MINICOURSE: *BANACH REPRESENTATIONS OF DYNAMICAL SYSTEMS*
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Contents

1. Introduction

Banach representations of dynamical systems is a relatively new theory already having several important applications. In the present course we study some hierarchies of topological dynamical systems and topological groups coming from Banach space theory. This allows to find new links between many different research lines. Among others: abstract topological dynamics, geometry of Banach spaces and the theory of Polish topological groups.

During this course I will expose some results and ideas mainly from recent joint works with Eli Glasner (Tel Aviv University). A part of the presented results are joint with Vladimir Uspenskij [30]. We mention also a major influence of several works of Vladimir Pestov. See, for instance, [59].

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You can download this lecture, some exercises, as well as, some related papers, from the site www.math.biu.ac.il/~megereli/seminar.html

First let us ask some intuitive questions.

Question 1.1. *What is common between:*

(1) *Grothendieck's Double Limit Property, (weak) almost periodicity and reflexive Banach spaces;*
(2) *Lack of* "butterfly effects", *averages of functions on topological groups and Asplund Banach spaces;*
(3) *Fibonacci binary sequence ("cutting sequences"), quasicrystals, monotonic maps and Rosenthal Banach spaces.*

By the Fibonacci binary sequence we mean the following particular case of a cutting binary sequence c_n of 0-s and 1-s (with the slope $\phi - 1$, ϕ= *golden ratio* $= \frac{1+\sqrt{5}}{2}$).

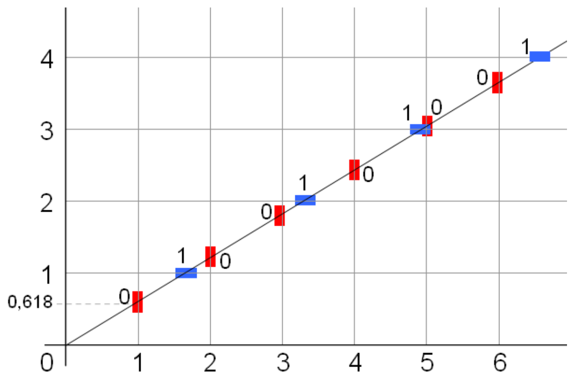


Figure 1. The Fibonacci binary sequence

It can be defined also by the finite blocks s_n using the Fibonacci substitution: $s_0 = 0, s_1 = 01, s_n = s_{n-1}s_{n-2}$. So, we have $0100101001001 \cdots$.

One may prove (Theorem 15.6) that the Fibonacci cutting sequence "lives in a Rosenthal Banach space" as a *generalized matrix coefficient*. That is, there exist: a Rosenthal Banach space V, a linear isometry $\sigma \in \mathrm{Iso}\,(V)$ and two vectors $v \in V$, $f \in V^*$ such that

$$c_n =< \sigma^n(v), f >= f(\sigma^n(v)) \qquad \forall n \in \mathbb{N}.$$

It is impossible to choose V reflexive or even Asplund.

Note that even "simple" bisequences sometimes are not reflexively representable. For example $c = \chi_{\mathbb{N}} : \mathbb{Z} \to \{0,1\}$

$$\cdots 000111 \cdots$$

is not reflexively representable but it is Asplund representable.

Our aim is to show that Questions like 1.1 can be studied by developing a relatively new tool: representations of dynamical systems on Banach spaces.

Like topological groups, compact dynamical systems, can be represented on (duals of) Banach spaces. We study dynamical analogs of Eberlein, Radon-Nikodým and weakly Radon–Nikodým compacta; that is the classes of dynamical systems which can be represented on reflexive, Asplund and Rosenthal Banach spaces. They correspond to important classes of compact metrizable dynamical systems: weakly almost periodic (WAP), hereditarily nonsensitive (HNS) and tame.

This approach naturally extends some classical research themes and at the same time opens new and sometimes quite unexpected directions. One of the examples is a connection between the lack of chaotic behavior (lack of *"butterfly effects"*) of a dynamical system (HNS systems) and the existence of weak-star continuous representations on the dual of Asplund Banach spaces. The topological concept of the *fragmentability* (originally coming from Banach spaces) and the famous factorization theorem of Davis-Figiel-Johnson-Pelczyński are the main tools in the present theory.

We provide the necessary background. Besides some new results we give soft geometric proofs of several classical results (like: Teleman's regular representations of topological groups, Ellis and Ellis-Lawson theorems; Helmer's theorem about WAP functions; Ryll-Nardzewski's fixed point theorem, etc.). At the same time we discuss perspectives of the theory and pose several open questions.

1.1. **Some concrete questions.** To every Banach space V one may associate several important structures. For example: compact spaces $X \subset B^* := (B_{V^*}, w^*)$, topological groups $G \leq \mathrm{Iso}\,(V)$ and continuous actions $G \times X \to X$, where $G \leq \mathrm{Iso}\,(V)$ and X is a G-subset of (V^*, w^*).

Let $\mathcal{K}$ be a nice subclass of Banach spaces. For example: Hilbert, reflexive, Asplund, Rosenthal. There are several good reasons explaining our interest just to these classes.

Question 1.2.

(1) *Which compact dynamical G-systems X can be represented on some $V \in \mathcal{K}$?*
(2) *Which topological groups can be embedded into* $\mathrm{Iso}\,(V)$ *where* $V \in \mathcal{K}$ *?*

Remark 1.3. A classical result of Teleman [69] (see also the survey of Pestov [59] for a detailed discussion (downloadable from the course website)) is that every (Hausdorff) topological group can be embedded into $\mathrm{Iso}\,(V)$ for some Banach space V (namely, one can take $V := \mathrm{RUC}(G)$). Furthermore, every continuous dynamical system (G, X) has a faithful representation on $V := C(X)$, where one can identify $x \in X$ with the point mass δ_x viewed as an element of $C(X)^*$. This is true also for semigroup actions. So, any compact dynamical S-system X is Banach representable (on $C(X)$). However, the Banach spaces $\mathrm{RUC}(G), C(X)$ very rarely are in a nice class.

For a given topological semigroup S, one way to measure the complexity of a compact dynamical S-system X is to investigate its representability on nice Banach spaces, [28, 27, 25]. Another way is to ask whether the points of X can be separated by a norm bounded S-invariant family $F \subset C(X)$ of continuous functions on X, such that F is "small" in some sense or another. More precisely, Questions 1.2 are closely related to the next question. For every $F \subset V$ and a weak-star compact subset $X \subset V^*$ one may consider the evaluation map

$$w : F \times X \to \mathbb{R}$$

induced by the canonical bilinear mapping $V \times V^* \to \mathbb{R}, (v, f) \mapsto f(v)$. For example, B_V separates the points of V^* (hence also of X). For every $F \subset C(X)$ the evaluation map $F \times X \to \mathbb{R}$ obviously is represented on $V := C(X)$.

Question 1.4. *Which abstract evaluation maps can be realized as a part of the canonical bilinear map on a Banach space $V \in \mathcal{K}$?*

Usually F is "small" means that the pointwise closure $\mathrm{cl}_p(\mathrm{F})$ of F (the *envelope of* F) in $\mathbb{R}^X$ is a "small" topological space. For example, (a) when $\mathrm{cl}_p(\mathrm{F}) \subset \mathrm{C(X)}$, or (b) when $\mathrm{cl}_p(\mathrm{F})$ consists of fragmented functions (Baire 1, when X is metrizable).

It turns out that the first case (a) characterizes the reflexively representable dynamical systems, (i.e., the dynamical analog of Eberlein compacta) or, for metric dynamical systems X, the class of Weakly Almost Periodic (in short: WAP) systems.

In the second case (b) we get the characterization of Rosenthal representable dynamical systems, or, for metric dynamical systems X, the class of tame systems (a Banach space V is said to be *Rosenthal* if it does not contain an isomorphic copy of the Banach space l_1.)

We have the natural intermediate case of Asplund representable or hereditarily non sensitive (HNS) systems. Namely, an S-system X is HNS iff there exists a separating bounded family $F \subset C(X)$ which is a fragmented family.

2. Preliminaries

2.1. **Notation.** The closure and the interior operators in topological spaces will be denoted by *cl* and *int*, respectively. "Compact" will mean "compact and Hausdorff".

As usual *hereditarily Baire* means that every closed subspace is a Baire space. A function $f : X \to Y$ is *Baire class 1 function* if the inverse image $f^{-1}(O)$ of every open set is F_σ in X. A topological space X is said to be *Polish* if it admits a complete separable metrizable metric. For Polish spaces X a function $f : X \to \mathbb{R}$ is Baire 1 iff f is a pointwise limit of a sequence of continuous functions.

Banach spaces and locally convex vector spaces are over the field $\mathbb{R}$ of real numbers.

When V is a Banach space we denote by B, or B_V, the closed unit ball of V. $B^* = B_{V^*}$ and $B^{**} := B_{V^{**}}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V respectively.

2.2. **From representations to compactifications.** Let X be a topological space, Y be a compact Hausdorff space and let $f : X \to Y$ be a function such that $f(X)$ is dense in Y. If f is continuous, then Y (more precisely, the pair (Y, f)) is called a *compactification of* X. If f is a homeomorphic embedding, then Y is called a *proper compactification of* X. Denote by $(\mathcal{C}(X), \leq)$ the partially ordered set of all compactifications of X up to the standard equivalence. For a topological space X denote by $C(X)$ the Banach algebra of real valued continuous and bounded functions equipped with the supremum norm. Recall that the unital closed subalgebras of $C(X)$ determine the compactifications of X.

Fact 2.1. (Gelfand-Kolmogoroff) *There exists a natural order preserving bijective correspondence between $\mathcal{C}(X)$ (different compactifications of X) and closed unital subalgebras of $C(X)$. In particular, $C(X)$ determines the greatest compactification $\beta : X \to \beta X$.*

Proof. (Sketch) Let $\mathcal{A}$ be a Banach unital subalgebra $\mathcal{A}$ of $C(X)$. Denote by $\mathcal{A}^*$ the dual Banach space of $\mathcal{A}$. Consider the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$, where $X^{\mathcal{A}} \subset B^* \subset \mathcal{A}^*$ is the Gelfand space (or, the *spectrum*) of the algebra $\mathcal{A}$. The map

$$\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}, \ x \mapsto \delta_x$$

is defined by the *Gelfand transform*, the evaluation at x multiplicative functional, that is $\alpha(x)(f) = \delta_x(f) = f(x)$ and $X^{\mathcal{A}}$ is the closure of $\alpha_{\mathcal{A}}(X)$ in $\mathcal{A}^*$ with respect to the weak* topology w^*. Therefore $X_{\mathcal{A}}$ is compact by Alaoglu Theorem.

Conversely, every compactification $\nu : X \to Y$ is equivalent to the *canonical* $\mathcal{A}_\nu$-*compactification* $\alpha_{\mathcal{A}_\nu} : X \to X^{\mathcal{A}_\nu}$, where the algebra $\mathcal{A}_\nu$ (corresponding to ν) is defined as the image $j_\nu(C(Y))$ of the natural embedding of Banach algebras

$$j_\nu : C(Y) \to C(X), \quad \phi \mapsto \phi \circ \nu.$$

If $\mathcal{A}_1 \subset \mathcal{A}_2$ then the adjoint operator induces the weak-star continuous onto map $\mathcal{A}_2^* \to \mathcal{A}_1^*$. Its restriction on $c_2(X)$ gives the desired morphism of compactifications $c_2 \to c_1$. □

Categorical view: the assignment $K \mapsto C(K)$ defines an important contravariant functor from the category **Comp** into the categories of Banach spaces **Ban** and Banach algebras.

For every (Tychonoff) space X and for the algebra $\mathcal{A} = C(X)$ we get the maximal (Chech-Stone) compactification.

$$\delta : X \to \beta(X) \subset B^* \subset C(X)^*$$

For every compact space K we have a topological embedding (Gelfand representation)

$$\delta : K \hookrightarrow C(K)^*, x \mapsto \delta_x.$$

Its image $\delta(K)$ affinely generates $P(K)$ (i.e. $\overline{co}^{w^*}(\delta(K)) = P(K)$), where

$$P(K) := \{\mu \in C(K)^* : \ \|\mu\| = \mu(\mathbf{1}) = 1\}$$

the weak-star compact set of all *probability measures* on K. We have $K := \delta(K) \subset P(K) \subset B_{C(K)^*}$.

2.3. **Topological prototypes.** An important direction in the classical study of (large) compact spaces went via the following general principle: Given a compact space X find a nice class $\mathcal{K}$ of Banach spaces such that there always is an element $V \in \mathcal{K}$ where X can be embedded into V^* equipped with its weak-star topology ?

Eberlein compacta in the sense of Amir and Lindenstrauss are exactly the weakly compact subsets in the class of all (equivalently, reflexive) Banach spaces. If X is a weak* compact subset in the dual V^* of an Asplund space V then, following Namioka [55], X is called a *Radon–Nikodým compactum* (in short: RN). In other words, reflexively representable compact spaces are the Eberlein compacta and Asplund representable compact spaces are the Radon–Nikodým compacta. Hilbert representable compacta are the so-called *uniformly Eberlein* compact spaces. Another interesting class of compact spaces, namely the *weakly Radon–Nikodým* (WRN) compacta, occurs by taking $\mathcal{K}$ to be the class of Rosenthal Banach spaces (i.e. those Banach spaces which do not contain an isomorphic copy of l_1). Comparison of the above mentioned classes of Banach spaces implies the inclusions of the corresponding classes of compact spaces:

$$(\mathbf{Comp} \cap \mathbf{Metr}) \subset uEb \subset Eb \subset RN \subset WRN \subset \mathbf{Comp}.$$

Note that this classification makes sense only for large compact spaces, where X is not metrizable. In fact, any compact metrizable space is even norm embeddable in a separable Hilbert space.

Example 2.2.

(1) For every Hilbert space H the (weak compact) unit ball (B_H, w) is uniformly Eberlein.
(2) 1-point compactification $A(\kappa)$ of a discrete space of cardinality κ is uniformly Eberlein.

A topological space (X, τ) is *scattered* if every (closed) subset $L \subset K$ has an isolated point in L. $A(\kappa)$ is Scattered. For every ordinal λ the linearly ordered compact space $[0, \lambda]$ is scattered.

Fact 2.3. *(Namioka-Phelps75) Let K be a compact space. The following are equivalent:*

(1) *K is scattered* [1].
(2) *$C(K)$ is an Asplund space.*

Example: $c = C(K)$ with $K = A(\omega)$.
Note that infinite dimensional space $C(K)$ never can be reflexive.

Corollary 2.4. *Every scattered compact space is RN.*

Remark 2.5.

(1) $A(\kappa) \in Eberlein$.
(2) $[0, \omega_1] \in RN \setminus Eberlein$.
(3) Two arrows space $D \in WRN \setminus RN$.
Indeed, every compact linearly ordered space is WRN (a recent result [29]). D is not RN by a result of Namioka [55, Example 5.9].
(4) $\beta\mathbb{N} \notin WRN$.
This was done by Todorčević (private communication).

One of the main directions taken in our research is the development of a dynamical analog, for compact S-dynamical systems (where S is a semigroup), of the above mentioned classification of large compact spaces (this is made precise in Definition 2.12 below).

Remark 2.6. Perhaps the first outstanding feature of this new theory is that, in contrast to the purely topological case (i.e., the case of trivial actions), for dynamical systems, the case of metrizable systems is "full of life". Moreover, the main interest of the dynamical theory is just within the class of metrizable dynamical systems. For example, even for $X := [0, 1]$, the unit interval, the action of the cyclic group $\mathbb{Z}$ on X generated by the map $f(x) = x^2$ is RN and not Eberlein. There exists a compact metric $\mathbb{Z}$-system which is reflexively but not Hilbert representable, i.e., Eberlein but not uniformly Eberlein. There are compact metric $\mathbb{Z}$-systems which are WRN but not RN, etc. See Example 9.1.

[1] *dispersed* in other terminology

It turns out that the corresponding classes of metric dynamical systems coincide with well known important classes whose study is well motivated by other independent reasons. For example we have, Eberlein = WAP (weakly almost periodic systems), RN = HNS (hereditarily non-sensitive), WRN = tame systems. The investigation of Hilbert representable (i.e., "uniformly Eberlein") systems is closely related to the study of unitary and reflexive representability of groups.

2.4. **Some connections to Banach space theory.** Another remarkable feature is the fact that the correspondence goes both ways. Thus, for example, to construct some nontrivial examples of Banach spaces. Every metric WRN but not RN $\mathbb{Z}$-system leads to an example of a separable Rosenthal Banach space which is not Asplund. One of the important questions in Banach space theory until the mid 70's was to construct a separable Rosenthal space which is not Asplund. The first counterexamples were constructed independently by James and Lindenstrauss-Stegall.

In view of the representation Theorem below we now see that a fruitful way of producing such distinguishing examples comes from dynamical systems. Just consider a compact metric tame G-system which is not HNS and then represent it on a (separable) Rosenthal space V. Then V is not Asplund (otherwise, (G, X) is HNS). We have several examples of dynamical systems of this type; e.g. $(H_+[0,1],[0,1])$, the Sturmian cascades, or the projective actions of $GL_n(\mathbb{R})$ on the sphere or the projective space.

One may make this result sharper by using representation theorem. There exists a separable Rosenthal space V without the *adjoint continuity property*. Indeed, the Polish group $G := H_+[0,1]$, which admits only trivial adjoint continuous representations (and, hence, trivial Asplund or reflexive representations), is however Rosenthal representable.

Finally, let us mention yet another potentially interesting direction, which may lead to a new classification inside Rosenthal Banach spaces induced by topological classification of Rosenthal compacta (Todorcevic trichotomy) applied to the Rosenthal compacta of the form $\mathcal{E}(V)$ (enveloping semigroups of Banach spaces V).

2.5. **Some typical applications.**

Theorem 2.7. [46, 49] *Let V be a reflexive space (remains true for PCP spaces). Then*

(1) *norm topology = weak topology on every orbit Gv for every $G \leq \mathrm{Iso}\,(V)$.*
(2) *WOP=SOP. The weak and the strong operator topologies coincide on* $\mathrm{Iso}\,(V)$.
(3) *Every weakly continuous (co)homomorphism $h : G \to \mathrm{Iso}\,(V)$ is strongly continuous.*

Proof. It is enough to show (1). Let $z \in X := Gv$. Denote by τ the weak topology on $X \subset V$. We have to show that for every $\varepsilon > 0$ there exists a τ-neighborhood $O(z)$ of z in X such that O is ε-small. Since X is $(\tau, norm)$-fragmented (*non-sensitivity* is enough), we can pick a *non-void* τ-open subset $W \subset X$ such that W is ε-small in V. Choose $g_0 \in G$ such that $g_0 z \in W$. Denote by O the τ-open subset $g_0^{-1}W$ of (X, τ). Then O is a τ-neighborhood of z and is ε-small. □

Theorem 2.8. *(Shtern, Megrelishvili) If P is a compact semitopological monoid then there exists a reflexive Banach space V and an embedding $P \hookrightarrow \Theta(V)_w$ into the* *compact semitopological monoid*

$$\Theta(V)_w := \{\sigma \in L(V,V) :\ ||\sigma|| \leq 1\}.$$

Theorem 2.9.

(1) *Let S be a compact semitopological monoid and G be its subgroup. Then G is a topological group.*
(2) *Ellis Theorem Every compact semitopological* ***group*** *G is a topological group.*

Proof. It is enough to show (1). Combine previous two theorems. By Theorem 2.8, $G \leq \mathrm{Iso}\,(V)_w \subset \Theta(V)_w$ for some reflexive V. By Theorem 2.7, WOP=SOP on $\mathrm{Iso}\,(V)$ for every $V \in PCP$. So, $G \leq \mathrm{Iso}\,(V)_w = \mathrm{Iso}\,(V)_s$ is a topological group. □

Theorem. ([27], 2012) $\mathbf{Ros_r} \neq \mathbf{Asp_r}$
The Polish topological group $H_+[0,1]$ is representable on a separable Rosenthal Banach space (and not representable on any Asplund space, [24]*, 2007).*

⇓
(Well known) There exists a separable Rosenthal Banach space which is not Asplund.

Remark 2.10. Well known but once it was a famous problem, resolving by James (JT space) and Lindenstrauss-Stegall (JF space).

Another corollary: There exists a separable Rosenthal space V without the *adjoint continuity property.*

2.6. **The hierarchy of Banach representations.** With every Banach space V one may naturally associate several structures which are related to the theories of topological dynamics, topological groups and compact right topological semigroups:

Definition 2.11.

(1) $\mathrm{Iso}\,(V)$ is the group of linear onto self-isometries of V. It is a topological (semitopological) group with respect to the strong (respectively, weak) operator topology. It is naturally included in the semigroup $\Theta(V) := \{\sigma \in L(V,V) : ||\sigma|| \leq 1\}$ of non-expanding linear operators. The latter is a topological (semitopological) monoid with respect to the strong (respectively, weak) operator topology. Notation: $\Theta(V)_s$, $\mathrm{Iso}\,(V)_s$ (respectively, $\Theta(V)_w$, $\mathrm{Iso}\,(V)_w$) or simply $\Theta(V)$ and $\mathrm{Iso}\,(V)$, where the topology is understood.
(2) For every subsemigroup $S \leq \Theta(V)^{op}$ the pair (S, B^*) is a dynamical system, where B^* is the weak star compact unit ball in the dual space V^*, and $\Theta(V)^{op}$ is the opposite semigroup (which can be identified with the adjoint) to $\Theta(V)$. The action is jointly (separately) continuous where S carries the strong (weak) operator topology.
(3) The enveloping semigroup $E(S, B^*)$ of the system (S, B^*) is a compact right topological semigroup (it can be identified with the pointwise closure of S in B^{*B^*}). In particular, $\mathcal{E}(V) := E(\Theta(V)^{op}, B^*)$ will be called the *enveloping semigroup of V*. Its topological center is just $\Theta(V)^{op}_w$ which is densely embedded into $\mathcal{E}(V)$. Note that $\mathcal{E}(V) = \Theta(V)^{op}$ iff V is reflexive.

