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# Group actions on treelike compact spaces

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**Abstract** We show that group actions on many treelike compact spaces are not too complicated dynamically. We first observe that an old argument of Seidler (1990) implies that every action of a topological group $G$ on a regular continuum is null and therefore also tame. As every local dendron is regular, one concludes that every action of $G$ on a local dendron is null. We then use a more direct method to show that every continuous group action of $G$ on a dendron is Rosenthal representable, hence also tame. Similar results are obtained for median pretrees. As a related result, we show that Helly's selection principle can be extended to bounded monotone sequences defined on median pretrees (for example, dendrons or linearly ordered sets). Finally, we point out some applications of these results to continuous group actions on dendrites.

**Keywords** amenable group, dendrite, dendron, fragmentability, median pretree, proximal action, Rosenthal Banach space, tame dynamical system

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## 1 Introduction and preliminaries

One of the motivations for writing the present work was to show that group actions on dendrons are Rosenthal representable, hence tame (see Definition 1.1 below). Representations on Banach spaces with "good" geometry lead to a natural hierarchy in the world of continuous actions $G \curvearrowright X$ of topological groups $G$ on topological spaces $X$. In particular, representations on Banach spaces without a copy of $l_1$ (we call them *Rosenthal* Banach spaces) play a very important role in this hierarchy. According to the Rosenthal $l_1$-dichotomy [34], and the corresponding dynamical Bourgain-Fremlin-Talagrand dichotomy [10, 12], there is a sharp dichotomy for metrizable dynamical systems; either their enveloping semigroup is of cardinality less than or equal to that of the continuum, or it is very large and contains a copy of the Stone-Čech compactification of $\mathbb{N}$ (the set of natural numbers).

When $X$ is compact metrizable, in the first case, such a dynamical system $(G, X)$ is called *tame*. By Köhler's definition [23], tameness means that for every continuous real-valued function $f : X \to \mathbb{R}$ the family of functions $fG := \{fg\}_{g\in G}$ is "combinatorially small"; namely, $fG$ does not contain an *independent sequence*.

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Recall the classical definition from [34]: a sequence $f_n$ of real-valued functions on a set $X$ is said to be *independent* if there exist real numbers $a < b$ such that

$$\bigcap_{n\in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n\in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets $P$ and $M$ of $\mathbb{N}$. As in [13,15] we say that a bounded family of real-valued functions is a *tame family* if it does not contain an independent sequence. For compact metrizable systems $X$ the two concepts discussed above coincide: $(G,X)$ is tame if and only if it is Rosenthal representable. The case of metrizable $X$ admits also an interesting enveloping semigroup characterization: $(G,X)$ is tame if and only if every element $p \in E(G,X)$ is a Baire class 1 function $p : X \to X$ (see [12,16]). Recall that the *enveloping semigroup* $E(G,X)$ of a compact $G$-system $X$ is the pointwise closure of the set of all $g$-translations $X \to X$ $(g \in G)$ in $X^X$.

We next recall the definition of a Banach representation of a dynamical system. For a Banach space $V$, denote by $Is(V)$ the topological group (equipped with the strong operator topology) of all linear isometries $V \to V$. The group $Is(V)$ acts on the dual space $V^*$ and its weak-star compact subsets.

**Definition 1.1** (See [10,12,30])**.** A *representation* of an action $G \curvearrowright X$ on a Banach space $V$ is a pair $(h,\alpha)$, where $h : G \to Is(V)$ is a strongly continuous co-homomorphism and $\alpha : X \to V^*$ is a weak-star continuous bounded map into the dual of $V$ such that $(h,\alpha)$ respect the original action and the induced dual action, i.e.,

$$\langle v, \alpha(gx)\rangle = \langle h(g)(v), \alpha(x)\rangle \quad \text{for all} \quad v \in V, \quad g \in G, \quad x \in X.$$

We say that $(G,X)$ is *Rosenthal representable*, or WRN (weakly Radon-Nikodym), if there exist a Rosenthal Banach space $V$ and a representation $(h,\alpha)$ as above such that $\alpha$ is a topological embedding. For compact $G$-systems $X$ this concept was defined in [11]. The class of all Rosenthal representable dynamical systems and its subclass of tame systems are quite large. When considering the trivial action of $G$ on a space $X$, this definition reduces to the purely topological notion of a WRN space.

For motivation, properties and examples, see, for example, [11,12,15] and the references therein. A relevant recent result is that, for every circularly (in particular, linearly) ordered compact space $K$, the action $H_+(K) \curvearrowright K$, of the topological group $H_+(K)$ of all order-preserving homeomorphisms of $K$ on $K$, is Rosenthal representable [14].

Recall that a *continuum* is a compact Hausdorff connected space. A continuum $D$ is said to be a *dendron* [42] if every pair of distinct points $u$ and $v$ can be separated in $D$ by a third point $w$. A metrizable dendron is called a *dendrite*. The class of dendrons is an important class of 1-dimensional treelike compact spaces. A compact space is said to be a *local dendron* if each of its points admits a closed neighborhood which is a dendron. Similarly, a compact space is called a *local dendrite* if each of its points admits a closed neighborhood which is a dendrite. Clearly, the circle is a local dendrite.

In a dendron $X$, for every pair of points $u$ and $v$ in $X$ one defines a "generalized arc"

$$[u,v] = \{x \in X : x \text{ separates } u \text{ from } v\} \cup \{u,v\}.$$

In dendrites the generalized arc $[u,v]$ is a *real arc*, i.e., a topological copy of an interval in the real line. This is not necessarily the case for dendrons. Indeed, note that any connected linearly ordered compact space, in its interval topology, is an example of a dendron.

A topological space $X$ is called *regular* if every point has a local base for its topology, each member of which has finite boundary. For compact Hausdorff spaces this is equivalent to saying that each open cover admits a finer (finite) open cover each member of which has finite boundary (see, for example, [24]). Some works refer to such compact spaces as being *rim-finite* (see, for example, [43]). Every (local) dendron is regular [43, Theorem 21]. For more information on dendrons and dendrites, see, for example, [6,7,42].

Our goal in the first part of this paper is to prove the following two results (see Theorems 2.3 and 3.14 for the proofs).

**Theorem 1.2.** *Any action of a group $G$ on a regular continuum is null, hence also tame.*

**Theorem 1.3.** *Let $D$ be a dendron. For every topological group $G$ and continuous action $G \curvearrowright D$, the dynamical $G$-system $D$ is Rosenthal representable, hence also tame.*

In the proof of Theorem 2.3 we use a method of Seidler [35]. Note that by a result of Kerr and Li [22] every null dynamical $G$-system is tame. This explains the tameness conclusion in Theorem 1.2.

In the proof of Theorem 3.14 we use the following useful characterization.

**Theorem 1.4** (See [11, Theorem 6.10]). *A dynamical $G$-system $X$ is Rosenthal representable if and only if there exists a $G$-invariant point-separating bounded family $F$ of continuous functions $X \to \mathbb{R}$ which is tame as a family of functions.*

We consider monotone (not necessarily, continuous) functions (see Definition 3.4) on dendrons $D$. In Theorem 3.10 we show that any such function is fragmented (Baire class 1, on dendrites). In order to apply Theorem 1.4 for dendrons $D$, in the role of the family $F$ we consider the set of all continuous monotone functions $D \to [0,1]$.

Later in Theorem 4.7 we will generalize Theorem 1.3 to *compact median pretrees* with monotone group actions. This approach provides the following corollary (see Corollary 4.8).

**Corollary 1.5.** *Let $X$ be a $\mathbb{Z}$-tree. Denote by* $\mathrm{Ends}(X)$ *the set of all its ends. Then for every monotone group action $G \curvearrowright X$ by homeomorphisms, the induced action of $G$ on the compact space $\widehat{X} := X \cup$* $\mathrm{Ends}(X)$ *is Rosenthal representable.*

As a related result we show, in Theorem 4.9, that Helly's selection principle can be extended to bounded monotone sequences of real-valued functions defined on median pretrees (e.g., dendrons or linearly ordered sets). Also in Theorem 4.13 we show that every monotone real-valued function $f: X \to \mathbb{R}$ on a compact Hausdorff pretree is fragmented.

In the second part of the paper, as applications of Theorems 1.2 and 1.3, and building on ideas and results from [7] and [28], we easily recover some old results, and prove some new ones, concerning dynamical systems defined on (local) dendrons.

