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INTRODUCTION TO TOPOLOGY (SOME ADDITIONAL BASIC EXERCISES)

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Abstract. We provide some additional exercises in the course Topology-8822205 (Bar-Ilan University). We are going to update this file several times (during the current semester).

Contents

1.	Metric spaces	1
2.	Topological spaces	5
3.	Topological products	8
4.	Compactness	9
5.	Quotients	11
6.	Hints and solutions	12
References		25

1. Metric spaces

Exercise 1.1. *Give an example of a metric space (X, d) containing two balls such that*

$$B(a_1, r_1) \subsetneq B(a_2, r_2) \text{ but } r_2 < r_1.$$

Hint: think about metric subspaces of $\mathbb{R}$.

Exercise 1.2. *Let (X, d) be a metric space and*

$$0 < r + d(a, b) < R.$$

Prove that $B(b, r) \subseteq B(a, R)$. Conclude that the ball $B(a, R)$ is an open subset of X.

Exercise 1.3. *Let X be a set. Define*

$$d_\Delta(x, y) := \begin{cases} 0 & \text{for } x = y \\ 1 & \text{for } x \neq y \end{cases}$$

Show that (X, d_Δ) is an ultrametric space and describe the balls and spheres according to their radii.

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Exercise 1.4. *Find non-isometric metric spaces X, Y such that there exist isometric embeddings $f : X \hookrightarrow Y$, $g : Y \hookrightarrow X$.*

Exercise 1.5. *Let $(V, ||\cdot||)$ be a normed space. Show that every translation*

$$T_z : V \to V,\ T_z(x) = z + x$$

is an isometry. Conclude that all open balls $B(a, r)$ with the same radius r (and $a \in V$) are isometric.

Exercise 1.6. *Give geometric descriptions of $B[v, r], B(v, r), S(v, r)$ in $\mathbb{R}^2$ for $v = (1, 2)$, $r = 3$ with respect to the following metrics: (a) Euclidean d; (b) d_1; (c) d_{max}.*

Exercise 1.7.

(1) *Consider the normed spaces $(C[0, 1], ||\cdot||_{max})$ and $(C[0, 1], ||\cdot||_1)$. Give an intuitive description of the "closed ball" $B_{max}[\theta, 3]$, $B_1[\theta, 3]$ with radius $r = 3$ and the center in the zero function $\theta : [0, 1] \to \mathbb{R}, x \mapsto 0$.*
(2) *Explain theoretically why $B_1[\theta, 3] \subset B_{max}[\theta, 3]$.*

Exercise 1.8. *Let (X, d) be a pseudometric space. Prove that the following conditions are equivalent:*

(1) *(X, d) is a metric space.*
(2) *Every finite subset $F \subset X$ is closed.*
(3) *For every converging sequence $f : \mathbb{N} \to X$, $x_n := f(n)$ the limit $\lim\limits_{n\to\infty} x_n$ is unique.*
(4) *(X, d) satisfies the Hausdorff property.*
(5) *$\cap\{B(a, r) |\ r > 0\} = \{a\}$.*

Exercise 1.9. ***(p-adic metric on $\mathbb{Z}$)***

For every given prime p define on the set $\mathbb{Z}$ of all integers the following metric

$$d_p(x, y) := \begin{cases} 0 & \text{for } x = y \\ \frac{1}{p^k} & \text{for } k = k(x, y) = \max\{i :\ p^i | (x - y)\} \end{cases}$$

Show that

(1) *Is an ultrametric and* $\operatorname{diam}(\mathbb{Z}, d_p) = 1$.
Hint: Observe that
$$k(x, z) \geq \min\{k(x, y), k(y, z)\}.$$
(2) *Every translation $T_z : \mathbb{Z} \to \mathbb{Z}$, $T_z(x) = z + x$ is an isometry.*
(3) *$\lim\limits_{n\to\infty} p^n = 0$. Conclude: $(\mathbb{Z}, d_p)$ does not contain isolated points.*
(4) *Every ball $B(0, r)$ (center is 0) is a subgroup of $(\mathbb{Z}, +)$. Every ball $B(a, r)$ is clopen.*
(5) *For every ball $B(a, r)$ and every $b \in B(a, r)$ we have $B(a, r) = B(b, r)$.*
That is, every point inside a ball is its center ! Try to generalize to every ultrametric space.
Remark: see the schematic pictures of $(\mathbb{Z}, d_3)$ and $(\mathbb{Z}, d_7)$ Figures 1 and 2.
(6) *Describe the "next stage" in the picture for $(\mathbb{Z}, d_3)$.*

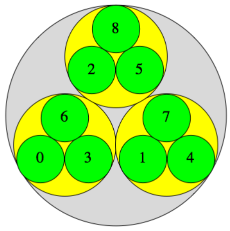

FIGURE 1. p=3

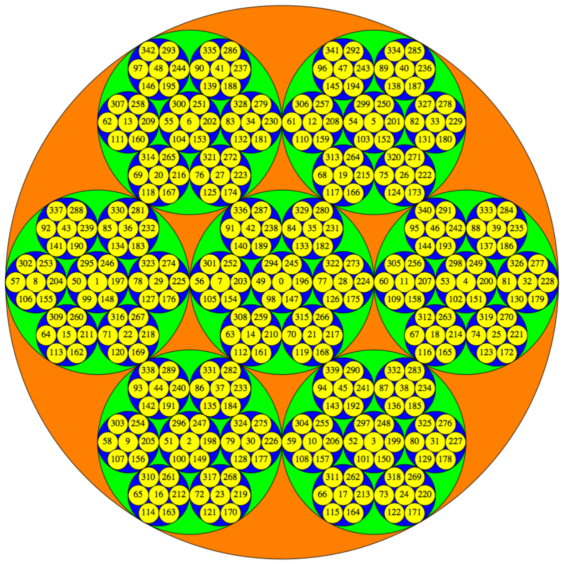

FIGURE 2. p=7

Exercise 1.10. ***(Cantor cube)*** *Let*

$$X = \{0,1\}^{\mathbb{N}} = \{x = (x_1, x_2, \cdots)| \quad x_k \in \{0,1\}\}$$

be the set of all binary sequences. Define $d : X \times X \to [0, \infty)$ *as*

$$d(x,y) := \begin{cases} 0 & \text{for } x = y \\ \frac{1}{k} & \text{for } k = k(x,y) = \min\{i : \; x_i \neq y_i\} \end{cases}$$

Show that

(1) d *is an ultrametric and* $\operatorname{diam}(X,d) = 1$.
Hint: Observe that

$$k(x,z) \geq \min\{k(x,y), k(y,z)\}$$

(2) *Give an example of a converging sequence with distinct members (hence, is not eventually constant).*
(3) *Every ball* $B(\theta, r)$ *(center is the zero-sequence* $\theta := (0,0,\cdots)$*) is a subgroup of the group* $(\{0,1\}^{\mathbb{N}}, +)$ *(where the sum* $+$ *of sequences is "modulo 2").*

Exercise 1.11. ***(Hilbert space*** l_2***)***

Define

$$l_2 := \{x = (x_1, x_2, \cdots)| \;\; x_i \in \mathbb{R} \quad \sum_{i=0}^{\infty} x_i^2 < \infty\}$$

$$||x|| := \sqrt{\sum_{i=0}^{\infty} x_i^2}$$

Then l_2 *is a normed space and* $\langle x, y\rangle := \sum_{i=0}^{\infty} x_i y_i$ *is an inner (scalar) product.*

Show that

(1) *There exists a sequence in* l_2 *which has no limit but converges coordinate-wise (in contrast to the Euclidean space* $\mathbb{R}^n$*).*

(2) *There exist isometric embeddings* $\mathbb{R}^n \hookrightarrow l_2$, $(\mathbb{N}, d_\Delta) \hookrightarrow l_2$.

Recall that $d_\Delta(x,y) := \begin{cases} 0 & \text{for } x = y \\ 1 & \text{for } x \neq y \end{cases}$

(3) *Find a linear isometric embedding* $f : l_2 \to l_2$ *which is not onto.*

(4) * *There exists a metric space* M *with 4 elements which cannot be embedded isometrically into* l_2 *(and hence into any* $\mathbb{R}^n$*). Show also that 4 is the minimum possible natural number with above property.*
Hint: when $d(x,z) = d(x,y) + d(y,z)$ *in* $\mathbb{R}^2$, $\mathbb{R}^n$ *?*

Exercise 1.12. *Let* (M, d) *be a metric space. Show that for every closed subset* $A \subseteq M$ *there exists a sequence* O_n *of open subsets in* M *such that* $A = \cap_n O_n$.

Exercise 1.13. *Show that the p-adic metric space* $(\mathbb{Z}, d_p)$ *(see Exercise 1.9) is not complete. What about the Cantor cube (from Exercise 1.10) ?*

Exercise 1.14. *Show that the normed space* $(C[-1,1], ||\cdot||_1)$ *defined by*

$$||f||_1 = \int_{-1}^{1} |f(x)|dx$$

is not a Banach space.

