THE PLATONIC GROUPS

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Abstract

There are five Platonic solids-regular polyhedra: the tetrahedron, octahedron, cube, dodecahedron, and icosahedron. Each defines a finite group of rotations that leave the solid set wise fixed. In this work we shall show those groups.

1 Preliminaries

Definition 1.1. Let G be a group. A G-space is a set S and a map $\tau : G \times S \longrightarrow S$, so that

$$\tau(e, s) = s$$

and

$$\tau(g, \tau(h, s)) = \tau(gh, s)$$

for all $g, h \in G$ and $s \in S$. We normally write $\tau_g(s) \equiv \tau(g, s)$ so $\tau_g : S \longrightarrow S$ and $\tau_g \tau_h = \tau_{gh}$.

Notice that $\tau_g \tau_{g^{-1}} = \tau_{g^{-1}} \tau_g = id$ so each τ is a bijection of S. Thus $g \to \tau_g$ is a group homomorphism of G into the bijection of S and every such homomorphism defines a G-space. τ is also called an action of G on S.

Definition 1.2. Let S be a G-space with action τ. Let $s \in S$. Then $\{\tau_x(s) | x \in G\}$ is called the orbit. If $O_s^\tau = S$, we say that the action is transitive.

Proposition 1.3. *Any G-space is a disjoint union of its orbits.*

Example 1.4. *The orbits under the action $\tau_x(y) = xyx^{-1}$, that is, $\{xyx^{-1} | x \in G, y \text{ fixed}\}$, are called conjugacy classes.*

Definition 1.5. Let S be a G-space and let $s \in S$. The isotropy subgroup I_s of s is $\{x \in G \mid \tau_x(s) = s\}$.

Theorem 1.6. *(Fundamental theorem of G-spaces). Let (S, τ) be a transitive G-space. Let I_s be the isotropy group of some $s \in S$. Then (S, τ) is isomorphic to G/I_s. In particular,*

$$\#(S) = \#(G)/\#(I_s).$$

Proof. *Fix s. Given $s' \in S$, define $Q_{s'} \subset G$ as*

$$Q_{s'} = \{x \in G \mid \tau_x(s) = s'\}.$$

Fix $x \in Q_{s'}$. Then $y \in Q_{s'}$ if and only if $x^{-1}y \in I_s$, that is, if and only if $y \in xI_s$. Thus, $Q_{s'} = xI_s$ and the $Q_{s'}$ are precisely elements of G/I_s. Let

$$\phi(s') = Q_{s'} \in G/I_s.$$

Then

$$\phi(\tau_x(s')) = xQ_{s'} = x\phi(s'),$$

so ϕ is the required isomorphism. □

Definition 1.7. An orthogonal matrix is $n \times n$ matrix A so that $\langle Av, Aw\rangle = \langle v, w\rangle$ for all $v, w \in \in R^n$ where $\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i$. It is easy to see that matrix products of such orthogonal matrices are orthogonal and $AA^t = A^tA = I$, so the set, $O(n)$, of $n \times n$ orthogonal matrices is a group called the orthogonal group. If A is orthogonal, one can show that $det(A)^2 = 1$ (since $det(A) = det(A^t)$ and $AA^t = I$) so $det(A) = \pm 1$. Thus $O(n)$ has a normal subgroup of index 2, $SO(n) = \{A \in O(n) | det(A) = 1\}$.

2 Finite subgroups of SO(3)

Our goal in this section is to find all finite subgroups of $SO(3)$, the group of three-dimensional rotations.

Proposition 2.1. *Let $A \in SO(3)$, $A \neq I$. Then there is a one-dimensional subspace $X_A = \{v \mid Av = v\}$ left invariant by A. In an orthogonal basis basis e_1, e_2, e_3 with $e_1 \in X_A$, A has the matrix representation*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}$$

θ is determined by A.