In Section 5 we show that when a group $G$ acts on a dendrite $X$ with no finite orbits and $M \subset X$ is the unique minimal set, then the action on $X$ is strongly proximal and the action on $M$ is extremely proximal. In Section 6 we show that for an amenable group $G$ every infinite minimal set in a dendrite system $(G, X)$ is almost automorphic. This result was strengthened recently by Shi and Ye [38], who have shown that it is actually equicontinuous. In the final section we comment on the special case where the acting group is the group of integers.

## 2 Actions of groups on a regular continuum are null

Following Goodman [19], for a sequence $S = \{s_1, s_2, \dots\} \subset G$ we define the topological sequence entropy of a dynamical system $(G, X)$, with respect to $S$ and a finite open cover $\mathbf{A}$ of $X$, by

$$h_{\mathrm{top}}(X, \mathbf{A}; S) = \lim_{n\to\infty} n^{-1} \log \left( N\left( \bigvee_{i=1}^{n} s_i(\mathbf{A}) \right)\right),$$

where $N(\cdot)$ denotes the minimal cardinality of a subcover. We say that $(G, X)$ is *null* if $h_{\mathrm{top}}(X, \mathbf{A}; S) = 0$ for all open covers $\mathbf{A}$ of $X$ and all sequences $S$ in $G$.

The proofs in this section are taken, almost verbatim, from Seidler's paper [35]. For a subset $A \subset X$ we denote its boundary by $\partial(A)$.

**Lemma 2.1.** *Let $\mathbf{A}$ be an open cover of a compact continuum $X$ and let $P$ be the set of boundary points of elements of $\mathbf{A}$. If $\mathbf{A}$ is a minimal cover then the number of elements of $\mathbf{A}$ is at most the number of elements in $P$.*

*Proof.* Assume that $\mathbf{A}$ does not have a proper subcover. Then $\mathbf{A}$ is a finite collection because $X$ is compact. Let $a$ be the number of elements of $\mathbf{A}$ so that $\mathbf{A} = \{A_i : 1 \leqslant i \leqslant a\}$. Let $A_j \in \mathbf{A}$, and let $V = \bigcup_{i\neq j} A_i$. Note that $V$ is open as a union of open sets. Because $\mathbf{A}$ does not have a proper subcover we see that $X \setminus V \neq \emptyset$, so that $\partial(V) \neq \emptyset$, since $X$ is connected. Furthermore, $\partial(V) \subset P$ because $V$ is a finite union of elements of $\mathbf{A}$. As $V$ is open, $\partial(V) \cap V = \emptyset$, so that $\partial(V) \subset A_j$. Thus, for each element $A_j$ of $\mathbf{A}$, there must exist a nonempty subset of $P$ contained in $A_j$ but disjoint from every other element of $\mathbf{A}$. This implies that the number of elements of $\mathbf{A}$ is at most the number of elements in $P$. □

**Lemma 2.2.** *Let $G$ be a countable infinite group acting on a regular compact continuum $X$. Let $S = \{s_0, s_1, s_2, \dots\}$ be a sequence of elements of $G$. Let $\mathbf{A}$ be a minimal open cover of $X$ of at least two elements such that every element of $\mathbf{A}$ has finite boundary. Let $L_{\mathbf{A}}$ be the total number of boundary points of elements of $\mathbf{A}$. For each positive integer $n$, let $\mathbf{M}_n$ be a subcover of minimum cardinality of $\bigvee_{i=0}^{n-1} s_i(\mathbf{A})$ and let $P_n$ be the collection of boundary points of elements of $\tilde{\mathbf{A}} = \bigcup_{i=0}^{n-1} s_i(\mathbf{A})$. Then*

(1) *For each positive integer $n$ every boundary point of an element of $\mathbf{M}_n$ is in $P_n$.*

(2) $N(\bigvee_{i=0}^{n-1} s_i(\mathbf{A})) \leqslant nL_{\mathbf{A}}$.

*Proof.* (1) Let $n$ be a positive integer and let $x$ be a boundary point of $M \in \mathbf{M}_n$. By the definition of $\mathbf{M}_n$, for each positive integer $0 \leqslant i < n$, there exists $V_i \in \tilde{\mathbf{A}}$ such that $M = \bigcap_{i=0}^{n-1} V_i$. Let $x$ be a boundary point of $M$ and let $B$ be an open set containing $x$; clearly $B \cap M \neq \emptyset$ so that $B \cap V_i \neq \emptyset$ for each $i$. For each $V_i$, this requires that either $x \in \partial(V_i)$ or $x \in V_i$. Suppose that every $V_i$ contains $x$. Then $x \in M$ by definition. But this contradicts $x \in \partial(M)$, because $M$ is open. Thus $x$ is a boundary point of at least one $V_i$ and thus $x \in P_n$.

(2) Let $n$ be a positive integer. Because each $s_j$ is a homeomorphism the number of boundary points of elements of $s_j(\mathbf{A})$ is $L_{\mathbf{A}}$ for every integer $j$. This requires that the number of elements of $P_n$ be at most $nL_{\mathbf{A}}$. As every boundary point of an element of $\mathbf{M}_n$ is in $P_n$ and as $\mathbf{M}_n$ does not have a proper subcover, Lemma 2.1 implies that there exist at most $nL_{\mathbf{A}}$ elements in $\mathbf{M}_n$. The desired result then follows from the definition of $\mathbf{M}_n$. □

**Theorem 2.3.** *Every action of a group $G$ on a regular continuum is null, hence a fortiori tame.*

*Proof.* Let $X$ be a regular compact space on which $G$ acts. Let $\mathbf{A}$ be a minimal open cover of $X$ containing at least two elements such that every element of $\mathbf{A}$ has finite boundary. Let $L_{\mathbf{A}}$ be the total number of boundary points of elements of $\mathbf{A}$. Given $S = \{s_1, s_2, \dots\} \subset G$, we have then, from Part (2) of Lemma 2.2, that

$$\begin{aligned} h_{\mathrm{top}}(X, \mathbf{A}; S) &= \lim_{n\to\infty} n^{-1} \log\left(N\left(\bigvee_{i=1}^{n} s_i(\mathbf{A})\right)\right) \\ &\leqslant \lim_{n\to\infty} n^{-1} \log(nL_{\mathbf{A}}) = 0. \end{aligned}$$

Thus $h_{\mathrm{top}}(X; S) = 0$ and this shows that the system $(G, X)$ is null. By a theorem of Kerr and Li [22] it is also tame. □

**Remark 2.4.** In [22, Theorem 12.2], Kerr and Li demonstrated with a simple proof, that every action of a convergence group $G$ on a compact space $X$ (in particular, any hyperbolic group acting on its Gromov boundary) is null.

## 3 Dendrons, monotone functions and group actions

### 3.1 Standard betweenness relations and dendrons

All the topological spaces in this work are assumed to be Hausdorff. Let $X$ be a connected topological space and $u, v \in X$. As usual, we say that a point $w$ *separates* $u$ and $v$ in $X$ if there exist in $X$ open disjoint neighborhoods $U$ and $V$ of $u$ and $v$ respectively such that $X \setminus \{w\} = U \cup V$.

For every $u, v$ in $X$ define the "generalized arc"

$$[u, v] = \{x \in X : x \text{ separates } u \text{ from } v\} \cup \{u, v\}.$$

By definition $[u, v] = [v, u]$.

**Definition 3.1.** Let $X$ be a topological space and $u, w, v \in X$. We say that $w$ is *between* $u$ and $v$ in $X$ if $w \in [u, v]$, i.e., $w$ separates $u$ and $v$ or $w \in \{u, v\}$. This defines a natural betweenness ternary relation on $X$. Denote by $R_B$ this ternary relation. Sometimes we write $\langle u, w, v\rangle$ instead of $(u, w, v) \in R_B$.

**Lemma 3.2.** *Let $D$ be a dendron.*

(1) (See [42]) *Then $D$ is locally connected and the intersection of arbitrary family of subcontinua of $D$ is either empty or is a continuum.*

(2) (See [42, Corollary 2.15.1]) *$[u, v]$ is the smallest subcontinuum of $D$ containing $u$ and $v$, i.e.,*

$$[u, v] = \bigcap\{C \subseteq D : u, v \in C \text{ and } C \text{ is a subcontinuum}\}.$$

(3) (See [3]) *$[a, b] \subseteq [a, c] \cup [c, b]$ for every $a, b, c \in D$.*

Every dendron with its standard betweenness relation is a *pretree* (see Section 4). This provides another explanation of Lemma 3.2(3). The following proposition can be derived from a result of Bankston [4, Theorem 3.1], which, in fact, shows that this assertion holds for every locally connected continuum. The direct proof given below, for dendrons, was explained to us by Nicolas Monod.