Exercise 1.15. *Give an example of a* ***linear map*** $f : E_1 \to E_2$ *between two normed spaces which is not continuous.*

Exercise 1.16. *(Cantor Set)*

Let C *be the set of real numbers that are sums of series of the form*

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \quad \text{where} \;\; k \in \{0, 2\}.$$

In other words, C *consists of the real numbers that have the form* $0.a_1a_2...a_k...$ *without the digit 1* ***in the number system with base 3****. Prove that*

(1) C *is contained in* $[0,1]$.

(2) C *does not meet* $(\frac{1}{3}, \frac{2}{3})$.

(3) C *does not meet* $(\frac{3s+1}{3^k}, \frac{3s+2}{3^k})$ *for every* $s, k \in \mathbb{N}$.

(4) *Find a geometric description of* C
(Hint: removing from $[0,1]$ *countably many open intervals).*

Remark: See for example [1], [5] *or/and wikipedia.*

Totally bounded metric spaces. A metric space (M,d) is said to be *totally bounded* (or, *precompact*) if for every $\varepsilon > 0$ there exists a **finite** subset A_ε of X such that A_ε is ε-dense in (M,d). A subset Y of M is said to be totally bounded if the metric subspace (Y,d_Y) is totally bounded.

Exercise 1.17. *Let (M,d) be a metric space.*

(1) *Every finite subset is totally bounded;*
(2) *Every totally bounded subset is bounded;*
(3) *Finite union of totally bounded subsets is totally bounded;*
(4) *If X is a totally bounded subset of M then every subset Y of X is also totally bounded.*
(5) *The closure $cl(Y)$ of a totally bounded subset Y is also totally bounded;*

Exercise 1.18.

(1) *Let (X,d) be a complete metric space and $Y \subset X$ is a closed subset. Then the metric subspace (Y,d_Y) is also complete.*
(2) *Let (Y,d_Y) is a metric subspace of (X,d). Show that if (Y,d_Y) is complete then Y is closed in X.*

2. Topological spaces

Exercise 2.1. *Prove that for every pseudometric space (X,d) the pair $(X,top(d))$ is a topological space.*

Exercise 2.2. *Prove that any intersection $\cap_i \tau_i$ of topologies τ_i on the same set X is a topology. Show that it is not true, in general, for unions.*

Exercise 2.3. *Let (X,τ) be a topological space. Show that the following conditions are equivalent:*

(1) $(X,\tau) \in T_1$*;*
(2) *Every singleton $\{a\}$ is closed;*
(3) *Every finite subset $F \subset X$ is closed;*
(4) $\tau_{cofin} \subset \tau$.

Exercise 2.4. *On $\mathbb{Z}$ define the following family $\tau_\leq$ of subsets*

$$\tau_\leq := \{\emptyset, \mathbb{Z}, O_k : k \in \mathbb{Z}\} \text{ where } O_k := \{x \in \mathbb{Z} : x \leq k\}.$$

Show that:

(1) $(\mathbb{Z},\tau_\leq)$ *is a connected topological space with property T_0.*
(2) *It is not (pseudo)metrizable.*
(3) *Every continuous map $(\mathbb{Z},\tau_\leq) \to \mathbb{R}$ into the reals is constant.*

Exercise 2.5. *On $\mathbb{Z}$ describe the smallest topology τ_{cofin} with the following property: every singleton $\{a\}$ is closed in $\mathbb{Z}$. Show that $(\mathbb{Z},\tau_{cofin})$ is connected, not Hausdorff and not (pseudo)metrizable.*

Exercise 2.6. *(generalized subspace topology) Let (X,τ) be a topological space and $f : Y \to X$ be a function. Define*

$$\tau_{f,Y} := \{f^{-1}(O) : \ O \in \tau\}.$$

Prove that $(Y, \tau_{f,Y})$ is a topological space.
Remark: In the case of a subset $Y \subset X$ and the inclusion $f = in : Y \to X$ we get the subspace topology on Y.

Exercise 2.7. *(heredity of the continuity)*
Let $f : X \to Y$ be a continuous map between topological spaces and $X_1 \subset X, Y_1 \subset Y$ be subspaces such that $f(X_1) \subset Y_1$. Show the continuity of the following induced map

$$f_* : X_1 \to Y_1, \quad x \mapsto f(x).$$

Conclude that, in particular, the following induced maps are continuous:

(1) $Y \to Z,\ y \mapsto f(y)$
(2) $Y \to f(Y),\ y \mapsto f(y)$

are also continuous.
Conclude also that if $f : X \to Z$ is a homeomorphism then also the full restriction $f^*_Y : Y \to f(Y)$ is a homeomorphism.

Exercise 2.8. *Show that for any topological space Y the following maps are always continuous:*

(1) $(X, \tau_{discr}) \to Y$.
(2) $Y \to (X, \tau_{triv})$.

Exercise 2.9. *Show that the following conditions are equivalent:*

(1) *X is not connected.*
(2) *There exists a clopen subset $A \subset X$ such that $\emptyset \neq A \neq X$.*
(3) *There exists a continuous map $f : X \to \mathbb{R}$ such that $f(X)$ is a two point set.*

Exercise 2.10. *Show (by giving corresponding examples) that the separation properties T_i, $i \in \{0, 1, 2, 3\}$ are not, in general, preserved by continuous onto maps.*

Exercise 2.11. *Which topological properties from the following list are hereditary:*

(1) *connectedness*
(2) T_i, $i \in \{0, 1, 2, 3\}$
(3) *compactness*
(4) B_2
(5) B_1
(6) *Metrizability.*

Remark: a property is said to be hereditary if every topological subspace inherits it.

Exercise 2.12. *Let $X = \mathbb{R} \cup \{p\}$, where $p \notin \mathbb{R}$. Define*

$$\tau := \{O \subset X \mid p \in O \Rightarrow X \setminus O \text{ is countable}\}.$$

Show that

(1) *(X, τ) is a Hausdorff topological space.*
(2) *There exists $A \subset X$ such that $scl(A) \neq cl(A)$.*
(3) *(X, τ) is not metrizable.*
(4) *(X, τ) is a normal topological space. That is, $(X, \tau) \in T_4$.*

Exercise 2.13. *Let C_1, C_2 be closed subsets of X and $f : X \to Y$ be a function such that $X = C_1 \cup C_2$ and the restrictions $f_{C_1} : C_1 \to Y$ and $f_{C_1} : C_1 \to Y$ are continuous (on subspaces C_1 and C_2, respectively). Show that f is continuous. Conclude that* *this remains true for FINITELY many closed subspaces (but not for infinitely many). Show also that it is not true in general if C_1, or C_2 is not closed.*

Exercise 2.14. *Let f_1, f_2 be two continuous functions $X \to Y$, where Y is a Hausdorff space, A be a dense subset in X such that $f_1(a) = f_2(a)$ for every $a \in A$. Show that $f_1(x) = f_2(x)$ for every $x \in X$.*

Exercise 2.15. *(H. Furstenberg's proof of the infinitude of primes)*

Define a topology τ_F on the integers $\mathbb{Z}$, by declaring a subset $U \subseteq Z$ to be an open set if and only if it is either the empty set, $\emptyset$ or it is a union of arithmetic sequences $S(a, b) := a\mathbb{Z} + b$ (for $a \neq 0$). In other words, U is open if and only if every $x \in U$ admits some non-zero integer a such that $S(a, x) \subseteq U$. Prove:

1. *$(\mathbb{Z}, \tau_F)$ is a topological space.*
2. *Finite nonempty subset of this topological space is not open.*
3. *$S(a, b)$ are clopen.*
4. ** Using (1,2,3) conclude that there are infinitely many prime numbers.*
   Hint: use the equality $\mathbb{Z} \setminus \{-1, 1\} = \cup\{S(p, 0) : \ p \text{ is prime}\}$.

Exercise 2.16. *Prove or disprove:*

1. *Let $A \subset B \subset X$. If A is dense in B and B is dense in X then A is dense in X.*
2. *Finite intersection $\cap_{i=1}^n A_i$ of dense subsets A_i in X is dense in X.*
3. *Finite intersection $\cap_{i=1}^n O_i$ of open dense subsets O_i in X is dense in X.*
4. *Countable intersection $\cap_{i \in \mathbb{N}} O_i$ of open dense subsets O_i in X is dense in X.*

Exercise 2.17. *Let $\partial(A) = \bar{A} \setminus int(A)$ be the boundary of a subset $A \subset X$ in a topological space X. Prove the following formula*

$$(\partial(A))^c = int(A^c) \cup int(A).$$

Exercise 2.18. *Let A be a subset of a topological space X. Define its indicator function*

$$\xi_A : X \to \{0, 1\}, \ \xi(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Show that the points of discontinuity of this function is just the boundary $\partial(A)$ of A. Conclude that ξ_A is continuous if and only if A is clopen.

