TOPOLOGICAL GROUPS AND TELEMAN'S THEOREM

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Abstract. We present the second part of Teleman's Theorem and Uspenkij Theorem and their proofs.

Contents

Date: May 22, 2006.

1. Introduction

In this paper, we will continue to discuss the subject of topological groups, and the way they act on topological spaces. The background to this paper was given in the last lecture, in which the first theorem of Teleman about embedding G-SPACES in the compact unit ball was given and proved.We will prove the second part of Teleman's Theorem and Uspenskij's Theorem which talk about the universality of $[0,1]^{\mathbb{N}}$

2. Background

2.1. **Banach Space.** Banach spaces are defined as complete normed vector spaces. This means that a Banach space is a vector space V over the real or complex numbers with a norm $||\cdot||$ such that every Cauchy sequence (with respect to the metric $d(x,y) = ||x-y||$) in V has a limit in V. Since the norm induces a topology on the vector space, a Banach space provides an example of a topological vector space.

2.2. **Hausdorff Space.** Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint . X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. This is why Hausdorff spaces are also called T2 spaces or separated spaces.

2.3. **Regular Space.** Suppose that X is a topological space. X is a regular space iff, given any closed set F and any point x that does not belong to F, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint. In fancier terms, this condition says that x and F can be separated by neighbourhoods. X is a T3 space if and only if it is both regular and Hausdorff.

2.4. **Completely Regular Space.** Suppose that X is a topological space. X is a completely regular space iff, given any closed set F and any point x that does not belong to F, there is a continuous function f from X to the real line R such that f(x) is 0 and f(y) is 1 for every y in F. In fancier terms, this condition says that x and F can be separated by a function.

2.5. **Tychonoff Space.** Suppose that X is a topological space. X is a Tychonoff space, or T3 space if and only if it is both completely regular and Hausdorff.

2.6. **Weak* Topology.** One may also define a weak* topology on $X^*, X \in Banach$ by requiring that it be the weakest topology such that for every x in X,the substitution map

$$\Phi_x : X^* \to \mathbb{R} \text{ or } \mathbb{C}$$

defined by

$$\Phi_x(\phi) = \phi(x) \text{ for all } \phi \in X^*.$$

bases

$$[\varphi_0, x_0, \varepsilon] \qquad \varepsilon > 0, x_0 \in X, \varphi_0 \in X^*$$

$$O_{[\varphi_0,x_0,\varepsilon]} := \{\varphi \in X^* : |\varphi(x_0) - \varphi_0(x_0)| < \varepsilon\} \subset X^*$$

2.7. **Banach-Alaoglu theorem.** (also known as Baurbaki Alaoglu's theorem) states that the closed unit ball of the dual space of a normed vector space is compact in the weak* topology. A common proof identifies the unit ball with the weak* topology as a closed subset of a product of compact sets with the product topology. As a consequence of Tychonoff's theorem, this product, and hence the unit ball within, is compact (The unit ball is closed in this topology, and a closed set in a compact space is compact.
Remark Every compact space can be embedded in V^*.

2.8. **Gelfand Map.** Suppose that X is a topological space.The map γ_x is identified as below

$$\gamma_x(f) = f(x), \quad x \in X, f \in C(X)$$

.

2.9. **G-Space.** We will be only considering Hausdorff topological groups, and Tychonoff topological spaces. A topological group G *acts* on a topological space X if there is a continuous mapping $\tau\colon G \times X \to X$, called an *action,* where the image of a pair $(g, x) \in G \times X$ is usually denoted either by $\tau_g x$ or simply by $g \cdot x$, having the properties that $g \cdot (h \cdot x) = (gh) \cdot x$ and $e_G \cdot x = x$ for every $g, h \in G$ and $x \in X$. Here and in the sequel, $e = e_G$ denotes the identity element of the acting group. The entire triple $\mathfrak{X} = (X, G, \tau)$ is called an (abstract) *topological dynamical system,* while X together with the action τ is referred to as a *G-space.* The triple $\mathfrak{X}$ is also known under the name a *topological transformation group.*

2.10. **Uniform continuous functions.** $C_\pi(G) = UC(G)$ 2.16 Uniform continuous functions in G spaces .
for example, if $G = \mathbb{R}$, $f \in UC(G)$ if

$$\text{for every } \varepsilon > 0, \text{ there exist } \delta > 0, s.t. |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

equals to :

$$\forall \varepsilon > 0, \text{ there exist a neighbourhood of 0, U} : |f(x+g) - f(x)| < \varepsilon$$
$$\forall x \in \mathbb{R} \text{ and } g \in U$$

2.11. **Separable space.** A topological space is called separable if it contains a countable dense subset. (There is a set with a countable number of elements whose closure is the entire space).