**Proposition 3.3.** *Let $D$ be a dendron. Then the betweenness relation $R_B$ on $D$ is closed.*

*Proof.* Suppose that in $D$ we have converging nets, $\lim u_i = u$, $\lim_i w_i = w$ and $\lim v_i = v$ such that $\langle u_i, w_i, v_i\rangle$ for every $i \in I$. We have to show that $\langle u, w, v\rangle$. Assuming the contrary, we have $w \notin [u, v]$. Since $[u, v]$ is compact there exist neighborhoods $U$, $W$, $V$ of $u$, $w$, $v$ respectively such that $W \cap (U \cup [u, v] \cup V) = \emptyset$. Since $D$ is locally connected, we can assume, in addition, that $U$ and $V$ are connected and closed. There exists $i_0$ such that $w_{i_0} \in W$, $u_{i_0} \in U$, $v_{i_0} \in V$. By Lemma 3.2(2) we have $[u_{i_0}, u] \subset U$, $[v, v_{i_0}] \subset V$. Then $[u_{i_0}, v_{i_0}] \subset [u_{i_0}, u] \cup [u, v] \cup [v, v_{i_0}]$ by Lemma 3.2(3). By our choice $W$ does not meet $[u_{i_0}, u] \cup [u, v] \cup [v, v_{i_0}]$, hence, $w_{i_0} \notin [u_{i_0}, v_{i_0}]$. This contradiction completes the proof. □

Note that in the *comb space* (which is not locally connected) the relation $R_B$ is not closed (see Bankston [4, Exercise 3.4(ii)]).

### 3.2 Monotone functions

**Definition 3.4.** Let us say that a (**not necessarily continuous**) map $f : X \to Y$ between two connected topological spaces is:

(1) *B-monotone* if it respects the betweenness relations $R_B$ (from Definition 3.1) of $X$ and $Y$, which means that $\langle u, w, v\rangle$ implies $\langle f(u), f(w), f(v)\rangle$. It is equivalent to the requirement that $f$ be *interval preserving*: $f[u, v] \subseteq [f(u), f(v)]$. Notation: $f \in M_B(X, Y)$. For $Y = \mathbb{R}$ we write $M_B(X)$.

(2) *C-monotone* (or, simply, *monotone*) if the preimage $f^{-1}(A)$ of every connected subset $A \subset Y$ is connected. Notation: $f \in M_C(X, Y)$. For $Y = \mathbb{R}$ we write $M_C(X)$.

For continuous maps on continua definition (2) is well known. See Kuratowski [24, Section 46]. Not every B-monotone continuous function is C-monotone. For a concrete example consider the distance (continuous) function

$$f : [0, 1]^2 \to \mathbb{R}, \quad x \mapsto d(x, K), \quad K := \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) \times [0, 1].$$

The fiber $f^{-1}(0) = K$ is not connected. So $f$ is not monotone. On the other hand, $f$ is B-monotone because $[0, 1]^2$ has no separating points.

**Lemma 3.5.** *Let $X$ be a connected space.*

(1) *Composition of B-monotone (C-monotone) functions is B-monotone (respectively, C-monotone).*

(2) *Let $G \curvearrowright X$ be an action of a group $G$ on $X$ by homeomorphisms. For every $g \in G$ and every $f \in M_B(X)$ ($f \in M_C(X)$) we have $fg \in M_B(X)$ (respectively, $fg \in M_C(X)$), where $(fg)(x) := f(gx)$.*

(3) *The set $M_B(X, D)$ is a pointwise closed (hence, compact) subset of $D^X$ for every dendron $D$.*

(4) *The set $M_B(X, [c, d])$ is a pointwise closed (hence, compact) subset of $[c, d]^X$.*

*Proof.* (1) and (2) are straightforward.

(3) Let $f : X \to D$ be the pointwise limit of the net $f_i : X \to D$, $i \in I$ where each $f_i \in M_B(X, D)$. We have to show that $f$ is also B-monotone, i.e., $\langle u, w, v\rangle$ in $X$ implies that $\langle f(u), f(w), f(v)\rangle$ in $D$. Since every $f_i$ is B-monotone we have $\langle f_i(u), f_i(w), f_i(v)\rangle$. As we already mentioned (see Proposition 3.3) the betweenness relation $R_B$ is closed in $D^3$. So, since $f$ is the pointwise limit of $f_i$ we get $\langle f(u), f(w), f(v)\rangle$.

(4) is a particular case of (3). □

**Lemma 3.6.** *For every dendrons $X, Y$ we have $M_C(X, Y) = M_B(X, Y)$ and $M_C(X) = M_B(X)$.*

*Proof.* (1) $M_C(X, Y) \subseteq M_B(X, Y)$.

Assuming the contrary let $f : X \to Y$ be C-monotone but not B-monotone. Then there exist $u, w, v \in X$ such that $\langle u, w, v\rangle$ but $\neg\langle f(u), f(w), f(v)\rangle$. This means that $w$ separates the points $u$ and $v$ but $f(w)$ does not separate the pair $f(u), f(v)$ in $Y$ and $f(w) \notin \{f(u), f(v)\}$. So,

$$f(w) \notin C := [f(u), f(v)].$$

Since $Y$ is a dendron the generalized arc $C = [f(u), f(v)]$ is connected (see Lemma 3.2). By the C-monotonicity of the function $f$ the preimage $f^{-1}(C)$ is connected in $X$. As $w$ separates $u$ and $v$ in $X$ we have

$$X \setminus \{w\} = U \cup V,$$

where $U$ and $V$ are disjoint open neighborhoods of $u$ and $v$, respectively. Then $u \in f^{-1}(C) \cap U$ and $v \in f^{-1}(C) \cap V$ are disjoint, nonempty (and open in $f^{-1}(C)$). The union of these subsets is $f^{-1}(C)$ (because $w \notin f^{-1}(C)$). So, we get that $f^{-1}(C)$ is not connected, a contradiction to the fact that $f$ is C-monotone.

(2) $M_C(X, Y) \supseteq M_B(X, Y)$.

Assuming the contrary let $f \in M_B(X, Y)$ such that $f \notin M_C(X, Y)$. Then there exists a connected subset $C \subset Y$ such that $f^{-1}(C)$ is not connected in $X$. So there exist (distinct) $u, v \in f^{-1}(C)$ such that the "generalized arc" $[u, v]$ (which is connected by Lemma 3.2 because $X$ is a dendron) is not contained in $f^{-1}(C)$. Therefore, there exists $w \in [u, v]$ such that $w \notin f^{-1}(C)$. Then $\langle u, w, v\rangle$ but it is not true that $\langle f(u), f(w), f(v)\rangle$ because $f(u), f(v) \in C \subset Y \setminus \{f(w)\}$, and $C$ is connected.

This proves $M_C(X, Y) = M_B(X, Y)$. In order to conclude that $M_C(X) = M_B(X)$ use the linear order equivalence $\mathbb{R} \to (0, 1) \subset Y := [0, 1]$, $x \mapsto \frac{x}{1+|x|}$. □

**Remark 3.7.** Lemma 3.6 suggests dropping the subscripts "C" and "B" and writing simply $M(D_1, D_2)$. We write $CM(D)$ for the set of continuous monotone real-valued functions on $D$.

Recall that the set $\mathcal{F}(X)$ of fragmented real-valued functions on $X$ is a vector space over $\mathbb{R}$. For the definition and properties of fragmented functions we refer to [10, 11, 30]. In the present paper we only use fragmentability in the case of real-valued functions $f : X \to \mathbb{R}$ defined on compact $X$. In this case the fragmentability of $f$ is equivalent to the PCP-property (see [10]), meaning that for every closed nonempty subset $Y \subseteq X$ the restriction $f|_Y : Y \to \mathbb{R}$ has a point of continuity. For Polish $X$, $\mathcal{F}(X)$ coincides with the set $\mathcal{B}_1(X)$ of Baire class 1 functions. A function $f : X \to Y$ is said to be a *Baire class* 1 *function* if the inverse image $f^{-1}(O)$ of every open set $O \subset Y$ is $F_\sigma$ in $X$.

