Topological dimensions, Hausdorff dimensions & fractals

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May 2006

Abstract

We present some basic facts about topological dimension, the motivation, necessary definitions and their interrelations. Finally we discuss the Hausdorff dimension and fractals.

1 Topological dimension

1.1 Motivation

Topological dimensions defines the basic difference between "related" topological sets such as I^n and I^m when $n \neq m$. The lack of that definition is especially highlighted because of the easy explanation of the geometric dimension.

When trying to define a dimension for a topological space, you might run into several difficulties - Unlike a vector space, you can not state that the dimension is the maximum of linear independent vector.

Therefore in order to get an intuitive definition for a topological dimension we should look for different properties that has the effects of the geometrical dimension.

1.2 General

A topological dimension has values in $-1, 0, 1, 2, 3, \ldots$, and is topological, i.e: if X and Y are homeomorphic then they have the same "dimension". It will also be nice, if $\mathbb{R}^n$ gets value n, for each n.
There are 3 commonly used definitions:

1. Small inductive dimension (ind)
2. Large inductive dimension (Ind)
3. Lebesgue covering dimension (dim)

All these definitions have the required properties, and we will see that they are the same for separable metrizable spaces (ind = Ind = dim).

Remainder: **metrizable space** is a topological space that is homeomorphic to a metric space.

2 The inductive dimension

2.1 Intuition

Lets look at a cube. What is the (geometrical) dimension of the cube? **3**, and what is the (geometrical) dimension of its boundary [square]? **2**.

Now let's look at the square. What is the dimension of its boundary? well its boundaries are lines, so their dimension is **1**.

And what about the boundary of the lines? Well their boundary is composed of dots, there dimension is **0**.

We get a pattern - the dimension of a spaces, is 1 + the dimension of its boundary. As a boundary is well defined in topology, this notion can be easily applied to topological spaces, to create a topological dimension.

3 The small inductive dimension

Definition 3.1. The small inductive dimension of X is notated $ind(X)$, and is defined as follows:

1. We say that the dimension of a space X ($ind(X)$) is -1 iff $X = \emptyset$
2. $ind(X) \leq$n if for every point $x \in X$ and for every open set U exists an open V, $x \in V$ such that $\overline{V} \subseteq U$, and $ind(\partial V) \leq n-1$. Where ∂V is the boundary of V.
3. $ind(X) = n$ if (2) is true for n, but false for $n-1$.
4. $ind(X) = \infty$if for every n, $ind(X) \leq n$ is false.

Remark 3.1. The ind dimension is indeed a topological dimension: X is homeomorphic to Y implies that $ind(X) = ind(Y)$.

This can be shown easily (and inductively) as the definition relays only of open\closed sets.

Remark 3.2. An equivalent condition to condition (2) is:

- The space has a base B, and every $U \in B$ has $ind(\partial U) \leq n-1$ [You can construct this base using the sets U from condition (2)].

Theorem 3.1. *As we would expect, if $Y \subseteq X$ then $ind(Y) \leq ind(X)$.*

Proof. By induction, it is true for $ind(X) = -1$. if $ind(X) = n$, for every point $y \in X$ there is an nbd V, with an open set $U \subseteq V$, such that $ind(\partial V) \leq n-1$. Note that $V_Y = Y \cap V$ is a nbd in Y, and U_Y is open in Y.

By induction, it is enough to show that $\partial U_Y \subseteq \partial U$, Because then $ind(\partial U_Y) \leq n-1$, and therefore $ind(Y) \leq n = ind(X)$ (by definition). And indeed $\overline{U_y} \subseteq \overline{U}, \partial U_Y = \overline{U_Y} \setminus U_Y \subseteq \overline{U} \setminus U = \partial U$ □

Example 3.1. Let's show that $ind(\mathbb{R})$ is 1.

for each $x \in \mathbb{R}$, lets select a nbd, V and a set $U = (a,b) \subseteq V$, $ind(\partial U) = ind(\{a,b\}) = 0$. This implies that $ind(\mathbb{R}) \leq 1$.

So, it is enough to prove that $ind(\mathbb{R})$ is not 0. But that is easy! if $ind(\mathbb{R}) = 0$, that means that a set U exists, such that: $ind(\partial U) = -1 \Leftrightarrow \partial U = \emptyset \Leftrightarrow U$ is clopen! but if U is clopen, $\mathbb{R}$ is disconnected! Because $\mathbb{R}$ is connected, we get that $ind(\mathbb{R}) > 0$.
So finally, $ind(\mathbb{R}) = 1 \diamond$.

The same proof can be done for the Sphere S^1 and the interval I^1.

Corollary 3.2. *A zero-dimensional space is disconnected.*

Corollary 3.3. *if for every nbd V of $x \in X$ exists a clopen set $U \subseteq V$ then $ind(X) = 0$*

Example 3.2. What is the dimension of C the cantor set?
Lets look at C in $\mathbb{R}$, C has no interior points (fact). That means, that for each nbd $U = (a, b)$ of $x \in C$, we can find two points c, d such that $a \leq c \leq x \leq d \leq b$, and $c, d \notin C$.