A *representation* of a semigroup S (with identity element e) on a Banach space V is a co-homomorphism $h : S \to \Theta(V)$, where $\Theta(V) := \{T \in L(V) : \ ||T|| \leq 1\}$ and $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \to V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \to \Theta(V)^{op}$ be a monoid homomorphism, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$. If $S = G$, is a group then $h(G) \subset \mathrm{Iso}\,(V)$, where $\mathrm{Iso}\,(V)$ is the group of all linear isometries from V onto V.

Definition 2.12. [49, 23, 24, 25] Let X be a dynamical S-system.

1. A *representation* of (S, X) on a Banach space V is a pair
$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$
where $h : S \to \Theta(V)$ is a weakly continuous representation (co-homomorphism) of semigroups and $\alpha : X \to V^*$ is a weak* continuous bounded S-mapping with respect to the dual action
$$S \times V^* \to V^*, \ (s\varphi)(v) := \varphi(h(s)(v)).$$
We say that the representation is *strongly continuous* if h is strongly continuous. A representation (h, α) is said to be *faithful* if α is a topological embedding.
2. If $S := G$ is a group then a representation of (G, X) on V is a pair (h, α), where α is as above and $h : G \to \mathrm{Iso}\,(V)$ is a group co-homomorphism.
3. If $\mathcal{K}$ is a subclass of the class of Banach spaces, we say that a dynamical system (S, X) is *(strongly) $\mathcal{K}$-representable* if there exists a weakly (respectively, strongly) continuous faithful representation of (S, X) on a Banach space $V \in \mathcal{K}$.
4. A dynamical system (S, X) is said to be *(strongly) $\mathcal{K}$-approximable* if it can be embedded in a product of (strongly) $\mathcal{K}$-representable S-spaces.

Remark 2.13. The notion of a reflexively (Asplund) representable compact dynamical system is a dynamical version of the purely topological notion of an *Eberlein* (respectively, a *Radon-Nikodym* (RN, in short)) compactum, in the sense of Amir and Lindenstrauss (respectively, in the sense of Namioka). As in [25], we call Rosenthal representable systems *Weakly Radon-Nikodym* (WRN) systems.

Remark 2.14.

1. Of course not every $\mathcal{K}$-approximable is $\mathcal{K}$-representable. Take for example, (S, X) with $S := \{e\}$ and $X := [0, 1]^{\mathbb{R}}$. Then (S, X) is clearly reflexively-approximable but not reflexively-representable (because X, as a compactum, is not Eberlein).
2. In some particular cases $\mathcal{K}$-approximability and $\mathcal{K}$-representability are equivalent. This happens for example if X is metrizable and $\mathcal{K}$ is closed under countable l_2-sums.
3. The classes of Eberlein, RN and WRN compact dynamical systems are closed under countable products.

   Proof. Use the l_2-sum of representations (h_n, α_n) of (S, X_n) on V_n where $||\alpha_n(x)|| \leq 2^{-n}$ for every $x \in X_n$ and $n \in \mathbb{N}$ with $V := (\Sigma_{n \in \mathbb{N}} V_n)_{l_2}$. $\square$

(4) ("building blocks" of dynamical systems) Let $S \times X \to X$ be a continuous action on a compact space X. Then the S-system X can be S-embedded into a topological S-product $\prod_{f \in F} X_f$ of "cyclic S-systems" X_f. Here $F \subset C(X)$ is a point separating family and X_f is defined as the Gelfand space of the algebra $\mathcal{A}_f =$ the smallest closed S-invariant subalgebra generated by fS in $C(X)$. It follows that if any X_f is s $\mathcal{K}$-approximated then X is $\mathcal{K}$-approximated, too.

3. Actions and semigroups

Definition 3.1. Let S be a semitopological semigroup with a neutral element e. Let $\pi : S \times X \to X$ be a *left action* of S on a topological space X. This means that $ex = x$ and $s_1(s_2x) = (s_1s_2)x$ for all $s_1, s_2 \in S$ and $x \in X$, where as usual, we write sx instead of $\pi(s,x) = \lambda_s(x) = \rho_x(s)$. Let $S \times X \to X$ and $S \times Y \to Y$ be two actions. A map $f : X \to Y$ between S-spaces is an *S-map* if $f(sx) = sf(x)$ for every $(s,x) \in S \times X$.

We say that X is a *dynamical S-system* (or an *S-space* or an *S-flow*) if the action π is separately continuous (that is, if all orbit maps $\rho_x : S \to X$ and all translations $\lambda_s : X \to X$ are continuous). If otherwise is not stated we assume that X is compact. We sometimes write it as a pair (S, X).

A *right system* (X, S) can be defined analogously. If S^{op} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left system (S^{op}, X) (and vice versa).

Fact 3.2. *(J. Lawson [41]) Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then every separately continuous action of G on a compact space X is continuous.*

Notation: *All semigroups S are assumed to be monoids, i.e, semigroups with a neutral element e. Also actions are* monoidal *(meaning $ex = x, \forall x \in X$) and separately continuous. We reserve the symbol G for the case when S is a group.*

Let $h : S_1 \to S_2$ be a semigroup homomorphism, S_1 act on X_1 and S_2 on X_2. A map $f : X_1 \to X_2$ is said to be *h-equivariant* if $f(sx) = h(s)f(x)$ for every $(s,x) \in S_1 \times X_1$. For $S_1 = S_2$ with $h = 1_S$, we say *S-map*. The map $h : S_1 \to S_2$ is a *co-homomorphism* iff $h : S_1 \to S_2^{op}$ (the same assignement) is a homomorphism.

Given $x \in X$, its *orbit* is the set $Sx = \{sx : s \in S\}$ A point x with $\mathrm{cl}\,(Sx) = X$ is called a *transitive point*, and the set of transitive points is denoted by X_{tr}. We say that the system is *point-transitive* when $X_{tr} \neq \emptyset$. The system is called *minimal* if $X_{tr} = X$. Equivalently: any nonempty closed S-subsystem of X is X.

By an (invertible) *cascade* on X we mean a continuous action $S \times X \to X$, where $S := \mathbb{N} \cup \{0\} := \mathbb{N}_0$ is the additive semigroup of all nonnegative integers (respectively, $S = (\mathbb{Z}, +)$).

3.1. **Some old and new classes of semigroups.**

Let S be a semigroup which is also a topological space. By $\lambda_a : S \to S, x \mapsto ax$ and $\rho_a : S \to S, x \mapsto xa$ we denote the left and right a-transitions. The subset $\Lambda(S) := \{a \in S : \ \lambda_a \text{ is continuous}\}$ is called the *topological center* of S.

Definition 3.3. A semigroup S with a given topology is called:

1. *right topological semigroup* if every ρ_a is continuous;
2. *admissible* if S is right topological and $\Lambda(S)$ is dense in S;
3. *topological* if the multiplication $S \times S \to S$ is continuous;
4. *topological group* if S, algebraically, is a group, topological semigroup and if the inversion $s \mapsto s^{-1}$ is continuous.
5. *semitopological* if the multiplication $m : S \times S \to S$ is separately continuous (i.e., if S is left and right topological);

Define two new classes of compact right topological semigroups.

Definition 3.4. [23, 25] A compact admissible right topological semigroup (P, τ) is said to be:

(1) [25] *tame* if the left translation $\lambda_a : (P, \tau) \to (P, \mu)$ is a fragmented map (Definition 12.1) for every $a \in P$.
(2) *HNS-semigroup* if $\{\lambda_a : P \to P\}_{a \in P}$ is a fragmented family of maps.

These classes are closed under factors. We have the inclusions:

$$\{\text{compact semitopological semigroups}\} \subset \{\text{HNS-semigroups}\} \subset \{\text{Tame semigroups}\}$$

3.2. **Functions with small orbits.** Let $S \times X \to X$ be an action. It induces the right action $C(X) \times S \to C(X)$ and a co-homorphism $h : S \to \Theta(C(X))$.

A (bounded) function $f \in C(X)$ is said to be:

(a) *Right uniformly continuous* if the orbit map $\tilde{f} : S \to C(X)$ is continuous; i.e.

$$\forall \varepsilon > 0 \ \forall s_0 \in S \ \exists U \in N(s_0) \ \ |f(sx) - f(s_0x)| < \varepsilon \ \ \forall x \in X, s \in U.$$

notation: $f \in \mathrm{RUC}(X)$. If the action is continuous and X is compact then $\mathrm{RUC}(X) = C(X)$.
(b) *Almost periodic* if the norm closure $cl(fS)$ of the orbit fS is norm compact in $C(X)$; notation: $f \in \mathrm{AP}(X)$.
(c) *Weakly almost periodic* if the weak closure $cl_w(fS)$ of the orbit fS is weak compact in $C(X)$; Notation: $f \in \mathrm{WAP}(X)$.

In particular we have $\mathrm{WAP}(S)$ for usual left action of S on itself. Note that for the right action of S on itself the corresponding (right) version of $\mathrm{WAP}(S)$ gives the equivalent definition [9, 6].

A compact S-space X is said to be *(weakly) almost periodic* if (resp., $C(X) = \mathrm{WAP}(X)$) $C(X) = \mathrm{AP}(X)$. For any S-space X the collections $\mathrm{WAP}(X)$ and $\mathrm{AP}(X)$ are S-invariant subalgebras of $C(X)$. The corresponding Gelfand spaces and compactifications define S-equivariant compactifications $u_{ap} : X \to X^{AP}$ and $u_{wap} : X \to X^{WAP}$. The compactification $S \to S^{WAP}$ (for $X := S$) is the universal semitopological semigroup compactification of S.

For every topological group G, treated as a G-space, the corresponding universal AP compactification is the classical *Bohr compactification* $b : G \to bG$, where bG is a compact topological group.

(d) Let X be a compact DS. We say that $f \in C(X)$ is *Asplund* if fS is a fragmented family (Definition 12.1) of maps $X \to \mathbb{R}$; notation: $f \in \mathrm{Asp}(X)$. It is equivalent saying that $cl_p(fS)$ is a fragmented family.
Define also, $\mathrm{Asp}(G) := \mathrm{Asp}(\beta_G G)$.
(e) Let X be a compact DS. We say that $f \in C(X)$ is *tame* if $cl_p(fS)$ is an *eventually fragmented* family of maps $X \to \mathbb{R}$; notation: $f \in \mathrm{Tame}(X)$. Or, equivalently, if fS does not contain an *independent subsequence* (Definition 12.7).
Another equivalent condition is: $cl_p(X) \subset \mathcal{F}(X)$, where $\mathcal{F}(X)$ is the set of all fragmented real valued functions on X. (See Theorem 12.10.)

3.3. **Semigroup compactifications.** A good introduction to semigroup compactifications of topological groups is a work of Uspenskij [73] (downloadable from the course website). See also a book of Berglund-Junghenn-Milnes [6].

Definition 3.5. Let S be a semitopological semigroup. [6, p. 105] A *right topological semigroup compactification* of S is a pair (γ, T) such that T is a compact right topological semigroup, and γ is a continuous semigroup homomorphism from S into T, where $\gamma(S)$ is dense in T and the left translation $\lambda_s : T \to T,\ x \mapsto \gamma(s)x$ is continuous for every $s \in S$, that is, $\gamma(S) \subset \Lambda(T)$.

It follows that the *associated action*

$$\pi_\gamma : S \times T \to T, \quad (s, x) \mapsto \gamma(s)x = \lambda_s(x)$$

is separately continuous.

Example 3.6.

(1) Maximal (jointly continuous) G-compactification $G \hookrightarrow \beta_G G := G^{\mathrm{RUC}}$ (the greatest ambit). The corresponding algebra is $\mathrm{RUC}(G)$.
(2) Universal semitopological compactification: $G \to wG := G^{\mathrm{WAP}}$. The corresponding algebra is $WAP(G)$.
(3) universal topological compact group compactification (Bohr compactification): $G \to bG = G^{AP}$.

$$G \hookrightarrow \beta_G G = G^{\mathrm{RUC}} \to G^{\mathrm{Tame}} \to G^{Asp} \to G^{\mathrm{WAP}} \to G^{AP} = bG$$

By [52], $q : G^{\mathrm{RUC}} \to G^{WAP}$ is a homeomorphism iff G is precompact.

3.4. **Enveloping semigroups.** For every (separately continuous) compact S-system X we have a (pointwise continuous) monoid homomorphism $j : S \to C(X, X)$, $j(s) = \tilde{s}$, where $\tilde{s} : X \to X, x \mapsto sx = \pi(s, x)$ is the *s-translation* ($s \in S$).

Definition 3.7. The *enveloping semigroup* $E(S, X)$ (or just $E(X)$) of the compact dynamical S-system X is defined as the pointwise closure $E(S, X) = \mathrm{cl}_p(j(S))$ of $\tilde{S} = j(S)$ in X^X.

The associated homomorphism $j : S \to E(X)$ is a right topological semigroup compactification (say, *Ellis compactification*) of S, $j(e) = id_X$ and the associated action $\pi_j : S \times E(X) \to E(X)$ is separately continuous. Furthermore, if the S-action on X is continuous then π_j is continuous. $E(X)$ is always a right topological compact monoid. Algebraic and topological properties of the families $j(S)$ and $E(X)$ reflect the asymptotic dynamical behavior of (S, X).

Exercise 3.8. (A concrete computation) Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{Z}$. Extend the usual addition by:

$$n + t = t + n = s + t = t \ \ n \in \mathbb{Z},\ s, t \in \{-\infty, \infty\}$$

Show:

(1) $(S, +)$ is a noncommutative compact right topological semigroup having dense topological centre $\Lambda(S) = \mathbb{Z}$.
(2) S is topologically isomorphic to the *enveloping semigroup* of the invertible cascade $(\mathbb{Z}, [0, 1])$ generated by the homeomorphism $\sigma : [0, 1] \to [0, 1], \sigma(x) = x^2$.

Proof. Let E be the enveloping semigroup of $(\mathbb{Z}, [0,1])$ and $j : \mathbb{Z} \to E$ be the corresponding compactification. Observe that besides the points $j(Z) = \{\sigma^n : n \in \mathbb{Z}\}$ the enveloping semigroup $E(X)$ contains two more points: a, b, where $a = \xi_{\{1\}}$ the characteristic function of $\{1\}$ and $b = 1 - \xi_0$, where $\xi_{\{0\}}$ is the characteristic function of $\{0\}$. □

Exercise 3.9. Is it true that E from Example 3.8 is: (a) semitopological; (b) HNS; (c) Tame ?

Hint: use the fact that the compactum E is countable.

Remark 3.10.

(1) {enveloping semigroups $E(S, X)$}= {compact right topological admissible semigroups}.
(2) (G, X) is AP (equiv., equicontinuous) iff E is a group of continuous maps.
(3) (Ellis-Nerurkar) [15]) X is a WAP system iff every $p \in E$ is a continuous map $X \to X$.
(4) Enveloping semigroup $E(\mathbb{Z}, \Omega)$ of the Bernoulli shift symbolic system $\Omega := \{0,1\}^{\mathbb{Z}}$ is $\beta\mathbb{Z}$.

    Sketch: To see this recall that the collection $\{\bar{A} : A \subset \mathbb{Z}\}$ is a basis for the topology of $\beta\mathbb{Z}$ consisting of clopen sets. Next identify Ω with the collection of subsets of $\mathbb{Z}$ in the obvious way: $A \longleftrightarrow \mathbf{1}_A$. Now define an "action" of $\beta\mathbb{Z}$ on Ω by:

$$p * A = \{g \in \mathbb{Z} : g^{-1}p \in \overline{A^{-1}}\}.$$

    This action extends the action of $\mathbb{Z}$ on Ω and defines an isomorphism of $\beta\mathbb{Z}$ onto $E(\Omega)$.
(5) If $\sigma : [0,1] \to [0,1], t \mapsto 4t(1-t)$ then for the corresponding cascade $\mathbb{N}_0 \times [0,1] \to [0,1]$ the enveloping semigroup E topologically contains $\beta\mathbb{N}$. (So, also in this case E is not a Frechet topological space.)

Below we answer the questions: for which compact metrizable dynamical systems the enveloping semigroup is: (a) metrizable; (b) Frechet.

3.5. **Enveloping semigroups of Banach spaces.**

Definition 3.11. Given a Banach space V we denote by $\mathcal{E}(V)$ the enveloping semigroup of the dynamical system $(\Theta(V)^{op}, B^*)$. We say that $\mathcal{E}(V)$ is the *enveloping semigroup of* V.

In the sequel whenever V is understood we use the following simple notations $\mathcal{E} := \mathcal{E}(V)$, $\Theta := \Theta(V)$, $\Theta^{op} := \Theta(V)^{op}$. By S_V we denote the unit sphere of V.

Lemma 3.12. [27] *For every Banach space V, every $v \in S_V$ and $\psi \in S_{V^*}$ we have*

(1) $\Theta v = B$.
(2) $v\mathcal{E} = B^{**}$.
(3) $cl_{w^*}(\Theta^{op}\psi) = B^*$.
(4) $\mathcal{E}\psi = B^*$.
(5) $\Lambda(\mathcal{E}) = \Theta^{op}$.
(6) *V is reflexive iff $\mathcal{E} = \Theta^{op}$ iff $\mathcal{E}$ is semitopological.*

Some DS results below will lead us to

Theorem 3.13. *Let V be a separable Banach space.*

1. *V is Asplund iff $\mathcal{E}$ is metrizable.*
2. *V is Rosenthal iff $\mathcal{E}$ is Frechet.*

Theorem 3.14. *Let V be a (not necessarily separable) Banach space.*

1. *V is Asplund iff $\mathcal{E}$ is HNS semigroup.*
2. *V is Rosenthal iff $\mathcal{E}$ is a tame semigroup.*

4. Representations of topological groups I

By a (*t-faithful*) representation of a topological group G on a Banach space V we mean a continuous homomorphism (resp., *topological group embedding*) $h : G \to \mathrm{Iso}\,(V)$ of G into the top. group $\mathrm{Iso}\,(V)$ of all linear onto isometries of V with SOT. Every (topological) group is topologically isomorphic to its opposite group by the assignment $g \mapsto g^{-1}$. Hence, G is representable on V iff it is co-representable on V.

Problem 4.1. *Which (Polish) groups can be represented on nice Banach spaces ?*

Let $\mathcal{K} \subset \boldsymbol{Ban}$ be a subclass of Banach spaces. We write: $G \in \mathcal{K}_r$ if $\exists$ a t-faithful representation of G on $V \in \mathcal{K}$.

$$\mathbf{TGr} = \mathbf{Ban_r} \supseteq \mathbf{Ros_r} \supset \mathbf{Asp_r} \supseteq \mathbf{Ref_r} \supset \mathbf{Hilb_r} \supset \{\textit{LC top. gr.}\}$$

Remark 4.2.

1. (Teleman's theorem) Any topological group G is Banach representable.
2. (Gelfand-Raikov) Every locally compact group is Hilbert representable. If G is a locally compact topological group then the regular representation of $H = L_2(G, \mu)$ (where μ is the Haar measure) defines an embedding $G \hookrightarrow \mathrm{Iso}\,(H)$.
3. (Me [48]) $G := L_4[0,1] \in \mathbf{Ref_r} \setminus \mathbf{Hilb_r}$ (For proofs see Section 11).
   (Glasner-Weiss 2012) $\exists G \in \mathbf{Ref_r} \setminus \mathbf{Hilb_r}$ s.t. $G = l_2/D$ is Polish monothetic.
4. Unknown if: $\mathbf{Asp_r} = \mathbf{Ref_r}$, $\mathbf{Ban_r} = \mathbf{Ros_r}$.
   It is an open question if every Polish group is Rosenthal representable (enough to examine the universal Polish group $G := \mathrm{Homeo}\,([0,1]^{\mathbb{N}})$).

Remark 4.3.

1. (Me 2001) $\mathbf{TGr} \neq \mathbf{Ref_r}$ $\quad H_+[0,1] \notin \mathbf{Ref_r}$
   $\forall$ representation $h : H_+ \to \mathrm{Iso}\,(V)$ is trivial $\forall\, V \in \mathbf{Ref}$.
   $\Downarrow$
   Every semitopological compactification of H_+ is trivial.
2. (Gl-Me 2007, independently also by Uspenskij)
   $\forall$ representation $h : H_+ \to \mathrm{Iso}\,(V)$ is trivial $\forall\, V \in \mathbf{Asp}$.
   $\Downarrow$
3. $\forall$ metrizable semigroup compactification of H_+ is trivial.
4. $\exists$ a proper semigroup compactification $H_+ \to S$ which is Fréchet (in fact, topologically $S \subset M_+([0,1],[0,1]$ = Helly compactum).
   $\Uparrow$

Theorem. (Gl-Me 2012) $\quad \mathbf{Ros_r} \neq \mathbf{Asp_r}$
The Polish topological group $H_+[0,1]$ is representable on a separable Rosenthal Banach space (and not representable on any Asplund space – 2007).