Exercise 2.19. *(Sorgenfrey line)*

Let $X = \mathbb{R}$. Define τ_s as follows:

$$\tau_s := \{O \subset \mathbb{R} : \ x \in O \Rightarrow \exists \varepsilon > 0 \ [x, x + \varepsilon) \subseteq O]\}$$

Show that

1. *$(\mathbb{R}, \tau_s)$ is a Hausdorff topological space (notation: $\mathbb{R}_s$).*
2. *$\{[a, b) : \ a, b \in \mathbb{R}\}$ is a basis of τ_s and $\dim \mathbb{R}_s = 0$.*
3. *$top(d) \subsetneq \tau_s$, where $top(d)$ is the usual topology on $\mathbb{R}$.*
4. *$\mathbb{R}_s$ is separable.*
5. *$\mathbb{R}_s \notin \boldsymbol{B}_2$ (i.e., there is no countable topological basis).*
6. *$\mathbb{R}_s$ is not metrizable.*

Exercise 2.20.

(1) *Characterize all intervals of* $\mathbb{R}$ *up to the homeomorphisms.*
(2) *For which intervals* X, Y *of* $\mathbb{R}$ *there exists a continuous onto function* $f : X \to Y$ *?*

Exercise 2.21.

(1) *Characterize all digits* $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ *in sans serif font up to the homeomorphisms.*
(2) *Characterize all capital letters of the English alphabet in sans serif font up to the homeomorphisms.*

Remark: ***sans serif font*** *is a font without additional "calligraphic tails". See* ***http://bueler.github.io/M404F16/letters.pdf***

Exercise 2.22.

(1) *Show that every nonempty open subset* O *of* $\mathbb{R}^n$ *is locally connected (but not necessarily connected).*
(2) ** Show that there exists a subspace of* $\mathbb{R}^2$ *which is (pathwise) connected but not locally connected.*

Remark: X is *locally connected* means that for every point $a \in X$ and every neighborhood $U \in N(a)$ there exists a neighborhood $V \in N(a)$ such that V is connected. That is, if there exists a local basis at every $a \in X$ which contains only connected members.

3. Topological products

Exercise 3.1. *Let* X *be a topological space. Prove that* X *is Hausdorff if and only if the diagonal is closed in* $X \times X$.

Exercise 3.2. *Let* $X = \prod_{i \in I} X_i$ *be a topological product. Prove that:*

(1) $X \in T_1$ *if and only if* $X_i \in T_1,\ \forall i \in I$.
(2) $X \in T_2$ *if and only if* $X_i \in T_2,\ \forall i \in I$.
(3) *If* A_i *is closed in* X_i *then* $\prod_{i \in I} A_i$ *is closed in* X.

Exercise 3.3. ** Let* $f : X \to Y$ *be a continuous function. Prove:*

(1) *The graph of this function* $Gr(f)$ *is homeomorphic to* X.
(2) *If, in addition,* Y *is Hausdorff then* $Gr(f)$ *is closed in* $X \times Y$.

Exercise 3.4. ** Show that the topological product* $\mathbb{R}_s \times \mathbb{R}_s$ *(called Sorgenfrey plane) is a separable space and contains a discrete subspace of cardinality* $|\mathbb{R}|$. *Conclude that the separability is not a heredity property in Hausdorff spaces.*

Exercise 3.5. *Let* α *be a prebase (that is, subbase) of a topological space* (Y, τ) *and* $f\colon X \to Y$ *be a map. Prove that* f *is continuous if and only if* $f^{-1}(O)$ *is open in* X *for every* $O \in \alpha$.

Exercise 3.6. *Let* $f_i : Y \to X_i$ *be continuous for every* $i \in I$. *Prove that the naturally defined "diagonal map"*

$$f : Y \to \prod_{i \in I} X_i,\ \ f(y) = (f_i(y))_{i \in I}$$

is continuous.

Hint use 3.5.

Exercise 3.7. *Let $f_i : Y_i \to X_i$ be continuous for every $i \in I$. Prove that the naturally defined "product map"*

$$f : \prod_{i\in I} Y_i \to \prod_{i\in I} X_i,\ \ f((y_i)_{i\in I}) = (f_i(x_i)_{i\in I})$$

is continuous. If every f_i is homeomorphism then f is homeomorphism.

Exercise 3.8.

(1) *Give an example of a topological product $X_1 \times X_2$ such that the projection $p_1 : X_1 \times X_2 \to X_1$ is not a closed map.*
(2) *Let $X = \prod_{i\in I} X_i$ be a topological product. Prove that every projection $p_i : X \to X_i$ is an open map.*

Exercise 3.9. *Let $X = X_1 \times X_2$ be a topological product and $a = (a_1, a_2)$ be a given point in X. Prove that*

$$i_1 : X_1 \to X, x \mapsto (x, a_2)$$

and

$$i_2 : X_1 \to X, y \mapsto (a_1, y)$$

are topological embeddings.

Remark: The similar result is true for infinite products.

Exercise 3.10. *Prove that $X = X_1 \times X_2$ is connected if and only if X_1, X_2 both are connected.*

Hint: One may use 3.9 .

Exercise 3.11. *Prove that:*

(1) $\mathbb{R} \setminus \{0\} \simeq \{1, 2\} \times \mathbb{R}$.
(2) $\mathbb{R}^2 \setminus \{0\} \simeq S_1 \times \mathbb{R}$.

4. Compactness

Exercise 4.1. *Prove:*

(1) *Every finite topological space is compact.*
(2) *a discrete space X is compact if and only if X is finite.*
(3) *(cofinite topology) Every set with the cofinite topology is compact.*
(4) *(cocountable topology[1]) In $\mathbb{R}$ with the cocountable topology compact subsets are exactly finite subsets.*
(5) ** There exists a T_1-space X which is not T_2 and every compact subset of X is closed in X.*
(6) *$\mathbb{Z}$ with the p-adic topology is not compact.*
(7) *Cantor set is compact.*
(8) *Cantor cube (see 1.10) is compact.*
(9) *The set $O_n := \{A \in GL_n(\mathbb{R}) : A^{-1} = A^t\}$ of all orthogonal $n \times n$ real matrices is a compact subset in the metric space $Mat_{n\times n}(\mathbb{R})$ of all $n \times n$ real matrices.*

[1]A subset is said to be *cocountable* if its complement is countable

Exercise 4.2. *Let C be the Cantor set (see 1.16 and 1.10). Show that:*

(1) $C \simeq \{0,1\}^{\mathbb{N}}$ *(where $\{0,1\}^{\mathbb{N}}$ is the topological product).*
(2) $C \simeq C^n \simeq C^{\mathbb{N}}$.
(3) ** C is a homogeneous topological space.*

Exercise 4.3.

(1) *Prove that $\mathbb{R}^n$ is locally compact for every $n \in \mathbb{N}$.*
(2) *Prove that $\mathbb{R}^{\mathbb{N}}$ is not locally compact.*

Remark: X is *locally compact* means that for every point $a \in X$ and every neighborhood $U \in N(a)$ there exists a neighborhood $V \in N(a)$ such that V is compact.

Exercise 4.4. *Prove:*

(1) *In the Banach space l_∞ the ball $B[v,r]$ is closed bounded but not compact for any vector $v \in l_\infty$ and any $r > 0$.*
(2) ** l_∞ is not locally compact.*

Exercise 4.5. *(1-point compactification)*

Let (X,τ) be a locally compact Hausdorff space which is not compact. Let $p \notin X$ and $X^* := X \cup \{p\}$. Define

$$\tau^* := \tau \cup \{X^* \setminus K : \quad K \text{ is compact in } X\}.$$

(1) *Prove that (X^*, τ^*) is a compact Hausdorff topological space and the natural injection $i : X \to X^*$ is a dense topological embedding.*
(2) *Show that (up to homeomorphisms):*
   a) $X^* = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ for $X = \mathbb{N}$ (or, every countable infinite X);
   b) $X^* = S_1$ for $X = \mathbb{R}$;
   c) $X^* = S_n$ for $X = \mathbb{R}^n$.
(3) *Conclude that every discrete space (X, τ_{discr}) is a subspace of a compact Hausdorff space.*

Exercise 4.6. ** Prove that in every compact metric space there are only countably many **clopen** subsets.*

Exercise 4.7. *Prove that every locally compact Hausdorff space is completely regular ($T_{3\frac{1}{2}}$).*

Exercise 4.8. *Let K be a compact subset of a metric space (X,d). Prove that there exist $k_1, k_2 \in K$ such that $diam(K) = d(k_1,k_2)$.*

Exercise 4.9. *Let X be a compact space. Prove that the following conditions are equivalent:*

(1) *There exists a topological embedding $i : X \to [0,1]^n$ into the n-dimensional cube $[0,1]^n$ for some $n \in \mathbb{N}$.*
(2) *There exists a family of continuous functions $f_i : X \to [0,1], i \in \{1, \cdots, n\}$ which separates the points*
(meaning that for every $x \neq y$ in X there exists i such that $f_i(x) \neq f_i(y)$).

Exercise 4.10. *Prove that the following conditions are equivalent:*

(1) $X \in T_{3.5}$.
(2) *X is embedded topologically into a Tychonoff cube $[0,1]^S$ (for some set S).*

(3) *X admits a compactification.*

Exercise 4.11.