In [13, 31] we proved that every linear order preserving function on a compact linearly ordered topological space is fragmented. The following Theorem 3.9 is a result in the same spirit.

**Definition 3.8.** Let $R$ be an abstract ternary relation on a Hausdorff topological space $(X, \tau)$.

(1) We say that $R$ is *$\tau$-stable* if for every pair of distinct points $u, v \in X$, there are, a point $w \in X$ with $w \in [u, v] \setminus \{u, v\}$ (where $[u, v] := \{x \in X : \langle u, w, v\rangle\} \cup \{u, v\}$), and neighborhoods $U$ and $V$ of $u$ and $v$ respectively, such that $w \in [x, y]$ for every $x \in U, y \in V$.

(2) We say that $R$ is *weakly $\tau$-stable* if for every infinite subset $K \subset X$ there exist: a pair of distinct points $u, v \in K$, a point $w \in X$ with $w \in [u, v]\setminus\{u, v\}$, and neighborhoods $U$ and $V$ of $u$ and $v$ respectively, such that $w \in [x, y]$ for every $x \in U \cap K, y \in V \cap K$.

Let $R$ be a ternary relation on $X$ and let $f : X \to \mathbb{R}$ respect $R$ and the standard betweenness relation of the reals. We will say that $f$ is an *R-monotone* function.

**Theorem 3.9.** *Let $(X,\tau)$ be a compact space and $R$ a ternary relation on $X$ which is weakly $\tau$-stable. Then every $R$-monotone function $f: X \to \mathbb{R}$ is fragmented.*

*Proof.* If $f$ is not fragmented then, by [41, Lemma 3.7] there exist a closed nonempty subspace $L \subset X$ and real numbers $\alpha < \beta$ such that the subsets $f^{-1}(-\infty,\alpha)\cap L$ and $f^{-1}(\beta,\infty)\cap L$ are dense in $L$, i.e.,

$$cl(f^{-1}(-\infty,\alpha)\cap L) = cl(f^{-1}(\beta,\infty)\cap L) = L. \tag{3.1}$$

$L$ is necessarily infinite (because $f^{-1}(-\infty,\alpha)\cap L$ and $f^{-1}(\beta,\infty)\cap L$ are disjoint). Since $X$ is weakly $\tau$-stable we may choose distinct points $u,v\in L$, $w\in X$ with $w\in[u,v]\setminus\{u,v\}$, and $\tau$-neighborhoods $U$ and $V$ of $u$ and $v$ respectively such that $\langle x,w,y\rangle$ for every $x\in U\cap L, y\in V\cap L$.

By Equation (3.1) there exist $u',u''\in U\cap L$ and $v',v''\in V\cap L$ such that

$$f(u')<\alpha, \quad f(v')<\alpha, \quad \beta<f(u''), \quad \beta<f(v''). \tag{3.2}$$

It is enough to show that $f$ is not R-monotone. There are two cases to check:

**Case 1.** $f(w)\leqslant\beta$. Then $\langle u'',w,v''\rangle$ but $f(w)$ is not between $f(u''), f(v'')$ in $\mathbb{R}$.

**Case 2.** $\beta<f(w)$. Then $\langle u',w,v'\rangle$ but $f(w)$ is not between $f(u'), f(v')$ in $\mathbb{R}$. □

**Theorem 3.10.** *Let $D$ be a dendron. Every monotone function $f: D\to\mathbb{R}$ is fragmented (Baire class 1, when $D$ is a dendrite). It follows that $M(D)\subset\mathcal{F}(D)$.*

*Proof.* For a dendron $D$ the standard betweenness relation (see Definition 3.1) is stable. Indeed, by the definition of dendrons for every distinct $u\neq v$ we have a separation by a point $w$. So, $D\setminus\{w\}=U\cup V$, where $U$ and $V$ are open disjoint neighborhoods of $u$ and $v$. Hence, $w$ separates any pair $x\in U, y\in V$. □

For dendrites we have $M(D)\subset\mathcal{B}_1(D)=\mathcal{F}(D)$.

**Corollary 3.11.** *For every dendron $D$ the family $F:=CM(D,[0,1])$ is tame.*

*Proof.* By Lemma 3.5(4), every function $\varphi: D\to[0,1]$ from the pointwise closure of $F$ in $[c,d]^D$ is a (not necessarily continuous) B-monotone function. By Lemma 3.6, $\varphi\in M(D,[0,1])$. By Theorem 3.10 we know that $M(D,[0,1])\subset\mathcal{F}(D)$ and we conclude that $cl_p(F)\subset\mathcal{F}(D)$. This means that $F$ is a Rosenthal family, in terms of [11]. This is the same as saying that $F$ does not contain an independent sequence (for a detailed proof see for example [13, Theorem 2.12]). □

In [42, Corollary 2.15] van Mill and Wattel proved the following remarkable result. We thank Jan van Mill for providing us the short proof below. For dendrites Theorem 3.12 is well known [6, Theorem 1.2].

**Theorem 3.12.** *For every dendron $D$ and every subcontinuum $C$ there exists a naturally defined continuous retraction $r_C: D\to C$. Moreover, this retraction is always monotone.*

*Proof.* The map $r_C: D\to C$ is defined by

$$r_C(x)=\bigcap\{[x,c]\cap C: c\in C\}.$$

We discuss only the monotonicity of $r_C$. Other details see in [42, Corollary 2.15]. Let $x\in D$ and $y=r_C(x)$. If $p\in[x,y]$ then $[p,y]$ is contained in $[x,y]$. Since $y\in C$, the formula for $r_C(p)$ gives us that $r_C(p)\in[p,y]\cap C=y$, hence $r_C(p)=y$. Thus all the points in the fiber of the point $y\in C$ can be connected to $y$ by a continuum in the fiber. So, $r_C^{-1}(y)$ is connected for every $y\in C$. Since $r_C: D\to C$ is a continuous closed map then by [24, Section 46, Subsection I, Theorem 9] the map $r_C$ is C-monotone (see Definition 3.4). The subcontinuum $C$ is also a dendron. So, $r_C$ is also $B$-monotone (see Lemma 3.6). □

This result leads to the following lemma.

**Lemma 3.13.** *On every dendron $D$ and every pair $u$ and $v$ of distinct points in $D$ there exists a continuous monotone function $f: D\to[0,1]$ such that $f(u)=0, f(v)=1$.*

*Proof.* By Theorem 3.12 for every pair of distinct points $u,v\in D$ we have a monotone continuous retraction $r_{[u,v]}: D\to[u,v]$. Now recall that the "generalized arc" $[u,v]$ is a linearly ordered compact connected space [42]. By results of Nachbin [32, pp. 48 and 113] we have an order-preserving (hence, monotone in the sense of Definition 3.4(1)) continuous map $h : [u,v] \to [0,1]$. The composition $f = h \circ r_{[u,v]}$ is the required continuous monotone map $D \to [0,1]$ which separates $u$ and $v$. □

**Theorem 3.14.** *Let $D$ be a dendron. For every topological group $G$ and continuous action $G \curvearrowright D$, the dynamical $G$-system $D$ is Rosenthal representable* (*hence, also tame*). *It follows that the topological group $H(D)$ is Rosenthal representable.*

*Proof.* By Lemma 3.5(2) we have $fg \in CM(D,[0,1])$ for every $g \in G$ and every $f \in CM(D,[0,1])$. So, if $F := CM(D,[0,1])$ then $FG = F$ is a $G$-invariant bounded family of continuous functions. By Corollary 3.11, $F$ is a tame family. By Lemma 3.13, $CM(D,[0,1])$ separates the points of $D$. So one may apply Theorem 1.4 and we obtain that the dynamical $G$-system $D$ is Rosenthal representable.

Finally, note that it is straightforward to see that Rosenthal representability of any compact dynamical $H(K)$-system $K$ implies that the topological group $H(K)$ is Rosenthal representable (for details see for example [13, Lemma 3.5]). □

**Theorem 3.15.** *Every monotone map $f : D_1 \to D_2$ between dendrons is fragmented* (*Baire class* 1 *map, if $D_1$ and $D_2$ are dendrites*).

*Proof.* By Lemma 3.13 there exists a family of continuous monotone maps $\{q_i : D_2 \to [0,1] : i \in I\}$ which separates the points of $D_2$. Then any composition $q_i \circ f : D_1 \to [0,1]$ is monotone, hence fragmented by Theorem 3.10. Now, using [11, Lemma 2.3.3] we obtain that the original map $f : D_1 \to D_2$ is also fragmented. □

The tameness of any continuous group actions on dendrons $G \curvearrowright D$ can be derived from Theorem 3.15 by the following corollary.