$\Downarrow$

(Well known) There exists a separable Rosenthal Banach space which is not Asplund.

Remark 4.4. (Ferri-Galindo [17]) The topological group c_0 is not reflexively representable.

Note that c_0 as a Banach space is Asplund (because, $c_0^* = l_1$ is separable). However the following concrete question is open

Question 4.5. *Is it true that c_0 is Asplund representable ?*

Question 4.6. *(Ferri-Galindo* [17]*) Is it true that for every abelian topological group G there exists a continuous injective representation $h : G \to \mathrm{Iso}\,(V)$ with reflexive V ?*

Question 4.7. *When a Polish topological group G can be embedded into a good right topological compact semigroup P ? For example: when P can be semitopological, metrizable, Frechet, ... ?*

The latter is equivalent to asking when the group G is representable on a reflexive, Asplund or Rosenthal space.

4.1. **Elementary observations.**

Exercise 4.8. Every locally compact topological group is closed in every Hausdorff topological group.

For example, $\mathbb{Z}$ and $\mathbb{R}$ cannot be embedded into compact groups. In particular, such groups do not admit finite dimensional orthogonal representations $h : G \hookrightarrow O_n(\mathbb{R})$ where h is an embedding. More precisely, we have

Fact 4.9. *Let G be a topological group. The following are equivalent:*

1. *G can be embedded into a compact topological semigroup.*
2. *G can be embedded into a compact group (i.e., the Bohr compactification $b : G \to bG$ is an embedding).*
3. *G is embedded into a product of finite dimensional orthogonal group $O_n(\mathbb{R})$.*

The equivalence of (1) and (3) is a consequence of Peter-Weyl theorem. The equivalence of (1) and (2) easily follows from the following

Exercise 4.10. If S is a compact topological semigroup and if G is a subgroup of S then $cl(G)$ is a (compact) topological group.

Exercise 4.11. Let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of G. Recall the topology

$$\tau_\infty := \tau \cup \{S \setminus K : \quad K \text{ is } \text{ compact } \text{ in } G\}.$$

Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not a topological semigroup.

Corollary 4.12. *Let G be a locally compact group. Then*

1. *G is embedded into a topological semigroup iff G is compact.*
2. *G is embedded into a compact semitopological semigroup.*

Theorem 4.13. *(Ellis thm) Every (locally) compact semitopological group is a topological group.*

4.2. **Examples of topological (semi)groups.**

Exercise 4.14. Prove that:

(1) for every metric space (M,d) the semigroup $S := \Theta(M,d)$ of all *contractive* maps [2] $f : X \to X$ (that is, $d(f(x), f(y)) \leq d(x,y)$) is a topological monoid with respect to the topology of pointwise convergence;
(2) the group $\mathrm{Iso}\,(M)$ of all onto isometries is a topological group;
(3) the evaluation map $S \times M \to M$ is a continuous monoidal action.

Exercise 4.15. Let Y be a compact space. Show that:

(1) The semigroup $C(Y,Y)$ endowed with the compact open topology is a topological monoid;
(2) The subset $H(Y)$ in $C(Y,Y)$ of all homeomorphisms $Y \to Y$ is a topological group;
(3) For every subsemigroup $S \subset C(Y,Y)$ the induced action $S \times Y \to Y$ is continuous;
(4) Furthermore, it satisfies the following remarkable *minimality property.* If τ_0 is an arbitrary topology on S such that $(S, \tau_0) \times Y \to Y$ is continuous then $\tau_{co} \subset \tau_0$.

4.3. **Operator topologies.**

Definition 4.16. Let V be a Banach space. The *strong operator topology* (SOT) on $L(V,V)$ is the pointwise topology inherited from $(V, ||\cdot||)^V$. That is, a net s_i converges to s iff $s_i(v)$ converges to $s(v)$ in the norm topology for every $v \in V$.

Replacing the norm topology of V by its weak topology we obtain the *weak operator topology* (WOT). A net s_i in $(L(V,V), WOT)$ converges to s iff $f(s_i(v))$ converges to $f(s(v))$ in $\mathbb{R}$ for every given pair of vectors $(v,f) \in V \times V^*$.

Exercise 4.17. Prove that:

(1) $\Theta(V)$ and $L(V,V)$ are semitopological monoids with respect to WOT.
(2) The semigroup $\Theta(V)$ endowed with the SOT is a topological monoid. The subspace $\mathrm{Iso}\,(V)_s$ of all linear onto isometries is a topological group.

Recall that (by Theorem 2.7) WOT=SOT on $\mathrm{Iso}\,(V)$ for reflexive V. Below in Theorem 10.2 we show this for a larger class of Banach spaces (PCP).

It is well known (de Leeuw-Glicksberg [12]) that $\Theta(V)_w$ is a compact (semitopological) semigroup for every reflexive V. One may show that this property characterizes reflexive spaces.

Lemma 4.18. *For any Banach space V and any given norm 1 vector $v \in S_V$ the map*

$$\Theta(V)_w \to (B, w), \ \ s \mapsto sv$$

is onto (and continuous).

Proof. Take $f \in S_{V^*}$ such that $f(v) = 1$. For every $z \in B$ define the rank 1 operator

$$z \otimes f : V \to V, \quad x \mapsto f(x)z.$$

Then $z \otimes f(v) = z$ and $z \otimes f \in \Theta$ since $||z \otimes f|| = ||f|| \cdot ||z|| = ||z|| \leq 1$. ☐

[2]in another terminology: *Lipschitz 1 maps*

Proposition 4.19. *The following are equivalent:*

(1) V *is reflexive;*
(2) $B := B_V$ *is weakly compact;*
(3) $\Theta(V)_w$ *is compact.*

Proof. (1) $\Leftrightarrow$ (2) is a well known criterion.
(2) $\Rightarrow$ (3) $\Theta(V) \subset (B, w)^B$ is a closed subset. Apply now Tychonoff Theorem.
(3) $\Rightarrow$ (2) Is a corollary of Lemma 4.18. □

Every left action $\pi : S \times X \to X$ induces the co-homomorphism $h_\pi : S \to C(X)$ and the right action $C(X) \times S \to C(X)$ where $(fs)(x) = f(sx)$. While the translations $\tilde{s} : V \to V$ are continuous, the orbit maps $\tilde{f} : S \to C(X)$ are not necessarily norm (even weakly) continuous and require additional assumptions for their continuity. As before denote by RUC(X), the set of all functions $f \in C(X)$ such that the orbit map $\tilde{f}$ is norm continuous.

For every normed space V the usual adjoint map

$$adj : L(V) \to L(V^*), \ s \mapsto s^* \qquad (< s(v), f >=< v, s^*(f) >)$$

is an injective co-homomorphism of monoids. Sometimes we write simply s instead of s^*.

Lemma 4.20. *For every normed space V the injective map*

$$\gamma : \Theta(V)^{op}_s \hookrightarrow C(B^*, B^*)$$

induced by the adjoint map $adj : L(V) \to L(V^*)$, is a topological (even uniform) monoid embedding. In particular,

$$\Theta(V)^{op} \times B^* \to B^*$$

is a jointly continuous monoidal action of $\Theta(V)^{op}_s$ on the compact space B^*.

Proof. The strong uniformity on $\Theta(V)$ is generated by the family of pseudometrics $\{p_v : v \in V\}$,where $p_v(s,t) = ||sv - tv||$. On the other hand the family of pseudometrics $\{q_v : v \in V\}$,where $q_v(s,t) = \sup\{|(fs)(v) - (ft)(v)| : \ f \in B^*\}$ generates the natural uniformity inherited from $C(B^*, B^*)$. Now observe that $p_v(s,t) = q_v(s,t)$ by the Hahn-Banach theorem. This proves that γ is a uniform (and hence, also, topological) embedding. □

Corollary 4.21. *Let V be a Banach space. Suppose that $\pi : V \times S \to V$ is a right action of a topologized semigroup S by linear contractive operators. The following are equivalent:*

(i) *The co-homomorphism $h : S \to \Theta(V), \ h(s)(v) := vs$ is strongly continuous.*
(ii) *The induced affine action $S \times B^* \to B^*, \ (s\psi)(v) := \psi(vs)$ is jointly continuous.*

For a compact space X we denote by $H(X)$ the topological group of all self-homeomorphisms of X endowed with the compact open topology.

Lemma 4.22. *Let X be a compact G-space, where G is a topological subgroup of $H(X)$. Assume that (h, α) is a faithful representation (that is, $\alpha : X \to (V^*, w^*)$ is an embedding) of (G, X) on a Banach space V. Then h is a topological group embedding.*

4.4. **Theorems of Teleman and Uspenskij.** Teleman's theorem below 4.25 shows that any topological group is represented on a Banach space by linear isometries..

For example, every finite group G can be represented on $\mathbb{R}^n$ with $n = |G|$. Indeed, take

$$G \hookrightarrow S_n \hookrightarrow \mathrm{Iso}\,(\mathbb{R}^n) = O_n(\mathbb{R}).$$

Exercise 4.23. Every discrete group G admits an effective isometric representation on the Banach space $l_\infty(G)$ and also on the Hilbert space $H := l_2(G)$. So, on $l_\infty := l_\infty(\mathbb{N})$ and on $l_2 := l_2(\mathbb{N})$ if G is countable.

For every compact space K we have the Gelfand representation $\delta : K \hookrightarrow C(K)^*$. Now let $S \times K \to K$ be a continuous action. The induced action $C(K) \times S \to C(K)$ is: linear, norm-preserving and continuous. We obtain a natural representation (h, δ), where $h : S \to \Theta(V)$ is a strongly continuous co-homomorphism of monoids. Call it the *Teleman's representation.* This is a dynamical version of Gelfand's representation.

Theorem 4.24. [69] *Let $S \times K \to K$ be a continuous action on a compact space K. Then*

1. *Teleman's representation (h, δ) of the dynamical system (S, K) on the Banach space $V = C(K)$ is faithful and strongly continuous. That is, $h : S \to \Theta(V)_s$ is continuous. If $S = G$ is a group then h is a co-homomorphism of groups $h : G \to \mathrm{Iso}\,(V)_s$;*
2. *Moreover, if S carries the compact open topology inherited from $C(K, K)$ then the homomorphisms $S \to C(B^*, B^*)$ and $h : S \to \Theta(V)_s^{op}$ are topological embeddings.*

Proof. (1) The induced right linear action $C(K) \times S \to C(K)$ is continuous (because the orbit maps are norm continuous). This action is contractive ($||fs|| \leq 1||$ for every $f \in C(X)$). It follows that $h : S \to \Theta(V)^{op}$ is a well defined strongly continuous homomorphism. Clearly, B^* is an S-subset under the dual action. By Lemma 4.20 we obtain that the action $S \times B^* \to B^*$ is continuous. Furthermore, h is injective (because, if $s_1 \neq s_2$ are distinct elements of $S \subset C(X, X)$ then $s_1x \neq s_2x$ for some $x \in X$. Choose $f \in C(X)$ such that $f(s_1x) \neq f(s_2x)$. Then $fs_1 \neq fs_2$). It is straightforward to see that $\alpha(sx) = h(s)(\alpha(x))$ for every $s \in X, x \in X$. So, (h, α) is equivariant.

(2) Let $S \subset C(K, K)$. Denote by τ_0 the induced topology on S. The action $S \times K \to K$ can be treated as a restriction of the bigger action $S \times B^* \to B^*$, where K naturally is embedded into B^* via Gelfand's map. Then the topology τ on S inherited from $C(B^*, B^*)$ majors the original topology τ_0. Hence, $\tau_0 \subset \tau$.

On the other hand, the continuity of $(S, \tau_0) \times B^* \to B^*$ easily implies (minimality property in Exercise 4.15.4) that $\tau \subset \tau_0$ on S. Summing up we conclude that $\tau = \tau_0$ on S. □

Theorem 4.25. *(Teleman's theorems) Let G be a topological group. Then*

1. *G is embedded into* Iso (V) *for some Banach space V.*
2. *G is embedded into* Iso (M, d) *for some metric space (M, d).*
3. *G is embedded into* Homeo (K) *for some compact space K.*

Proof. Clearly, (1) ⇒ (2). By Theorem 4.24, (3) ⇒ (1). So it is enough to show (3).

The left action $G \times G \to G$ is G-compactifiable. The algebra $\mathrm{RUC}(G)$ separates the points and closed subsets. Then the maximal G-compactification $G \to \beta_G G$ is an embedding and $G \hookrightarrow H(K)$, where $K := \beta_G G$. $\square$

Theorem 4.26. [50] *(universal small actions) Let* $K := [0,1]^{\mathbb{N}}$ *be the Hilbert cube. Denote by* $U := C(K,K)$ *the topological monoid and the natural action* $U \times K \to K$. *Then this action is universal for all monoidal actions on compact metrizable spaces. The pair* $(H(K), K)$ *is universal for continuous group actions on compact metrizable spaces.*

[That is, for every compact metrizable X *and a topological submonoid* S *of* U *there exists an equivariant pair* $(h, \alpha) : (S, X) \rightrightarrows (U, I^{\omega})$ *such that* $h : S \hookrightarrow U$ *is an embedding of topological semigroups and* $\alpha : X \hookrightarrow I^{\omega}$ *is a topological embedding.]*

Proof. Use Equivariant Teleman's representation on $V := C(K)$ and the fact that by Keller's theorem $B^* = (B_{V^*}, w^*)$ is homeomorphic to the Hilbert cube K for every separable V. $\square$

Corollary 4.27. *(Uspenskij* [72]*)* $H([0,1]^{\mathbb{N}})$ *is a universal Polish topological group.*

5. Matrix coefficients

For every $h : S \to L(V,V)$ and any pair of vectors $v \in V$ and $\psi \in V^*$, we have a canonically associated *(generalized) matrix coefficient*

$$m_{v,\psi} : S \to \mathbb{R}, \quad s \mapsto < vs, \psi > = < v, s\psi >$$

$$\begin{array}{ccc} S & \xrightarrow{m_{v,\psi}} & \mathbb{R} \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle \psi} \\ L(V,V) & \xrightarrow{\tilde{v}} & V \end{array}$$

Easy to adopt this definition for any bilinear mappings $V \times W \to \mathbb{R}$ and a pair of compatible (co)-homomorphisms from S to $L(V,V)$ and to $L(W,W)$.

Remark 5.1. In order to justify the name "matrix coefficient" note the following. For $V = \mathbb{R}^n$ (rows $1 \times n$) consider V^* (columns $n \times 1$) and $v = e_i \in V$ and $w = e_j^t \in V^*$ taken from the standard basis.

Then for a matrix $A = (a_{ij})_{n \times n}$ we have

$$a_{ij} = < e_i \cdot A, e_j^t > = e_i \cdot A \cdot e_j^t$$

(where $< v, w > := v \cdot w$ matrix multiplication).

Exercise 5.2. Let $h : S \to \Theta(V)$ be a co-homomorphism. The following are equivalent:

(1) h is weakly continuous;
(2) The action $S \times B^* \to B^*$ is separately continuous (where $B^* := (B^*, w^*)$);
(3) Every (*matrix coefficient*) $m_{v,\psi} : S \to \mathbb{R}, \ s \mapsto < vs, \psi >$ is continuous for any $(v, \psi) \in V \times V^*$.

It is natural to expect that matrix coefficients reflect good properties of flow representations. We recall two well-known facts. The first example is the case of Hilbert representations. Let $h : G \to \mathrm{Iso}\,(H)$ be a continuous homomorphism, where H is Hilbert with its scalar product $H \times H \to \mathbb{R}$. Then the corresponding matrix coefficient $m_{u,v}$ is the so-called *Fourier-Stieltjes functions* on G. If $u = v$, then we get *positive definite functions* on G. The converse is also true: every continuous positive definite function comes from some continuous Hilbert representation. Recall that a continuous bounded function $f : G \to \mathbb{R}$ is said to be *positive definite* if $\sum \alpha_i \alpha_j f(g_i^{-1} g_j) \geq 0$ for all $\alpha_1, \cdots, \alpha_n \in \mathbb{R},\ g_1, \cdots, g_n \in G$. Every positive definite function is WAP.

The second example comes from Eberlein (see [6, Examples 1.2.f]). If V is reflexive, then every bounded V-representation (h, α) and arbitrary pair (v, ψ) lead to a weakly almost periodic function $m_{v,\psi}$ on S. This follows easily by the (weak) continuity of the natural operators defined by the following rule. For every fixed $\psi \in V^*$ $(v \in V)$ define

$$L_\psi : V \to C(S) \text{ and } R_v : V^* \to C(S), \text{where } L_\psi(v) = R_v(\psi) = m_{v,\psi}.$$

We say that a vector $v \in V$ is strong (weak) continuous if the corresponding orbit map $\tilde{v} : S \to V, \tilde{v}(s) = vs$, defined through $h : S \to \Theta(V)$, is strongly (weakly) continuous.

Fact 5.3. *Let $h : S \to \Theta(V)$ be a weakly continuous co-homomorphism (homomorphism). Then*

(1) *$L_\psi : V \to C(S)$ is a linear bounded S-operator between right (left) S-actions.*
(2) *If $v \in V$ is strong continuous, then $m_{v,\psi}$ is right uniformly continuous on S. If ψ is norm continuous then $m_{v,\psi}$ is left uniformly continuous on S.*

Proof. (1) Is straightforward.

(2) In order to establish that $m_{v,\psi} \in \mathrm{RUC}(S)$, observe that

$$|m_{v,\psi}(st) - m_{v,\psi}(s_0 t)| = |< vst, \psi > - < vs_0 t, \psi >| =$$

$$|< vs, t\psi > - < vs_0, t\psi >| \leq ||vs - vs_0|| \cdot ||t\psi|| = ||vs - vs_0||.$$

The second case is similar. □

Fact 5.4. *Let $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B^*)$ be an equivariant pair with weak* continuous α.*

(i) *The map $T : V \to C(X), v \mapsto T(v)$, where $T(v) : X \to \mathbb{R}$ is defined by*

$$T(v)(x) =< v, \alpha(x) >$$

is a linear S-operator (between right S-actions) with $||T|| \leq 1$.

(ii) *$T(v_0) \in \mathrm{RUC}(X)$ for every strongly continuous vector v_0 in V. Hence, if h is strongly continuous then $T(V) \subset \mathrm{RUC}(X)$.*
(iii) *If V is reflexive, then $T(V) \subset \mathrm{WAP}(X)$.*

Proof. (i) Is straightforward.

(ii) Observe that $||\alpha(x)|| \leq 1$ for every $x \in X$. We get

$$||T(v_0)s - T(v_0)s_0|| = \sup\{|< v_0 s - v_0 s_0, \alpha(x) >| : x \in X\} \leq$$

$$\leq ||v_0 s - v_0 s_0|| \cdot ||\alpha(x)|| \leq ||v_0 s - v_0 s_0||.$$

This implies that $T(v_0) \in \mathrm{RUC}(X)$.

(iii) If V is reflexive, the orbit vS is relatively weakly compact for each $v \in V$. By the (weak) continuity of the S-operator T, the same is true for the orbit of $T(v)$ in $C(X)$. Therefore we get $T(v) \in \mathrm{WAP}(X)$. $\square$

Proposition 5.5. *For every S-flow X the following are equivalent:*

1. $f \in \mathrm{RUC}(X)$.
2. *There exist: a Banach space V, a strongly continuous antihomomorphism $h : S \to \Theta(V)$, a weak* continuous equivariant map $\alpha : X \to B^*$, and a vector $v \in V$ such that*

$$f(x) =< v, \alpha(x) >= T(v)(x).$$

Proof. (1) $\Longrightarrow$ (2) The function f belongs to an S-invariant Banach subalgebra $\mathcal{A}$ of $RUC_S(X)$. The right action of S on $V := \mathcal{A}$ is jointly continuous. Then by Corollary 4.21, corresponding left action of S on the dual ball (B^*, w^*) is jointly continuous. Then the naturally associated map $\alpha : X \to B^*$ and the vector $v := f$ satisfy the desired property.

(1) $\Longleftarrow$ (2) Immediate by Fact 5.4 (ii). $\square$

Proposition 5.6. *For every semitopological monoid S the following are equivalent:*

1. $f \in \mathrm{RUC}(S)$.
2. *There exist: a Banach space V, a strongly continuous antihomomorphism $h : S \to \Theta(V)$, and a pair of vectors $v \in V$ and $\psi \in V^*$ such that $f = m_{v,\psi}$.*

   If G is a topological group then $h(G) \subset Is(V)$.

Proof. (1) $\Longrightarrow$ (2) Consider the Gelfand compactification $u_R : S \to S^R$ defined by $\mathrm{RUC}(S) = C(S^R)$. Then the action $S \times S^R \to S^R$ is jointly continuous. Now define: $V := C(S^R)$, corresponding strongly continuous $h : S \to \Theta(V)$ (induced by the right action of S on $C(S^R)$), $v := f \in V$ and $\psi = u_R(e) \in V^*$.

(1) $\Longleftarrow$ (2) Immediate by Fact 5.3.2. $\square$

So we see that every right uniformly continuous function on a (semi)group can be represented as a matrix coefficient $m_{v,\psi}$ of some strongly continuous *Banach representation.* We mentioned also that a positive definite function on a topological group G is a matrix coefficient of some Hilbert representation. One of our aims is to understand the role of matrix coefficients for reflexive, Asplund and Rosenthal representations. We show that wap functions are exactly the *reflexive matrix coefficients.* In the "Asplund case" this approach leads to a definition of *Asplund functions.* For "Rosenthal case" we will get the so-called tame (regular (Kohler [40]) functions.