(1) *Let X be compact and metrizable. Show:*
$X \in B_2$, $X \in Sep$, $card(X) \leq card(\mathbb{R})$.
(2) *Give an example of a topological space X which is compact Hausdorff but not metrizable.*

5. Quotients

Exercise 5.1. *Let $f : X \to Y$ be a continuous onto function.*

(1) *Give an example of f which is not a quotient.*
(2) *If f is open then f is a quotient.*
(3) *If f is closed then f is a quotient.*
(4) *Define the induced equivalence relation on X: $a \sim b$ if and only if $f(a) = f(b)$. Then we have the following commutative ($f = \widetilde{f} \circ \alpha$) diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\alpha"'] & Y \\ & \widetilde{X} \arrow[u, "\widetilde{f}"'] \end{tikzcd}$$

where $\alpha(x) = [x]$ (the equivalence class of x) and $\widetilde{f}([x]) = f(x)$ is the induced map. Define on $\widetilde{X}$ the factor-topology. Now prove that $\widetilde{f}$ is a continuous 1-1 onto map. Moreover f is quotient if and only if $\widetilde{f}$ is a homeomorphism.

Exercise 5.2. *Let $f : X \to Y$ be an onto continuous map, Y is Hausdorff and X is compact. Prove that f is a quotient.*

Exercise 5.3. *("Gluing points")*

(1) *On the interval $X = [0, 1]$ consider the equivalence relation induced by the identification: $0 \sim 1$. In the quotient set $\widetilde{X}$ define the quotient topology. Show that $\widetilde{X}$ is homeomorphic to the circle S_1.*
(2) *On the square $X = [0, 1] \times [0, 1]$ consider the equivalence relation induced by the identifications:*

$$(0, t) \sim (1, t) \quad \forall\ 0 \leq t \leq 1.$$

Show that the quotient space $\widetilde{X}$ is homeomorphic to the cylinder $S_1 \times [0, 1]$.
(3) ** On the square $X = [0, 1] \times [0, 1]$ consider the equivalence relation induced by the identifications:*

$$(0, t) \sim (1, t), \quad (t, 0) \sim (t, 1) \quad \forall\ 0 \leq t \leq 1.$$

Show that the quotient space $\widetilde{X}$ is homeomorphic to the torus $S_1 \times S_1$.

Exercise 5.4. *Prove that if the composition $f_2 \circ f_1 : X \to Z$ of continuous maps $f_1 : X \to Y$ and $f_2 : Y \to Z$ is a quotient map, then $f_2 : Y \to Z$ is a quotient map.*

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f_1"'] & Z \\ & Y \arrow[u, "f_2"'] \end{tikzcd}$$

As a corollary conclude: let $f : Y \to Z$ be a continuous onto map. If X is a subspace of Y and the restriction $f_X : X \to Z$ is an onto quotient map then f is a quotient.

Exercise 5.5.

(1) *Let Y be a subset of a topological space X and $f : X \to Y$ be a continuous onto retraction (that is, $f(y) = y$ for every $y \in Y$). Then $f : X \to Y$ is a quotient.*
(2) *("complex projective line") In the unit circle $T := \{z \in \mathbb{C} |\ ||z|| = 1\}$ define the equivalence relation as follows*

$$v \sim -v \quad \forall v \in T$$

Show that the corresponding quotient space $T/\sim$ is homeomorphic to the circle.

Exercise 5.6. *Let $X := \{(x, y) \in \mathbb{R}^2 |\ 0 \leq x \text{ or } y = 0\}$ show that the projection $p_1 : X \to \mathbb{R},\ (x, y) \mapsto x$ is a quotient map which is not open and not closed.*

6. Hints and solutions

Subsection 1

1.1 Consider $X = [1, 2] \cup \{3\}$ (as a metric subspace of $\mathbb{R}$).
Define $a_1 = 1, r_1 = 2.5, a_2 = 2, r_2 = 2.5$. Then

$$B(a_1, r_1) = [1, 2] \subsetneq B(a_2, r_2) = [1, 2] \cup \{3\} \text{ but } r_2 < r_1.$$

1.2 Let $x \in B(b, r)$. This means that $d(b, x) < r$. Now by the axiom (m_3) we get

$$d(a, x) \leq d(a, b) + d(b, x) < R - r + r = R.$$

Therefore, $x \in B(a, R)$.

1.3 $B(a, r) = \begin{cases} \{a\} & \text{for } r \leq 1 \\ X & \text{for } r > 1 \end{cases}$ $B[a, r] = \begin{cases} \{a\} & \text{for } r < 1 \\ X & \text{for } r \geq 1 \end{cases}$ $S(a, r) = \begin{cases} \emptyset & \text{for } r \neq 1 \\ X \setminus \{a\} & \text{for } r = 1 \end{cases}$

1.4 Possible answers:

(1) $X = \mathbb{N} \setminus \{4\}, \quad Y = \mathbb{N}$
(2) $X = [1, \infty), \quad Y = (1, \infty)$

1.5 Observe that $T_z(B(\theta, r) = B(z, r)$.

1.6 See Lecture Notes.

1.7 (1) $B_{max}[\theta, 3]$={all $f \in C[0,1]$ such that the graph belongs to the rectangle bounded by the lines $x = 0, x = 1, y = 3, y = -3$}

$B_1[\theta, 3]$={all $f \in C[0,1]$ such that its graph (together with the lines $x = 0, x = 1$) bounds an area ≤ 3}.

(2) Observe that $||f||_1 \leq ||f||_{max}$ for every $f \in C[0,1]$. Hence, $B_{max}[h, r] \subset B_1[h, r]$ for every $h \in C[0,1]$.

1.8 Straightforward. For (5) observe that for every **pseudometric** d we have

$$\cap\{B(a, r)|\ r > 0\} = \{x \in X : d(a, x) = 0\}.$$

1.9

(3) $d_p(0, p^n) = \frac{1}{p^n} \to 0$.

(4) We have to show that $B(0, r)$ is a subgroup of $\mathbb{Z}$. It is equivalent to show that $x - y \in B(0, r)$ for every $x, y \in B(0, r)$. Observe that

$$d(0, x - y) = d(0 + y, x - y + y) = d(y, x) \leq \max\{d(y, 0), d(0, x)\} < r.$$

Open ball $B_r(a)$ is also closed. One proof follows from the equality $B_r(a) = B_{p^m}[a]$ for every $\frac{1}{p^m} < r \leq \frac{1}{p^{m-1}}$.

Another proof (for every ultrametric space): for every given $r > 0$ the relation $x \sim y$ if and only if $d(x, y) < r$ is an equivalence relation. Each equivalence class $[a]$ of a is the open ball $B_r(a)$. In particular, any union of such subsets are open. Now observe that the complement $B_r(a)^c$ is open (being a union of other equivalence classes).

(5) Let $b \in B(a, r)$ (so, $d(a, b) < r$). We have to show that $B(a, r) = B(b, r)$. First we show that $B(b, r) \subseteq B(a, r)$. Indeed, for every $x \in B(b, r)$ we have $d(b, x) < r$. Then we get

$$d(a, x) \leq \max\{d(a, b), d(b, x)\} < r.$$

This means that $x \in B(a, r)$. Hence, $B(b, r) \subseteq B(a, r)$. Similarly you can check $B(a, r) \subseteq B(b, r)$.

Remark: This proof remains true for every ultrametric space.

1.10 Some arguments are similar to the case of 1.9.

1.11

(1) Consider for example, the following sequence $e_1, e_2, \cdots$ in l_2. Then this sequence is coordinate-wise converging to the zero element $\theta := (0, 0, \cdots)$ of l_2 but $\{e_n\}_{n \in \mathbb{N}}$ is not converging with respect to the metric not being even Cauchy. Indeed, $||e_i - e_j|| = \sqrt{2},\ \forall i \neq j$.

(2) $(\mathbb{N}, d_\Delta) \hookrightarrow l_2, \quad n \mapsto \frac{1}{\sqrt{2}} e_n\}$.

(3) Consider (for example) the following linear shift

$$f : l_2 \to l_2, \quad (x_1, x_2, \cdots) \mapsto (0, x_1, x_2, \cdots)$$

(4) On $X := \{A, B, C, D\}$ define

$$d(A,B) = d(B,A) = d(B,C) = d(C,B) = d(C,D) = d(D,C) = d(A,D) = d(D,A) = 1$$

$$d(A,C) = d(C,A) = d(B,D) = d(D,B) = 2$$

Then (X, d) is a metric space. It cannot be embedded isometrically into any Euclidean space $\mathbb{R}^n$. Indeed, otherwise B should be the midpoint of the interval AC. Similarly, D should be the midpoint of the same interval AC. This implies that B and D are the same points, a contradiction.

In fact, (X, d) cannot be embedded isometrically into l_2. Indeed, otherwise it can be embedded into a 4-dimensional linear subspace of l_2. On the other hand, every such subspace of l_2 is isometric to the Eucledean space $\mathbb{R}^4$.

1.12 $A = f^{-1}(0)$, where $f : X \to \mathbb{R}, x \mapsto d(x, A)$. Clearly, $\{0\} = \cap_{n\in\mathbb{N}}(-\infty, \frac{1}{n})$. Now observe that

$$A = \cap_n O_n, \quad O_n := f^{-1}(-\infty, \frac{1}{n})$$

Equivalent proof:

$$A = \cap_n O_n, \quad O_n := B(A, \frac{1}{n}) = \{x \in X : d(A, x) < \frac{1}{n}\}$$

1.13

For example, the sequence $a_n := 1+p+p^2+\cdots+p^n$ is a Cauchy sequence in $(\mathbb{Z}, d_p)$ which is not converging. This sequence is Cauchy because

$$d(a_n, a_{n+i}) \leq \frac{1}{p^n} \to 0$$

for every $i \in \mathbb{N}$.