**Corollary 3.16.** *Let $G$ act on a dendron $D$. Then every element of the enveloping semigroup $E(G,D)$ is fragmented, and hence the system $(G,D)$ is tame.*

*Proof.* Using Lemma 3.5(3) we obtain that every element $p \in E(G,D)$ is a monotone map $p : D \to D$. By Theorem 3.15, $p$ is a fragmented map and it follows that the $G$-system $D$ is tame by the enveloping semigroup characterization of tameness [11]. □

Theorem 3.14 implies also the following purely topological nontrivial fact.

**Corollary 3.17.** *Every dendron $D$, as a topological space, is Rosenthal representable, i.e., $D$ is WRN.*

As a related result note that by [42, Theorem 6.6] a Hausdorff space can be embedded in a dendron if and only if it possesses a *cross-free* (see [42] for the definitions) closed subbase.

**Remark 3.18.** Theorem 3.14 remains true also for continuous *monotone* monoid actions $S$ on $D$ such that for all $s \in S$ the corresponding $s$-translation $D \to D$ is monotone. Clearly continuous group actions on dendrons are always monotone.

## 4 Monotone actions on median pretrees

In this section we consider actions on a pretree, a useful treelike structure that naturally generalizes several important structures including linear orders and the betweenness relation on dendrons. By a *pretree* (see for example [5,27]) we mean a pair $(X,R)$, where $X$ is a set and $R$ is a ternary relation on $X$ (we write $\langle a,b,c\rangle$ to denote $(a,b,c) \in R$) satisfying the following three axioms:

(B1) $\langle a,b,c\rangle \Rightarrow \langle c,b,a\rangle$.

(B2) $\langle a,b,c\rangle \wedge \langle a,c,b\rangle \Leftrightarrow b = c$.

(B3) $\langle a,b,c\rangle \Rightarrow \langle a,b,d\rangle \vee \langle d,b,c\rangle$.

In [2] such a ternary relation is called a *B-relation*.

It is convenient to use also an interval approach. For every $u,v \in X$ define

$$[u,v] := \{x \in X : \langle u,x,v\rangle\}.$$

In the list of properties below the first four conditions (A0), (A1), (A2) and (A3), as a system of axioms, are equivalent to the above definition via (B1), (B2) and (B3) (see [27]).

**Lemma 4.1.** *In every pretree* $(X,R)$ *we have*

(A0) $[a,b] \supseteq \{a,b\}$.

(A1) $[a,b]=[b,a]$.

(A2) *If* $c \in [a,b]$ *and* $b \in [a,c]$ *then* $b=c$.

(A3) $[a,b] \subseteq [a,c] \cup [c,b]$ *for every* $a,b,c \in X$.

(A4) $[a,b] = [a,c] \cup [c,b]$ *for every* $a,b \in X$, $c \in [a,b]$.

(A5) *If* $b \in [a,c]$ *and* $c \in [a,d]$ *then* $c \in [b,d]$.

Following [27] we define the so-called *shadow topology* $\tau_s$ on $(X,R)$. Given an ordered pair $(u,v) \in X^2, u \neq v$, let

$$S_u^v := \{x \in X : u \in [x,v]\}$$

be the *shadow* in $X$ defined by the ordered pair $(u,v)$. Pictorially, the shadow $S_u^v$ is cast by a point $u$ when the light source is located at the point $v$. The family $\mathcal{S} = \{S_u^v : u,v \in X, u \neq v\}$ is a subbase for the closed sets of the topology $\tau_s$.

In the case of a linearly ordered set we get the interval topology. In general, for an abstract pretree the shadow topology is often (but not always) Hausdorff. Furthermore, by [27, Theorem 7.3] a pretree equipped with its shadow topology is Hausdorff if and only if, as a topological space, it can be embedded into a dendron. So, by Corollary 3.17 we can deduce the following:

**Corollary 4.2.** *Every Hausdorff pretree* (*for example, linearly ordered topological space*) *is a WRN topological space.*

For every triple $a,b,c$ in a pretree $X$ the *median* $m(a,b,c)$ is the intersection

$$m(a,b,c) := [a,b] \cap [a,c] \cap [b,c].$$

When it is nonempty the median is a singleton (see for example [5, p. 14]). A pretree $(X,R)$ for which this intersection is always nonempty is called a *median pretree*.

A *median algebra* (see for example [5, p. 14] or [40]) is a pair $(X,m)$, where the function $m: X^3 \to X$ satisfies the following three axioms:

(M1) $m(x,x,y)=x$.

(M2) $m(x,y,z)=m(x,y,z)=m(y,z,x)$.

(M3) $m(m(x,y,z),u,v)=m(x,m(y,u,v),m(z,u,v))$.

**Remark 4.3.** (1) Every median pretree is a *median algebra*.

(2) A map $f: X_1 \to X_2$ between two median algebras is monotone (i.e., interval preserving) if and only if $f$ is median-preserving ([40, p. 120]) if and only if $f$ is convex ([40, p. 123]). Convexity of $f$ means that the preimage of a convex subset is convex.

(3) Every median pretree is Hausdorff in its shadow topology [27, Theorem 7.3].

A *compact* (*median*) *pretree* is a (median) pretree $(X,R)$ for which the shadow topology $\tau_s$ is compact.

**Example 4.4.** (1) Every dendron $D$ is a compact median pretree with respect to the standard betweenness relation $R_B$. Its shadow topology is just the given compact Hausdorff topology on $D$ (see [27, 42]).

(2) Every linearly ordered set is a median pretree. Its shadow topology is just the interval topology of the order.

(3) Let $X$ be a $\mathbb{Z}$-*tree* (a median pretree with finite intervals $[u,v]$). Denote by $\mathrm{Ends}(X)$ the set of all its ends. According to [27, Section 12] the set $X \cup \mathrm{Ends}(X)$ carries a natural $\tau_s$-*compact* median pretree structure.

**Proposition 4.5.** *Let* $(X,R)$ *be a median pretree. Then the retraction map*

$$\phi_{u,v}: X \to [u,v], \quad x \mapsto m(u,x,v)$$

*is monotone and continuous in the shadow topology for every* $u,v \in X$.

*Proof.* Note that always $c \in [a,b]$ if and only if $\mathrm{med}(a,c,b) = c$. So, $\phi_{u,v}(x) = x$ for every $x \in [u,v]$. This means that $\phi_{u,v}$ is a retraction. A well-known property of median algebras, namely [39, Equation (8.7)], directly implies that $m(m(u,x_1,v), m(u,x_2,v), m(u,x_3,v)) = m(u, m(x_1,x_2,x_3), v)$. This means that $m(\phi(x_1), \phi(x_2), \phi(x_3)) = \phi(m(x_1,x_2,x_3))$. So every $\phi_{u,v}$ is a median preserving map (hence, monotone, as a map between pretrees).

Now we check the continuity of $\phi_{u,v}$. If $u = v$ then $\phi_{u,v}$ is constant. So, we can suppose that $u \neq v$. Every interval $[a,b]$ is a convex subset of $X$. Hence its interval topology and its topological subspace topology of $(X, \tau_s)$ are the same (see [27, Proposition 6.5]). It is enough to show that the preimage of a closed subbase element in the space $[u,v]$ is closed in the shadow topology. We prove that in fact

$$\phi_{u,v}^{-1}[u,w] = S_w^v, \quad \forall\, w \in [u,v) \quad \text{and} \quad \phi_{u,v}^{-1}[w,v] = S_w^u, \quad \forall\, w \in (u,v].$$

First we show that $\phi_{u,v}^{-1}[u,w] \subseteq S_w^v$.

Let $x \in \phi_{u,v}^{-1}[u,w]$. This means that $m := m(u,x,v) \in [u,w]$. Since $w \in [u,v]$, by (A5) we have $w \in [m,v]$. So, $[m,w] \cup [w,v] = [m,v]$ by (A4). Again, by (A4) we obtain $[x,m] \cup [m,v] = [x,v]$. So, $w \in [x,v]$. Hence, $x \in S_w^v$.