6. Reflexive spaces and WAP systems

6.1. **Double Limit Property.** Let F, X, Y be topological spaces and $w : F \times X \to Y, w(f,x) := f(x)$ be a function. We say that F has the Double Limit Property (DLP) on X if for every sequence $\{f_n\} \subset F$ and every sequence $\{x_m\} \subset X$ the limits

$$\lim_n \lim_m f_n(x_m) \quad \text{and} \quad \lim_m \lim_n f_n(x_m)$$

are equal whenever they both exist.

Example 6.1. Let V be a reflexive space. Then B has DLP on B^*.

Proof. By Eberlein-Shmulian theorem B and B^* in their weak topologies are sequentially compact. □

Theorem 6.2. *(Raynaud* [62, Prop. 1.1]*, Krivine-Maurey* [39, Theorem II.3] *for metrizable X, F) Let $w : F \times X \to \mathbb{R}$ be a separately continuous bounded function with compact spaces F and X. Then it can be represented on a reflexive space. That is, there exists a reflexive space V and weak continuous maps $\nu : F \to V$, $\alpha : X \to V^*$ such that $< \nu(f), \alpha(x) >= w(f,x)$.*

Remark 6.3. One may refine these results (even keeping the general action setting) as follows. The fundamental DFJP-factorization construction from [10] has an "isometric modification" [42]. Taking into account this modification note that we can prove a little bit more. Namely, if the given family $F \subset C(X)$ is *bounded by constant* 1, then we can assume that $\nu(F) \subset B$ and $\alpha(X) \subset B^*$. Hence the following sharper diagram commutes:

$$\begin{array}{ccc} F \times X & \longrightarrow & [-1,1] \\ \nu \downarrow \quad \downarrow \alpha & & \downarrow id \\ B \times B^* & \longrightarrow & [-1,1] \end{array}$$

For more details see [27].

Corollary 6.4. *Let F and X are compact spaces and $w : F \times X \to \mathbb{R}$ be a separately continuous bounded function. Then:*

1. *F has DLP on X;*
2. *The induced (bounded) images $j_1(F) \subset C_p(X)$ and $j_2(X) \subset C_p(F)$ are Eberlein compacta (hence, Frechet and sequentially compact).*

Note that (1) admits also a direct proof easily reducing the proof to the case of metrizable F, X.

Lemma 6.5. (Grothendieck) *Let X be a compact space. Then a bounded subset F of $C(X)$ is weakly compact iff F is pointwise compact.*

Proof. By Lebesgue dominated convergence theorem it follows that any pointwise converging bounded sequence in $C(X)$ is weakly converging. So, $id : (F,p) \to (F,w)$ is sequentially continuous. The evaluation map $(F,p) \times X \to \mathbb{R}$ is separately continuous. Therefore, (F,p) is a Frechet compactum (Corollary 6.4). So we obtain that the pointwise and weak topologies on $F \subset C(X)$ are the same. □

Lemma 6.6. *(Grothendieck; see for example* [6, Appendix A]*) Let F be a bounded subset in a Banach space V. The following are equivalent:*

1. *The weak closure of F in V is weakly compact;*
2. *F has DLP on B^*.*

Theorem 6.7. *Let V be a Banach space. The following conditions are equivalent:*

1. *V is reflexive.*
2. *B has DLP on B^*.*
3. *every bounded subset $F \subset V$ has DLP on every bounded $X \subset V^*$.*
4. *$B \subset V$ is weakly compact.*

Proof. (1) ⇒ (2) As in Example 6.1 use Eberlein-Shmulian theorem.

(2) ⇒ (3) Is trivial.

(3) ⇒ (4) Apply Lemma 6.6.

(4) ⇒ (1) B_V is w^*-closed in V^{**}. By Goldstine's theorem the w^*-closure of B_V in V^{**} is $B^{**} := B_{V^{**}}$. Hence, $B = B^{**}$. This implies directly that $V = V^{**}$. □

Exercise 6.8. Let X be a compact space and $F \subset C(X)$ be a bounded subset. Show that F has DLP on X iff F has DLP on B^*, where $B^* = B_{C(X)^*}$.

Lemma 6.9. *(Grothendieck) A bounded function $f \in C(G)$ is wap iff*

$$\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m)$$

whenever all the limits exist.

6.2. **WAP dynamical systems.** Given a function $f \in C(X)$ we consider its orbit $fS := \{f \circ \tilde{s} : s \in S\} \subset C(X)$. For every $f \in C(X)$ the function $E \to \mathbb{R}^X, s \mapsto fs$ is pointwise continuous. So we have $fE = cl_p(fS)$.

One may estimate the dynamical complexity of f is by considering the pointwise compact subset $\mathrm{cl}_{\mathrm{p}}(\mathrm{fS})$ in $\mathbb{R}^X$. Various kinds of "smallness" of this compactum leads to a natural hierarchy. The classical example is (weakly) almost periodic functions.

Definition 6.10. Let X be a compact S-system.

(1) $f \in C(X)$ is said to be WAP if one of the following equivalent conditions is satisfied:
  (a) fS is weakly precompact in $C(X)$;
  (b) $\mathrm{cl}_{\mathrm{p}}(\mathrm{fS}) \subset \mathrm{C(X)}$;
  (c) fS has DLP on X.

(2) (S, X) is said to be WAP if one of the following equivalent conditions is satisfied:
  (a) every member $p \in E(S, X)$ is a continuous function $X \to X$;
  (b) $\mathrm{WAP}(X) = C(X)$.

The equivalences can be verified using Grothendieck's classical results. See for example, [6, Theorem A4] and [6, Theorem A5]. If X is metrizable (or, sequentially compact) then (a) and (b) in (2) are equivalent to the condition: (c) $S \times X \to X$ has DLP.

Theorem 6.11. *(Ellis and Nerurkar* [15]*) Let X be a compact S-dynamical system. The following conditions are equivalent.*

(1) *(S, X) is WAP.*

(2) *The enveloping semigroup $E(S, X)$ consists of continuous maps. That is, $E(S, X) \subset C(X, X)$.*

Proof. (1) ⇒ (2) By Definition 6.10, $\mathrm{cl}_{\mathrm{p}}(\mathrm{fS}) \subset \mathrm{C(X)}$ for every $f \in C(X) = \mathrm{WAP}(X)$. Therefore, $fp : X \to \mathbb{R}$ is continuous for every $f \in C(X)$ and $p \in E(X)$. Since X is compact this guarantees that every $p : X \to X$ is continuous.

(2) ⇒ (1) If $E \subset C(X, X)$ then $fE = cl_p(fS) \subset C(X)$. By Grothendieck's Lemma 6.5, $cl_p(fS)$ is weakly compact. Hence, $f \in \mathrm{WAP}(X)$. □

Corollary 6.12. *When (S, X) is WAP the enveloping semigroup $E(X)$ is a semitopological semigroup. The converse holds if in addition we assume that (S, X) is point transitive.*

Example 6.13.

(1) (Eberlein, see for example [6, Examples 1.2.f]) If V is reflexive, then every weakly continuous representation (h, α) of an S-system X on V and every pair $(v, \psi) \in V \times V^*$ lead to a weakly almost periodic function $m_{v,\psi}$ on S. This follows easily by the (weak) continuity of the bounded operator $L_\psi : V \to C(S)$, where $L_\psi(v) = m_{v,\psi}$.

(2) Analogously, every $v \in V$, with reflexive V, defines a wap function $T_v : X \to \mathbb{R}$ on the G-system X which naturally comes from the given dynamical system representation (h, α). Precisely, define

$$T(V) = T_v : X \to \mathbb{R}, \quad x \mapsto \langle v, \alpha(x) \rangle.$$

Then the set of functions $\{T_v\}_{v \in V}$ is a subset of $\mathrm{WAP}(X)$.

Proof. (1) If the orbit of vS is relatively weakly compact in V. Then $L_\psi(vS) = m_{v,\psi}S$ is relatively weakly compact in $C(S)$. Thus, $m_{v,\psi} \in \mathrm{WAP}(S)$.

For the case when h is a "homomorphism" recall (see [9] or [6]) that fS is weakly precompact iff Sf is weakly precompact in $C(S)$.

(2) Is similar. □

If in (2) α is an embedding (which implies that X is reflexively representable) then it follows that the collection $\{T_v\}_{v \in V}$ (and hence also $\mathrm{WAP}(X)$) separates the points of X. If, in addition, X is compact it follows that $\mathrm{WAP}(X) = C(X)$ (because $\mathrm{WAP}(X)$ is always a closed subalgebra of $C(X)$). That is, in this case (S, X) is WAP in the sense of Ellis and Nerurkar.

The converses of Example 6.13 is also true as we show below.

Theorem 6.14. [49] *Let $S \times X \to X$ be a separately continuous action of a semitopological semigroup S on a compact space X. For every $f \in \mathrm{WAP}(X)$ there exist: a reflexive space V, a functional $\phi \in V^*$ and an equivariant pair*

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V), B_V)$$

such that $h : S \to \Theta(V)$ is a weakly continuous homomorphism, $\alpha : X \to B_V$ is a weakly continuous S-map, and $f(x) = \langle \phi, \alpha(x) \rangle = \phi(\alpha(x))$ for every $x \in X$.

If $S = G$ is a semitopological group then one can assume in addition that $h(G) \subset \mathrm{Iso}\,(V)$ and $h : G \to \mathrm{Iso}\,(V)$ is strongly continuous.

Theorem 6.15. [49, section 4] *Let S be a semitopological semigroup.*

(1) *A compact (continuous) S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively approximable.*
(2) *A compact (continuous)* metric *S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively representable.*
(3) *Every $f \in \mathrm{WAP}(S)$ is a matrix coefficient of a reflexive representation.*

It is important to take into account the following characterization of reflexive spaces.

Lemma 6.16. *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is reflexive.*
(2) *The semitopological semigroup $\Theta(V)_w$ is compact.*
(3) *The compact right topological semigroup $\mathcal{E}(V)$ is semitopological.*

(4) *(Θ^{op}, B^*) is a WAP system.*

Proof. (1) $\Rightarrow$ (2) Always, $\Theta(V)$ is a closed subset of the product $(B, w)^B$. So, if V is reflexive then by Theorem 6.7 $(B, w)^B$ is compact. Hence, we obtain by Tychonoff theorem that $\Theta(V)$ is compact.

(2) $\Rightarrow$ (3) Use the fact that $\Theta(V)^{op}_w$ is dense in $\mathcal{E}(V)$ (Lemma 3.12).

(3) $\Rightarrow$ (4) One may apply Corollary 6.12 because $(\Theta(V)^{op}, B^*)$ is transitive.

(4) $\Rightarrow$ (1) Choose any $v \in S_V$ and treat it as a (continuous) function on the dynamical Θ^{op}-system B^*. Then $v \in \mathrm{WAP}(B^*)$. Then its orbit has DLP on B^*. So, $v\Theta^{op} = \Theta v = B$ has DLP on B^*. □

Another consequence of Theorem 6.15 (taking into account Lemma 6.16) is

Theorem 6.17. ([67] and [46]) *Every compact semitopological semigroup S can be embedded into $\Theta(V)$ for some reflexive V.*

Thus, compact semitopological semigroups S can be characterized as closed subsemigroups of $\mathcal{E}(V)$ for reflexive Banach spaces V.

6.3. DFJP factorization for WAP dynamical systems.

Theorem 6.18. *Let X be a compact S-space and $F \subset C(X)$ a norm bounded S-invariant subset of $C(X)$. The following are equivalent:*

(1) *(F, S, X) admits a reflexive representation.*
(2) $\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \subset \mathrm{C}(\mathrm{X})$.
(3) *F has DLP on X.*

Proof. (1) $\Rightarrow$ (3) If V is a reflexive space then every bounded subset F of the dual V^* has DLP on every bounded subset $X \subset V$. This follows from the Eberlein-Šmulian theorem. See Theorem 6.7.

(2) $\Rightarrow$ (1) Theorem 6.20 below.

(2) $\Leftrightarrow$ (3) It can be found for example in [6, Theorem A.4]. □

Theorem 6.19. *(S, X) is a WAP (continuous) system if and only if (S, X) is weakly (respectively, strongly) reflexively-approximable. If the compactum X is metrizable then "approximable" can be replaced by "representable".*

Proof. The "only if" part: Use the fact that (Θ^{op}, B^*) is a WAP system (Theorem 14.4) for every reflexive space V.

The "if" part: (1) For every $f \in C(X) = \mathrm{Tame}(X)$ the orbit fS has DLP (being weakly precompact) family for X. Applying Theorem 15.4 below we conclude that every $f \in C(X) = \mathrm{Tame}(X)$ on a compact S-space X comes from a reflexive representation. Since continuous functions separate points of X, this implies that reflexive representations of (S, X) separate points of X. So, it is enough to prove the following result which gives a proof of Theorem 6.18.

Theorem 6.20. *Let X be a compact S-space and let $F \subset C(X)$ be a bounded S-invariant pointwise compact family. Then there exist: a reflexive Banach space V, an injective S-equivariant mapping $\nu : F \to B_V$ and a representation*

$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$

of (S,X) on V such that h is weakly continuous, α is a weak* (in fact, weakly) continuous map and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\, f \in F \quad \forall\, x \in X.$$

Thus the following diagram commutes

(6.1)
$$\begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ {\scriptstyle\nu}\downarrow \quad \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle id_{\mathbb{R}}} \\ V \times V^* & \longrightarrow & \mathbb{R} \end{array}$$

If X is metrizable then in addition we can suppose that V is separable.
If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous.
If $S = G$ is a group then $h(G) \subset \mathrm{Iso}\,(V)$.
If F separates points of X then $\alpha : X \to (V^*, w^*)$ is a topological embedding.

Proof. **Step 1:** The construction of V.

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F; that is, $W := \mathrm{co}\,(F \cup -F)$. Then W is convex and symmetric. Consider the sequence of sets

(6.2)
$$M_n := 2^n W + 2^{-n} B.$$

We apply the construction of Davis-Figiel-Johnson-Pelczyński [10] as follows. Let $\|\ \|_n$ be the Minkowski functional of the set M_n, that is,

$$\|v\|_n = \inf\,\{\lambda > 0 \;\big|\; v \in \lambda M_n\}.$$

Then $\|\ \|_n$ is a norm on $\mathcal{A}$ equivalent to the given norm of $\mathcal{A}$. For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and let } V := \{v \in \mathcal{A} \;\big|\; N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \to \mathcal{A}$ is a continuous linear injection and

(6.3)
$$W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B)$$

Indeed, if $v \in W$ then $2^n v \in M_n$, hence $\|v\|_n \leq 2^{-n}$ and $N(v)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$. This proves $W \subset j(B_V)$. In order to prove the second inclusion recall that the norms $\|\cdot\|_n$ on $\mathcal{A}$ are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, for every $n \in \mathbb{N}$, $v \in \lambda_n M_n$ for some $0 < \lambda_n < 1$. By the construction M_n is a convex subset containing the origin. This implies that $\lambda_n M_n \subset M_n$. Hence $j(v) = v \in M_n$ for every $n \in \mathbb{N}$.

Step 2: The construction of the representation (h, α) of (S, X) on V.

The given action $S \times X \to X$ induces a natural linear norm preserving continuous right action $C(X) \times S \to C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S-invariant subsets in $\mathcal{A}$. This implies that V is an S-invariant subset of $\mathcal{A}$ and the restricted natural linear action $V \times S \to V, \ (v,g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \to \Theta(V), \ h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \to V^*$ be the adjoint map of $j : V \to \mathcal{A}$. Define $\alpha : X \to V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V.

By the construction, $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\, f \in F \ \ \forall\, x \in X. \tag{6.4}$$

Step 4: V is a reflexive space.

Proof. By Grothendieck Lemma F is weakly compact. Now use that by Krein-Smulian theorem $W := \mathrm{co}\,(F \cup -F)$ is relatively weakly precompact. Now follow the arguments of [10]. Sketch: $j^{**}(B_{V^{**}}) \subset \mathcal{A}$. Hence, j^{**} is 1-1 and $(j^{**})^{-1}(\mathcal{A}) = V$. It follows that $V^{**} \subset V$ (reflexivity).

Step 3: Weak continuity of $h : S \to \Theta(V)$.

By our construction $j^* : C(X)^* \to V^*$, being the adjoint of the bounded linear operator $j : V \to C(X)$, is a norm and weak* continuous linear operator. By [16, Lemma 1.2.2] we obtain that $j^*(C(X)^*)$ is norm dense in V^*. Since V (being reflexive) is Rosenthal, Haydon's theorem (Fact 12.11.4) gives $Q := cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \to X$ is separately continuous, every orbit map $\tilde{x} : S \to X$ is continuous, and each orbit map $\widetilde{j^*(x)} : S \to j^*(X)$ is weak* continuous. Then also $\widetilde{j^*(z)} : S \to V^*$ is weak* continuous for each $z \in cl_{norm}(co(j^*(X))) = Q$. It is well known that $P(X)$ generates $C(X)^*$ (even algebraically). So, $sp(Q = j^*(P(X)))$ is norm dense in V^*. Since $||h(s)|| \leq 1$ for each $s \in S$, it easily follows that $j^*\tilde{(z)} : S \to V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N, we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on $\mathcal{A}$ is equivalent to the given norm on $\mathcal{A}$).

□

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable. □

This proves Theorem 15.4 and hence also Theorem 15.3.1. □

7. Applications

Theorem 7.1. *(***WAP Representation Theorem***) Let X be a compact semitopological S-system and $f \in C(X)$. The following conditions are equivalent:*

(i) *$f : X \to \mathbb{R}$ is weakly almost periodic.*

(ii) *There exist: a representation* (h, α) *of* (S, X) *into reflexive* V *with a weak continuous antihomomorphism* $h : S \to \Theta(V)$*, weak (eq., weak-star) continuous* $\alpha : X \to B^*$*, and a vector* $v \in V$ *such that* $f(x) =< v, \alpha(x) >$*.*

If either: a) $S = G$ *is a semitopological group; or b)* X *is compact and the action* $S \times X \to X$ *is jointly continuous, then in (ii) we can suppose that* h *is strongly continuous.*

Proof. Recall that a continuous function $f \in C(X)$ is WAP iff the orbit fS is relatively weakly compact in $C(X)$. That is, $cl_w(fS)$ is weakly compact. Apply Theorem 6.20 for $F := cl_w(fS)$. □

Theorem 7.2. *A compact* S*-system* X *is WAP iff* X *is reflexively approximable.*

Proof. If X is REFL-approximable then X is wap by Fact 5.4 (iii) (in fact, WAP separates points but since WAP(X) is an algebra and X is compact it is enough).

The nontrivial part follows from Theorem 7.1 because if X has sufficiently many wap functions, then (S, X) has sufficiently many reflexive representations. □

Corollary 7.3. *Every metrizable compact WAP system is Eberlein.*

Theorem 7.4. *For every semitopological monoid* S *the function* $f : S \to \mathbb{R}$ *is wap iff* f *is a matrix coefficient of a weak continuous antihomomorphism* $S \to \Theta(V)$ *for a reflexive* V*. That is, there exist* $v \in V$ *and* $\psi \in V^*$ *such that* $f(s) =< vs, \psi >$*.*

If $S = G$ *is a group then* $h(G) \subset \mathrm{Iso}\,(V)$ *(and* h *is strongly continuous).*

Proof. Apply Theorem 7.1 to the flow (S, S). Then for $f \in WAP(S)$ there exists a reflexive V and a representation $h : S \to \Theta(V)$, $\alpha : S \to B(V^*)$ such that $f(s) =< v, \alpha(s) >$ for a suitable $v \in V$. Denote by e the identity of S. Then $f = m_{v,\psi}$ where $\psi = \alpha(e)$. □

If we wish to get a *homomorphism*, just consider $h : S \to \Theta(V)^{opp} = \Theta(V^*)$.

Fact 7.5. *(* [67] *and* [46]*) Let* S *be a semitopological semigroup. The following are equivalent:*

(i) *S is embedded into a compact semitopological monoid.*
(ii) *There exists a reflexive space* E *such that* S *is embedded (as a semitopological subsemigroup) into* $\Theta(E)_w$*.*

Therefore, compact semitopological semigroups are exactly the class of all closed subsemigroups of $\Theta(E)_w$ *for some reflexive* V*.*

Proof. (i) $\Longrightarrow$ (ii) We can suppose that S is a monoid. Consider $X := S^W$ the universal semitopological compactification of S. Then the corresponding universal map $u_W : S \to S^W$ is a topological embedding by (i) and hence, the action (S, S^W) is *left strict*. That is, there is no strictly coarser topology on S under which S is a semitopological semigroup and S^W is still a semitopological S-flow. By Theorem 7.1 there exists a separating family (h_i, α_i) of reflexive V_i-representations $(i \in I)$ of (S, S^W). Then the l_2-sum of these representations defined on the Banach space $V := (\sum_i V_{i \in I})_{l_2}$ will induce a weakly continuous antihomomorphism $h : S \to \Theta(V)$. Since the original action is left strict, it is easy to show that h must be a topological embedding. Define $E := V^*$. It is clear that the antihomomorphism h defines the desired homomorphism $h : S \to \Theta(V)^{opp} = \Theta(V^*) = \Theta(E)$.
(ii) $\Longrightarrow$ (i) Use Lemma 6.16. □

By Theorem 2.7, $\mathrm{Iso}\,(V)_s = \mathrm{Iso}\,(V)_w$ for every reflexive V. Therefore we obtain

Fact 7.6. *Let G be a topological group. The following are equivalent:*

(i) *$G \to G^W$ is an embedding.*
(ii) *G is a topological subgroup of the group $Is(V)_s$ (endowed with the strong operator topology) of all linear isometries for a suitable reflexive V.*

We next recall a version of Lawson's theorem and a soft geometric proof using representations of dynamical systems on reflexive spaces.

