, then we say that so does E.

Lemma 10.9 Let $F \le E$ be a dense large subspace of E. If F has the Haydon property, then so does E.

Proof Suppose that *F* has the Haydon property, and let $M \in \text{eqc}(E^*)$. Consider the inclusion map $i: F \hookrightarrow E$ and its adjoint $i^*: E^* \to F^*$, the restriction map. By Lemma 2.7, i^* is a strong isomorphism. Moreover, i^* is weak-star continuous (Fact 2.6). Since *M* is weak-star compact, i^* is a closed map on *M*. Therefore:

$$i^*(\overline{co}^{w^*}(M)) = \overline{co}^{w^*}(i^*(M)) = \overline{co}(i^*(M)) = i^*(\overline{co}(M)).$$

Since i^* is a bijection (and therefore injective), we conclude that $\overline{co}^{w^*}(M) = \overline{co}(M)$. Note that we used the fact that *F* has the Haydon property.

Lemma 10.10 Let E be a locally convex space and let $M \subseteq E^*$ be a disked, equicontinuous, weak-star compact subset. If M satisfies Haydon's property, then it also satisfies (\mathbf{R}_1^*).

Proof By contradiction, suppose that M does not satisfy (\mathbf{R}_1^*) . By Lemma 6.8, there exist an embedding $T: V \to E$ where V is a dense normed subspace of l^1 and $\delta > 0$ such that $\delta B_{V^*} \subseteq T^*(M)$. Recall that l^1 does not satisfy the Haydon property. By virtue of Lemma 10.9, neither does V. By definition, there exists $N' \in eqc(B_{V^*})$ such that:

$$\overline{\operatorname{co}}^{w^*}(N') \supsetneq \overline{\operatorname{co}}(N').$$

Without loss of generality, we may assume that $N' \subseteq \delta B_{V^*}$. We know that $T^*(M) \supseteq \delta B_{V^*}$ and therefore

$$T^*(M) \supseteq \delta B_{V^*} \supseteq N'.$$

Define $N := M \cap ((T^*)^{-1}(N')) \subseteq M$. Then $T^*(N) = T^*(M) \cap N' = N'$. The adjoint T^* is strongly continuous and therefore:

$$T^*(\overline{\operatorname{co}} N) \subseteq \overline{\operatorname{co}}(T^*(N)).$$

Since T^* is also weak-star continuous (Fact 2.6), N is a closed subspace of M, and therefore equicontinuous and weak-star compact. Since T^* is a closed map over weak-star compact M, we get

$$T^*(\overline{\operatorname{co}} N) \subseteq \overline{\operatorname{co}}(T^*(N)) = \overline{\operatorname{co}}(N') \subsetneq \overline{\operatorname{co}}^{w^*}(N') = \overline{\operatorname{co}}^{w^*}(T^*(N)) = T^*(\overline{\operatorname{co}}^{w^*}N).$$

In particular, $T^*(\overline{\operatorname{co}} N) \subseteq T^*(\overline{\operatorname{co}}^{w^*} N)$, and therefore $\overline{\operatorname{co}} N \subseteq \overline{\operatorname{co}}^{w^*} N$. By definition, M does not satisfy Haydon's property. This contradiction completes the proof. п

The following is a locally convex analogue of the equivalence (1) \Leftrightarrow (2) in Lemma 2.27.

Proposition 10.11 Let E be a lcs.

- E is tame if and only if every $x^{**} \in E^{**}$ is weak-star fragmented over every $M \in \operatorname{eqc}(E^*).$
- a weak-star compact, equicontinuous $M \in eqc(E^*)$ is co-tame if and only if every $x^{**} \in E^{**}$ is weak-star fragmented on M;
- a bounded $B \subseteq E$ is tame on M if and only if every $x^{**} \in \overline{B}^{w^*}$ is weak-star fragmented on M (the closure here is taken with respect to the weak-star topology of $E \subseteq E^{**}$).

