ABOUT CHAOS AND SENSITIVITY IN TOPOLOGICAL DYNAMICS

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Abstract. In this work we unify and generalize some results about chaos and sensitivity.

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1. Symbolic Dynamics

Definition 1.1. The sequence space on two symbols is the set

$$\Sigma = \{(s_0, s_1, s_2, ...) : s_j = 0 \quad or \quad 1\}$$

Proposition 1.2. *The distance d on Σ is defined by $d(s,t) = \Sigma \frac{|s_i - t_i|}{2^i}$.*

Proof. Let $s = (s_0, s_1, s_2, ...)$, $t = (t_0, t_1, t_2, ...)$ and $u = (u_0, u_1, u_2, ...)$. Clearly, $d(s,t) \geq 0$ and $d(s,t) = 0$ if and only if $s = t$. Since $|s_i - t_i| = |t_i - s_i|$, it follows that $d(s,t) = d(t,s)$. Finally, for any three real numbers s_i, t_i, u_i we have the usual triangle inequality

$$|s_i - t_i| + |t_i - u_i| \geqq |s_i - u_i|$$

from which we deduce that

$$d(s,t) + d(t,u) \geq d(s,u).$$

This completes the proof. □

Theorem 1.3. *Let $s, t \in \Sigma$ and suppose $s_i = t_i$ for i=0,1,2,..n. Then $d(s,t) \leq \frac{1}{2^n}$. Conversely, if $d(s,t) < \frac{1}{2^n}$, then $s_i = t_i$ for $i \leq n$.*

Proof. If $s_i = t_i$ for $i \leq n$, then

$$d(s,t) = \Sigma_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \Sigma_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$$

On the other hand, if $s_i \neq t_i$ for some $i \leq n$, then we must have $d(s,t) \geq \frac{1}{2^i} \geq \frac{1}{2^n}$. Consequently if $d(s,t) < \frac{1}{2^n}$, then $s_i = t_i$ for $i = 0, 1, 2, ..n$. □

Definition 1.4. The shift map $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(s_0, s_1, s_2, ...) = (s_1, s_2, s_3, ...)$.

Our first observation about this map is that the subset of Σ that consist of all periodic points in Σ is a dense subset. To see why this is true, we must show that, given any point $s = (s_0, s_1, s_2, ...)$ in Σ, we can find a periodic point arbitrarily close to s. So let $\epsilon > 0$.

Let's choose an integer number n so that $\frac{1}{2^n} < \epsilon$. We may now write down an explicit periodic point within $\frac{1}{2^n}$ units of s. Let $t_n = (s_0, s_1, ..., s_n, \overline{s_0, s_1, ..., s_n})$. The first $n+1$ entries of s and t_n are the same. By Theorem 1.3 this means that

$$d(s, t_n) \leq \frac{1}{2^n} < \epsilon.$$

Clearly t_n is a periodic point of period $n+1$ for $\sigma : \Sigma \to \Sigma$. Since ϵ and s were arbitrary, we have succeeded in finding a periodic point arbitrarily close to any point of Σ. Note that the sequence (of sequences) $\{t_n\}$ converges to s in Σ as $n \to \infty$.

A second and even more interesting property of σ is that there is a point whose orbit is dense in Σ. That is we can find an orbit which comes arbitrarily close to any point of Σ. Clearly, this kind of orbit is far from periodic or eventually periodic. As above, we can write down such an orbit explicitly for σ. Consider the point

$$\hat{s} = (0100011011000001......).$$

In other words, $\hat{s}$ is the sequence which consists of all possible blocks of 0's and 1's of length, followed by all such blocks of length 2, then length 3, and so forth. The point $\hat{s}$ has an orbit that forms a dense subset of Σ. To see this, we again choose an arbitrary $s = (s_0, s_1, s_2, ...) \in \Sigma$ and an $\epsilon > 0$. Again choose n so that $\frac{1}{2^n} < \epsilon$. Now we show that the orbit of $\hat{s}$ comes within $\frac{1}{2^n}$ units of s. Far to the right in the expression for $\hat{s}$, there is a block of length $n+1$ that consists of the digits $s_0 s_1 ... s_n$. Suppose the entry s_0 is at the k-th place in the sequence. Now apply the shift map k times to $\hat{s}$. Then the first n+1 entries of $\sigma^k(\hat{s})$ are precisely $s_0 s_1 ... s_n$. So by Theorem 1.3 we get

$$d(\sigma^k(\hat{s}), s) \leq \frac{1}{2^n} < \epsilon.$$

There is a dynamical notion that is intimately related to the property of having a dense orbit. This is the concept of *sensitivity.*