Theorem 7.7. (Ellis-Lawson Joint Continuity Theorem) *Let G be a subgroup of a compact semitopological monoid S. Suppose that $S \times X \to X$ is a separately continuous action with compact X. Then the action $G \times X \to X$ is jointly continuous (and G is a topological group).*

Proof. A sketch of the proof from [49]: We show the joint continuity of $G \times X \to X$ (for the last part take $X := S$ and the natural action $G \times S \to S$). It is easy to see by Grothendieck's Lemma (Theorem 6.5) that $C(X) = \mathrm{WAP}(X)$. Hence (S, X) is a weakly almost periodic system. By Theorem 6.15 the proof can be reduced to the particular case where $(S, X) = (\Theta(V)^{op}, B_{V^*})$ for some reflexive Banach space V with $G := \mathrm{Iso}\,(V)$, where $\Theta(V)^{op}$ is endowed with the weak operator topology. By Theorem 2.7, the weak and strong operator topologies coincide on $\mathrm{Iso}\,(V)$ for reflexive V. In particular, G acts continuously on B_{V^*}. □

As a corollary one gets the classical result of Ellis.

Theorem 7.8. (Ellis' Theorem) *Every compact semitopological group is a topological group.*

8. Some parallel hierarchies

8.1. **Representation theorems.**

Theorem 8.1. *(Small families of functions) Let X be a compact S-space and let $F \subset C(X)$ be a norm bounded S-invariant subset of $C(X)$.*

(1) *(F, S, X) admits a Rosenthal representation iff F is an eventually fragmented family iff $\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \subset \mathcal{F}(\mathrm{X})$ iff F does not contain an independent subsequence.*
(2) *(F, S, X) admits an Asplund representation iff F is a fragmented family iff the envelope $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ of F is a fragmented family.*
(3) *(F, S, X) admits a reflexive representation iff $\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \subset \mathrm{C}(\mathrm{X})$ iff F has DLP on X.*

Proof. (3) Already was proved in Section 6.3.
(1) and (2): The "only if part" is a consequence of the characterizations of Asplund and Rosenthal spaces in terms of fragmented and eventually fragmented families, Theorems 12.5.4 and 12.11.4. □

Recall the definitions of HNS and tame compact dynamical systems.

Definition 8.2. We say that a compact S-system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

(1) For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.
(2) The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.
(3) $E(S, X)$ is a fragmented family of maps from X into itself.

It is equivalent to the condition that fS is fragmented for every $f \in C(X)$.

Definition 8.3. A compact separately continuous S-system X is said to be *tame* if the translation $\lambda_a : X \to X,\ x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.

It is equivalent to saying that fS is free of independent subsequences for every $f \in C(X)$ or that fS is eventually fragmented.

Theorem 8.4.
(1) *(S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.*
(2) *(S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.*

If X is metrizable then in (1) and (2) "approximable" can be replaced by "representable".

Theorem 8.5. *A compact S-system X is RN (WRN, Eberlein) iff there exists a bounded S-invariant X-separating family $F \subset C(X)$ which is fragmented (resp.: eventually fragmented, DLP).*

In the following table we encapsulate some features of the trinity: dynamical systems, enveloping semigroups, and Banach representations. Here X is a compact metrizable G-space and $E(X)$ denotes the corresponding enveloping semigroup. The symbol f stands for an arbitrary function in $C(X)$ and $fG = \{f \circ g : g \in G\}$ denotes its orbit. Finally, $\mathrm{cl}\,(fG)$ is the pointwise closure of fG in $\mathbb{R}^X$.

DS	**Dynamical characterization**	**Enveloping semigroup**	**Banach representation**
WAP	$\mathrm{cl}\,(fG)$ is a subset of $C(X)$	Every element is continuous	Reflexive
HNS	$\mathrm{cl}\,(fG)$ is metrizable	$E(X)$ is metrizable	Asplund
Tame	$\mathrm{cl}\,(fG)$ is Fréchet	Every element is Baire 1	Rosenthal

Table 1. The hierarchy of Banach representations

9. Some examples

Example 9.1.
(1) Let $X = [0, 1]$ be the unit interval. Consider the cascade $(\mathbb{Z}, X)$ generated by the homeomorphism $\sigma(x) = x^2$. Then $(\mathbb{Z}, X)$, as a dynamical system, is RN and not Eberlein (not WAP). To see this observe that the pair of sequences $x_n = 1 - \frac{1}{n}$ in $X = [0, 1]$ and $\sigma^m \in G$ with $\sigma^m(x) = x^{2^m}$ does not satisfy DLP. The corresponding limits are 0 and 1. This means that $(\mathbb{Z}, [0, 1])$ is not Eberlein. The enveloping semigroup $E(\mathbb{Z}, [0, 1])$ is metrizable

being homeomorphic to the two-point compactification of $\mathbb{Z}$. Hence, by [30], $(\mathbb{Z}, [0,1])$ is RN. The sequence $\{\sigma^m : [0,1] \to [0,1]\}_{m\in\mathbb{N}}$ is a fragmented family which does not satisfy DLP.

(2) $\chi_{\mathbb{N}} : \mathbb{Z} \to \{0,1\}$ is not a WAP function. Indeed, it does not satisfy DLP. Choose $s_n := n, x_m := -m$. Then

$$\lim_m \lim_n f(n-m) = 1 \neq 0 = \lim_n \lim_m f(n-m)$$

The "simple signal"

$$\cdots 000111 \cdots$$

is not reflexively representable (as a matrix coefficient). However it can be represented on an Asplund space.

(3) Let P_0 be the set $[0,c)$ and P_1 the set $[c,1)$; let z be a point in $[0,1)$ (identified with $\mathbb{T}$) via the rotation R_α we get the binary bisequence u_n, $n \in \mathbb{Z}$ defined by $u_n = 0$ when $R^n_\alpha(z) \in P_0, u_n = 1$ otherwise. These are called *Sturmian codings*. With $c = 1-\alpha$ we retrieve the previous example. For example, when $\alpha := \frac{\sqrt{5}-1}{2}$ and $c = 1-\alpha$ the corresponding sequence, computed at $z = 0$, is called the *Fibonacci bisequence*.

(4) The Sturmian symbolic dynamical $\mathbb{Z}$-system $(\sigma, X) \subset \Omega = \{0,1\}^{\mathbb{Z}}$ is WRN but not RN (being tame but not HNS). The sequence $\{\sigma^n : X \to X\}_{n\in\mathbb{Z}}$ is an eventually fragmented but not fragmented family. As a (nontrivial) corollary:

Every (irrational slope) "cutting bisequence" (like Fibonacci) is Rosenthal representable (but not Asplund representable).

The corresponding enveloping semigroup of (σ, X) topologically is a copy of $E = \mathbb{Z} \cup Y$, where Y is the double arrow space. So, E topologically is Frechet but not metrizable ([23], Pikula [61], Aujogue [2]). This means that (σ, X) is tame but not HNS.

(5) The natural action of the Polish group $H_+[0,1]$ of all increasing homeomorphisms of $[0,1]$ on $[0,1]$ is tame but not HNS. The family $H_+[0,1]$ of functions (or its dense subsequence) is eventually fragmented but not fragmented.

(6) The Bernoulli shift system $(\mathbb{Z}, \{0,1\}^{\mathbb{Z}})$ is not WRN (equivalently, nontame). In fact, it is well known that the enveloping semigroup of this system can be identified with $\beta\mathbb{Z}$. Now use the dynamical version of BFT dichotomy (Fact 14.8).

Another way to see that the shift system is not tame is the well known fact that the sequence of projections

$$\{\pi_m : \{0,1\}^{\mathbb{Z}} \to \{0,1\}\}_{m\in\mathbb{Z}}$$

is independent. Hence by Theorem 12.10 this family fails to be eventually fragmented.

(7) In his paper Ellis (93), following Furstenberg's classical work (63), investigates the projective action of $GL(n,\mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$. It follows from his results that the corresponding enveloping semigroup is not first countable. However, in a later work Akin (98) studies the action of $G = GL(n,\mathbb{R})$ on the sphere $\mathbb{S}^{n-1}$ and shows that here the enveloping semigroup is first countable (but not metrizable). It follows that the dynamical systems $D_1 = (G, \mathbb{P}^{n-1})$ and $D_2 = (G, \mathbb{S}^{n-1})$ are tame but not HNS. Note that $E(D_1)$ is Fréchet, being a continuous image of a first countable compact space, namely $E(D_2)$.

10. More applications of fragmentability

10.1. **When does weak imply strong ?** A not necessarily compact G-system X is called *quasiminimal* if $\mathrm{int}(\mathrm{cl}\,(Gz)) \neq \emptyset$ for every $z \in X$. 1-orbit systems and compact minimal G-systems are quasiminimal.

Theorem 10.1. *Let $G \leq \mathrm{Iso}\,(V)$ be a subgroup, X a bounded, $(weak, norm)$-fragmented G-invariant subset of a Banach space V. Then for every, not necessarily closed, quasiminimal G-subspace (e.g., the orbits) Y of X the weak and norm topologies coincide.*

Proof. □

This result together with a characterization of PCP (Theorem 12.12) yield:

Theorem 10.2. [46, 49] *Let V be a Banach space with PCP (e.g., reflexive, RNP, or the dual of Asplund). Then*
1. *norm topology = weak topology on every orbit Gv for every $G \leq \mathrm{Iso}\,(V)$.*
2. *The weak and the strong operator topologies coincide on $\mathrm{Iso}\,(V)$.*
3. *Every weakly continuous (co)homomorphism $h : G \to \mathrm{Iso}\,(V)$ is strongly continuous.*

10.2. **Adjoint continuity property of Asplund spaces.**

Theorem 10.3. *Let V be an Asplund space. Then for every subgroup $G \subset \mathrm{Iso}\,(V)$ the dual action $V \times G \to V$ is norm continuous.*

Proof. Similar to Theorem 10.2. Use the characterization of Asplund spaces in terms of fragmentability. □

More generally, this is true for any continuous linear topological group action (not necessarily by isometries).

Theorem 10.4. [49, Corollary 6.9] *Let V be an Asplund Banach space and $\pi : V \times G \to V$ a linear jointly continuous action. Then the dual action $\pi^* : G \times V^* \to V^*$ is also jointly continuous.*

The regular representation $\mathbb{T} \to \mathrm{Iso}\,(V)$ of the circle group $G := \mathbb{T}$ on $V := C(\mathbb{T})$ is continuous but not adjoint continuous. Consider the Banach space $V := l_1$ and the topological subgroup $G := S(\mathbb{N})$ ("permutations of coordinates") of $\mathrm{Iso}\,(l_1)$. Then we have a natural continuous representation of the symmetric topological group $S(\mathbb{N})$ on l_1 which is not adjoint continuous.

10.3. **Helmer's theorem.** One more application is a quick proof of $\mathrm{WAP}(G) \subset UC(G) := \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$, Helmer's theorem. In fact, we can show more.

Theorem 10.5. [49] $\mathrm{WAP}(G) \subset \mathrm{Asp}(G) \subset UC(G)$ *for every topological group G.*

Proof. Let $f \in \mathrm{Asp}(G)$. The function f coincides with a matrix coefficient $m_{v,\psi}$ for a suitable strongly continuous antihomomorphism $h : G \to \mathrm{Iso}\,(V)_s$, where V is Asplund. In particular, v is a norm continuous vector. By Theorem 10.4 the orbit $G\psi$ is light. Hence, ψ is a norm continuous vector. By Fact 5.3.2, $f = m_{v,\psi}$ is both left and right uniformly continuous. □

10.4. **Ryll-Nardzewski's fixed point Theorem.**

Theorem 10.6. (Ryll-Nardzewski) *Let V be a locally convex vector space equipped with its uniform structure ξ. Let (Q,τ) be an affine compact S-system such that*

(1) *(Q,τ) is a weakly compact subset in V.*
(2) *S is ξ-distal on Q.*

Then Q contains a fixed point.

Proof. (Sketch) We can suppose that $Q = cl(coX)$, where X is a compact minimal S-system. Weakly compact set X is ξ-fragmented (generalized Namioka's Theorem). Using the fragmentability one may lift the distality of X from the topology ξ to τ. So, (S,X) is distal. Therefore, the proof can be reduced to the following theorem of H. Furstenberg.

Theorem 10.7. (Furstenberg) *Every distal compact dynamical system admits an invariant probability measure.*

□

11. Representations of topological groups II

Theorem 11.1.

(1) $L_4[0,1] \in \mathbf{Ref_r}$;
(2) $L_4[0,1] \notin \mathbf{Hilb_r}$.

[Chaatit 1996] The additive group of every separable *stable* (Krivine-Maurey [39]) Banach space belongs to $\mathbf{Ref_r}$.

$\Longrightarrow$ (1)

(direct) proof of (1):

Lemma 11.2. *G is reflexively representable iff $WAP(G)$ separates points and closed subsets.*

Lemma 11.3. $\phi : L_{2k} \to \mathbf{R} \quad v \mapsto e^{-\|v\|}$ *is wap.*

(Grothendieck's DLP) A bounded function $f \in C(G)$ is wap iff $\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m)$ whenever all the limits exist.

Exercise 11.4. If G admits a left (or, right) invariant metric with DLP then $G \in \mathbf{Ref_r}$.

- $(L_{2k}(\mu), \|\|) \quad (k \in \mathbf{N})$ has the DLP.

$$\|u_n + v_m\|^{2k} = \|u_n\|^{2k} + \sum_{i=1}^{2k-1} C_{2k}^i \int u_n^{2k-i} v_m^i dt + \|v_m\|^{2k}$$

$$\int u_n^{2k-i} v_m^i dt =< u_n^{2k-i}, v_m^i >$$

$$u_n^{2k-i} \in L_{\frac{2k}{2k-i}} \quad v_m^i \in L_{\frac{2k}{i}} = L^*_{\frac{2k}{2k-i}}.$$

- Use DLP (Theorem 6.7) for $B(V) \times B(V^*) \to [-1, 1]$ with reflexive V.

(2) $L_4[0,1] \notin \mathbf{Hilb_r}$

Theorem 11.5 (Aharoni-Maurey-Mityagin 1985)**.** *For $2 < p < \infty$, an infinite-dimensional $L_p(\mu)$ space is not uniformly embedded into a Hilbert space.*

Lemma $\forall$ metric subgroup G of $\mathrm{Iso}\,(H)_s$
$\exists\ (G, \mathcal{L}) \overset{unif}{\hookrightarrow} H.$

(a): $\|v_n\| = \frac{1}{2^n}$
(b): $\{\tilde{v}_n : \mathrm{Iso}\,(H) \to H, \quad g \mapsto g v_n\}_{n \in \mathbb{N}}$ generates the left uniformity on $\mathrm{Iso}\,(H)$.

$$\mathrm{Iso}\,(H)_s \overset{unif}{\hookrightarrow} \prod_n B_{\frac{1}{2^n}} \overset{unif}{\hookrightarrow} (\sum_n (H)_n)_{l_2} \longleftrightarrow H$$

$$\Longrightarrow \quad \boxed{L_p[0,1] \notin \mathbf{Hilb_r} \quad p > 2} \quad \Longrightarrow (2)$$

11.1. **About** $H_+[0,1]$.

Theorem 11.6. *The group $G := H_+[0,1]$ is Rosenthal representable.*

Proof. Consider the natural action of G on the closed interval $X := [0,1]$ and the corresponding enveloping semigroup $E = E(G,X)$. Every element of G is a (strictly) increasing self-homeomorphism of $[0,1]$. Hence every element $p \in E$ is a nondecreasing function $[0,1] \to [0,1]$. So, Baire 1 function. This means that the G-system X is tame. By Theorem 15.3 we have a faithful representation (h, α) of (G, X) on a separable Rosenthal space V. Therefore we obtain a G-embedding $\alpha : X \hookrightarrow (V^*, w^*)$. It follows, by the minimality properties of the compact open topology (Lemma 4.22), that h is an embedding. Thus $h \circ \mathrm{inv} : \mathrm{G} \to \mathrm{Iso(V)}$ is the required topological group embedding. $\square$

Theorem 11.7. $H_+[0,1]$ *is Asplund-trivial (hence, also reflexively-trivial) .*

Proof. Denote by $j : G \to G^{Asp}$ and $i : G \to G^{UC}$ the G-compactifications (i necessarily is proper because $UC(G)$ separates the points and closed subsets) induced by the Banach G-algebras $Asp(G) \subset UC(G)$ (Theorem 10.5). There exists a canonical onto G-map $\pi : G^{UC} \to G^{Asp}$ such that the following diagram of compact G-systems is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\ i\ } & G^{UC} \\ & \searrow^{j} & \downarrow{\scriptstyle \pi} \\ & & G^{Asp} \end{array}$$

We have to show that G^{Asp} is trivial for $G = H_+[0,1]$. One of the main tools for the proof is the following identification.

Lemma 11.8. [Uspenskij] *The dynamical system G^{UC} is isomorphic to the G-space (G, Ω). Here Ω denotes the compact space of all curves in $[0,1] \times [0,1]$ which connect the points $(0,0)$ and $(1,1)$ and "never go down", equipped with the Hausdorff metric. These are the relations $\omega \subset [0,1] \times [0,1]$ where for each $t \in [0,1]$, $\omega(t)$ is either a point or a vertical closed segment.*

Moreover, the natural action of $G = H_+[0,1]$ on Ω is $(g\omega)(t) = g(\omega(t))$ (by composition of relations on $[0,1]$).

We first note that every "zig-zag curve" (i.e. a curve z which consists of a finite number of horizontal and vertical pieces) is an element of Ω. In particular the curves γ_c with exactly one vertical segment defined as $\gamma_c(t) = 0$ for every $t \in [0,c)$, $\gamma_c(c) = \{c\} \times [0,1]$ and $\gamma(t) = 1$ for every $t \in (c,1]$, are elements of $\Omega = G^{UC}$. Note that the curve γ_1 is a fixed point for the left G action. We let $\theta = \pi(\gamma_1)$ be its image in G^{Asp}. Of course θ is a fixed point in G^{Asp}. We will show that $\theta = j(e)$ and since the G-orbit of $j(e)$ is dense in G^{Asp} this will show that G^{Asp} is a **singleton**.

The idea is to show that zig-zag curves are "Asp-proximal" in G^{UC}. Meaning that their images in G^{Asp} coincide. Choosing a sequence z_n of zig-zag curves which converges in the Hausdorff metric to $i(e)$ in G^{UC} we will have $\pi(z_n) = \pi(\gamma_1) = \theta$ for each n. This will imply that indeed $j(e) = \pi(i(e)) = \pi(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} \pi(z_n) = \theta$.

First we show that $\pi(\gamma_1) = \pi(\gamma_c)$ for any $0 < c < 1$. Assuming the contrary there exists $f \in C(G^{Asp})$ such that $f(\pi(\gamma_1)) \neq f(\pi(\gamma_c))$. Dynamical system (G, G^{Asp}) is Asplund. So, there exists an Asplund representation (F, h, α) of $F := fG$ on an Asplund space V. So, there exists $v \in V$ such that

$$f(x) = < v, \alpha(x) > \qquad \forall\ x \in G^{Asp}.$$

Let $p \in G^{UC}$ be the curve defined by $p(t) = t$ in the interval $[0,c]$ and by $p(t) = c$ for every $t \in [c,1)$. Pick a sequence s_n of elements in G such that s_n converges to e and $s_n c < c$. It is easy to choose a sequence g_n in G such that $g_n s_n c$ converges to 0 and $g_n c$ converges to 1. Then $g_n p \to \gamma_c$, $g_n s_n p \to \gamma_1$ (see the picture below). **Since V is Asplund the dual action $G \times V^* \to V^*$ is norm continuous.** (Theorem 10.3) In particular, $||s_n \alpha(\pi(p)) - \alpha(\pi(p))|| \to 0$. This implies that

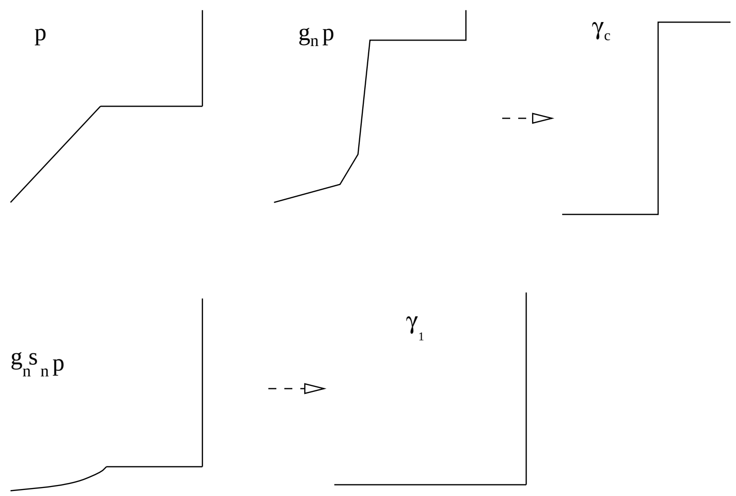
$$|f(\pi(g_n s_n p)) - f(\pi(g_n p))| =$$

$$= | < v, \alpha(\pi(g_n s_n p)) > - < v, \alpha(\pi(g_n p)) > | =$$
$$= | < v g_n, s_n \alpha(\pi(p)) - \alpha(\pi(p)) > | \leq$$
$$\leq ||v g_n|| \cdot ||s_n \alpha(\pi(p)) - \alpha(\pi(p))|| \to 0.$$

Hence, $f(\pi(\gamma_1)) = f(\pi(\gamma_c))$.