This sequence is not converging in $(\mathbb{Z}, d_p)$. Indeed, assuming the contrary let $x \in \mathbb{Z}$ such that $x = \lim_n a_n$ (with respect to the p-adic metric). Then for every given $k \in \mathbb{N}$ there exists sufficiently large $n \in \mathbb{N}$ such that

$$d(x, a_m) < \frac{1}{p^k} \quad \forall m \geq n.$$

Since $d(x, a_m) < \frac{1}{p^k}$ we have $a_m - x \equiv 0 \mod p^k$. Therefore,

$$a_m - x = 1 + p + p^2 + \cdots + p^{k-1} + (p^k + \cdots + p^m) - x \equiv 0.$$

Clearly, $p^k | (p^k + \cdots + p^m)$. So,

$$(1 + p + p^2 + \cdots p^{k-1} - x) \equiv 0 \mod p^k$$

This means that $p^k | (1 + p + p^2 + \cdots p^{k-1} - x)$.

On the other hand, since $x \in \mathbb{Z}$ is a given constant we can suppose that for sufficiently big $k \in \mathbb{N}$ we have

$$0 < 1 + p + p^2 + \cdots p^{k-1} - x < p^k$$

Then p^k cannot divide $1 + p + p^2 + \cdots p^{k-1} - x$.

We get a contradiction.

About Cantor cube (from Exercise 1.10). Later we show that it is compact (being a topological copy of the Cantor set). So, being a compact metric space it is complete.

1.14 See Lecture Notes.

1.15 $id : (C[-1,1], || \cdot ||_1) \to (C[-1,1], || \cdot ||_{max})$.

1.18

(1) Let $\{y_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (Y, d_Y). We have to show that $\{y_n\}_{n\in\mathbb{N}}$ converges in (Y, d_Y). Then $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is complete there exists a limit $\lim\limits_{n\to\infty} y_n = x \in X$. Since Y is closed then it is sequentially closed. So, $x \in Y$. This implies that $\{y_n\}_{n\in\mathbb{N}}$ converges in (Y, d_Y).

(2) By a characterization of closed subsets in metric spaces it is equivalent to show that Y is sequentially closed in X. Let $\{y_n\}_{n\in\mathbb{N}}$ be a converging sequence in X. Then there exists $\lim\limits_{n\to\infty} y_n = x \in X$. This sequence is Cauchy in (X, d). Since d_Y is a restriction of d we obtain that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y), too. Since it is complete we obtain that there exists $\lim\limits_{n\to\infty} y_n = y \in Y$ in Y. Then necessarily $x = y$.

Section 2

2.1 Check for example, the axiom (t_2). Let $O_1, O_2 \in top(d)$. For every $x \in O_1 \cap O_2$ (if it exists) one may choose $\varepsilon_1, \varepsilon_2$ such that $B(x, \varepsilon_1) \subseteq O_1$ and $B(x, \varepsilon_2) \subseteq O_2$. Then $B(x, \varepsilon) \subseteq O_1 \cap O_2$ for $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Hence, $O_1 \cap O_2 \in top(d)$.

2.2 Counterexample for unions:

$$X := \{a, b, c\},\ \tau_1 := \{\emptyset, \{a\}, \{a, b, c\}\},\ \tau_2 := \{\emptyset, \{b\}, \{a, b, c\}\}$$

Then τ_1, τ_2 are topologies on X but not $\tau_1 \cup \tau_2$.

2.3

(1) $\Rightarrow$ (2). Let us show that $\{a\}$ is closed. By the axiom T_1 for every $x \neq a$ one may choose an open nbd $O_x \in N(x)$ such that $a \notin O_x$. Then $\cup\{O_x : x \neq a\} = X \setminus \{a\}$ is open (by t_3, the union of open subsets). Therefore, its complement, $\{a\}$ is closed.

(2) $\Rightarrow$ (3). Every finite set is an union of finitely many singletons.

(3) $\Rightarrow$ (4). Every finite set F is closed. Hence, its complement $X \setminus F$ is open. This implies that all co-finite sets are open in (X, τ). Therefore, $\tau_{cofin} \subset \tau$.

(4) $\Rightarrow$ (1). Let $a \neq b$. Since the cofinite subsets are open. In particular, $X \setminus \{b\}, X \setminus \{b\} \in \tau$. Define $U := X \setminus \{b\}, V := X \setminus \{b\}$. Then $b \notin U \in N(a)$ and $a \notin V \in N(b)$.

2.4

(1) It is straightforward to check the axioms $(t_1), (t_2), (t_3)$. So, $(\mathbb{Z}, \tau_{\leq}) \in TOP$.

Since $\mathbb{Z}$ is linearly ordered, for every distinct $m, n \in \mathbb{Z}$ we have $m < n$ or $n < m$. Let $m < n$ (the second case is similar). Then $m \in O_m$ and $n \notin O_m$. Hence, $(\mathbb{Z}, \tau_{\leq}) \in T_0$.

For every pair U, V of nonempty open subsets $U \cap V$ contains one of the O_k for some $k \in \mathbb{Z}$. So, $U \cap V$ is nonempty. This implies that there is no topological partition of $(\mathbb{Z}, \tau_{\leq})$. This means that $(\mathbb{Z}, \tau_{\leq}) \in Conn$.

(2) Indeed, if yes, then there exists a (pseudo)metric d on $\mathbb{Z}$ such that $top(d) = \tau_{\leq}$. We have two cases (a) $d(0,1) = 0$; or b) $d(0,1) > 0$. If (a) then every open set containing 0 contains also 1 and vice versa. Contradiction to T_0 property established in (1).

If (b) then $B(0,r) \cap B(1,r) = \emptyset$ for every $r < \frac{1}{2}d(0,1)$. On the other hand by definition of $\tau_{\leq}$ $\boxed{\text{every open set containing 1 contains also 0.}}$ So we get a contradiction.

(3) Assuming the contrary let $f : (\mathbb{Z}, \tau_{\leq}) \to \mathbb{R}$ be a continuous function such that $f(m) \neq f(n)$ for some $m < n$. Take disjoint open neighborhoods U, V of $f(m)$ and $f(n)$ in $\mathbb{R}$, respectively. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open neighborhoods of m and n in $(\mathbb{Z}, \tau_{\leq})$. On the other hand, $\boxed{\text{every open set containing } n \text{ contains also } m.}$ This contradiction finishes the proof.

2.5 $\tau_{cofin} := \{\emptyset\} \cup \{F^c : F \text{ is a finite subset of } X\}$.

Straightforward to prove that $(X, \tau_{cofin}) \in TOP$. Every singleton $\{a\}$ is closed because $\{a\}^c$ is open being cofinite.

$(X, \tau_{cofin}) \in Conn$ for every **infinite** set X. Indeed, let U, V be **open** subsets such that $U \neq X, V \neq X, U \neq \emptyset, V \neq \emptyset$. Then U and also U^c both are cofinite. Then U^c, U both are **finite** but this implies that X is finite, a contradiction !

As before, intersection of two cofinite subsets is not empty (otherwise, X is finite). This implies that $(X, \tau_{\leq})$ is not Hausdorff for infinite X.

Finally, $(\mathbb{Z}, \tau_{cofin})$ is not (pseudo)metrizable. Indeed, since the intersection of two cofinite subsets is not empty we obtain that $(\mathbb{Z}, \tau_{cofin})$ is not Hausdorff. Therefore, $(\mathbb{Z}, \tau_{cofin})$ is not metrizable. Assuming that it is pseudometrizable with some d we necessarily have $d(a,b) = 0$ for some distinct $a, b \in \mathbb{Z}$. Then $\{a\}$ is not closed (because its closure contains also b). On the other hand, $(X, \tau_{cofin}) \in T_1$ (every singleton is closed). This contradiction finishes the proof.

2.6 Check the axioms for $\tau_{f,Y}$:
(t_1) Observe that $f^{-1}(X) = Y$, $f^{-1}(\emptyset) = \emptyset$. Hence $Y, \emptyset \in \tau_{f,Y}$.
(t_2) Use $f^{-1}(O_1) \cap f^{-1}(O_2) = f^{-1}(O_1 \cap O_2)$ (and the fact that τ is a topology)
(t_3) Use $\cup_{i \in I} f^{-1}(O_i) = f^{-1}(\cup_{i \in I} O_i)$.

2.7 See Lecture Notes.

2.8 See the homeworks.

2.10 Take $(X, \tau) \in TOP$ such that $(X, \tau) \notin T_0$. Now consider $id : (X, \tau_{discr}) \to (X, \tau)$. This map is continuous (2.8) and $(X, \tau_{discr}) \in T_0$. Similarly, for other cases.

2.11 Which topological properties from the following list are hereditary:

(1) connectedness
NOT
(2) T_i, $i \in \{0, 1, 2, 3\}$

YES

(3) compactness

NOT

(4) B_2

YES

(5) B_1

YES

(6) Metrizability.