Definition 1.5. A metric dynamical system $((X,d),F)$ depends *sensitively on initial conditions*, if there is a $\beta > 0$ such that for any $x \in X$ and any $\epsilon > 0$ there is $k \in N$ and $y \in X$ with $d(x,y) < \epsilon$ such that

$$d(F^k(x), F^k(y) \geq \beta$$

To see that shift map depends sensitivity on initial conditions, we choose $\beta = 1$. For any $s \in \Sigma$ and $\epsilon > 0$ one can again pick $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \epsilon$. Suppose $t \in \Sigma$ satisfies $d(s,t) < \frac{1}{2^n}$ but $t \neq s$. Then we know that $t_i = s_i$ for $i = 0, 1, 2, 3, ..n$. However, since $t \neq s$ there is $k > n$ such that $s_k \neq t_k$. So $|s_k - t_k| = 1$. Now consider the sequence $\sigma^k(s)$ and $\sigma^k(t)$. The initial entries of each of these sequences are different, so we have

$$d(\sigma^k(s), \sigma^k(t)) \geq \frac{|s_k - t_k|}{2^0} + \Sigma \frac{0}{2^i} = 1.$$

This proves sensitivity for the shift.

2. Chaos by R.L.Devaney

Definition 2.1. A dynamical system $F : X \to X$ is chaotic if

P_1 Periodic points for F are dense.
P_2 F is transitive.
P_3 F depends sensitivity on initial conditions.

Theorem 2.2. *The shift map $\sigma : \Sigma \to \Sigma$ is a chaotic dynamical system.*

Theorem 2.3. *The doubling map f is chaotic on the unit circle.*

Proof. Let $\mathbb{S}^1$ be the unit circle $\{(x,y) : x^2 + y^2 = 1\}$. In the complex analysis

$$\mathbb{S}^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

1).Let $e^{i\theta} \in \mathbb{S}^1$ and U is an open neighborhood of $e^{i\theta}$. Let A be an open arc in U containing $e^{i\theta}$, too. Note that $f^{(n)}(A)$ is an arc 2^n-th longer as A. There exists $n \in \mathbb{N}$ such that $f^{(n)}(a)$ is a cover of $\mathbb{S}^1$. Denote this iteration by N. There are two points x, y with

$$d(f^{(N)}(x), d(f^{(N)}(y)) = 1.$$

2). Let U, V are open sets in S^1. If we precede as above, then for n large enough we got that $f^{(n)}(U)$ covers S^1 and therefore intersects V.
3). For points of the form $e^{i\theta}$ with period n, the following equation holds $e^{i2^n\theta}$. This means that there periodic points are unit roots with order $2^n - 1$. The set of all there points is dense in S^1. $\square$

Theorem 2.4. *Let's suppose that $g \circ T = T \circ f$, and T is continuous and subjective map. If f is transitive or periodic on X, then g is also transitive or periodic on Y.*

Theorem 2.5. *The function $g(x) = x^2 - 2$ chaotic order the interval* $[-2, 2]$

Proof. Let $T : S^1 \longrightarrow [-2, 2]$ be defined as $T(e^{i\theta}) = 2cos\theta$. It's clear to see that T is continuous and subjective and

$$g \circ T(e^{i\theta}) = T \circ f(e^{i\theta}) = 2cos2\theta.$$

f is chaotic on S^1, then by theorem 2.5, the function g is chaotic too. □

The following theorem which relates to chaos definition was published in 1992.

Theorem 2.6. *Let (X, d) be a metric space that is include an infinite set of points. If the mapping $f : X \longrightarrow Y$ is continuous and transitive and if a set of periodic points is dense in X, then f is sensitive dependent on initial conditions.*

Proof. Let choose two periodic points q_1, q_2 such that $O(q_1) \cap O(q_2) = \emptyset$. Let

$$\delta_0 = d(O(q_1), O(q_2)).$$

We'll show that the sensitive dependent on the initial conditions holds when $\delta = \frac{\delta_0}{8}$. Notice that $\delta_0 > 0$, and for every $x \in X$, other $d(x, O(q_1)) > \frac{\delta_0}{2}$ or $d(x, O(q_2)) > \frac{\delta_0}{2}$.

Let $x \in X$ and U be an open set that includes x. Let $B_\delta(x)$ be an open sphere with radius δ and center x. Let p be a periodic point in $W = U \cap B_\delta(x)$ with period n. From this we conclude that one of the points q_1, q_2 (denoted by q) has an orbit, for which $d(x, O(q)) > 4\delta$. Let's define

$$V = \cap_{i=0}^{n} f^{(-i)}(B_\delta(f^{(i)}(q))).$$

The set V is non empty, because $q \in V$ and V is open. From the transitivity of f exists a point $y \in W$ and integer number k such that $f^{(k)}(y) \in V$.

Let j be an integer part of $\frac{k}{n} + 1$. Consequently, $\frac{k}{n} + 1 = j + r$, when r is the rest, $0 \leq r < 1$. Clearly, $nj - k = n - rn$. It follows that $0 \leq nj - k \leq n$.