We could say instead of these concrete computations that, as we know, every matrix coefficient on Asplund space, being an Asplund function, is (right and) left uniformly continuous. Here we have a MATRIX COEFFICIENT – "tango" pairing two vectors $v \in V$ and $\alpha\pi(p) \in V^*$; i.e.,

$$m(v, \alpha(\pi(p)) : G \to \mathbb{R}, \ g \mapsto f(g\alpha(\pi(p)).$$



Denote $\theta = \pi(\gamma_1) = \pi(\gamma_c)$. Using similar arguments construct a sequence $z_n \in G^{UC}$ of zig-zag curves which converges to $i(e)$ and such that $\pi(z_n) = \theta$ for every n.

In view of the discussion above this construction completes the proof of the theorem.

□

Remark 11.9. Note that the WAP-triviality of some other natural groups were established very recently. For the homeomorphisms group of the pseudo-arc and Lelek's fan, see [3]. For the orientation preserving homeomorphisms of the circle $H_+(\mathbb{T})$ (in fact, Asplund-triviality), see [29].

Remark 11.10. Some additional useful references: [64, 19, 18, 74, 4].

12. Appendix A: Fragmentability and Banach spaces

The concept of fragmentability originally comes from Banach space theory and has several applications in Topology, and more recently also in Topological Dynamics.

Definition 12.1. Let (X, τ) be a topological space and (Y, μ) a uniform space.

(1) [36] X is (τ, μ)-*fragmented* by a (typically, not continuous) function $f : X \to Y$ if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε-small in Y. We also say in that case that the function f is *fragmented*. Notation: $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$ then we write simply $\mathcal{F}(X)$.
(2) [23] We say that a *family of functions* $F = \{f : (X, \tau) \to (Y, \mu)\}$ is *fragmented* if condition (1) holds simultaneously for all $f \in F$. That is, $f(O \cap A)$ is ε-small for every $f \in F$.
(3) [29] We say that F is an *eventually fragmented family* if every infinite subfamily $C \subset F$ contains an infinite fragmented subfamily $K \subset C$.

In Definition 12.1.1 when $Y = X, f = id_X$ and μ is a metric uniform structure, we get the usual definition of fragmentability (more precisely, (τ, μ)-fragmentability) in the sense of Jayne and Rogers [37]. Implicitly it already appears in a paper of Namioka and Phelps [56].

Remark 12.2. [23, 25]

(1) It is enough to check the condition of Definition 12.1 for closed subsets $A \subset X$ and for $\epsilon \in \mu$ from a *subbase* γ of μ (that is, the finite intersections of the elements of γ form a base of the uniform structure μ).
(2) When X and Y are Polish spaces, $f : X \to Y$ is fragmented iff f is a Baire class 1 function.
(3) When X is compact and (Y, ρ) metrizable uniform space then $f : X \to Y$ is fragmented iff f has a *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point).
(4) When Y is compact with its unique compatible uniformity μ then $p : X \to Y$ is fragmented if and only if $f \circ p : X \to \mathbb{R}$ has a point of continuity property for every $f \in C(Y)$.
(5) A topological space (X, τ) is scattered iff X is (τ, μ)-fragmented, for every uniform structure μ on the set X.
(6) Let (X, τ) be a separable metrizable space and (Y, ρ) a pseudometric space. Suppose that $f : X \to Y$ is a fragmented onto map. Then Y is separable.

Lemma 12.3.

(1) *Suppose F is a compact space, X is Čech-complete, Y is a uniform space and we are given a separately continuous map $w : F \times X \to Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \to Y\}_{f \in F}$ is fragmented, where $\tilde{f}(x) = w(f, x)$.*
(2) *Suppose F is a compact metrizable space, X is hereditarily Baire and M is separable and metrizable. Assume we are given a map $w : F \times X \to M$ such that every $\tilde{x} : F \to M, f \mapsto w(f, x)$ is continuous and $y : X \to M$ is continuous at every $\tilde{y} \in Y$ for some dense subset Y of F. Then the family $\tilde{F}$ is fragmented.*
(3) (version of Osgood's theorem) *Let $f_n : X \to \mathbb{R}$ be a pointwise convergent sequence of continuous functions on a hereditarily Baire space X. Then $F := \{f_n\}_{n \in \mathbb{N}}$ is a fragmented family.*

Proof. (1): There exists a collection of uniform maps $\{\varphi_i : Y \to M_i\}_{i \in I}$ into metrizable uniform spaces M_i which generates the uniformity on Y. Now for every closed subset $A \subset X$ apply Namioka's joint continuity theorem to the separately continuous map $\varphi_i \circ w : F \times A \to M_i$ and take into account Remark 12.2.1.

(2): Since every $\tilde{x} : F \to M$ is continuous, the natural map $j : X \to C(F, M)$, $j(x) = \tilde{x}$ is well defined. Every closed nonempty subset $A \subset X$ is Baire. By [30, Proposition 2.4], $j|_A : A \to C(F, M)$ has a point of continuity, where $C(F, M)$ carries the sup-metric. Hence, $\tilde{F}_A = \{\tilde{f} \restriction_A : A \to M\}_{f \in F}$ is equicontinuous at some point $a \in A$. This implies that the family $\tilde{F}$ is fragmented.

(3) follows from (2) applied to the evaluation map $w : F \times X \to \mathbb{R}$, where $F := \{f\} \cup \{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ with $f := \lim f_n$, the pointwise limit.

□

Remark 12.4. Let us briefly describe one of the ideas linking fragmentability and dynamical systems. Suppose that X is a weak* compact dual ball of some Banach space V. One of the major themes in Banach space theory is the study of the relationship between the norm and the weak* topologies on $X \subset V^*$. When these two coincide, we say that X is a *Kadec subset* of V^*. If moreover X is a subsystem (under some action by linear isometries) then X, as a dynamical system, is equicontinuous. In general, as an attempt to measure "the level of equicontinuity", we can ask how close are the two natural topologies on X inherited from V^*. A more concrete, but sufficiently flexible, question is: for which dynamical system representations is the natural mapping $1_X : (X, weak^*) \to (X, norm)$ fragmented ? The latter means that every nonempty subset of X admits relatively weak* open nonempty subsets with arbitrarily small diameters. Every point of continuity of 1_X is a point of equicontinuity of the dynamical system X.

12.1. **Banach spaces defined by fragmentability.** We recall the definitions of three important classes of Banach spaces: Asplund, Rosenthal and PCP. Each of them can be characterized in terms of fragmentability.

12.1.1. *Asplund Banach spaces.* Recall that a Banach space V is an *Asplund space* if the dual of every separable linear subspace is separable.

In the following result the equivalence of (1), (2) and (3) is well known and (4) is a reformulation of (3) in terms of fragmented families.

Theorem 12.5. [56, 55] *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is an Asplund space.*
(2) *V^* has the Radon-Nikodým property.*
(3) *Every bounded subset A of the dual V^* is (weak*,norm)-fragmented.*
(4) *B is a fragmented family of real valued maps on the compactum B^*.*

Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. By [56] the Banach space $C(K)$ for compact K is Asplund iff K is a scattered compactum (see also Lemma 12.2.4). Namioka's Joint Continuity Theorem implies that every weakly compact set in a Banach space is norm fragmented, [55]. This explains why every reflexive space is Asplund.

12.1.2. *Banach spaces not containing l_1.*

Definition 12.6. Let $f_n : X \to \mathbb{R}$ be a uniformly bounded sequence of functions on a *set* X. Following Rosenthal we say that this sequence is an *l_1-sequence* on X if there exists a real constant $a > 0$ such that for all $n \in \mathbb{N}$ and choices of real scalars $c_1, \dots, c_n$ we have

$$a \cdot \sum_{i=1}^{n} |c_i| \leq || \sum_{i=1}^{n} c_i f_i||.$$

This is the same as requiring that the closed linear span in $l_\infty(X)$ of the sequence f_n be linearly homeomorphic to the Banach space l_1. In fact, in this case the map

$$l_1 \to l_\infty(X), \quad (c_n) \to \sum_{n \in \mathbb{N}} c_n f_n$$

is a linear homeomorphic embedding.

Definition 12.7. A sequence f_n of real valued functions on a set X is said to be *independent* if there exist real numbers $a < b$ such that

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{m \in M} f_m^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

Definition 12.8. A Banach space V is said to be *Rosenthal* if it does not contain an isomorphic copy of l_1.

Every Asplund space is Rosenthal (because l_1^* is the nonseparable space l_∞).

Definition 12.9. Let X be a topological space. We say that a subset $F \subset C(X)$ is a *Rosenthal family* (for X) if F is norm bounded and the pointwise closure $\mathrm{cl}_{\,\mathrm{p}}(F)$ of F in $\mathbb{R}^X$ consists of fragmented maps, that is, $\mathrm{cl}_{\,\mathrm{p}}(F) \subset \mathcal{F}(X)$.

The following useful result synthesizes some known results.

It is based on results of Rosenthal [65], Talagrand [68, Theorem 14.1.7] and van Dulst [11]. In [25, Prop. 4.6] we show why eventual fragmentability of F can be included in the following list.

Theorem 12.10. *Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

(1) *F does not contain an independent subsequence.*
(2) *F does not contain a subsequence equivalent to the unit basis of l_1.*
(3) *Each sequence in F has a pointwise convergent subsequence in $\mathbb{R}^X$.*
(4) *F is a Rosenthal family for X.*
(5) *F is an eventually fragmented family.*

Theorem 12.11. *Let V be a Banach space. The following conditions are equivalent:*

(1) *V is a Rosenthal Banach space.*
(2) (E. Saab and P. Saab [66]) *Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^*. Equivalently, $B^{**} \subset \mathcal{F}(B^*)$.*
(3) *B is a* Rosenthal family *for the weak* compact unit ball B^*.*
(4) *B is an eventually fragmented family of maps on B^*.*
(5) (Haydon [34, Theorem 3.3]) *For every weak* compact subset $Y \subset V^*$ the weak* and norm closures of the convex hull $co(Y)$ in V^* coincide: $cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$.*

Condition (2) is a reformulation (in terms of fragmented maps) of a criterion from [66] which was originally stated in terms of the point of continuity property. The equivalence of (1), (3) and (4) follows from Theorem 12.10.

12.1.3. *Banach spaces with PCP.* A Banach space V is said to have the *point of continuity property* (PCP for short) if every bounded weakly closed subset $C \subset V$ admits a point of continuity of the identity map $(C, weak) \to (C, norm)$ (see for example Edgar-Wheeler [13] and [37]). Every Banach space with RNP has PCP. In particular, this is true for the duals of Asplund spaces and for reflexive spaces. This concept was studied, among others, by Bourgain and Rosenthal. They show, for instance, that there are separable Banach spaces with PCP which do not satisfy RNP.

Theorem 12.12. *(Jayne and Rogers* [37]*) Let V be a Banach space. The following conditions are equivalent:*

(1) *V has PCP.*
(2) *Every bounded subset $A \subset V$ is $(weak, norm)$-fragmented.*

12.2. **More properties of fragmented families.** Here we demonstrate a general principle: the fragmentability of a family of continuous maps on a compact space is "countably-determined". Formally the following theorem is new, though its proof (the part (3) ⇒ (1)) is inspired by a result of Namioka [55, Theorem 3.4].

Theorem 12.13. *Let $F = \{f_i : X \to Y\}_{i \in I}$ be a bounded family of* ***continuous*** *maps from a compact (not necessarily metrizable) space (X, τ) into a pseudometric space (Y, d). The following conditions are equivalent:*

(1) *F is a fragmented family of functions on X.*
(2) *Every* countable *subfamily K of F is fragmented.*
(3) *For every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where*

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} d(f(x_1), f(x_2)).$$

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3): Let K be a countable subfamily of F. Consider the natural map

$$\pi : X \to Y^K, \pi(x)(f) := f(x).$$

By (2), K is a fragmented family. Thus by Lemma 12.2.6 the map π is (τ, μ_K)-fragmented, where μ_K is the uniformity of d-uniform convergence on $Y^K := \{f : K \to (Y, d)\}$. Then the map π is also (τ, d_K)-fragmented, where d_K is the pseudometric on Y^K defined by

$$d_K(z_1, z_2) := sup_{f \in K} d(z_1(f), z_2(f)).$$

Since d is bounded, $d_K(z_1, z_2)$ is finite and d_K is well-defined. Denote by (X_K, τ_p) the subspace $\pi(X) \subset Y^K$ in pointwise topology. Since $K \subset C(X)$, the induced map $\pi_0 : X \to X_K$ is a continuous map onto the compact space (X_K, τ_p). Denote by $i : (X_K, \tau_p) \to (Y^K, d_K)$ the inclusion map. So, $\pi = i \circ \pi_0$, where the map π is (τ, d_K)-fragmented. Then by Lemma 12.2.7 we obtain that i is (τ_p, d_K)-fragmented. It immediately follows that the identity map $id : (X_K, \tau_p) \to (X_K, d_K)$ is (τ_p, d_K)-fragmented.

Since K is countable, $(X_K, \tau_p) \subset Y^K$ is metrizable. Therefore, (X_K, τ_p) is second countable (being a metrizable compactum). Now, since d_K is a pseudometric on Y^K, and $id : (X_K, \tau_p) \to (X_K, d_K)$ is (τ_p, d_K)-fragmented, we can apply Lemma 12.2.5. It directly implies that the set X_K is a separable subset of (Y^K, d_K). This means that $(X, \rho_{K,d})$ is separable.

(3) ⇒ (1) : Suppose that F is not fragmented. Thus, there exists a non-empty closed subset $A \subset X$ and an $\varepsilon > 0$ such that for each non-empty open subset $O \subset X$ with $O \cap A \neq \emptyset$ there is some $f \in O$ such that $f(O \cap A)$ is not ε-small in (Y, d). Let V_1 be an arbitrary non-empty relatively open subset in A. There are $a, b \in V_1$ and $f_1 \in F$ such that $d(f_1(a), f_1(b)) > \varepsilon$. Since f_1 is continuous we can choose relatively open subsets V_2, V_3 with $\mathrm{cl}\,(\mathrm{V_2} \cup \mathrm{V_3}) \subset \mathrm{V_1}$ such that $d(f(x), f(y)) > \varepsilon$ for every $(x, y) \in V_2 \times V_3$.

By induction we can construct a sequence $\{V_n\}_{n \in \mathbb{N}}$ of non-empty relatively open subsets in A and a sequence $K := \{f_n\}_{n \in \mathbb{N}}$ in F such that:

(i) $\mathrm{cl}\,(\mathrm{V_{2n}} \cup \mathrm{V_{2n+1}}) \subset \mathrm{V_n}$ for each $n \in \mathbb{N}$;
(ii) $d(f_n(x), f_n(y)) > \varepsilon$ for every $(x, y) \in V_{2n} \times V_{2n+1}$.

We claim that $(X, \rho_{K,d})$ is not separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} d(f(x_1), f(x_2)).$$

In fact, for each *branch*

$$\alpha := V_1 \supset V_{n_1} \supset V_{n_2} \supset \cdots$$

where for each $i, n_{i+1} = 2n_i$ or $2n_i + 1$, by compactness of X one may choose an element

$$x_\alpha \in \bigcap_{i \in \mathbb{N}} \mathrm{cl}\,(V_{n_i}).$$

If $x = x_\alpha$ and $y = x_\beta$ come from different branches, then there is an $n \in \mathbb{N}$ such that $x \in \mathrm{cl}\,(V_{2n})$ and $y \in \mathrm{cl}\,(V_{2n+1})$ or (vice versa). In any case it follows from (ii) and the continuity of f_n that $d(f_n(x), f_n(y)) \geq \varepsilon$, hence $\rho_{K,d}(x, y) \geq \varepsilon$. Since there are uncountably many branches we conclude that A and hence also X are not $\rho_{K,d}$-separable. □

Definition 12.14. [16, 49] Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X. Then F is said to be an *Asplund family for* X if for every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} |f(x_1) - f(x_2)|.$$

Any Asplund family for a compact space X can be viewed, by [16, Lemma 1.5.3], as a particular case of the more general concept of an *Asplund set* in the Banach space $C(X)$.

Corollary 12.15. *Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X. Then F is fragmented if and only if F is an Asplund family for X.*

Theorem 12.16. *Let $F = \{f_i : X \to Y\}_{i \in I}$ be a family of continuous maps from a compact (not necessarily metrizable) space (X, τ) into a pseudouniform space (Y, μ). Then F is fragmented if and only if every countable subfamily $A \subset F$ is fragmented.*

Proof. The proof can be reduced to Theorem 12.13. Every pseudouniform space can be uniformly approximated by pseudometric spaces. Using Lemma 12.2.1 we can suppose that (Y, μ) is pseudometrizable; i.e. there exists a pseudometric d such that $\text{unif}(d) = \mu$. Moreover, replacing d by the uniformly equivalent metric $\frac{d}{1+d}$ we can suppose that $d \leq 1$. □

12.3. **The natural affine extension map** $T : b\mathcal{B}_1(X) \to b\mathcal{B}_1(B^*)$.

Definition 12.17. Let X be a topological space.

(1) A function $f : X \to \mathbb{R}$ is said to be Baire 1 if $f^{-1}(O)$ is an F_σ in X for every open $O \subset X$. Notation: $f \in \mathcal{B}_1(X)$.
(2) Denote by $\mathcal{B}_1^l(X)$ the set of all pointwise limits of sequences of continuous functions on X.
(3) Bounded functions in $\mathcal{B}_1(X)$ and $\mathcal{B}_1^l(X)$ are denoted by $b\mathcal{B}_1(X)$ and $b\mathcal{B}_1^l(X)$.

Always, $\mathcal{B}_1^l(X) \subset \mathcal{B}_1(X)$ (van Dulst p. 137 for every X) and $\mathcal{B}_1(X) \subset \mathcal{F}(X)$, for every hereditarily Baire space [7, Lemma 1C(c)].

Below X be a compact space. It naturally is embedded into $(C(X)^*, w^*)$. This embedding induces a natural injective map

$$T : b\mathcal{B}_1^l(X) \to b\mathcal{B}_1^l(B^*),$$

where B^*, as before, is the weak* compact unit ball of $C(X)^*$. In the definition of T below we will use Riesz representation theorem and Lebesgue's Dominated Convergence Theorem.

For compact X we have $\mathcal{F}(X) = B_r^{\S}$ in terms of [7]. Each $f \in \mathcal{F}(X)$ is universally measurable for every compact space X (see for example [7, Prop. 1F])). Therefore, for every measure $\mu \in B^*$ we can define

$$(Tf)(\mu) := \int f d\mu.$$

This map is well defined. Indeed, first note that when $f \in C(X)$, $T(f) = i(f)$, where

$$i : C(X) \hookrightarrow C(B^*), \quad i(f)(\mu) := \langle f, \mu \rangle = \int f d\mu$$

is the canonical isometric inclusion of the corresponding Banach spaces and

$$\langle \cdot, \cdot \rangle : C(X) \times C(X)^* \to \mathbb{R}$$

is the canonical bilinear mapping. Now if $f \in b\mathcal{B}_1^l(X)$ then by definition f is a pointwise limit of a sequence of continuous functions $h_n \in C(X)$. Since $f : X \to \mathbb{R}$ is a bounded function we can assume in addition that the sequence h_n is uniformly bounded. By Lebesgue's Convergence Theorem it follows that $T(f)$ is a pointwise limit of the sequence $T(h_n) = i(h_n)$, $n \in \mathbb{N}$. Since every $i(h_n) \in C(B^*)$ we conclude (by Definition 12.17) that $T(f) \in \mathcal{B}_1^l(B^*)$. The sequence $i(h_n)$ is uniformly bounded in $C(B^*)$ hence $T(f)$ is a bounded function. This means that $T(f) \in b\mathcal{B}_1^l(B^*)$.

The map T is injective because $T(f)(\delta_x) = f(x)$ for every point mass $\delta_x \in B^*$ $(x \in X)$.

Remark 12.18. Each $T(f)$ for $f \in b\mathcal{B}_1^l(X)$ can be treated as an element of the second dual $C(X)^{**}$ of $C(X)$. Moreover the pointwise topology of $\mathcal{B}_1^l(B^*)$ and the weak*-topology on $C(X)^{**}$ agree on $T(b\mathcal{B}_1^l(X))$.

Lemma 12.19. *Let X be a compact space. For every uniformly bounded subset $A \subset b\mathcal{B}_1^l(X)$ the restriction $T|_A$ of the natural injective map*

$$T : b\mathcal{B}_1^l(X) \to b\mathcal{B}_1^l(B^*) \cap C(X)^{**}$$

on A is sequentially continuous. Furthermore, $T(A)$ is also uniformly bounded.