Now we show $\phi_{u,v}^{-1}[u,w] \supseteq S_w^v$. Let $x \in S_w^v$. This means that $w \in [x,v]$. We have to show that $m \in [u,w]$ (where $m := m(u,x,v)$). Assuming the contrary, suppose $m \notin [u,w]$. Clearly, $m \in [u,v]$. By (A3), $[u,v] \subseteq [u,w] \cup [w,v]$. So, we have $m \in (w,v]$. Since $w \in [v,x]$ and $m \in [v,w]$, by (A5) we get $w \in [m,x] = [x,m]$. Similarly, since $m \in [v,w]$ and $w \in [v,u]$, by (A5) we get $w \in [m,u]$.

Now taking into account that $m \neq w$ by (A2) we have $m \notin [x,w]$ and also $m \notin [w,u]$. So, $m \notin [x,w] \cup [w,u]$. Then (A3) guarantees that $m \notin [x,u]$. This contradicts the fact that $m$, being the median of the triple $x,u,v$, belongs to $[x,u]$.

Similarly we can check also that $\phi_{w,v}^{-1}[u,w] = S_w^u$. □

**Theorem 4.6.** *Let $X$ be a median pretree. Then every pair of monotone (equivalently, convex) real-valued functions $f_i : X \to \mathbb{R}$, $i \in \{0,1\}$ is not independent.*

*Proof.* Assuming that $\{f_1, f_2\}$ is an independent pair, there exist real numbers $a < b$ such that

$$A_1 \cap A_2 \neq \emptyset, \quad A_1 \cap B_2 \neq \emptyset, \quad B_1 \cap A_2 \neq \emptyset, \quad B_1 \cap B_2 \neq \emptyset,$$

where $A_i = f_i^{-1}(-\infty, a)$, $B_i = f_i^{-1}(b, \infty)$, $i \in \{0,1\}$.

Choose four points

$$a \in A_1 \cap B_2, \quad b \in A_1 \cap A_2, \quad c \in A_2 \cap B_1, \quad d \in B_1 \cap B_2.$$

By the monotonicity of the functions $f_1$ and $f_2$ the preimages of convex subsets are convex. So, $A_i$ and $B_i$ are convex. Then

$$[a,b] \subset A_1, \quad [b,c] \subset A_2, \quad [c,d] \subset B_1, \quad [d,a] \subset B_2.$$

Consider the median $m := \mathrm{med}(c,a,d) = [a,c] \cap [c,d] \cap [d,a]$. Then $m \in [a,c] \cap B_1 \cap B_2$. Using (A3) we get $[a,c] \subset [a,b] \cup [b,c] \subset A_1 \cup A_2$. Therefore, $m \in (A_1 \cup A_2) \cap (B_1 \cap B_2)$. Clearly, $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$. So, $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \emptyset$, a contradiction. □

For a compact median pretree $X$ we denote by $H_+(X)$ the topological group of R-monotone (equivalently, median-preserving) homeomorphisms. We treat $H_+(X)$ as a topological subgroup of the full homeomorphism group $H(X)$.

The following result generalizes Theorem 3.14. In the case of a dendron $D$ we have $H_+(D) = H(D)$.

**Theorem 4.7.** *For every compact median pretree $X$ and its automorphism group $G = H_+(X)$ the action of the topological group $G$ on $X$ is Rosenthal representable. It follows that the topological group $H_+(X)$ is Rosenthal representable.*

*Proof.* Recall again that a median pretree is Hausdorff in its shadow topology (see Remark 4.3(3)). By Proposition 4.5 the retraction map $\phi_{u,v} : X \to [u,v]$, $x \mapsto m(u,x,v)$ is monotone and continuous in the shadow topology for every $u, v \in X$. Every $[u,v]$ is a linearly ordered set with respect to the order $x \leqslant y$

whenever $\langle x, y, v\rangle$ (see for example [5, p. 14]). Moreover $[u, v] = \phi(X)$ is compact in the subspace topology which coincides with the interval topology of the linear order. So, as in Lemma 3.13, using Nachbin's result we see that the set $CM(X)$ of continuous monotone functions separates the points. Also, $CM(X)$ is $G$-invariant because the action $G \curvearrowright X$ is monotone. Theorem 4.6 guarantees that $CM(X)$ is a tame family. A median pretree is always Hausdorff in its shadow topology (see Remark 4.3(3)). The rest is similar to the proof of Theorem 3.14. □

By Example 4.4(3), Theorem 4.7 applies when $X$ is a $\mathbb{Z}$-tree and we get the following corollary.

**Corollary 4.8.** *Let $X$ be a $\mathbb{Z}$-tree. Denote by* $\mathrm{Ends}(X)$ *the set of all its ends. Then for every monotone group action $G \curvearrowright X$ with continuous transformations the induced action of $G$ on the compact space $\widehat{X} := X \cup \mathrm{Ends}(X)$ is Rosenthal representable.*

Such compact spaces $\widehat{X}$ as in Corollary 4.8 are often zero-dimensional. So, at least, formally this case cannot be deduced from the dendron's case.

### 4.1 Generalized Helly's selection principle

**Theorem 4.9.** *Let $X$ be a median pretree and $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a bounded sequence of convex* (*equivalently, monotone*) *real-valued maps. Then there exists a pointwise converging subsequence.*

*Proof.* Combine Theorem 4.6 with the following form of Rosenthal's theorem: let $f_n : S \to \mathbb{R}$ be a bounded sequence of functions on a set $S$. Then it has a subsequence which is pointwise converging or has a subsequence which is independent; see Theorem 1 in Rosenthal's classical paper [34]. In Rosenthal's original formulation he states a weaker statement "$l_1$-subsequence" (instead of "independent"). However, as Rosenthal's proof shows he proves in fact a little bit more (see the text above Lemma 5 on page 2413 of [34]), namely that there exists an independent subsequence. □

**Corollary 4.10.** *Let $X$ be either a dendron or a linearly ordered set. Then the pointwise compact set of all monotone maps $M(X, [c, d])$ into the real interval $[c, d]$ is sequentially compact.*

**Remark 4.11.** For linearly ordered sets Theorem 4.9 can be extended to sequences of real-valued functions with bounded total variation [31]. This suggests the idea that one should search for a right analog for bounded variation functions, defined on dendrons, or more generally on (median) pretrees.

### 4.2 Fragmentability of monotone functions on compact Hausdorff pretrees

**Lemma 4.12.** *If the shadow topology $\tau_s$ on a pretree $(X, R)$ is Hausdorff* (*for example, median pretree*), *then the pretree relation $R$ is weakly $\tau_s$-stable* (*see Definition* 3.8(2)).

*Proof.* We have to show that for every infinite subset $K \subset X$ there exist: a pair $u, v \in K$, a point $w \in X$ with $w \in [u, v] \setminus \{u, v\}$, neighborhoods $U$ and $V$ of $u$ and $v$ respectively, such that $w \in [x, y]$ for every $x \in U \cap K$ and $y \in V \cap K$. In fact, we show this for every $x \in U$ and $y \in V$.

First of all note that there exist distinct $u, v \in K$ and $w \in X$ such that $w \in [u, v] \setminus \{u, v\}$. If not, then $K$ is a *star subset* in terms of [27]. Since $\tau_s$ is Hausdorff there is no *infinite* star subset in $X$ by [27, Theorem 7.3].

Consider the standard shadow topology neighborhoods $u \in U := X \setminus S^u_w$ and $v \in V := X \setminus S^v_w$. We are going to show that $w \in [x, y]$ for every $x \in U$ and $y \in V$. By (B3) and $\langle u, w, v\rangle$ we have $\langle u, w, x\rangle \vee \langle x, w, v\rangle$. By our choice of $x \in U = X \setminus S^u_w$ it is impossible that $w \in [x, u]$. So we necessarily have the second condition $\langle x, w, v\rangle$. Now apply again (B3) but now for the triple $\langle x, w, v\rangle$ and a point $y \in V$. Then by the definition of $V := X \setminus S^v_w$ it is impossible that $\langle y, w, v\rangle$. Therefore, we necessarily have $\langle x, w, y\rangle$. □

Lemma 4.12 and Theorem 3.9 imply the following theorem.