By construction,

$$f^{(nj)}(y) = f^{(nj-k)}(f^{(k)}) \in f^{(nj-k)}(V) \subset B_\delta(f^{(nj-k)}(q)).$$

Let

$$a = f^{(nj)}(y),$$
$$b = f^{(nj-k)}(q).$$

Note that $d(a, b) < \delta$. Let us use the triangle inequality for points p, a, b and x, p, b:

$$d(p, b) \leq d(p, a) + d(a, b),$$

$$d(x, b) \leq d(x, p) + d(p, b).$$

When

$$d(x, b) \leq d(x, p) + d(p, a) + d(a, b),$$

or

$$d(p, a) \geq d(x, b) - d(x, p) - d(a, b).$$

By construction

$$d(x, b) = d(x, f^{(nj-k)}(q)) \geq d(x, O(q)) \geq 4\delta.$$

Since $p \in B_\delta(x)$, then $d(x, p) < \delta$. From this it follows that

$$d(a, b) > 4\delta - \delta - \delta,$$

or

$$d(f^{(nj)}(p), f^{(nj)}(y)) > 2\delta.$$

Applying the triangle inequality to the following points $f^{(nj)}(x), f^{(nj)}(p), f^{(nj)}(y)$, we get that:

$$d(f^{(nj)}(x), f^{(nj)}(p)) > \delta \quad or \quad d(f^{(nj)}(x), f^{(nj)}(y)) > \delta.$$

$\square$

3. Topological transitivity, minimality

Given a dynamical system (X, S) we let sx denote the image of $x \in X$ under the homeomorphism corresponding to the element $s \in S$. Let $O_S x$ be the orbit of x; i.e. the set $\{sx : s \in S\}$.

$\overline{O_S x}$ will denote the orbit closure of x. If $(X, S)$is a system and Y a closed S-invariant subset, then we say that (Y, S), the restricted action, a subsystem of (X, S).

Definition 3.1. The dynamical system (X, S) is called **topologically transitive** or just **transitive**, if for every pair of non-empty open sets U, V in X there exists $s \in S$ with $sU \bigcap V \neq \varnothing$.

Definition 3.2. The dynamical system (X, S) us called **point transitive**, if there exists point $x_0 \in X$ with $\overline{O_S x} = X$. Such x_0 is called a **transitive point**.

Example 3.3. If $X = S^1$ and $S = \{f^{(n)} : n \in \mathbb{N}, f(z) = z^2\}$, then this dynamical system is transitive.

Definition 3.4. We dynamical system (X, S) is called **minimal**, if $\overline{O_S x} = X$ for every $x \in X$.

Definition 3.5. A point x in dynamical system (X, S) is called **minimal**(or **almost periodic**), if the subsystem $\overline{O_S x}$ is minimal.

Definition 3.6. If set of minimal points is dense in X, we say that (X, S) satisfies the **Bronstein condition**. If, in addition, the system (X, S) is transitive, we say that it is an M**-system**.

Definition 3.7. A point $x \in X$ is a **periodic point**, if $O_S x$ is finite set. If (X,S) is a transitive system and set of periodic point is dense in X, then we say that it is an P**-system**.

For a system (X, S) and subsets $A, X \subset X$, we use the following natation $N(A, B) = \{s \in S : sA \bigcap B \neq \varnothing\}$. In particular, for $A = \{x\}$ we write $N(x, B) = \{s \in S : sx \in B\}$.

Definition 3.8. A subset $P \subset S$ is (left) **syndetic**, if there exists a finite set $F \subset S$ such that $FP = S$.

Theorem 3.9. *The following are equivalent:*

1 (X, S) *is minimal.*

2 (X, S) *is a transitive and for every* $x \in X$ *and neighborhood* U *of* x*, the set* $N(x, U) = \{s \in S : sx \in B\}$ *is syndetic in* S*.*

Proof. $(a) \Longrightarrow (b)$: If (X, S) is a minimal system then for every non-empty set U in X there exists a finite subset $F = \{s_1, s_2, ..., s_k\}$ in S with $\cup s_i U = X$. Then $\cup_j N(x, s_i U) := S$. But $N(x, s_j U) = s_j N(x, U)$, then $N(x, U)$ us syndetic.
$(a) \Longleftarrow (b)$: For every $x \in X$ and every neighborhood U a set $\cup s_j U$ is covered X. Then for all open set V exists $s_i \in S$ such that $s_i U \cap V \neq \varnothing$. Because X is a metric space we get that (X, S) is a minimal system. $\square$

4. Equicontinuity and almost equicontinuity

Definition 4.1. The system (X, S) is **equicontinuous** if the semigroup S acts equicontinuously on X; for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $d(sx_1, sx_2) < \epsilon$, for every $s \in S$.