Proof First note that the second assertion follows from the third since every $x^{**} \in E^{**}$ is contained in the weak-star closure of some bounded $B \subseteq E$ ([52, Thm. 5.4 p. 143]).

The rest of the proposition is a consequence of Lemma 2.24.

Theorem 10.12 (Generalized Haydon Theorem) For a locally convex space E, the following are equivalent:

- (i) E is tame.
- (ii) *E* satisfies (\mathbf{R}_1) .
- (iii) Every weak-star compact, equicontinuous convex subset of E^* is the strong closed convex hull of its extreme points.
- (iv) For every weak-star compact, equicontinuous subset T of E^* , we have:

$$\overline{\operatorname{co}}^{w^*}(T) = \overline{\operatorname{co}}(T).$$

(Local version) Specifically, if $M \in eqc(E^*)$, then the following are equivalent:

(1) M is co-tame.

(2) *M* satisfies (\mathbf{R}_1^*) .

(3) For every weak-star closed, convex $N \subseteq \overline{\operatorname{acx}}^{w^*}(M)$ we have:

 $N = \overline{\operatorname{co}}(\operatorname{ext} N).$

(4) $\overline{\operatorname{acx}}^{w^*}(M)$ has the Haydon property. Explicitly, for every weak-star closed $N \subseteq \overline{\operatorname{acx}}^{w^*}(M)$, we have:

$$\overline{\operatorname{co}}^{w^*}(N) = \overline{\operatorname{co}}(N).$$

(5) Every $x^{**} \in E^{**}$ is a fragmented map over M.

Proof

(2) \Rightarrow (1) By definition, there is no l^1 -sequence with respect to ρ_M . Suppose that $B \subseteq E$ is bounded, and let $r: E \rightarrow C(M)$ be the restriction map. As a consequence, r(B) contains no l^1 sequences in C(M).

By Lemma 2.24, r(B) is tame on M. This is true for every bounded $B \subseteq E$ so M is indeed co-tame.

- (1) \Rightarrow (3) Proposition 10.7 and Lemma 7.12.
- (3) \Rightarrow (4) Suppose that $N \subseteq \overline{acx}^{w^*}(M)$ is a weak-star closed subset. As a consequence, it is also weak-star compact. Write $N' := \overline{co}^{w^*}(N)$. By [42, Lemma 9.4.5], the extreme points of the closed convex hull of a set lies in the closure of the original set. Thus:

$$\operatorname{ext} N' \subseteq \overline{N}^{w^*} = N.$$

Note that N' is a closed convex subset of $\overline{acx}^{w^*}(M)$ and therefore we can apply (3):

 $N' = \overline{\operatorname{co}}(\operatorname{ext} N') \subseteq \overline{\operatorname{co}}(N) \subseteq \overline{\operatorname{co}}^{w^*}(N) = N'.$

In particular, for every closed $N \subseteq \overline{\operatorname{acx}}^{w^*}(M)$,

$$\overline{\operatorname{co}}(N) = \overline{\operatorname{co}}^{w^*}(N),$$

as required.

- (4) \Rightarrow (2) By Lemma 10.10, $\overline{\operatorname{acx}}^{w^*}(M)$ satisfies (\mathbb{R}_1^*). It is easy to see that so does $N \subseteq \overline{\operatorname{acx}}^{w^*}(M)$.
- (1) \Leftrightarrow (5) Proposition 10.11.

Second part of Theorem 10.12 (for Banach spaces, in particular) gives a local version of Haydon's theorem involving co-tameness. There are some other localization results for Banach spaces. Among others: [20, Proposition 3.8], [47, Thm. 9 and 11], [50, Thm. 1] and [46, Theorem. 4].

11 Representations of group actions

Representations of dynamical systems on lcs. We apply properties of tame locally convex spaces to the theory of representations of dynamical systems. Let *G* be a topological group and $G \times X \rightarrow X$ be a continuous action of *G* on a topological space *X*. Then we say that *X* is a *G*-space. If, in addition, *X* is compact, then we say that *X* is a *dynamical G*-system. By

the approach of A. Köhler [30], a dynamical *G*-system *X* is said to be *tame* (*regular*, in the original terms of Köhler) if the orbit $fG = \{fg : g \in G\}$, as a family of functions on *X*, is tame. Or, equivalently, if *fG* is a tame subset in the Banach space C(X).