**Theorem 4.13.** *Let $(X, \tau_s)$ be a compact Hausdorff pretree* (*for example, median pretree*). *Then every monotone real-valued function $f : X \to \mathbb{R}$ is fragmented.*

**Remark 4.14.** By a result of Malyutin [27, Theorem 7.3] every $\tau_s$-Hausdorff pretree $X$ can be topologically embedded into a dendron $D$. This leads to a natural question if there exists an *equivariant* *version* of this embedding. Consider the automorphism group $G := \mathrm{Aut}(X, R)$ (of all monotone homeomorphisms) as a topological subgroup of $H(X, \tau_s)$. Is it true that there exists an equivariant embedding of $(G, X)$ into $(H(D), D)$ with some dendron $D$?

It seems that this is true under an additional assumption that the shadow subbase $\mathcal{S}$ of $\tau_s$ is *connected* in the sense of [42]. Then it is possible to use the results from [42] and *superextensions* (as in [27, Theorem 7.3]).

## 5 Some consequences of the tameness of actions on dendrites

In the following sections we would like to apply Theorem 1.3 in order to strengthen several known results on actions of groups on dendrites, mainly from [7,28,33]. For the following definitions, see for example [18].

**Definition 5.1.** Let $(G, X)$ be a dynamical system.

(1) A pair of points $x, y \in X$ is said to be *proximal* if there is a net $g_i \in G$ and a point $z \in X$ with $\lim g_i x = \lim g_i y = z$. The system $(G, X)$ is *proximal* if every pair of points in $X$ is proximal.

(2) The system $(G, X)$ is *strongly proximal* if for every probability measure $\mu$ on $X$ there is a net $g_i \in G$ and a point $z \in X$ with $\lim g_i = \delta_z$.

(3) An infinite minimal dynamical system $(G, X)$ is said to be *extremely proximal* if for every nonempty closed subset $A \subsetneq X$ and any nonempty open subset $U \subset X$ there is an element $g \in G$ with $gA \subset U$.

An action of a group $G$ on a dendron $D$ is called *dendro-minimal* if every $G$-invariant subdendron $C \subset D$ is either all of $D$ or the empty set. By Zorn's lemma every group action on a dendron admits a (nonempty) dendro-minimal subdendron. An arc $C$ in a dendrite $D$ is a *free arc* if it contains no end points: $C \cap \mathrm{End}(D) = \emptyset$.

Except for the extreme proximality claim, the following results were proven (independently) in [7, 26, 36, 37].

**Theorem 5.2.** *Suppose $G$ acts on a dendrite $X$ with no finite orbits.*

(1) *There is a unique infinite minimal set $M \subset X$.*

(2) *If the action is dendro-minimal and $X$ has no free arc then $M = X$.*

(3) *The action on $M$ is extremely proximal.*

(4) *The action on $X$ is proximal.*

*Proof.* (1) is proved in [28, Corollary 4.3]. In fact, it suffices to assume that there is no fixed point (see Remark 4.2 in [7]).

For (2) see Remark 4.7 in [7].

(3) By Lemma 4.4 of [7] the set of end points of $X$ is contained (and is dense) in $M$. If $x$ is an endpoint then it has a basis for its neighborhoods which consists of connected open sets $U$ with $|\partial(U)| = 1$ (see Lemma 2.3 in [7]). Let $K \subsetneq M$ be a closed subset. Thus there is an $x \in \mathrm{End}(X) \cap K^c$ and a connected open neighborhood $x \in U$ with $|\partial(U)| = 1$. As $X$ is a dendrite we have that $\overline{U^c}$ is also a dendrite and, by Lemma 4.3 of [7] there is $g \in G$ with $gK \subset gU^c \subset U$.

(4) Given $x, y \in X$ there is a sequence $g_j \in G$ such that the limits $g_j x = x'$ and $\lim g_j y = y'$ exist and are elements of $M$. Since $x'$ and $y'$ are proximal, it follows that $x$ and $y$ are proximal as well. □

As corollaries we obtain the following result.

**Corollary 5.3.** *Suppose $G$ acts on a dendrite $X$ with no finite orbits and let $M \subset X$ be the unique minimal set, and then the following hold*:

(1) *The action of $G$ on $M$ is strongly proximal.*

(2) *$G$ contains a free group on two generators. In particular, $G$ is not amenable.*

(3) (See [7, Corollary 8.3]) *The amenable radical of $G$ acts trivially on $M$.*

*Proof.* It is shown in [17] that extreme proximality implies strong proximality and that a group admitting a nontrivial extremely proximal action has a free subgroup on two generators (see also [18]). To prove Part (3), consider the action of the amenable radical, say, $R \lhd G$ on $M$. This admits an invariant probability measure, say $\mu$. By strong proximality there is a sequence $g_i \in G$ and a point $y \in M$ such that $\lim g_i\mu = \delta_y$. As $R$ is a normal subgroup, it follows that each translation $g_i\mu$ as well as the limit $\delta_y$ is all $R$-invariant measures. Next, we deduce similarly that every point mass $gy$, $g \in G$, is an $R$ fixed point, and finally conclude that $R$ acts trivially on $M$. □

**Proposition 5.4.** *Let $(G, X)$ be a metric tame dynamical system.*

(1) *For any element $p \in E(G, X)$, the set $pX$ is an analytic, hence universally measurable, subset of $X$.*

(2) *For every $p \in E(G, X)$ and any probability measure $\mu$ on $X$ we have $\mu(pX) = 1$.*

*Proof.* (1) The system $(G, X)$ being tame, by the enveloping semigroup characterization of metric tame systems [16], we have that every element $p \in E(G, X)$ is Baire class 1, hence Borel measurable. It follows that the set $pX$ is an analytic, hence a universally measurable, subset of $X$.

(2) As $(G, X)$ is tame there is a **sequence** $g_n \in G$ with $\lim g_n \to p$ in $E(G, X)$. Given any function $f \in C(X)$ we have $\lim f(g_nx) = f(px)$ for every $x \in X$. We also have that the function $f \circ p$ is Baire class 1, hence Borel measurable. By Lebesgue's bounded convergence theorem it follows that

$$\int f\,d\mu = \lim \int (f \circ g_n)\,d\mu = \int \lim(f \circ g_n)\,d\mu = \int (f \circ p)\,d\mu = 1.$$

Taking $f = 1_X$ we conclude that $\mu(pX) = 1$. □

The next theorem strengthens Theorem 10.1 of [7].

**Theorem 5.5.** *Suppose $G$ acts on a dendrite $X$ with no finite orbits. Then the system $(G, X)$ is strongly proximal.*

*Proof.* It follows from Theorem 5.2 that $E(G, X)$ has a unique minimal ideal $I \subset E(G, X)$ and that $uX$ is a singleton for every minimal idempotent $u \in I$. Conversely, for every $x \in M$, there is a minimal idempotent $u \in I$ with $uX = \{x\}$. Applying Part (2) of Proposition 5.4 and Theorem 1.3 to $u$ and denoting $uX = \{x\}$, we have $\delta_x(uX) = 1$, hence $u_*\mu = \delta_x$. □

**Example 5.6.** (1) For each one of Ważewski's universal dendrites $X = D_n$, $n = 3, 4, \ldots, \infty$ (see [44]), the dynamical system $(H(X), X)$, where $H(X)$ is the group of homeomorphisms of $X$, is minimal, tame and extremely, hence also strongly, proximal. Recently Kwiatkowska gave a description of the universal minimal $G$-system $M(G)$ for the topological group $G = H(X)$ (see [25]). It would be interesting to check if $M(G)$ as a dynamical $G$-system is tame. This space $M(G)$ is not a dendrite (being zero-dimensional) but perhaps it admits some suitable betweenness relation.

(2) Let $T$ be the $\mathbb{R}$-tree built on the Cayley graph of the free group on two generators $F_2$ with $S = \{a, b, a^{-1}, b^{-1}\}$ as a set of generators. Let $X = T \cup Y$ be the natural compactification of $T$ obtained by adding the boundary $Y$ comprising the infinite reduced words on the generators $\{a, b, a^{-1}, b^{-1}\}$. Then $X$ is a dendrite and the corresponding dynamical system $(F_2, X)$ is tame, with $Y$ as its unique minimal subset, and the system $(F_2, Y)$ is extremely proximal.

(3) Let $T$ be the $\mathbb{R}$-tree built on the increasing array of finite groups $\mathbb{Z}/2^k\mathbb{Z}$ and let $Y = \lim_{\leftarrow} \mathbb{Z}/2^k\mathbb{Z}$ be the inverse limit of this array which can be identified with the dyadic adding machine. Let $X = T \cup Y$ be the corresponding compactification of $T$. The dynamical system $(\mathbb{Z}, X)$ is a $\mathbb{Z}$-action on a dendrite, hence tame, with the (equicontinuous) adding machine $Y$ as its unique minimal subset.