2.12. **Metrizable space.** A metrizable space is a topological space that is homeomorphic to a metric space. That is, a topological space (X, τ) is said to be metrizable if there is a metric $d : X \to [0, \infty)$.

2.13. **Second-countable space.** A second-countable space is a topological space satisfying the "second axiom of countability". Specifically, a space is said to be second-countable if its topology has a countable base.
For example, Euclidean space ($\mathbb{R}^n$) with its usual topology is second-countable.Although the usual base of open balls is not countable, one can restrict to the set of all open balls with rational radii and whose centers have rational coordinates. This restricted set is countable and still forms a base.

2.14. **Topological Base.** A base (or basis) B for a topological space X with topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of B.

2.15. **Topological dimension.** Topological dimension of a topological space is defined to be the minimum value of n, such that any open cover has a refinement in which no point is included in more than n+1 elements. We say that $dimX = 0$ if there is a base to the topology so that every group in the base is both close and open.

2.16. **Brook's Lemma.** For every Hausdorff topological group the algebras $C_\pi(G)$ and $UC(G)$ of (right) uniformly continuous functions are the same.

2.17. **Hilbert Cube.** *It is named after the German mathematician David Hilbert.*

Hilbert cube $I^{\aleph_0}$ may be defined as the product of countably infinitely many copies of the unit interval [0,1]. That is, it is the cube of countably infinite dimension 2.13. As a product of compact Hausdorff spaces, the Hilbert cube is itself a compact Hausdorff space as a result of the Tychonoff theorem (product of any collection of compact topological spaces is compact).
Since l_2 is not locally compact, no point has a compact neighbourhood, so one might expect that all of the compact subsets are finite-dimensional. The Hilbert cube shows that this is not the case. But the Hilbert cube fails to be a neighbourhood of any point p because its side becomes smaller and smaller in each dimension, so that an open ball around p of any fixed radius $e > 0$ must go outside the cube in some dimension.

3. Teleman's Theorem II

Reminder: Teleman's Theorem I. For every compact G-space X, there is a G-embedding onto a compact space (B^*, w^*) for some Banach space V.

3.1. **Teleman's Theorem II.** For every topological group G the next arguments are true and equivalent:

(1) There exist a Banach space V so that $G \hookrightarrow IS_L(V)$
(2) There exist a Compact space X so that $G \hookrightarrow Homeo(X)$

Proof. Let G be a topological group. $V := C_\pi(G) = UC(G)$. According to Brook's lemma 2.16, there is a sufficient number of functions in $C_\pi(G)$ so that we can separate points and close sets. $V := C_\pi(G)$ is a close sub-Algebra which is $G - Invarient$ of $C(G)$. This is closed linear sub-set of a Banach space,there for it is also a Banach space. As we recall, G acts on $C(G)$ with left shift. We got an action $V \times G \to V$ which is well defined by $(g, f) \to f(g^{-1}(x))$.
Is this actions continuous?

Lemma 3.1.1. *Let G be a group which acts on metric space M with isometries $\alpha : G \times M \to M$. For every $g \in G$ there is a shift $\tilde{g}$ which is an isometric of M, then the next claims are equivalent:*

(1) *the action is continues for every $m \in M$.*

(2) *Let $m \in M$, there is a function $\tilde{m} : G \to M$ defined by $\tilde{m}(g) = gm$.*

In our proof we will look at $C(G)$ with the *supremom* metric (we will ignore function for which the orbit function isn't continues). We got a function

$$h : G \to Is(V), g \to \tilde{g}$$

We will show that this function topologically embeds G in $Is(V)$.

(1) This function is well defined because $\tilde{g}(f) = gf(x)$
(2) This is homeomorphism. For every $g_1, g_2 \in G, g_1 \neq g_2$, therefor

$$f(g_1) \neq f(g_2) \Rightarrow h(g_1) \neq h(g_2)$$

We proved that $G \hookrightarrow$ in $Is(V)$. Now we will show that the second part of the theorem.
We will defined $X := B^* \in Comp$ (with the weak* topology 2.6).