YES

2.12 For (1), (2), (3) see Lecture Notes.

(4) Let A, B are disjoint closed subsets. If one of them, say A, contains p then $p \notin B$. Therefore, B necessarily is a subset of $\mathbb{R}$. Moreover, B is countable because $p \in B^c \in \tau$ and B^c is cocountable. Now observe that $U := X \setminus B \in N(A)$ and $V := B \in N(B)$ are disjoint (open) neighborhoods of A and B.

If $A \subset \mathbb{R}, B \subset \mathbb{R}$ then choose $U := A, V := B$.

2.15 (1) Let's check (t_2). If $x \in U_1 \cap U_2$. Then $S(a_1, x) \subseteq U_1, S(a_2, x) \subseteq U$. Then $S(a_1 a_2, x) \subseteq U_1 \cap U_2$.

(2) Every $S(a, b)$ (with $a \neq 0$) is infinite.

(3) The basis sets $S(a, b)$ are both open and closed. They are open by definition, and we can write $S(a, b)$ as the complement of an open set as follows:

$$S(a,b) = \mathbb{Z} \setminus \bigcup_{j=1}^{a-1} S(a, b+j).$$

(4) Assume in contrary that the number of primes is finite. Then the **finite union** $\cup\{S(p, 0) :\ p \text{ is prime}\}$ of closed sets (using (3)) is closed. Now using $\mathbb{Z} \setminus \{-1, 1\} = \cup\{S(p, 0) :\ p \text{ is prime}\}$. we conclude that $\mathbb{Z} \setminus \{-1, 1\}$ is closed. Then $\{-1, 1\}$ is open, a contradiction (by (2)).

2.16

(1) YES. The closure $cl_X(A)$ of A in X contains B (because, $cl_X(A)) \supseteq cl_B(A) = B$). Clearly, in general $cl_X(cl_X(A)) = cl_X(A)$. On the other hand, B is dense in X, so $cl_X(B) = X$. Summing up we get $cl_X(A) \supseteq cl_X(B) = X$.

(2) NOT. The disjoint subsets $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$ are both dense in $\mathbb{R}$.

(3) YES. Let U, V are open dense subsets in X. Our aim is to show that $cl(U \cap V) = X$. It is equivalent to prove that $O \cap (U \cap V) \neq \emptyset$ for every open nonempty subset O in X. Since $cl(U) = X$ we have $O \cap U \neq \emptyset$. The intersection $O \cap U$ is a nonempty open set. Since $cl(V) = X$ we have $(O \cap U) \cap V \neq \emptyset$.

(4) NOT. Consider the metric space $\mathbb{Q}$. For every element $q \in \mathbb{Q}$ its complement $O_q := \mathbb{Q} \setminus \{q\}$ in $\mathbb{Q}$ is open in $\mathbb{Q}$. However, the (countable !) intersection $\cap\{O_q : q \in \mathbb{Q}\}$ is empty.

2.17

$$(\partial(A))^c = (\bar{A} \setminus int(A))^c = (\bar{A} \cap (X \setminus int(A)))^c = (\bar{A})^c \cup int(A) = int(A^c) \cup int(A)$$

2.18 It is convenient to use the following formula from 2.17

$$(\partial(A))^c == int(A^c) \cup int(A)$$

which implies that we have the following "disjoint union"

$$X = int(A^c) \cup \partial(A) \cup int(A).$$

Let $x \in int(A)$. Then $\xi_A(x) = 1$. $\{1\}$ is an open nbd of 1 in the two element space $Y := \{0, 1\}$. The preimage $\xi^{-1}(1) = A$ is a neighborhood for every $x \in int(A)$ (note that always, $A \in N(x) \ \ \forall x \in int(A)$). This proves (by local definition of continuity points) the continuity of $\xi_A : X \to \{0, 1\}$ at points $x \in int(A)$.

Completely similar proof shows the continuity of $\xi_A : X \to \{0, 1\}$ at points $x \in int(A^c)$.

Assume now that $x \in \partial(A)$. We are going to show that ξ_A is discontinuous at x. We have two cases: $\xi_A(x) = 1$ or $\xi_A(x) = 0$. In the first case: $\xi_A(x) = 1$, observe that for every nbd $V \in N(x)$ its image is not a subset of $U := \{1\} \in N(1)$. That is, $\xi_A(V) \nsubseteq \{1\}$. Indeed, as we know $\partial(A) = \bar{A} \cap \bar{A^c}$. So, $V \cap A^c$ is not empty. For $y \in V \cap A^c$ we have $\xi_A(y) = 0$. Therefore, $\xi_A(V) \nsubseteq \{1\}$. The second case $\xi_A(x) = 0$ is similar.

2.19 See Lecture Notes.

2.20
1. There are three classes up to homeomorphisms
(enough to use homeomorphisms coming from elementary functions ...)
$\mathbb{R} \simeq (a, b) \simeq (a, \infty) \simeq (-\infty, b)$
$[a, b) \simeq (c, d] \simeq [a, \infty) \simeq (-\infty, b]$
$[a, b]$
2. So by (1) the question on continuous images it is enough compare only three representatives: $\mathbb{R}$, $[a, b)$, $[a, b]$.
$\mathbb{R} \to (a, b] \to [a, b]$ the corresponding continuous onto functions are quite easy.
$(a, b] \to \mathbb{R}$ this case is not so easy. Take for example $f : (0, 1] \to \mathbb{R},\ f(x) = \frac{1}{x} sin(\frac{1}{x})$

2.21 (1) There are four equivalence classes up to the homeomorphisms:
{1,2,3,5,7} are homeomorphic to the closed unit segment $[0, 1]$
(warning: take into account that 1 has no "bottom tail" in sans serif font)
{4,6,9}
{8}
{0}

(2) There are following nine classes up to the homeomorphisms:
C I J L M N S U V W Z
D O
E F G T Y
H K
A R
B
P
Q

X

Hint (one of the typical cases): T and K are not homeomorphic. K has a point k_0 such that $K \setminus \{k_0\}$ has four Conn-components while T has no such a point. Another approach: K has four end-points (= not separating points) while T has three end-points.

Remark: For detailed explanations see the following file of Rafael Lopez:
`https://arxiv.org/pdf/1410.3364.pdf`

2.22 (1) Hint: recall that $\mathbb{R}^n$ is homeomorphic to any open ball $B(v, \varepsilon)$, $v \in \mathbb{R}^n$.

With more details: Let O be a nonempty open subset of $\mathbb{R}^n$ and $v \in O$. Every open neighbourhood of v in the space O looks like $U \cap O$ where U is an open neighborhood of v in $\mathbb{R}^n$. Hence, $U \cap O$ is open in $\mathbb{R}^n$ (and in O). Now choose $\varepsilon > 0$ small enough such that $B(v, \varepsilon) \subset U \cap O$. Clearly, $B(v, \varepsilon)$ is connected being homeomorphic to $\mathbb{R}^n$. So, O contains arbitrarily small connected neighborhoods, as desired.

Note that O is not necessarily connected. Indeed, take for example, $O = (0, 1) \cup (3, 5)$ in $\mathbb{R}$.

(2) Define X (as a subspace of $\mathbb{R}^2$) union of countably many closed intervals as follows:

$$X = I_0 \cup I_1 \cup \cdots I_n \cup \cdots, \quad n \in \mathbb{N}$$

where each I_n is the standard linear interval in $\mathbb{R}^2$ between the vectors $(1, 0)$ and $(0, \frac{1}{n})$ and I_0 is the interval between $(1, 0)$ and $(0, 0)$. These (connected, of course) intervals have a unique common point $(1, 0)$. Clearly, X is connected (and even pathwise connected). However, X is not locally connected at $(0, 0)$. Take a neighborhood U of $(0, 0)$ in X such that $(1, 0) \notin U$. Then every neighborhood V of $(0, 0)$ in X such that $V \subseteq U$ is not connected. Indeed, observe that V meets almost all intervals I_n (use that the sequence $(0, \frac{1}{n})$ converges to $(0, 0)$). So, there exists $n_0 \in \mathbb{N}$ such that $V \cap I_n \neq \emptyset$, $\forall n \geq n_0$. However I_n is not contained in V for every n (because $(1, 0) \notin V$). It follows that $V \cap I_{n_0}$ is a clopen subset in the space V. Since $\emptyset \neq V \cap I_{n_0} \neq V$ we conclude that V is not connected.

2.14 Assuming the contrary let $f_1(z) \neq f_2(z)$ for some $z \in X$. Since Y is Hausdorff we may choose **disjoint** open neighborhoods $U \in N(f_1(z))$ and $V \in N(f_2(z))$. The continuity of f guarantees that $O_1 := f_1^{-1}(U) \in N(z), O_2 := f_2^{-1}(V) \in N(z)$ are also open neighborhoods. The subset A is dense in X. Therefore, there exists $a \in A \cap O_1 \cap O_2$. Then $f_1(a) \neq f_2(a)$ (because $f_1(a) \in U$, $f_2(a) \in V$). This contradiction completes the proof.

Section 3

3.1 See the homeworks.

3.2 (1) is a particular case of (3) (because T_1 is equivalent to the closedness of the singletons). No problem to prove this also directly similar to (2).