Proof. *Let $P(x) \equiv det(xI - A)$. Then $P(0) = det(-A) = -det(A) = -1$ since $A \in SO(3)$. Since $lim_{x\to\infty} = +\infty$, $P(x)$ has a zero in $(0, \infty)$ and so A has a positive eigenvalue which must be 1. Let $X_A = \{v \mid Av = v\}$. We have* *just shown that $dim(X_A) > 0$. If $dim(X_A) = 2$, then $X_A^\perp$ is left invariant by A, so contains ϕ with $A\phi = -\phi$ and $det(A) = -1$. It follows that $dim(X_A)$ is 1 or 3 if $A \in SO(3)$. $A \neq I$ means the dimension is not 3.*

Thus a 3D-rotation $A \neq 1$ has an axis which it leaves fixed and a rotation angle θ about that axis. Now let $G \subset SO(3)$ be a fixed, finite subgroup. Let $\alpha \in \mathbb{R}^3$, $\alpha \neq 0$, be an axis of rotation for some $x \in G, x \neq e$, that is, $x\alpha = \alpha$. Let $G^{(\alpha)} = \{x \in G \mid x\alpha = \alpha\}$. $G^{(\alpha)}$ is a subgroup of G. Let θ be the minimal positive angle of rotation of x's in $G^{(\alpha)}$. We claim that $\theta = 2\pi/n$ for some n for, if not, there is multiple of θ in $(2\pi, 2\pi + \theta)$ and so a smaller angle than θ. Thus each $G^{(\alpha)} = \{x \mid x\alpha = \alpha;\ x$ is an angle $0, 2\pi, 2(2\pi/n), ..., (n-1)(2\pi/n)\} \cong \mathbb{Z}_n$. α is called an n-fold point.

Let $S = \{\alpha \in S^2 \mid x\alpha = \alpha$, some $x \in G, x \neq e\}$ be the set of points on the sphere $\S^2 = \{\beta \in \mathbb{R}^3 \mid |\beta| = 1\}$ which are on axis of rotation for some nontrivial $x \in G$. We claim that S is a G-space; explicitly, if $y \in G$ and $\alpha \in S$, then $y\alpha \in S$. For if $x\alpha = \alpha$, then $(yxy^{-1})y\alpha = y\alpha$. Let S consist of k-orbits $O_1, ..., O_k$. the yxy^{-1} argument shows that if α is an n-fold axis, so is $y\alpha$; that is, the orbits have integers $n_1, ..., n_j$ associated with them so that O_j is a set of points lying on n_j-fold axes. We have the following counting result:

Proposition 2.2.
$$\sum_{i=1}^{k}(1 - \frac{1}{n_i}) = 2 - \frac{2}{o(G)}.$$

proof. Consider the set of pairs, $P = \{(\alpha, x) \mid \alpha \in S, x \in G, x \neq e, x\alpha = \alpha\}$. For each x, there are exactly two points on the intersection of its axis of rotation and S^2, that is, $\#(P) = 2(o(G) - 1)$.

on the other hand, for each $\alpha \in O_j$, there are clearly $n_j - 1$ x's in G with $x \neq e$ and $x\alpha = \alpha$, so

$$\#(p) = \sum_{i=1}^{k} \#(O_i)(n_i - 1).$$

But O_i is an orbit and the isotopy group for $\alpha \in O_i$ is exactly $\mathbb{Z}_{n_i}$, so by the fundamental counting principle (Theorem 1.6), $\#(O_i) = \frac{o(G)}{n_i}$. Thus

$$\#(P) = \sum_{i=1}^{k} \frac{o(G)}{n_i}(n_i - 1).$$

Equating the two formulas finish the proof. $\square$

We seek solutions with $n_i \geq 2, o(G) \geq 2$, and $n_i \leq o(G)$.

- Since $o(G) \geq 2$, $2 - \frac{2}{o(G)}$ lies in $[1, 2)$ and each $(1 - \frac{1}{n_i})$ lies $[\frac{1}{2}, 1)$. It follows that $k = 2$ or $k = 3$ (for if $k = 1$, the sum is strictly below 1 and if $k \geq 4$, the sum is at least 2).

- Consider the case $k = 2$. Then
$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{2}{o(G)},$$
so $n_i \leq o(G)$. implies that $n_1 = n_2 = o(G)$

- From now on, we take $k = 3$ and remember so $n_1 \leq n_2 \leq n_3$. If $n_1 \geq 3$, then each $1 - \frac{1}{n_i} \geq \frac{2}{3}$ so the sum is ≥ 2, which isn't possible; so $n_1 = 2$. If $n_2 \geq 4$, then the sum is at least $\frac{1}{2} + \frac{3}{4} + \frac{3}{4} = 2$, again impossible; so $n_2 = 2$ or 3.