Proof. Lebesgue's Convergence Theorem implies that T is sequentially continuous. The boundedness of $T(A)$ is easy. □

Namioka [55] gave a kind of duality between uniform separability and pointwise metrizability.

Theorem 12.20. (Namioka [55, Theorem 4.1]) *Let $F \subset \mathbb{R}^X$ be a bounded set of maps. The following are equivalent:*

(1) *The pseudometric space (X, ρ_F) is separable, where*

$$\rho_F(x_1, x_2) := \sup\{|f(x_1) - f(x_2)| : \ f \in F\}.$$

(2) *The pointwise closure* $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ *is a metrizable subspace of* $\mathbb{R}^X$.

Proof. (1) ⇒ (2) F is an equicontinuous family on the pseudometric space (X, ρ_F). Then its pointwise closure $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ is also equicontinuous. So, it follows that the topology of pointwise convergence on X for $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ is the same as the topology of pointwise convergence on a countable ρ_F-dense subset of X. Hence, $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ is metrizable.

(2) ⇒ (1) Let $K := \mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ be pointwise metrizable. Then $C(K)$ is norm separable. Denote by $\varphi : X \to C(K)$ the induced map defined by $\varphi(x)(f) := f(x)$ for every $f \in C(K)$. Then $\varphi(X) \subset C(K)$ is norm separable. In particular (since $F \subset K$) we obtain that (X, ρ_F) is separable. □

Lemma 12.21. *(Namioka* [55, Theorem 4.4]*) Let $F \subset C(K)$ be a bounded family of continuous functions on a compact space K. The following are equivalent:*

(1) *F is a fragmented family of functions on K.*
(2) *The pseudometric space (K, ρ_A) is separable for every countable $A \subset F$.*
(3) *The pointwise closure* $\mathrm{cl}\,(\mathrm{A})$ *is a metrizable subspace of* $\mathbb{R}^K$ *for every countable $A \subset F$.*

Proof. Combine Theorem 12.21 and Corollary 12.15. □

Proposition 12.22. *If $F \subset C(X)$ is a countable bounded fragmented family on a compact space X then*

(1) $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ *is a metrizable subset of* $b\mathcal{B}_1^l(X)$.
(2) *the restriction of T on* $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ *induces a homeomorphism*

$$\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \to \mathrm{cl}_{\mathrm{p}}(\mathrm{T}(\mathrm{F})) \subset \mathrm{b}\mathcal{B}_1^{\mathrm{l}}(\mathrm{B}^*) \cap \mathrm{C}(\mathrm{X})^{**}.$$

Proof. (1) $\mathrm{cl}_p(F)$ is a uniformly bounded subset of $\mathbb{R}^X$ because F is bounded. Since F is fragmented we obtain that (X, ρ_F) is separable. So, by Lemma 12.21, $\mathrm{cl}_p(F)$ is metrizable in $\mathbb{R}^X$. Therefore, every $\phi \in \mathrm{cl}_p(F)$ is a pointwise closure of a subsequence in F. Hence, $\phi \in b\mathcal{B}_1^l(X)$.

(2) In view of Lemma 12.19 the restricted (injective!) map $T : \mathrm{cl}_p(F) \to b\mathcal{B}_1^l(B^*)$ is sequentially continuous. This restriction is even continuous because $\mathrm{cl}_p(F)$ is metrizable by (1). We conclude that the map $T : \mathrm{cl}_p(F) \to b\mathcal{B}_1^l(B^*)$ is a continuous injection, and therefore a homeomorphism, of $\mathrm{cl}_p(F)$ onto its image in $b\mathcal{B}_1^l(B^*)$. □

Proposition 12.23. *Let X be a compact space and $F \subset C(X)$ be bounded family. The following conditions are equivalent:*

(1) *F is a (eventually) fragmented family for X iff F_{B^*} is a (resp., eventually) fragmented family for B^*.*
(2) *F is a Rosenthal family for X iff F_{B^*} is a Rosenthal family for B^*.*

Proof. (1) Let A be a countable subfamily of F. Then since A is fragmented, Lemma 12.21 implies that $\mathrm{cl}\,(\mathrm{A_X}) \subset \mathrm{b}\mathcal{B}_1^{\mathrm{l}}(\mathrm{X})$ is metrizable. By Proposition 12.22.2 $\mathrm{cl}\,(\mathrm{A_{B^*}})$ is homeomorphic to $\mathrm{cl}\,(\mathrm{A_X})$. Therefore, $\mathrm{cl}\,(\mathrm{A_{B^*}}) \subset \mathrm{b}\mathcal{B}_1^{\mathrm{l}}(\mathrm{B^*})$ is metrizable, too. Now again by Lemma 12.21 we obtain that A_{B^*} is fragmented. It is true for every countable subfamily of F_{B^*}. Thus, F_{B^*} is fragmented. This proves the "fragmented case". The "eventually fragmented case" is verbatim the same.

(2) is a reformulation of the eventually fragmented case of (1). □

Of course if F_{B^*} is fragmented then $F_{P(X)}$ is fragmented, too.

Corollary 12.24. *$C(K)$ is Asplund iff the compact space K is scattered.*

Proof. Let K is scattered. Then it is fragmented by any uniformity, in particular with respect to the norm of $C(K)^*$. Then Proposition 12.23 guarantees that B^* is also fragmented by the norm. Therefore, $C(K)$ is Asplund.

Second direction comes from the following Exercise. □

Exercise 12.25. Let K be a compact space which is norm-fragmented in $C(K)^*$. Show that K is scattered.

Hint: the norm uniformity on $X \subset C(K)^*$ is discrete.

Theorem 12.26. *Let $F \subset V$ be a norm bounded subset and $K \subset V^*$ be a weak-star compact subset. Then F is a fragmented family on K iff F is a fragmented family on $Q := \overline{co}(K)$.*

Proof. Consider the restriction operator

$$r_K : V \to C(K),\ r_K(v)(x) =< v, x > \quad \forall x \in K$$

$F_K := r_K(F)$ is a fragmented family on K. Then by Proposition 12.23, F_Q is also fragmented on $P(K) \subset B^*$. Now consider the adjoint $r_K^* : C(K)^* \to V^*$. Then $r_K^*(P(K) = Q$. □

Corollary 12.27. *(Namioka* [55, Thm 2.5]*) Let $K \subset V^*$ be a weak-star compact subset which is norm fragmented. Then $\overline{co}(K)$ is also norm fragmented.*

Proof. Take $F := B_V$. □

Lemma 12.28. *(Fabian's Lemma* [16, Lemma 1.5.3]*) Let K be a compact space, let $F \subset B_{C(K)}$ be a nonempty set, and consider on $C(K)^*$ the pseudometric ρ_F defined as*

$$\rho_F(\lambda, \mu) = \sup_{f \in F} < \lambda - \mu, f > \qquad \lambda, \mu \in C(K)^*.$$

Assume that (K, ρ_F) (as a subset of $C(K)^*$) is separable. Then $(C(K)^*, \rho_F)$ is also separable.

Proof. By Theorem 12.20, $cl_p(F) \subset b\mathcal{B}_1^l(X)$ is metrizable. Since $T : cl_p(F) \subset b\mathcal{B}_1^l(X) \to b\mathcal{B}^l(B^*)$ is sequentially continuous we obtain that this map is a homeomorphic embedding. Hence, $T(cl_p(F)) \subset b\mathcal{B}^l(B^*)$ is also metrizable. Again by Theorem 12.20 we obtain that (B^*, ρ_F) is separable. This implies that $(C(K)^*, \rho_F)$ is also separable. □

Lemma 12.29. *Let F be a fragmented family of real valued functions on (X, μ). Then $co(F)$ is also fragmented.*

Proof. If $f_i(D)$ is ε-small for every $i = 1, \cdots, n$ and $\sum_{i=1}^n c_i = 1,\ c_i > 0$ then $\sum_{i=1}^n c_i f_i(D)$ is ε-small. □

13. Appendix B: HNS and tame systems

13.1. **Some classes of right topological semigroups and dynamical systems.** To the basic classes of right topological semigroups listed in 3.3 above, we add the following two which have naturally arisen in the study of tame and HNS dynamical systems.

Definition 13.1. [23, 25] A compact admissible right topological semigroup P is said to be:

(1) [25] *tame* if the left translation $\lambda_a : P \to P$ is a fragmented map for every $a \in P$.
(2) *HNS-semigroup* if $\{\lambda_a : P \to P\}_{a \in P}$ is a fragmented family of maps.

These classes are closed under factors. We have the inclusions:

{compact semitopological semigroups} $\subset$ {HNS-semigroups} $\subset$ {Tame semigroups}

Lemma 13.2.

(1) *Every compact semitopological semigroup P is a HNS-semigroup.*
(2) *Every HNS-semigroup is tame.*

(3) *If P is a* metrizable *compact right topological admissible semigroup then P is a HNS-semigroup.*

Proof. (1) Apply Lemma 12.3.1 to $P \times P \to P$.
(2) is trivial.
(3) Apply Lemma 12.3.2 to $P \times P \to P$. □

If P is Fréchet, as a topological space, then P is a tame semigroup by Corollary 14.6 below.

13.2. **HNS-semigroups, dynamical systems and Asplund Banach spaces.**

Definition 13.3. We say that a compact S-system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

(1) For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.
(2) The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.
(3) $E(S, X)$ is a fragmented family of maps from X into itself.

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2). As to the implication (2) $\Rightarrow$ (3), observe that the pointwise closure of a fragmented family is again a fragmented family, [25, Lemma 2.8].

Note that if $S = G$ is a group then in Definition 13.3.1 one may consider only closed subsets A which are G-invariant (see the proof of [23, Lemma 9.4]).

Lemma 13.4.

(1) *For every S the class of HNS compact S-systems is closed under subsystems, arbitrary products and factors.*
(2) *For every HNS compact S-system X the corresponding enveloping semigroup $E(X)$ is HNS both as an S-system and as a semigroup.*
(3) *Let P be a HNS-semigroup. Assume that $j : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $j(S) \subset \Lambda(P)$. Then the S-system P is HNS.*
(4) *{HNS-semigroups}={enveloping semigroups of HNS systems}.*

Theorem 13.5. *Let V be a Banach space. The following are equivalent:*

(1) *V is an Asplund Banach space.*
(2) *(Θ^{op}, B^*) is a HNS system.*
(3) *$\mathcal{E}$ is a HNS-semigroup.*

Proof. (1) $\Rightarrow$ (2): Use Definition 13.3.2 and the following well known characterization of Asplund spaces: V is Asplund iff B^* is $(w^*, norm)$-fragmented (Fact 12.5).

(2) $\Rightarrow$ (1) By Fact 12.5 we have to show that B is a fragmented family for B^*. Choose a vector $v \in S_V$. Since Θ^{op} is a fragmented family of self-maps on B^* and as $v : B^* \to \mathbb{R}$ is uniformly continuous we get that the system $v\Theta^{op} = \Theta v$ of maps from B^* to $\mathbb{R}$ is also fragmented. Now recall that $\Theta v = B$ by Lemma 3.12.1.

(2) $\Rightarrow$ (3): Follows from Lemma 13.4.2 and the fact that $\mathcal{E}$ is the enveloping semigroup $E(\Theta^{op}, B^*)$.

(3) ⇒ (2): $\Lambda(\mathcal{E}) = \Theta^{op}$ (Lemma 3.12.5) and $\mathcal{E}$ is a HNS-semigroup. So, $(S, \mathcal{E})$ is HNS by Lemma 13.4.3 with $S = \Theta^{op}$. Take $\psi \in B^*$ with $||\psi|| = 1$. The map $q : \mathcal{E} \to B^*$, $p \mapsto p\psi$ defines a continuous homomorphism of Θ^{op}-systems. By Lemma 3.12.4, we have $\mathcal{E}\psi = B^*$. So q is onto. Now observe that the HNS property is preserved by factors of S-systems (Lemma 13.4.1). □

Our next two theorems are based on ideas from Glasner-Megrelishvili-Uspenskij [30].

Theorem 13.6. *Let V be a Banach space. The following are equivalent:*

(1) *V is a separable Asplund space.*
(2) *$\mathcal{E}$ is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ (for infinite-dimensional V).*
(3) *$\mathcal{E}$ is metrizable.*

Proof. (1) ⇒ (2) Since $\mathcal{E}$ is a compact affine subset in the Fréchet space $\mathbb{R}^{\mathbb{N}}$ we can use Keller's Theorem [8, p. 100].

(2) ⇒ (3) Is trivial.

(3) ⇒ (1) $\mathcal{E}$ is a HNS-semigroup by Lemma 13.2.3. Now Theorem 13.5 implies that V is Asplund. It is also separable; indeed, by Lemma 3.12.4, B^* is a continuous image of $\mathcal{E}$, so that B^* is also w^*-metrizable, which in turn yields the separability of V. □

Theorem 13.7. *Let X be a compact S-system. Consider the following assertions:*

(a) *$E(X)$ is metrizable.*
(b) *(S, X) is HNS.*

Then:

(1) (a) ⇒ (b).
(2) *If X, in addition, is metrizable then* (a) ⇔ (b).

Proof. (1) By Definition 13.3 we have to show that $E(X)$ is a fragmented family of maps from X into itself. The unique compatible uniformity on the compactum X is the weakest uniformity on X generated by $C(X)$. Using Remark 12.2.1 one may reduce the proof to the verification of the following claim: $E^f := \{f \circ p : p \in E(X)\}$ is a fragmented family for every $f \in C(X)$. In order to prove this claim apply Lemma 12.3.2 to the induced mapping $E(X) \times X \to \mathbb{R}$, $(p, x) \mapsto f(px)$ (using our assumption that $E(X)$ is metrizable).

(2) If X is a metrizable HNS S-system then by Theorem 15.3 below, (S, X) is representable on a separable Asplund space V. We can assume that X is S-embedded into B^*. The enveloping semigroup $E(S, B^*)$ is embedded into $\mathcal{E}$ The latter is metrizable by virtue of Theorem 13.6. Hence $E(S, X)$ is also metrizable, being a continuous image of $E(S, B^*)$. □

Theorem 13.8. *For a compact metric S-space X the following conditions are equivalent:*

(1) *the dynamical system (S, X) is RN (that is, Asplund representable);*
(2) *(S, X) is HNS;*
(3) *the enveloping semigroup $E(S, X)$ is metrizable.*

Theorem 13.9. *Every* scattered *(e.g., countable) compact S-space X is HNS*

Proof. Recall that $C(X)$ is Asplund if (and only if) the compactum X is scattered. □

14. Appendix C: Tame systems and dynamical BFT dichotomy

Definition 14.1. A compact separately continuous S-system X is said to be *tame* if the translation $\lambda_a : X \to X,\ x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.

Lemma 14.2. *Every WAP system is HNS and every HNS is tame.*

Proof. If (S, X) is WAP then $E(X) \times X \to X$ is separately continuous. By Lemma 12.3.1 we obtain that E is a fragmented family of maps from X to X. In particular, its subfamily $\{\tilde{s} : X \to X\}_{s \in S}$ of all translations is fragmented. Hence, (S, X) is HNS.

Directly from the definitions we conclude that every HNS is tame. □

Another proof of Lemma 14.2 comes also from Banach representations theory for dynamical systems because every reflexive space is Asplund and every Asplund is Rosenthal.

By [29], a compact metrizable S-system X is tame iff S is *eventually fragmented* on X, that is, for every infinite (countable) subset $C \subset G$ there exists an infinite subset $K \subset C$ such that K is a fragmented family of maps $X \to X$.

Lemma 14.3.

1. *For every S the class of tame S-systems is closed under closed subsystems, arbitrary products and factors.*
2. *For every tame compact S-system X the corresponding enveloping semigroup $E(X)$ is tame both as an S-system and as a semigroup.*
3. *Let P be a tame right topological compact semigroup and let $\nu : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $\nu(S) \subset \Lambda(P)$. Then the S-system P is tame.*
4. *$\{$tame semigroups$\}$=$\{$enveloping semigroups of tame systems$\}$.*

Theorem 14.4. *Let V be a Banach space. The following are equivalent:*

1. *V is a Rosenthal Banach space.*
2. *(Θ^{op}, B^*) is a tame system.*
3. *$p : B^* \to B^*$ is a fragmented map for each $p \in \mathcal{E}$.*
4. *$\mathcal{E}$ is a tame semigroup.*

Proof. (2) $\Leftrightarrow$ (3): Follows from the definition of tame flows because $\mathcal{E} = E(\Theta^{op}, B^*)$.

(2) $\Rightarrow$ (4): Since $\mathcal{E} = E(\Theta^{op}, B^*)$, Lemma 14.3.2 applies.

(4) $\Rightarrow$ (2): By our assumption, $\mathcal{E}$ is a tame semigroup. Then by Lemma 14.3.3 the system $(\Theta^{op}, \mathcal{E})$ is tame. Its factor (Lemma 3.12.4) (Θ^{op}, B^*) is tame, too.

(2) $\Rightarrow$ (1): By a characterization of Rosenthal spaces [25, Prop. 4.12] (see also Fact 12.11) it suffices to show that $B^{**} \subset \mathcal{F}(B^*)$. Since (Θ^{op}, B^*) is tame, $p : B^* \to B^*$ is fragmented for every $p \in E(\Theta^{op}, B^*) = \mathcal{E}$. Pick an arbitrary $v \in B_V$ with $\|v\| = 1$. Then $v\mathcal{E}$ is exactly B^{**} by Lemma 3.12.2. So every $\phi \in B^{**}$ is a composition $v \circ p$, where p is a fragmented map. Since $v : B^* \to \mathbb{R}$ is weak* continuous we conclude that $\phi : B^* \to B^*$ is fragmented.

(1) $\Rightarrow$ (3): We have to show that $\mathcal{E} \subset \mathcal{F}(B^*, B^*)$ for every Rosenthal space V. Let $p \in \mathcal{E}$. Then $p \in \Theta(V^*)$. That is, p is a linear map $p : V^* \to V^*$ with norm ≤ 1. Then, for every vector $f \in V$, the composition $f \circ p : V^* \to \mathbb{R}$ is a linear bounded

functional on V^*. That is, $f \circ p \in V^{**}$ belongs to the second dual. Again, by the above mentioned characterization of Rosenthal spaces, the corresponding restriction $f \circ p|_{B^*} : B^* \to \mathbb{R}$ is a fragmented function for every $f \in V$. Since V separates points of B^* we can apply [25, Lemma 2.3.3]. It follows that $p : B^* \to B^*$ is fragmented for every $p \in \mathcal{E}$. □

14.1. **A dynamical BFT dichotomy.** Recall that a topological space K is a *Rosenthal compactum* [33] if it is homeomorphic to a pointwise compact subset of the space $\mathcal{B}_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compactum is the *Helly compact* of all nondecreasing selfmaps of $[0,1]$ in the pointwise topology. Recall that a topological space K is *Fréchet* (or, *Fréchet-Urysohn*) if for every $A \subset K$ and every $x \in cl(A)$ there exists a sequence of elements of A which converges to x. Every Rosenthal compact space K is Fréchet by a result of Bourgain-Fremlin-Talagrand [7, Theorem 3F], generalizing a result of Rosenthal.

Theorem 14.5. *If the enveloping semigroup $E(X)$ is a Fréchet (e.g., Rosenthal) space, as a topological space, then (S,X) is a tame system (and $E(X)$ is a tame semigroup).*

Proof. Let $p \in E(X)$. We have to show that $p : X \to X$ is fragmented. By properties of fragmented maps [25, Lemma 2.3.3] it is enough to show that $f \circ p : X \to \mathbb{R}$ is fragmented for every $f \in C(X)$. By the Fréchet property of $E(X)$ we may choose a sequence s_n in S such that the sequence $j(s_n)$ converges to p in E(X). Hence the sequence of continuous functions $f \circ s_n = f \circ j(s_n)$ converges pointwise to $f \circ p$ in $\mathbb{R}^X$. Apply Lemma 12.3.2 to the evaluation map $F \times X \to \mathbb{R}$, where $F := \{f \circ p\} \cup \{f \circ j(s_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^X$ carries the pointwise topology. We conclude that F is a fragmented family. In particular, $f \circ p$ is a fragmented map. ($E(X)$ is a tame semigroup by Lemma 14.3.2.) □

Corollary 14.6. *Let P be a compact right topological admissible semigroup. If P is Fréchet (e.g., when it is Rosenthal), as a topological space, then P is a tame semigroup.*

Proof. Applying Theorem 14.5 to the system (S,P), with $S := \Lambda(P)$ we obtain that $E(S,P) = P$ is a tame semigroup. □

The following theorem is due to Bourgain-Fremlin-Talagrand [7, Theorem 3F], generalizing a result of Rosenthal. The second assertion (BFT dichotomy) is presented as in the book of Todorčević [70] (see Proposition 1 of Section 13).

Theorem 14.7. (1) *Every Rosenthal compact space K is Frechet.*

(2) *(BFT dichotomy) Let X be a Polish space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous real valued functions on X which is bounded. Then, either the sequence $\{f_n\}_{n \in \mathbb{N}}$ contains a pointwise convergent subsequence, or it contains a subsequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.*

The following result was proved in [23, Theorem 3.2] using the Bourgain-Fremlin-Talagrand (BFT) dichotomy in the setting of continuous group actions. The same arguments work also for separately continuous semigroup actions. For the sake of completeness we include a simplified proof.