## 6 Actions of amenable groups on dendrites

Our starting point is the structure theorem for minimal metric tame dynamical systems of amenable groups [9] (see also [8, 20, 22]).

**Theorem 6.1.** *Let $\Gamma$ be any group and $(G, X)$ a metric tame minimal system that admits an invariant probability measure $\mu$. Then $X$ is almost automorphic, i.e., it has the structure $X \stackrel{\iota}{\to} Z$, where $Z$ is equicontinuous and $\iota$ is an almost one-to-one extension. Moreover, $\mu$ is unique and the map $\iota$ is a measure theoretical isomorphism $\iota : (X, \mu, \Gamma) \to (Z, \lambda, \Gamma)$, where $\lambda$ is the Haar measure on the homogeneous space $Z$.*

*Thus, when $\Gamma$ is amenable, since every $G$-system admits an invariant probability measure, the claim above holds for any minimal metric tame $G$-system.*

**Theorem 6.2.** *Let $(G,X)$ be a tame dynamical system. Let $E=E(G,X)$ denote its enveloping semigroup and $I\subset E$ be any minimal left ideal in $E$. Let $\mathcal{M}$ be the collection of minimal subsets of $X$ and set*

$$\tilde{M}=\bigcup\{M\subset X : M\in\mathcal{M}\}.$$

*Let $P_G(X)$ be the space of $G$-invariant probability measures on $X$. Then*

(1) *For any element $p\in E$ the set $pX$ is an analytic, hence universally measurable, subset of $X$.*

(2) *$\mu(pX)=1$ for every $\mu\in P_G(X)$ and $p\in E$.*

(3) *For any element $p\in I$ the set $pX$ is contained in $\tilde{M}$.*

(4) *If $\mu\in P_G(X)$ is ergodic then there exists $M\in\mathcal{M}$ with $\mu(M)=1$. Thus the minimal subsystem $(M,\mu,G)$ is uniquely ergodic and satisfies the conclusions of Theorem* 6.1.

(5) *For any $\mu\in P_G(X)$ the measure dynamical system $(X,\mu,G)$ has a discrete spectrum, i.e., $L_2(\mu)$ is spanned by the collection of matrix coefficients of finite dimensional unitary representations of $G$.*

*Proof.* For the claims (1) and (2) repeat the proofs of the corresponding claims in Proposition 5.4.

Claim (3) is clear.

(4) Suppose $\mu\in P_G(X)$ is ergodic. Let $p$ be any element of a minimal ideal in $E(G,X)$. By (2) and (3), $\mu(pX)=1$ and we can find a $\mu$-generic point $x\in pX$. Let $M=\overline{Gx}\in\mathcal{M}$[1]. Then clearly $\mu(M)=1$. As the subsystem $(G,M)$ is minimal and tame, it follows from Theorem 6.1 that it is uniquely ergodic and satisfies the conclusion of that theorem.

(5) Let $\mu\in P_G(X)$ be any invariant measure. As in the proof of (2) we see that the map $V_p : f\mapsto f\circ p$ is a linear operator on $L_2(\mu)$ of norm less than or equal to 1. Since $\mathbf{1}\circ p=\mathbf{1}$ it follows that $\|V_p\|=1$. Moreover, the map $V:p\mapsto V_p$ is a continuous semigroup homomorphism from $E$ into the semigroup of linear contractions of $L_2(\mu)$ equipped with its strong operator topology. Let $u$ be an idempotent in $I$. Then by (2), $\mu(uX)=1$ and for every $x\in uX$ and every $p\in E$ we have $px=(pu)x$. It follows that the image of $E$ under $V$ coincides with the image of $I$, $\{V_p : p\in E\}=\{V_p : p\in I\}$. Moreover, for every $p\in I$ we have $\mu(pX\cap uX)=1$, hence $px=(up)x$, $\mu$-a.e. Thus $V_p=V_{up}$ and we conclude that $\{V_p : p\in E\}=\{V_{up} : p\in I\}$. Now $uI$ is a **group** and it follows that this image, which is also the closure of the Koopman group $\{V_g : g\in G\}$, is a compact group of unitary operators. The Peter-Weyl theorem completes the proof. □

Now using the results of this section and Theorem 1.3 we get the following corollary.

**Corollary 6.3.** *Suppose an amenable group $G$ acts on a dendrite $X$. Then*

(1) *Each minimal subset $M\subset X$ is as described in Theorem* 6.1 (*but see Theorem* 6.5 *below*).

(2) *Thus an ergodic invariant probability measure on $X$ is either a uniform distribution on a finite set, or it is the uniquely ergodic measure on a minimal infinite almost automorphic $M\subset X$.*

**Remark 6.4.** (1) In [21] Huang et al. showed that every tame cascade satisfies the "Möbius disjointness conjecture". Their proof is based on the fact that tame cascades have discrete spectrum.

(2) In [1] it was shown that every monotone cascade on a local dendrite satisfies the "Möbius disjointness conjecture".

(3) Theorem 2.3 shows that (2) can be derived from (1), at least for every invertible cascade on a local dendrite.

In the first version of our work, posted on the arXiv on June 26, 2018, we posed the following question:

**Question.** *Is there an amenable group $G$, an action of $G$ on a dendrite $X$, and a minimal subset $Y\subset X$, such that the system $(G,Y)$ is almost automorphic but not equicontinuous*?

On July 4, 2018, Shi and Ye [38] provided a negative answer:

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1) For an amenable group $G$ this has the usual meaning. When $G$ is not amenable we can still consider a probability measure $m$ on the discrete group $G$ such that $\mu(\{g\})>0$ for every $g\in G$, and then find a generic point for the Markov operator on $L_1(\mu)$ defined by $Kf=m*f$.

**Theorem 6.5.** *Let $G$ be an amenable group acting on a dendrite $X$. Suppose $K$ is a minimal set in $X$. Then $(G,K)$ is equicontinuous, and $K$ is either finite or homeomorphic to the Cantor set.*

## 7 Cascades on dendrites

Let $(T,X)$ be a cascade, i.e., a $\mathbb{Z}$-dynamical system where $T: X \to X$ is the homeomorphism of $X$ which corresponds to $1 \in \mathbb{Z}$.

We recall the following results of Naghmouchi [33]; in all of them $f: X \to X$ is a monotone dendrite map. The sets $P(f)$, $R(f)$ and $UR(f)$ denote, respectively, the set of periodic points, recurrent points and uniformly recurrent points of $f$. The set $\Lambda(f)$ is the union of the $\omega$-limit sets.

**Theorem 7.1.** *For any $x \in X$, we have*:

(1) *$\omega_f(x)$ is a minimal set.*

(2) *$\omega_f(x) \subset \overline{P(f)}$.*

**Theorem 7.2.** *For any $x \in X$, $\omega_f(x)$ is either a finite set or a minimal Cantor set. In particular, $f$ is not transitive.*

**Theorem 7.3.** *The following equalities hold*: $UR(f) = R(f) = \Lambda(f) = \overline{P(f)}$.

**Theorem 7.4.** *The restriction map $f \upharpoonright R(f)$ is a distal homeomorphism.*

Of course every self homeomorphism $T: X \to X$ of a dendrite $X$ is monotone.

We can now discuss some of Naghmouchi's results in [33] as follows (see also [28, 29]):

**Theorem 7.5.** *Let $T: X \to X$ be a self homeomorphism of a dendrite $X$ and consider the $\mathbb{Z}$-system $(T,X)$.*

(1) *If $M \subset X$ is a minimal subset then it is either finite or an adding machine.*

(2) *The union of minimal sets*

$$\tilde{M} = \bigcup \{M \subset X : M \text{ is minimal}\}$$

*is closed.*

(3) *Every point in $X \setminus \tilde{M}$ is asymptotic to $\tilde{M}$, i.e., $\omega_T(x) \subset \tilde{M}$.*

*Proof.* (1) By Theorem 1.3, the system $(T,X)$ is tame, and so is $M$. By Theorem 6.1 $M$ is almost automorphic. By Theorem 7.4 it is distal, and by Theorem 7.2 it is a Cantor set. These facts put together imply that $M$ is a minimal equicontinuous cascade on a Cantor set, i.e., an adding machine.

(2) follows from Theorem 7.3.

(3) follows from Theorem 7.1. □

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