We defined

$$Is(V) \hookrightarrow Homeo(B^*) \text{ by } \psi : \tilde{g} \to \hat{g}$$

where $\hat{g} : B^* \to B^*$ homeomorphism.
We got that $G \hookrightarrow Is(V) \hookrightarrow Homeo(B^*) \simeq [0,1]^{\mathbb{N}}$ By $g \to \tilde{g} \to \hat{g}$

4. Universality and Uspenskij Theorem

4.1. **Definition :Universality of a group.** There is a topological group namely $Homeo(I^{\aleph_0}) = U$ such that for every second countable topological group G there exists a topological subgroup G' of U such that $G \sim G'$ (topologically Isomorphism)

Examples:

(1) $\mathbf{C} := \{0,1\}^{\mathbb{N}}$ This is "Cantor cube". Universal 4.1 for Metric and separable spaces with dim=0 2.15. (Is also homogeneous and compact.)
(2) $\mathbf{B} := \mathbb{N}^{\mathbb{N}}$ (homeomorphic to $\mathbb{R} - \mathbb{Q}$) is universal for metrizabel spaces with complete metric which are also separable and with dim=0. Those spaces are homeomorphic to close sub set of $\mathbf{B}$.

4.2. **Uspenskij Theorem.** Every topological group G which is metrezabil 2.12 and Separable 2.11 there exist embedding of topological groups so that $G \hookrightarrow Homeo([0,1]^{\mathbb{N}})$ in other words: $Homeo([0,1]^N)$ is universal4.1 second-countable 2.13 topological group.

Before we prove that theorem, we will look at the next one:

Theorem 4.2.1. ***Keller's Theorem*** *Every convex closed subset in a separable Banach space is homeomorphic to the Hilbert cube, and therefore is second countable.*

Proof. We will consider the action of the group G on itself (left translation) $G \times G \to G$. There exists a $G - Compactification$ of the $G - SpaceG$.according to Brook's lemma 2.16, $C_\pi(G) = UC(G)$ and there are sufficient functions in order to separate points and close sets. There exists a $G - AlgebraA$ in $UC(G)$ such that A is separable.

The compactification that suits A is second countable $i : G \hookrightarrow Y \subseteq B^* \in C_\pi$. i Is $G - embedding$. We will defined $B^* := B(A)$ with the weak topology 2.6.
We got that

$$\begin{array}{ccc} G \times B^* & \rightarrow & B^* \\ \uparrow & & \uparrow \\ G \times G & \rightarrow & G \end{array}$$

Eventually :

$$G \hookrightarrow Homeo(B^*) \cong Homeo([0,1]^{\mathbb{N}})$$

Remark 4.2.2. The same idea was shown more generally by Megrelishvili: Every compact topological G-space X, where X and G are both second-countable 2.13, can be embedded into the pair $(Homeo([0,1]^{\mathbb{N}}), [0,1]^{\mathbb{N}})$

Another example of a universal group with countable base is shown next:

4.3. **Urysohn metric spaces and their group of isometries.**

Definition 4.3.1. Urysohn space A metric space M is called an Urysohn space if for every finite metric space X and every finite subspace Y of X, every isometric embedding $Y \hookrightarrow M$ can be extended to an isometric embedding of $X \hookrightarrow M$.

Every separable Urysohn metric space M contains an isometric copy of every other separable metric space, and if M is in addition complete, it is unique up to isometry; we will denote it by $\mathbb{U}$.

Definition 4.3.2. n-homogeneous space A metric space X is called $n-homogeneous$, where n is a natural number, if every isometry between two subspaces of X ,containing at most n elements each, extends to an isometry of X onto itself. If X is $n - homogeneous$ for every natural n, then it is said to be $\omega - homogeneous$.
The complete separable Urysohn space $\mathbb{U}$ is $\omega - homogeneous$ and moreover enjoys the stronger property: every isometry between two compact subspaces of X extends to an isometry of X onto itself. At the same time, nonseparable Urysohn metric spaces need not have this property.

Theorem 4.3.3. ***Uspenskij, 1990*** *The topological group $Iso(\mathbb{U})_s$ is a universal second-countable 2.13 group.*

A more complicated argument, was established by Uspenskij 8 years later:

Definition 4.3.4. Weight The weight $w(X)$ of a topological space X is defined to be $\tau(X)\aleph_0$, where $\tau(X)$ denotes the minimal cardinality of a base for X.

Theorem 4.3.5. *(Uspenskij, 1998) Every topological group G embeds as a topological subgroup into the isometry group of a suitable generalized Urysohn space M that is $\omega - homogeneous$ and has the same weight as G.*