Theorems 11.4 and 11.5 establish the close relation between tame dynamical systems and representations on tame locally convex spaces. A compact *G*-system *X* is representable on a tame lcs iff (*G*, *X*) is tame as a dynamical system. Similarly, by Theorem 11.6, a compact *G*-system *X* is representable on an NP (reflexive) lcs *E* iff *X* is hereditarily nonsensitive (weakly almost periodic). Such results are well-known (see [14, 15, 34]) for *metrizable* tame (hereditarily nonsensitive, weakly almost periodic) *G*-systems with Rosenthal (Asplund, reflexive) Banach spaces *V*.

Recall that a (*proper*) representation of a *G*-space *X* on a Banach space $(V, || \cdot ||)$ is a pair (h, α) , where $h: G \to \text{Iso}(V)$ is a strongly continuous co-homomorphism and $\alpha: X \to V^*$ is a weak-star continuous bounded *G*-mapping (resp. *embedding*) with respect to the *dual action*

 $G \times V^* \to V^*, \ (g\varphi)(v) := \varphi(h(g)(v)) = \langle vg, \varphi \rangle = \langle v, g\varphi \rangle.$

Fact 11.1 Let X be a compact metrizable G-space. X admits a proper representation on

- (1) [34] a reflexive Banach space iff X is a WAP dynamical G-system;
- (2) [14] an Asplund Banach space iff X is a HNS dynamical G-system;
- (3) [15] a Rosenthal Banach space iff X is a tame dynamical G-system.

Many details about these results, as well as the definitions of HNS (hereditarily nonsensitive) and WAP (weakly almost periodic) dynamical systems, can be found in [18].

These results, in the framework of a unified approach representing "small subsets", appear in [36, 39].

In this section, we extend Fact 11.1 to nonmetrizable dynamical systems and suitable locally convex spaces. More definitions and remarks are in order. For every lcs E denote by GL(E) the group of all continuous linear automorphisms of V.

(1) The right action of G (induced by h: G → Iso (V)) on (V, ||·||) and the corresponding dual action on the dual Banach space V* are equicontinuous. Moreover, one may attempt to give a slightly more general definition. Namely, defining h: G → GL(V) as a co-homomorphism which is equicontinuous. Then one may modify the norm getting again the "isometric version" but under the new norm. Namely, define

$$||v||_{new} := \sup\{||vg|| : g \in G\}.$$

Since the action is (uniformly) equicontinuous, this norm generates the same topology.

(2) If X is compact then $\alpha(X)$ is weak-star compact in V*. Since V is a Banach space then $\alpha(X)$ is automatically bounded and equicontinuous.

Definition 11.2 By a (proper) *representation* of a *G*-space *X* on a lcs *E* we mean a pair (h, α) , where $h: G \to GL(E)$ is a strongly continuous co-homomorphism, h(G) is equicontinuous and $\alpha: X \to V^*$ is a weak-star continuous *G*-mapping (resp. embedding) with respect to the *dual action*

$$G \times E^* \to E^*, \ (g\varphi)(v) := \varphi(h(g)(v)) = \langle vg, \varphi \rangle = \langle v, g\varphi \rangle$$

such that $\alpha(X)$ is equicontinuous.

Remarks 11.3

- (1) The right action $E \times G \rightarrow E$ is continuous. It is equivalent to strong continuity of *h* (because the action is equicontinuous).
- (2) It is well-known that the dual action G × E^{*} → E^{*}, in general, is not continuous (where E^{*} carries its standard strong dual topology), even for Banach spaces and linear isometric actions. A sufficient condition is that E is an Asplund Banach space (see [33]). The same is not true for Rosenthal Banach spaces.
- (3) If E is barreled and X is compact, we can omit the condition that $\alpha(X)$ is equicontinuous.
- (4) It is easy to see that if X ∈ eqc (E*) is G-invariant, then the induced action G × X → X is continuous. For Banach spaces E it is well-known; see, for example, the proof in [35] (for the isometric representation).