Example 4.2. Every isometric system is equicontinuous. It's clear to see that we must take $\epsilon = \delta$.

Definition 4.3. Let (X, S) be a dynamical system. A point $x_0 \in X$ is called an **equicontinuity point** if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that for every $y \in U$ and every $s \in S$, $d(sx_0, sy) \leq \epsilon$.

Proposition 4.4. *A system (X,S) is equicontinuous iff every $x \in X$ is an equicontinuity point.*

Proof. $\Rightarrow$: It's clearly.
$\Leftarrow$: Given $\epsilon > 0$, let $\Im := \{U_x : x \in X\}$ be a collection of neighborhoods as in the definition of equicontinuity points. Any *Lebesgue* number δ for the open cover $\Im$ will serve for the equicontinuity condition. □

Definition 4.5. The dynamical system (X, S) is called **almost equicontinuous** (or is an AE-**system**) if the subset $EQ(X)$ of equicontinuity points is a dense subset of X.

Proposition 4.6. *A minimal almost equicontinuous system is equicontinuous.*

Proof. Let x_0 be a transitive point and $x \in EQ(X)$. We shall show that also $x_0 \in EQ(X)$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x', x'' \in B_\delta(x)$, $d(x', x'') < \epsilon$ for all $s \in S$. Since x_0 is a transitive point, there exists $s' \in S$ and $\eta > 0$ such that $s'B_\eta(x_0) \subset B_\delta(x)$. Thus for every $z \in B_\eta(x_0)$ and every $s \in S$ we have $d(ss'z, ss'x_0) < \epsilon$ foe all $s \in S$; i.e. $x_0 \in EQ(X)$ and we conclude that the set of transitive points is contained in $EQ(X)$. Then a minimal almost equicontinuous system is equicontinuous. □

5. Sensitive dynamical system

Definition 5.1. We shell say that a system (X, S) is **sensitive** if it satisfies the following condition(sensitive dependence on initial condition): there exists an $\epsilon > 0$ such that for all $x \in X$ and all $\delta > 0$ there are some $y \in B_\delta(x)$ and $s \in S$ with $d(sx, sy) > \epsilon$. We say that (X, S) is **non-sensitive** otherwise.

Proposition 5.2. *A transitive dynamical system is almost equicontinuous iff it is non-sensitive.*

Proof. Clearly an almost equicontinuous system is non-sensitive. Conversely, being non-sensitive means that for every $\epsilon > 0$ there exists $x_\epsilon \in X$ and $\delta_\epsilon > 0$ such that for all $y \in B_{\delta_\epsilon}(x_\epsilon)$ and every $s \in S$, $d(sx_\epsilon, sy) < \epsilon$. For $m \in \mathbb{N}$ set $V_{1/m} = B_{\delta_{1/m}}(x_{1/m})$, $U_m = SV_{1/m}$ and let $R = \cap_{m \in \mathbb{N}} U_m$. Suppose $x \in R$ and $\epsilon > 0$. Choose m so that $2/m < \epsilon$, then $x \in U_m$ implies the exists $s_0 \in S$ and $x_0 \in V_{1/m}$ such that $s_0 x_0 = x$. Put $V = s_0 V_{1/m}$. We now see that for all $y \in V$ and every $s \in S$

$$d(sx, sy) = d(ss_0x_0, ss_0y_0) < 2/m < \epsilon, \quad y_0 \in V_{1/m}.$$

Thus the dense G_δ set R consists of equicontinuity points. □

In this proposition we have seen that minimality and almost equicontinuity imply equicontinuity. We easily get a stronger result.

Theorem 5.3. *An almost equicontinuous M-system (X, S) is minimal and equicontinuous. Thus an M-system (hence also P-system) which is not minimal equicontinuous is sensitive.*

Proof. Every transitive point is a equicontinuity point. Let $x_0 \in X$ be an equicontinuity and transitive point . Given $\epsilon > 0$ there exists a $0 < \delta < \epsilon$ such that $x \in B_\delta(x_0)$ implies $d(sx_0, sx) < \epsilon$ for every $s \in S$. Let $x' \in B_\delta(x_0)$ be a minimal point. It that follows that $T = \{s \in S : d(sx_0, sx) \leq \epsilon\}$ is a syndetic subset of S. Collecting these estimations we get, for every $s \in S$,

$$d(sx_0, x_0) \leq d(sx_0, sx') + d(sx', x_0) \leq 2\epsilon.$$

Thus for each $\epsilon > 0$ the set $N(x_0, B_\epsilon(x_0)$ is a syndetic, whence x_0 is a minimal. It follows that X is a minimal, hance also equicontinuous by Proposition 4.6. Now, by proposition 5.2 we have that dynamical system which not minimal or equicontinuous is sensitive. $\square$

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