(2) For every distinct pair of points $u, v \in X$ there exists $i \in I$ such that $u_i \neq v_i$. Since X_i is Hausdorff one may choose $u_i \in U_i \in \tau_i$ and $v_i \in V_i \in \tau_i$ such that $U_i \cap V_i \neq \emptyset$. Then $p_i^{-1}(U_i)$ and $p_i^{-1}(V_i)$ are disjoint neighborhoods of u and v in X.

(3) Homeworks 9.6. In fact, a stronger result is true:

$$cl(\prod_{i \in I} A_i) = \prod_{i \in I} cl(A_i)$$

for every family $A_i \subseteq X_i$.

3.3 (1) It is enough to show that

$$h : Gr(f) \to X, h(x, f(x)) := x = p_1(x, f(x))$$

(the restriction of the projection $p_1 : X \times Y \to X$ on $Gr(f)$) is a homeomorphism. This function is 1-1. Indeed, $(x_1, f(x_1)) = (x_2, f(x_2)) \Leftrightarrow x_1 = x_2$.

This function is onto. Indeed, clearly, $p_1(x, f(x)) = x$ for every $x \in X$.

This function is continuous (as a restriction of the continuous projection).

We need only to show that the inverse function

$$h^{-1} : X \to Gr(f), x \mapsto (x, f(x))$$

is continuous. It is equivalent to show that

$$h^{-1} : X \to X \times Y, x \mapsto (x, f(x))$$

is continuous. Let's use the local criterion of continuity. Namely, we have to show that h^{-1} is continuous at every given $x_0 \in X$. Consider any neighborhood O of $(x_0, f(x_0)) \in X \times Y$ in the product $X \times Y$. We need to prove that there exists $U_0 \in N(x_0)$ in X such that $h^{-1}(U_0) \subset O$. Since the "open rectangles" is a base of the product topology we can suppose (without restriction of generality) that $O = V \times W$ (where $V \in N(x_0)$ and $W \in N(f(x_0))$. By the continuity of f there exists $U \in N(x_0)$ in X such that $f(U) \subset W$. Define $U_0 := U \cap V$. Then $U_0 \in N(x_0)$ and $f(U_0) \subseteq f(U) \subseteq W$. Hence, $h^{-1}(U_0) \subset V \times W$.

(2) Assume, in addition, that Y is Hausdorff. We have to show that $Gr(f)$ is closed in $X \times Y$. It is equivalent to check that the complement is open. Take any point in the complement $(x, y) \notin Gr(f)$. Then $y \neq f(x)$. Since Y is Hausdorff we may choose disjoint open neighborhoods

$$U \in N(y), V \in N(f(x)), U \cap V = \emptyset.$$

By the continuity of f there exists an open neighborhood $O \in N(x)$ in X such that $f(O) \subset V$. Observe now that $O \times U$ is an open neighborhood of (x, y) in $X \times Y$ such that $(O \times U) \cap Gr(f) = \emptyset$. This proves that the complement of $Gr(f)$ is open.

3.4 $\mathbb{Q} \times \mathbb{Q}$ is dense in the "Sorgenfrey plane" $\mathbb{R}_s \times \mathbb{R}_s$. Hence, $\mathbb{R}_s \times \mathbb{R}_s$ is separable. The topological subspace

$$X := \{(x, -x) : x \in \mathbb{R}_s\} \subset \mathbb{R}_s \times \mathbb{R}_s$$

Is discrete, as a topological subspace. Indeed, for every $x_0 \in \mathbb{R}_s$ consider $O := [x_0, x_0 + 1) \times [-x_0, -x_0 + 1]$. Then O is an open neighborhood of $x_0, -x_0)$ in $\mathbb{R}_s \times \mathbb{R}_s$. On the other hand, the intersection $O \cap X = \{x_0\}$ is just the singleton $\{x_0\}$. Hence, x_0 is an isolated point in X_0.

Clearly, X, being discrete and uncountable, is not separable (because discrete space is separable if and only if it is countable).

3.5 The preimage f^{-1} preserves intersections and unions ...

3.6 The "elementary open boxes" is the standard subbase of the product topology. By 3.5 it is enough to show that the preimage $f^{-1}(p_i^{-1}(O_i))$ is open in Y for every $O_i \in \tau_i$. Now observe that $f^{-1}(p_i^{-1}(O_i)) = f_i^{-1}(O_i)$ (because $p_i \circ f = f_i$) and apply the continuity of $f_i : Y \to X_i$.

3.7 As in 3.6 use 3.5.

3.8 (1) $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not a closed map. Indeed, $A := \{(x, \frac{1}{x}) : x > 0\}$ is closed in $\mathbb{R} \times \mathbb{R}$ but $p_1(A) = \{x \in \mathbb{R} : x > 0\}$ is not closed in $\mathbb{R}$.

(2) Let $X = \prod_{i \in I} X_i$ be a topological product.

We have to prove that every projection $p_{i_0} : X \to X_{i_0}$ is an open map. Since every map preserves the union (and the union of open sets is open) it is enough to show that the image of every basic open box is open. Every basic open box O in X looks like

$$O = \prod_{i \in I} O_i$$

where there exists a finite $J \subseteq I$ such that $O_i = X_i$ for every $i \notin J$. Now it is clear that $p_{i_0}(O) = p_{i_0}(\prod_{i \in I} O_i) = O_{i_0}$.

3.9 Straightforward using the definitions (product and subspace ...)

3.10 Let $u = (u_1, u_2), v = (v_1, v_2) \in X_1 \times X_2)$. It is enough to show that they belong to the same component. Equivalently, that there exists a connected subset $A \subset X_1 \times X_2)$ such that $u, v \in A$. Now define

$$A := (X_1 \times \{u_2\}) \cup (\{v_1\} \times X_2).$$

Now observe that each of these two subsets are connected by 3.9 and they have a nontrivial intersection. Namely the point (v_1, u_2).

3.11 (1) $\mathbb{R} \setminus \{0\} \simeq \{1, 2\} \times \mathbb{R}$.

Indeed, the function

$$h : \mathbb{R} \setminus \{0\} \to \{1, 2\} \times \mathbb{R}, \quad h(x) = \begin{cases} (1, ln(x)) & \text{for } x > 0 \\ (2, ln(-x) & \text{for } x < 0, \end{cases}$$

is a desired homeomorphism.

(2) $\mathbb{R}^2 \setminus \{0\} \simeq S_1 \times \mathbb{R}$.

Hint:

$$\mathbb{C}^* \to S_1 \times (0, \infty), \; rcis(\alpha) \mapsto (cis(\alpha), r)$$

is a homeomorphism, where $\mathbb{C}^*$ is the space of nonzero complex numbers.

Section 4

4.1 (2) Hint: the covering $\{\{x\} : \ x \in X\}$ of X is open if and only if X is discrete.

(3) Hint: every cofinite subset already covers almost all elements.

(4) If $F \subseteq X$ is finite then F is always compact for any topology on X.

If $A \subseteq X$ is not finite then it contains a sequence $B := \{b_n : n \in \mathbb{N}\}$ with distinct elements. Consider the following family of subsets in X:

$$\alpha := \{O_n : \ n \in \mathbb{N}\}, \quad O_n := \{b_1, b_2, \cdots, b_n\} \cup (A \setminus B).$$

Since X carries the cocountable topology and every $O_n^c = \{b_{n+1}, b_{n+2}, \cdots\}$ is countable, then α is an open family in X which covers A. No finite subfamily of α can cover A. This implies that A is not a compact subset in the subspace topology.

(5) Take $X := (\mathbb{R}, \tau_{cocount})$ the reals in the cocountable topology. Then X is T_1 because every singleton is closed (in fact, every countable subset is closed). At the same time X is not T_2 (because any two cocountable subsets necessarily meet). Finally, observe that the compact subsets of X are exactly all finite subsets as it follows from (4).

(6) $\mathbb{Z}$ with the p-adic metric topology is not compact because $(\mathbb{Z}, d_p)$ is not complete.

(7) Cantor set is a bounded closed subset of $[0, 1]$ by the construction.

(8) It is enough to show that the Cantor cube (see 1.10) is homeomorphic to the Cantor set. Observe that Cantor set C is exactly the set of points in $[0, 1]$, where in the ternary representation we allow to use only 0 or 2 (but not 1). That is,

$$C = \{\sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } a_k \in \{0, 2\}\}.$$

The function

$$f : \{0, 2\}^{\mathbb{N}} \to C, \ (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

is continuous onto 1-1. By Tychonoff theorem the topological product $\{0, 2\}^{\mathbb{N}}$ is compact. So, f is a homeomorphism. Now observe that $\{0, 2\}^{\mathbb{N}}$ is naturally homeomorphic to the product $\{0, 1\}^{\mathbb{N}}$ which in turn is homeomorphic to the Cantor cube from 1.10.

(9) Hint: Observe that the metric space of all $n \times n$ real matrices under the natural matric is isometric to the Euclidean space $\mathbb{R}^{n^2}$ and use Heine-Borel theorem.