Theorem 11.4 *Every compact tame dynamical G-system admits a proper representation on a tame lcs.*

Proof Let X be a compact tame dynamical G-system. As we already know by [15], X is Rosenthal-approximable. That is, there exists a G-embedding of X into a G-product $\prod_{i \in I} X_i$ of Rosenthal-representable G-systems X_i . Let (h_i, α_i) be a proper representation of (G, X_i) on a Rosenthal Banach space V_i . Then the lcs direct sum $V := \bigoplus_{i \in I} V_i$ is a tame lcs according to Theorem 5.5.4.

Indeed, first of all note that algebraically the dual V^* is the product $\prod_{i \in I} V_i^*$ with the corresponding duality

$$V \times V^* \to \mathbb{R}, \ (v, u) \mapsto \sum_{i \in I} \langle v_i, u_i \rangle.$$

Furthermore, the compact space $\prod_{i \in I} X_i$ naturally is embedded into V^* with the weak-star topology. One of the main steps here is to show that $\prod_{i \in I} X_i$ is an equicontinuous subset of V^* . This follows by Remark 3.11.2.

We have a naturally defined coordinate-wise linear action of G on $V = \bigoplus_{i \in I} V_i$. Using the above mentioned description of the topology on $V = \bigoplus_{i \in I} V_i$, it is easy to show that this action is equicontinuous. So, we have an equicontinuous strongly continuous co-homomorphism $h: G \to GL(V)$. Finally, observe that the embedding $\alpha: \prod_{i \in I} X_i \hookrightarrow V^*$ is weak-star continuous and equivariant.

Note that if in Theorem 11.4 the compact G-space X in addition is metrizable, then we can suppose that V is a Banach space. However, it is not true if X is not metrizable (even for trivial G-actions).

Theorem 11.5 Every compact G-space X which admits a proper representation on a tame lcs is tame as a dynamical system.

Proof Let (h, α) be a proper representation of the *G*-system *X* on a tame lcs *E*, where $h: G \to GL(E)$ is a strongly continuous co-homomorphism, h(G) is equicontinuous and $\alpha: X \to V^*$ is a weak-star continuous *G*-embedding with respect to the *dual action* $E^* \times G \to E^*$, such that $\alpha(X)$ is equicontinuous. For every $v \in V$, we have the induced continuous function

$$f_v: X \to \mathbb{R}, x \mapsto \langle v, \alpha(x) \rangle.$$

Then the orbit vG is bounded because h(G) is an equicontinuous subset of GL(E). Since E is a tame lcs, its bounded subsets are tame. Hence, vG is a tame family for every equicontinuous

compact subset in the dual. In particular, it is true for $\alpha(X)$. This implies that the orbit $f_v G$ of f_v is a tame family on X. Therefore, f_v is a tame function on X (in the sense of [15]). Since V separates the points of $\alpha(X)$, we obtain that Tame(X) separates points of X. This means that the G-system X is tame in the sense of [15].

Similarly, making use of Remark 3.11, the following theorem can be proved.

Theorem 11.6 A compact dynamical G-system X is representable on

(1) $E \in (\mathbf{NP})$ if and only if (G, X) is hereditarily nonsensitive;

(2) $E \in (\mathbf{DLP})$ if and only if (G, X) is weakly almost periodic.

Note that, like (**T**), the classes (**NP**), (**DLP**) and reflexive lcs also are closed under lc direct sums. So, we can assume in (2) that E is a reflexive lcs. For brevity, we omit the details.

Acknowledgements We are very grateful to Saak Gabriyelyan for many important suggestions and corrections. We thank the referees for their useful contribution.

Funding This research was supported by a grant of the Israel Science Foundation (ISF 1194/19).

Data Availability Data sharing is not applicable as no datasets were generated or analyzed during the current study.

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