- Consider the case $n_1 = n_2 = 2$. Then $\frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{n_3} = 2 - \frac{2}{o(G)}$ so $n_3 = \frac{o(G)}{2}$. Thus, a general solution is $o(G) = 2n, n_1 = n_2 = 2, n_3 = n$.

- If $n_1 = 2, n_2 = 3$, then $n_3 < 6$ for $\frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 2$. Each of the possibilities $n_3 = 3, 4, 5$ yields integral $o(G)$.

So, we have

k	n_1	n_2	n_3	o(G)	Name
2	n	n	-	n	C_n
3	2	2	n	2n	D_{2n}
3	2	3	3	12	$T \cong A_4$
3	2	3	4	24	$C \cong S_4$
3	2	3	5	60	$I \cong A_5$

The first group is isomorphic to $\mathbb{Z}_n$.
There are two orbits, each with only one point. So $\#(S) = 2$. There is a single axis of rotation which is by angle $2\pi/n$. The group is isomorphic to $\mathbb{Z}_n$, the cyclic group.

The second group is isomorphic to $D_{2n} = \langle a, b \mid a^n = e, b^2 = e, ba = a^{-1}b \rangle$. There is an orbit of n-fold points and with $2n/n = 2$ points. There are two orbits of 2-fold points, each with $2n/2$ points. Since each axis has two points, there is one n-fold axis and n 2-fold axes. The 2-fold rotations must interchange the two ends of the n-fold axis so the n-fold axis is orthogonal to the 2-fold axes. This group is isomorphic to $\mathbb{Z}_n \otimes \mathbb{Z}_2$.

The remaining cases correspond to the symmetries of the five Platonic solids.

3 The Platonic groups

There are five Platonic solids-regular polyhedra: tetrahedron, octahedron, cube, dodecahedron, and icosahedron. Each defines a finite group of rotations: namely, if one places the origin at their centroid, the group consists of all rotations that leave the solid set wise fixed.

- Consider first the tetrahedron: the familiar triangular pyramid with four equilateral triangles for faces. It has four vertices and six edges. It is evident that the axis that runs from each vertex to the centroid of the opposite face is a 3-fold axis of symmetry; so the symmetry group T has four 3-fold axes. The edges come in opposite pairs and the axis between the midpoints of those edges is a 2-fold axis of symmetry. In fact, the product of two 3-fold rotations about distinct axes is one of those 2-fold rotations! Thus, T has $4 \times 2 = 8$ 3-fold rotations, $3 \times 1 = 3$ 2-fold rotations, and the identity for a 12-element group. It leads precisely to the G-space we called T.

- The cube has three 4-fold axes, four 3-fold axes, and six 2-fold axes. It yield a G-space with $n_1 = 2, n_2 = 3, n_3 = 4$, and $o(G) = 3 \times 3 + 4 \times 2 + 6 + 1 = 24$. It leads precisely to the G space we called C.

- The icosahedron has ten 3-fold axes, six 5-fold axes, and fifteen 2-fold axes. Thus it yields a G-space with $n_1 = 2, n_2 = 3, n_3 = 5$, and $o(G) = 10 \times 2 + 6 \times 4 + 15 + 1 = 60$. It leads precisely to the G space we called I.

- The cube: take a cube and consider the six centers of its square faces. Join two such centers when the corresponding faces meet in an edge. Thus, each center is joined to four others. The twelve lines from eight triangles and we see an octahedron ambedded in the cube as a "dual solid." This shows that the symmetries of the cube and the octahedron are the same.

- The dodecahedron and icosahedron are dual and have the same symmetry group.

Thus, we need only prove uniqueness. Start with the case T, that is, a G-space of points on the sphere with the T orbit structure. We want to prove that the group is just the symmetries of some tetrahedron. By construction, there are eight 3-fold points (in two orbits) and so four 3-fold axes. Consider one orbit of such 3-fold points (a, b, c, d) and the axis through a. A priori, its other end could be any of b, c, d (or it could be in the other orbit). If it were b, we'd have a problem since the 3-fold rotation would have to yield an order three permutation of (c, d). Thus, the other hand of the axis is in the other orbit, and b, c, d are cyclically permuted by the rotation about the axis. It follows that b, c, d lie on an equilateral triangle orthogonal to the a axis. But we can do the same analysis for any vertex, so a, b, c, d lie at the vertices of a tetrahedron, G is its symmetry group. This proves uniqueness in case T.