Fact 14.8 (A dynamical BFT dichotomy)**.** [23, Theorem 3.2] *Let X be a compact metric dynamical S-system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

(1) *E is a separable Rosenthal compact, hence card $E \leq 2^{\aleph_0}$; or*
(2) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence card $E = 2^{2^{\aleph_0}}$.*

The first possibility holds iff X is a tame S-system.

Proof. For every $f \in C(X)$ define $E^f := \{f \circ p : p \in E\}$. Then E^f is a pointwise compact subset of $\mathbb{R}^X$, being a continuous image of E under the map $q_f : E \to E^f$, $p \mapsto f \circ p$. Since X is metrizable by the separability of E there exists a sequence $\{s_m\}_{m=1}^{\infty}$ in S such that $\{j(s_m)\}_{m=1}^{\infty}$ is dense in $E(X)$. In particular, the sequence of real valued functions $\{f \circ s_m\}_{m=1}^{\infty}$ is pointwise dense in E^f.

Choose a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C(X)$ which separates the points of X. For every pair s, t of distinct elements of E there exist a point $x_0 \in X$ and a function f_{n_0} such that $f_{n_0}(sx_0) \neq f_{n_0}(tx_0)$. It follows that the continuous diagonal map

$$\Phi : E \to \prod_{n\in\mathbb{N}} E^{f_n}, \qquad p \mapsto (f_1 \circ p, f_2 \circ p, \dots)$$

separates the points of E and hence is a topological embedding. Now if for each n the space E^{f_n} is a Rosenthal compactum then so is $E \cong \Phi(E) \subset \prod_{n=1}^{\infty} E^{f_n}$, because the class of Rosenthal compacta is closed under countable products and closed subspaces. On the other hand if at least one $E^{f_n} = cl_p(\{f_n \circ s_m\}_{m=1}^{\infty})$ is not Rosenthal then, by BFT-dichotomy it contains a homeomorphic copy of $\beta\mathbb{N}$ and it is easy to see that so does its preimage E. In fact if $\beta\mathbb{N} \cong Z \subset E^{f_n}$ then any closed subset Y of E which projects onto Z and is minimal with respect to these properties is also homeomorphic to $\beta\mathbb{N}$.

Now we show the last assertion. If X is tame then every $p \in E(X)$ is a fragmented self-map of X. Hence every $f \circ p \in E^f$ is fragmented. By Remark 12.2.2 this is equivalent to saying that every $f \circ p$ is Baire 1. So $E^f \subset \mathcal{B}_1(X)$ is a Rosenthal compactum. Therefore, $E \cong \Phi(E) \subset \prod_{n\in\mathbb{N}} E^{f_n}$ is also Rosenthal. Conversely, if E is a Rosenthal compactum then (S, X) is tame by Theorem 14.5. □

Theorem 14.9 (BFT dichotomy for Banach spaces)**.** *Let V be a separable Banach space and let $\mathcal{E} = \mathcal{E}(V)$ be its (separable) enveloping semigroup. We have the following alternative. Either*

(1) *$\mathcal{E}$ is a Rosenthal compactum, hence card $\mathcal{E} \leq 2^{\aleph_0}$; or*
(2) *the compact space $\mathcal{E}$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence card $\mathcal{E} = 2^{2^{\aleph_0}}$.*

The first possibility holds iff V is a Rosenthal Banach space.

Proof. Recall that $\mathcal{E} = E(\Theta^{op}, B^*)$. By Theorem 14.4, V is Rosenthal iff (Θ^{op}, B^*) is tame. Since V is separable, B^* is metrizable. So we can apply Fact 14.8. □

The correspondence between $\mathcal{E}(V)$ and the dynamical system (Θ^{op}, B^*) suggest to try some new classes of Banach spaces which correspond to known classes of dynamical systems. As well as to try find new classes of dynamical systems.

14.2. **Some topological corollaries.** Answering a question of Talagrand [68, Problem14-2-41], R. Pol [58] gave an example of a separable compact Rosenthal space K which cannot be embedded in $\mathcal{B}_1(X)$ for any compact metrizable X. In [25] we say that a compact space K is *strongly Rosenthal* if it is homeomorphic to a subspace of $\mathcal{B}_1(X)$ for a compact metrizable X; and that it is *admissible* if there exists a metrizable compact space X and a bounded subset $Z \subset C(X)$ with $K \subset \mathrm{cl}_p(Z)$, such that the pointwise closure $\mathrm{cl}_p(Z)$ of Z in $\mathbb{R}^X$ consists of Baire 1 functions. Clearly every admissible compactum is strongly Rosenthal.

Theorem 14.10. [25] *Let X be a compact metrizable S-system. Then (S, X) is tame iff the compactum $K := E(X)$ is Rosenthal iff $E(X)$ is admissible.*

Thus, Pol's separable compactum mentioned above cannot be of the form $E(X)$. We do not know if every separable strongly Rosenthal space is admissible. If the answer to this question is in the negative, then this will yield another topological obstruction on being an enveloping semigroup.

Finally, as a consequence of the representation theorem 15.3.1 below we obtain the following result: *A compact space K is an admissible Rosenthal compactum iff it is homeomorphic to a weak* closed bounded subset in the second dual of a separable Rosenthal Banach space V.*

Essentially the same result (using different terminology and setting) was obtained earlier by Marciszewski (see Marciszewski [43, Section 6.2] and also Marciszewski-Pol [44, Theorem 8.2]).

14.3. **Some classes of functions.**

Definition 14.11. Let $f \in C(X)$ on a (not necessarily, compact) S-system X.

(1) We say that f *comes* from the S-compactification $q : X \to Y$ (where the action of S on Y is at least separately continuous) if there exists a continuous function $f' : Y \to \mathbb{R}$ such that $f = f' \circ q$.
(2) We say that $f \in C(X)$ is *RMC* (right multiplicatively continuous) if f comes from some S-compactification $q : X \to Y$. For every compact S-system X we have $\mathrm{RMC}(X) = C(X)$.
(3) If we consider only jointly continuous S-actions on Y then the functions $f : X \to \mathbb{R}$ which come from such G-compactifications $q : X \to Y$ are *right uniformly continuous*. Notation: $f \in \mathrm{RUC}(X)$.
(4) f is said to be: a) *WAP*; b) *Asplund*; c) *tame* if f comes from an S-compactification $q : X \to Y$ such that (S, Y) is: WAP, HNS or tame respectively. For the corresponding classes of functions we use the notation: $\mathrm{WAP}(X), \mathrm{Asp}(X), \mathrm{Tame}(X)$, respectively. Each of these is a norm closed S-invariant subalgebra of the S-algebra $\mathrm{RMC}(X) \subset C(X)$ and
$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X).$$
For more details see [27, 28].
(5) Note that as a particular case of (3) we have defined the algebras $\mathrm{WAP}(S)$, $\mathrm{Asp}(S)$, $\mathrm{Tame}(S)$ corresponding to the left action of S on $X := S$.

Definition 14.12. [23, 27] We say that a compact dynamical S-system X is *cyclic* if there exists $f \in C(X)$ such that (S, X) is topologically S-isomorphic to the Gelfand space X_f of the S-invariant unital subalgebra $\mathcal{A}_f \subset C(X)$ (generated by the orbit fS).

Remark 14.13. Let X be a (not necessarily compact) S-system and $f \in \mathrm{RMC}(X)$. Then, as was shown in [27], there exist: a cyclic S-system X_f, a continuous S-compactification $\pi_f : X \to X_f$, and a continuous function $\tilde{f} : X_f \to \mathbb{R}$ such that $f = \tilde{f} \circ \pi_f$; that is, f comes from the S-compactification $\pi_f : X \to X_f$. The collection of functions $\tilde{f}S$ separates points of X_f. Finally, $f \in \mathrm{RUC}(X)$ iff the action of S on X_f is jointly continuous.

The cyclic S-systems X_f provide "building blocks" for compact S-systems. That is, every compact S-space can be embedded into the S-product of S-spaces X_f, where $f \in C(X)$.

Proposition 14.14. *Let X be a compact S-space and $f \in C(X)$.*
1. *$f \in WAP(X)$ iff fS has DLP on X.*
2. *$f \in Asp(X)$ iff fS is a fragmented family.*
3. *$f \in \mathrm{Tame}(X)$ iff fS is eventually fragmented iff fS does not contain an l_1-sequence.*

15. Appendix D: Representations of HNS and tame systems

Next we deal with the representability of families of real-valued functions on compact systems. This topic is closely related to the "smallness" of the family F in terms of its pointwise closure in the spirit of Theorem 12.10.

Definition 15.1. Let $\mathcal{K} \subset \mathbf{Ban}$ be a subclass of Banach spaces.

(1) Let X be a compact S-system and (h, α) a representation of (S, X) on a Banach space V. Let $F \subset C(X)$ be a bounded S-invariant family of continuous functions on X and $\nu : F \to V$ a bounded mapping. We say that (ν, h, α) is an *F-representation* of the triple (F, S, X) if ν is an S-mapping (i.e., $\nu(fs) = \nu(f)s$ for every $(f, s) \in F \times S$) and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\, f \in F, \ \ \forall\, x \in X.$$

In other words, the following diagram commutes

$$\begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ \nu \downarrow \quad \downarrow \alpha & & \downarrow id_{\mathbb{R}} \\ V \times V^* & \longrightarrow & \mathbb{R} \end{array} \tag{15.1}$$

(2) We say that a family $F \subset C(X)$ is *$\mathcal{K}$-representable* if there exists a Banach space $V \in \mathcal{K}$ and a representation (ν, h, α) of the triple (F, S, X). A function $f \in C(X)$ is said to be *$\mathcal{K}$-representable* if the orbit fS is $\mathcal{K}$-representable.

Note that we do not assume in (1) or (2) that α is injective. However, when the family F separates points on X it follows that the map α is necessarily an injection.

(3) In particular, we obtain the definitions of reflexively, Asplund and Rosenthal representable families of functions on dynamical systems.

Clearly, every bounded S-invariant $F \subset C(X)$ on an S-system X is Banach representable via the canonical representation on $V = C(X)$.

15.1. **Representation theorems.** Let S be a semitopological semigroup and X a compact S-system with a separately continuous action.

Theorem 15.2. *(Small families of functions)*
Let $F \subset C(X)$ be a norm bounded S-invariant subset of $C(X)$.

1. *(F, S, X) admits a Rosenthal representation iff F is an eventually fragmented family iff* $\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \subset \mathcal{F}(\mathrm{X})$.
2. *(F, S, X) admits an Asplund representation iff F is a fragmented family iff the envelope* $\mathrm{cl}_{\mathrm{p}}(\mathrm{F})$ *of F is a fragmented family.*
3. *(F, S, X) admits a reflexive representation iff* $\mathrm{cl}_{\mathrm{p}}(\mathrm{F}) \subset \mathrm{C}(\mathrm{X})$ *iff F has DLP on X.*

Proof. (3) Already was proved in Section 6.3.

(1) and (2): The "only if part" is a consequence of the characterizations of Asplund and Rosenthal spaces in terms of fragmented and eventually fragmented families, Theorems 12.5.4 and 12.11.4. □

Theorem 15.3.

1. *(S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.*
2. *(S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.*

If X is metrizable then in (1) and (2) "approximable" can be replaced by "representable". Moreover, the corresponding Banach space can be assumed to be separable.

Proof. "only if" part: For (1) use the fact that (Θ^{op}, B^*) is a tame system (Theorem 14.4) for every Rosenthal V and for (2), the fact that (Θ^{op}, B^*) is HNS (Theorem 13.5) for Asplund V.

"if" part: (1) For every $f \in C(X) = \mathrm{Tame}(X)$ the orbit fS is a Rosenthal family for X (Proposition 14.14). Applying Theorem 15.4 below we conclude that every $f \in C(X) = \mathrm{Tame}(X)$ on a compact G-space X comes from a Rosenthal representation. Since continuous functions separate points of X, this implies that Rosenthal representations of (S, X) separate points of X. So, for (1) it is enough to prove the following result which gives a proof of Theorem 15.2.1. The proof of (2) is similar.

Theorem 15.4. *Let $F \subset C(X)$ be a Rosenthal family (Asplund family) for X such that F is S-invariant; that is, $fS \subset F \quad \forall f \in F$. Then there exist: a Rosenthal (respectively, Asplund) Banach space V, an injective mapping $\nu : F \to B_V$ and a representation*

$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$

of (S, X) on V such that h is weakly continuous, α is a weak* continuous map and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\ f \in F \ \ \forall\ x \in X.$$

Thus the following diagram commutes

(15.2)
$$\begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ \nu \downarrow \quad \downarrow \alpha & & \downarrow id_{\mathbb{R}} \\ V \times V^{*} & \longrightarrow & \mathbb{R} \end{array}$$

If X is metrizable then in addition we can suppose that V is separable.
If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous.

Proof. **Step 1:** The construction of V.

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F; that is, $W := \mathrm{co}\,(F \cup -F)$. Consider the sequence of sets

(15.3)
$$M_n := 2^n W + 2^{-n} B.$$

Then W is convex and symmetric. We apply the construction of Davis-Figiel-Johnson-Pelczyński [10] as follows. Let $\| \ \|_n$ be the Minkowski functional of the set M_n, that is,

$$\|v\|_n = \inf \{\lambda > 0 \mid v \in \lambda M_n\}.$$

Then $\| \ \|_n$ is a norm on $\mathcal{A}$ equivalent to the given norm of $\mathcal{A}$. For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \text{ and let } V := \{v \in \mathcal{A} \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \to \mathcal{A}$ is a continuous linear injection and

(15.4)
$$W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B)$$

Indeed, if $v \in W$ then $2^n v \in M_n$, hence $\|v\|_n \leq 2^{-n}$ and $N(v)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$. This proves $W \subset j(B_V)$. In order to prove the second inclusion recall that the norms $\|\cdot\|_n$ on $\mathcal{A}$ are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, for every $n \in \mathbb{N}$, $v \in \lambda_n M_n$ for some $0 < \lambda_n < 1$. By the construction M_n is a convex subset containing the origin. This implies that $\lambda_n M_n \subset M_n$. Hence $j(v) = v \in M_n$ for every $n \in \mathbb{N}$.

Step 2: The construction of the representation (h, α) of (S, X) on V.

The given action $S \times X \to X$ induces a natural linear norm preserving continuous right action $C(X) \times S \to C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S-invariant subsets in $\mathcal{A}$. This implies that V is an S-invariant subset of $\mathcal{A}$ and the restricted natural linear action $V \times S \to V$, $(v, g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \to \Theta(V)$, $h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \to V^*$ be the adjoint map of $j : V \to \mathcal{A}$. Define $\alpha : X \to V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V.

By the construction, $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall\, f \in F \ \ \forall\, x \in X. \tag{15.5}$$

Step 3: V is a Rosenthal space.

Proof. (In [25, Theorem 6.3] we have a different proof.) If F is an eventually fragmented family for X then $W := \mathrm{co}\,(F \cup -F)$ is an eventually fragmented family for X and even for B^* (Proposition 12.23). By the construction of DFJP ([16, Lemma 1.2.2]) we get $j : V \to \mathcal{A}$ such that $cl_{norm} j^*(\mathcal{A}^*) = V^*$. Denote by $M := \cap\{2^n W + \frac{1}{2^n} B\}$. Then $j(B_V) \subset M$. Since $\mathbb{R}$ is a metrizable uniform space by diagonal arguments we obtain that M is eventually fragmented on X. Therefore M is eventually fragmented also on B^* (again Proposition 12.23). In order to show that V is Rosenthal it is equivalent to show that B_V is eventually fragmented on B_{V^*} (Fact 12.11). It is equivalent to show that for every infinite subset C_0 of B_V there exists a (countable) infinite subset $C \subset C_0$ which is fragmented on B_{V^*}, or equivalently, that (B_{V^*}, ρ_C) is separable. Equivalently, that (V^*, ρ_C) is separable. Since $j(C_0) \subset M$ is an infinite subset (recall that j is injective) we obtain that there exists an infinite subset $j(C)$ which is a fragmented family on $B_{\mathcal{A}^*}$. Equivalently, $(B_{\mathcal{A}^*}, \rho_{j(C)})$ is separable. Equivalently, $(\mathcal{A}^*, \rho_{j(C)})$ is separable (Theorem 12.13). By the definition of the adjoint operator $(< j(c), v^* >=< c, j^*(v^*) >)$ we obtain that $(j^*(\mathcal{A}^*), \rho_C)$ is separable. Then its closure is also separable. That is, $cl_{\rho_C}(j^*(\mathcal{A}^*))$ is also ρ_C-separable. Since C is a bounded subset, clearly, $cl_{\rho_C}(j^*(\mathcal{A}^*)) \supset cl_{norm}(j^*(\mathcal{A}^*)) = V^*$. Therefore, $(V,^*, \rho_C)$ is separable, as desired.

Step 4: Weak continuity of $h : S \to \Theta(V)$.

By our construction $j^* : C(X)^* \to V^*$, being the adjoint of the bounded linear operator $j : V \to C(X)$, is a norm and weak* continuous linear operator. By [16, Lemma 1.2.2] we obtain that $j^*(C(X)^*)$ is norm dense in V^*. Since V is Rosenthal, Haydon's theorem (Fact 12.11.4) gives $Q := cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \to X$ is separately continuous, every orbit map $\tilde{x} : S \to X$ is continuous, and each orbit map $\widetilde{j^*(x)} : S \to j^*(X)$ is weak* continuous. Then also $\widetilde{j^*(z)} : S \to V^*$ is weak* continuous for each $z \in cl_{norm}(co(j^*(X))) = Q$. Since $sp(Q)$ is norm dense in V^* (and $||h(s)|| \leq 1$ for each $s \in S$) it easily follows that $\widetilde{j^*(z)} : S \to V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N, we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on $\mathcal{A}$ is equivalent to the given norm on $\mathcal{A}$). $\square$

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable. □

This proves Theorem 15.4 and hence also Theorem 15.3.1.

For the "Asplund case" (when F is fragmented on X) use

Step 3': V is an Asplund space.

The main idea is that the corresponding results of [49, Section 7] and [23, Section 9] can be adopted here, thus obtaining a modification of Theorem 15.4 which replaces a Rosenthal space by an Asplund space, and a "Rosenthal family F" for X by an "Asplund set". The latter means that for every countable subset $A \subset F$ the pseudometric ρ_A on X defined by

$$\rho_A(x, y) := \sup_{f \in A} |f(x) - f(y)|, \ \ x, y \in X$$

is separable. By [16, Lemma 1.5.3] this is equivalent to saying that $(C(X)^*, \rho_A)$ is separable. Now $co(F \cup -F)$ is an Asplund set for B^* by [16, Lemma 1.4.3]. The rest is similar to the proof of [49, Theorem 7.7]. Checking the weak continuity of h one can apply a similar idea (using again Haydon's theorem as in (1)).

Finally note that if X is metrizable then in (1) and (2) "approximable" can be replaced by "representable" using an l_2-sum of a sequence of separable Banach spaces. □

Note that, in Definition 15.1, when the family F separates points on X it follows that the map α is necessarily an injection. In view of this remark, Theorem 15.2 implies Theorem 15.3 and also the following useful result.

Theorem 15.5. *A compact S-system X is RN (WRN, Eberlein) iff there exists a norm bounded S-invariant fragmented (resp.: eventually fragmented, DLP) family $F \subset C(X)$ which separates points of X.*

Similar to Theorem 7.4 one may show

Theorem 15.6. *Let G be a topological group and $f \in \mathrm{RUC}(G)$. The following are equivalent:*

(1) *$f \in \mathrm{Asp}(G)$ iff f is a matrix coefficient of a strongly continuous Asplund co-representation of G.*
(2) *$f \in \mathrm{Tame}(G)$ iff f is a matrix coefficient of a strongly continuous Rosenthal co-representation of G.*

15.2. **The purely topological case.** Note that the definitions and results of Section 15 (for instance, Theorem 15.2) make sense in the purely topological setting, for trivial $S = \{e\}$ actions, yielding characterizations of "small families" of functions, and of RN, WRN and Eberlein compact spaces.

The "only if" parts of these results, in the cases of Eberlein and RN compact spaces (with trivial actions), are consequences of known characterizations of reflexive and Asplund spaces. The Eberlein case yields a well-known result: a compact space X is Eberlein iff there exists a pointwise compact subset $Y \subset C(X)$ which separates the points of X. The RN case is very close to some results of Namioka [55] (up to some reformulations). The case of WRN spaces seems to be new.

Recall that (answering to a question posed by Lindenstrauss) by a classical result of Benjamini-Rudin-Wage, continuous surjective maps preserve the class of Eberlein compact spaces. The same is true for uniformly Eberlein (that is, Hilbert representable) compacta. Recently, Aviles and Koszmider [1] proved that this is not the case for the class RN of Asplund representable compacta, answering a long standing open problem posed by Namioka [55]. The following question seems to be interesting.

Question 15.7. *Is it true that the class of WRN (e.g., Rosenthal representable) compact spaces is closed under continuous onto maps ?*

Remark 15.8.

(1) An example of a compact space which is not WRN is $\beta\mathbb{N}$. This was done by Todorčević (private communication).
(2) Two arrows space is WRN but not RN. More precisely, as we recently established, every compact linearly ordered space is WRN. On the other hand, two arrows space is not RN as it was established by Namioka [55, Example 5.9].
(3) One may show that a compact space K is WRN iff the Banach space $C(K)$ is *Rosenthal generated* (meaning that there exists a Rosenthal space V and a linear dense (injective) operator $V \to C(K)$). It is a WRN analog of Stegall's result for RN compacts.

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