4.2 (3) * Hint: look at $\{0, 1\}^{\mathbb{N}}$ as a group under the addition of binary sequences modulo 2. Now show that every translation $T_a : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is a homeomorphism.

4.3 (1) Hint: By Heine-Borel theorem every "n-dimensional cube" $[a, b]^n$ is compact. Every point of $\mathbb{R}^n$ has a neighborhood which is homeomorphic to $[a, b]^n$.

(2) In order to show that $\mathbb{R}^{\mathbb{N}}$ is not locally compact note that every neighborhood U of any point in $\mathbb{R}^{\mathbb{N}}$ contains a basic box $\prod_{k \in \mathbb{N}} O_k$, where $O_k = \mathbb{R}$ for almost all k. So, some projection of U is not compact. This means that U itself is not compact.

4.4 (1) Hint: Consider $D := \{e_n : \ n \in \mathbb{N}\} \subset l_\infty$. Then D is discrete as a subspace. Since D is infinite we get that D is not compact (4.1.2). On the other hand, D is a bounded closed subset of l_∞. Moreover, D is a closed subset of $B[\theta, 1]$ (where θ is the zero element). This implies that $B[\theta, 1]$ is not compact.

(2) Hint: Similar to (1) show first that $B[\theta, r]$ for every $r > 0$ and observe that every neighborhood of θ contains $B[\theta, \varepsilon]$ for some $\varepsilon > 0$.

4.5 (1) Homeworks.
(2) (a) $X^* \simeq \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ for $X = \mathbb{N}$;

Hint: consider the function

$$f : \mathbb{N}^* \to \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\},\ f(p) = 0, f(n) = \frac{1}{n}$$

Observe that f is continuous, onto and 1-1. In fact, a homeomorphism because it is a function from a compact space into a Hausdorff...

(b) and (c) Use the stereographic projection homeomorphism $S_n \setminus \{p\} \simeq \mathbb{R}^n$. Observe that the 1-point compactification is unique (up to the homeomorphisms).

At least, for the case $n = 1$ one may give an alternative proof: $X^* \simeq S_1$ for $X = \mathbb{R}$.

Since S_1 is homeomorphic to $\mathbb{T}$ it is enough to show $X^* \simeq \mathbb{T}$. Consider the function

$$f : \mathbb{R}^* \to \mathbb{T},\ f(x) = cis(\frac{2\pi x}{1 + |x|}) \text{ for } x \in \mathbb{R}, f(p) = (1, 0).$$

The rest as in (a).

4.7 Hint: by 4.5, X topologically is embedded into the one-point compactification X^* which is a compact Hausdorff space. Therefore, X^* is T_4. Hence, also $T_{3.5}$. Now check that $T_{3.5}$ is a hereditary property. So, X also is $T_{3.5}$.

4.8 Hint: By Tychonoff theorem $X \times X$ is compact. Now show that the distance function $d : X \times X \to [0, \infty)$ is continuous and apply generalized Weierstrass theorem.

4.9
(1) $\Rightarrow$ (2) Take $f_i = p_i,\ i \in \{1, \cdots, n\}$ the projections. Observe that the family of all projections always separate the points for any product.

(2) $\Rightarrow$ (1) Suppose that $f_i : X \to [0, 1],\ i \in \{1, \cdots, n\}$ are continuous and separate the points of X. Consider the following "diagonal function":

$$f : X \to [0, 1]^n,\ f(x) = (f_i(x))_{i \in I}.$$

This function is continuous (see 3.6). Also, f is 1-1. Indeed, if $x \neq y$ in X then $f_i(x) \neq f_i(y)$ for some i. Therefore, the vectors $f(x)$ and $f(y)$ are distinct in the cube. Since X is compact and $[0, 1]^n$ is Hausdorff we know that f is a closed map. Since f is also 1-1 we obtain (by theorem about topological embeddings) that $f : X \to [0, 1]^n$ is a topological embedding.

4.10 (1) $\Rightarrow$ (2): $X \in T_{3.5}$. So, by definition there exists a family of functions $S = \{f_s : X \to [0, 1] :\ s \in S\}$ which separates points and closed subsets of X (for example, $S = C(X, [0, 1])$). Consider the diagonal function

$$f : X \to [0, 1]^S,\ f(x) = (f_s(x))_{s \in S}.$$

Then f is a topological embedding.

(2) $\Rightarrow$ (3): Let $\nu : X \to [0,1]^S$ be a topological embedding. Denote by Y the closure of the image. That is, $Y := f(X)$. Then $Y \in Cop \cap T_2$ and the induced map $\nu_* : X \to Y$ is dense. Therefore, $\nu_* : X \to Y$ is a compactification of X.

(3) $\Rightarrow$ (1): Let $f : X \to Y$ be a compactification. Then X is homeomorphic to $f(X)$. So, it is equivalent to show that $f(X) \in T_{3.5}$. This follows from the hereditary property of $T_{3.5}$ taking into account that $f(X) \subset Y$ and $Y \in Comp \cap T_2) \subset T_4 \subset T_{3.5}$.

4.11 See Lecture notes.

Section 5

5.1 (1) Consider, for example, any 1-1 onto continuous map which is not homeomorphism. Say, $f : (\mathbb{R}, \tau_{discr}) \to (\mathbb{R}, \tau(d))$.

(2) Let $f : X \to Y$ be an onto continuous open map. We have to show that f is a quotient. Equivalently, we need to check that $f^{-1}(A)$ is open in X implies that A is open in Y. Now, if $f^{-1}(A)$ is open then $f(f^{-1}(A))$ is also open because f is open. On the other hand, $f(f^{-1}(A)) = A$ (because f is onto).

(3) Let $f : X \to Y$ be an onto continuous closed map. We have to show that f is a quotient. Equivalently, we need to check that $f^{-1}(A)$ is open in X implies that A is open in Y. Turning to the complement, it is equivalent to show: $f^{-1}(A)$ is closed in X implies that A is closed in Y. Now, if $f^{-1}(A)$ is closed then $f(f^{-1}(A))$ is also closed because f is closed. On the other hand, $f(f^{-1}(A)) = A$.

(4) If $\tilde{f} : \widetilde{X} \to Y$ is a homeomorphism then it is a quotient map. Now, $f : X \to Y$ is a quotient as a composition $\tilde{f} \circ \alpha$ of two quotient maps.

Conversely, let f be a quotient. We have to show that $\tilde{f} : \widetilde{X} \to Y$ is a homeomorphism. It is continuous 1-1 and onto. Therefore, it is enough to show that this map is open. Let O be open in $\widetilde{X}$. We have to prove that $\tilde{f}(O)$ is open in Y.

Since $\tilde{f}$ is 1-1 we have $O = \tilde{f}^{-1}(\tilde{f}(O)) = \tilde{f}^{-1}(A)$, where $A := \tilde{f}(O)$. Then $\alpha^{-1}(\tilde{f}^{-1}(A)) = f^{-1}(A)$ is open in X (because α is continuous). Since f is a quotient we can conclude that A is open. On the other hand, $A = \tilde{f}(O)$.

5.2 $f : X \to Y$ is an onto continuous map, Y is Hausdorff and X is compact. Therefore, f is a closed map. Hence, f is a quotient by 5.1.3.

5.3 (1) Consider the function

$$f : X := [0,1] \to Y := S_1, \ f(x) = cis(2\pi x) = (cos 2\pi x, sin 2\pi x)$$

This function is a quotient by 5.2. On the other hand, the equivalence relation $\sim$ induced by f on $[0,1]$ is just $0 \sim 1$. So by 5.1.4 we can conclude that $\widetilde{X} \sim S_1$.

(2) Consider the function

$$f : X := [0,1] \times [0,1] \to Y := S_1 \times [0,1], \ f(x,y) = (cis(2\pi x), y)$$

The rest as in (1).

(3) Consider the function

$$f : X := [0,1] \times [0,1] \to Y := S_1 \times S_1, \ f(x,y) = (cis(2\pi x), cis(2\pi y))$$

The rest as in (1).

5.4 We have to show that $f_2 : Y \to Z$ is a quotient. Let $f_2^{-1}(A)$ is open in Y. By the continuity of f_1 we obtain that $f_1^{-1}(f_2^{-1}(A))$ is open in X. Since $f = f_2 \circ f_1$ is a quotient and $f^{-1}(A) = f_1^{-1}(f_2^{-1}(A))$ we can conclude that A is open.

5.5 (a) Observe that the restriction of f on Y is the identity function. So, f is the quotient by 5.4.

(b) The function $f : T \to T, f(t) = t^2$ is continuous onto from a compact space, T is Hausdorff. So, this map is closed and onto. We obtain that f is a quotient.

5.6 The map $p_1 : X \to \mathbb{R},\ (x, y) \mapsto x$ is continuous (as a restriction of the projection) and onto. Its restriction to the x-axis $\{0\} \times \mathbb{R}$ is a homeomorphism. So, our map is a quotient.

It is **not open**: the subset $[0, 1) \times (2, 3)$ is open in X but its image $[0, 1)$ is not open in $\mathbb{R}$.

It is **not closed**: the subset $\{(x, \frac{1}{x}) :\ x > 0\}$ is closed in X but its image $(0, \infty)$ is not closed in $\mathbb{R}$.

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