Next look at case C. There are eight 3-fold points. The 4-fold axes are disjoint from these point, so consider one of them. any orbit of a 4-fold axis starting with a point not on the axis forms the vertices of a square orthogonal to the axis. Thus, the eight 3-fold points lie in two squares orthogonal to the first 4-fold axis.

These squares do not lie on the equator relative to this first 4-fold axis because if they did, there couldn't be any other 4-fold axis. Thus, these 3-fold axes must have ends. one in each square. Consider one of those 3-fold axes, X. The remaining six 3-fold points must be the vertices of two equilateral triangles orthogonal to that axis (by the argument we used for 4-fold axes). All three points can't lie in the same square (because the three points lie on an equivalent triangle!), so two lie in one square and one on the other. Since all points are equidistant from the end points of X, they must be neighbors, and so form a 90^o angle at X.

By the 3-fold symmetry, the other connections are at 90^o; that is, one of the second square lies on a line perpendicular to the first square, through a vertex at a distance equal to the square length. Using the 4-fold symmetry, we see that the eight 3-fold points are the vertices of a cube and the group is C.

Finally, consider the $G - space$ we called I. It has twelve 5-fold points. Consider a pair (a, b) at opposite ends of a 5-fold axis. the remaining ten points must form two regular pentagons (P_1, P_2) orthogonal to ab. They can't lie on the equator or else there couldn't be additional 5-fold axes. Thus, any other 5-fold axis c, d must have c and d in different pentagons, implying that two pentagons have a common side, D. Suppose $c \in P_1$.

As with (a, b), the (c, d) axis is orthogonal to two pentagons made out of the remaining points. Consider the pentagon P_3 closest to c. It must have two points from P_1 closest to c, either a or b, and two points from P_2. It follows that the distance from c to the vertices in P_3 is also D and that the two nearest points in P_2 to c have distance D. If you join together all pairs a distance P apart, you get twenty triangles, five each coming together at a and b, five with a single vertex in P_1 and two in P_2, and five with two vertices in P_1 and one in P_2. Thus, they form a regular icosahedron.

We have there for proven:

Theorem 3.1. *Every finite subgroup of $SO(3)$ is conjugate to one of*

1. *C_n, the symmetry of a single n-fold axis.*
2. *D_2n, the symmetry of a single n-fold axis and n 2-fold axes orthogonal ti it.*
3. *The symmetries of a tetrahedron, a cube, or an icosahedron (denoted T,C,I).*

We want to give geometric explanations of the isomorphism:

1. $T \cong A_4$
2. $C \cong S_4$
3. $I \cong A_5$

Example 3.2. *($T \cong A_4$). A tetrahedron has four vertices. Every symmetry defines a permutation of those vertices and the permutation clearly determines the symmetry. This has two kinds of non identity elements: the 3-fold rotations which correspond to a 3-cycle which lie in A_4, and the 2-fold axes which have two 2-cycles and so also lie in A_4. Every element of A_4 is one or the other, so we have the isomorphy.*

Example 3.3. *($C \cong S_4$). A cube has eight vertices, six faces, and twelve edges. There are, less obviously, four objects to permute. In fact the cube has four body diagonals. Each element of C defines a permutation of these four diagonals. It is not a priori evident that this map of C to S_4 is either one-one or onto. We'll show it onto so that, since $o(C) = 24 = o(S_4)$, it is one-one also.*

Consider any pair of body diagonals. They define a plane which contains two opposite edges. The 2-fold axis through the center of these edges permutes the two given body diagonals. The other two body diagonals are orthogonal to the 2-fold axis and so are left set wise invariant by the 2-fold rotation. Hence, any 2-cycle lies in the image of the map $C \to S_4$. Since the 2-cycle generated S_4, the image is everything.

Example 3.4. *($I \cong A_5$). Among the twenty faces, twelve vertices, and thirty edges, we will find a set of five objects as follows. There are fifteen 2-fold axes. They fall naturally into five sets. In each set, the three axes are mutually orthogonal like the standard x, y, z frame. Equivalently, there are sets of six edges (which cover each vertex). This defines a map of I into S_5, which is one-one and onto A_5.*