Lets look at C as a subspace, $V = (c, d)$ is open, and closed $\Rightarrow ind(C) = 0$

Note that these results are not exactly what we expect. The cantor set has the same 'size' as $\mathbb{R}$: $2^{\aleph_0}$, so we should expect it to have the same dimension. But then again, the cantor set has no interval in it. Its dimension should be then, somewhere between 0 and 1. We shall later learn of a dimension definition that suits our expectations.

It is worth to mention, that the cantor set is universal 0-dimensional space of separable metrizable spaces. meaning that every other space separable metric space $X, ind(X) = 0$, is homeomorphic to a sub-space of the cantor set.
Remember that a number is in the Cantor set $\Leftrightarrow$ It has the form $\sum_{i \in \mathbb{N}} \frac{x_i}{3^i}, x_i \in \{0, 2\}$.
for example: $\frac{2}{3} + \frac{0}{9} + \frac{2}{27} + \frac{0}{81} + \cdots$.
We can display this number simply as a fraction of radix 3 0.2020.... This set of fractions has one-to-one and onto mapping to the set of binary fractions 0.1010.... We simply replaced all the 2's with 1's. The latter set of fractions is more familiar to topology, it is the space:$D^{\aleph_0}, D = \{0, 1\}$, D with the discrete topology.

Proposition 3.4. *The $D^{\aleph_0}$ is homeomorphic to the Cantor set.*

Now it is enough to show that every zero dimensional separable metrizable space is homeomorphic to a sub-spaces of $D^{\aleph_0}$.

Remark 3.3. Every zero dimensional spaces has a clopen base (by the alternative definition).
If the space is also metrizable and separable, it has a countable clopen base (because then every base has a countable base contained in it).

Theorem 3.5. *The Cantor set is a universal space for all the zero-dimensional metrizable separable space.*

Proof. Let X be a separable metrizable space. It is enough to show that X is homeomorphic to a sub-set of $D^{\aleph_0}$.

Let $B = \{B_i\}$ be its clopen countable base. We define f to be: $f_i(x) = \begin{cases} 1 & x \in B_i \\ 0 & otherwise \end{cases}$. Define $f(x) = (f_1(x), f_2(x), \ldots)$. f is the homeomorphism that we wanted. $\diamond$ $\square$

An interesting question we can ask about this dimension, is wether $ind(\mathbb{R}^n) = n$. The answer is yes,we will start by showing that $ind(\mathbb{R}^n) \leq n$.

Proposition 3.6. *The dimension of* $\mathbb{R}^n, S^n, I^n \leq n$.

We can show this by induction. we have already shown that the dimension of $\mathbb{R}^1, S^1, I^1$is 1.
For every point $x \in \mathbb{R}^n, S^n, I^n$, there is a nbd U with an open set V which is homeomorphic to S^{n-1}, and by induction the proof is complete $\diamond$

In order to show that $ind(\mathbb{R}^n) \geq n$, We shall have some definitions and theorems. Let's start by defining a partition between sets:

Definition 3.2. A Partition L on between A and B exists, if there are open sets U, W, $A \subseteq U, B \subseteq W$, such that $W \cap U = \emptyset$ and $L = U^c \cap W^c$.

Theorem 3.7. $ind(X) \leq n$ $\Leftrightarrow$*For every point x and every closed set B there is a partition L, such that:*$ind(L) \leq n-1$.

Proof. $(\Rightarrow)$
If $x \in X$ and B is closed, Then there is a set V, such that $\overline{V} \cap B = \emptyset$. By definition of ind, there is a $U \subseteq V$, $x \in U$, $ind(\partial U) \leq n-1$ now, let $W = \overline{U}$ and $L = \partial U$.

$(\Leftarrow)$
Let $x \in X$ and V an nbd of x. The set $B = V^c$ is closed, and therefore there are U, W such that $x \in U$, $B \subseteq W$. Note that by definition of B, $U \subseteq V$.

Now W^c is closed, therefore $\overline{U} \subseteq W^c$, and obviously $\partial U \subseteq \overline{U} \subseteq W^c$, and by definition $\partial U = \overline{U} \setminus U = \overline{U} \setminus U \subseteq U^c$.

Therefore $\partial U \subseteq U^c \cap W^c = L$ $ind(U) \leq n-1$ □

Theorem 3.8. *If X is a metric space and Z is a zero dimensional separable subspace of X, then for all closed set A, B of X $A \cap B = \emptyset$, there is a partition between them L, such that $L \cap Z = \emptyset$*

Theorem 3.9. *X is a separable metrizable space, then $ind(X) \leq n \Leftrightarrow X$ is the union of two sub spaces Y, Z such that $ind(Y) \leq n-1, ind(Z) \leq 0$.*

From this we can easily see, that for a separable metrizable space X $ind(X) \leq n \Leftrightarrow X$ is the union of $Z_1, \ldots, Z_{n+1}$, $ind(Z_i) \leq 0$.

A nice result of this theorem, gives us an estimate on the dimension of the sum of two spaces.

Theorem 3.10. ***(The addition theorem)*** *If X, Y , $ind(X) \leq n, ind(Y) \leq m$ are separable metric spaces, then $X \cup Y$ can be represented with $Z_{x,1}, \ldots Z_{x,n+1}, Z_{Y,1} \ldots, Z_{Y,m+1}$. And therefore $ind(X \cup Y) \leq n+1+m+1-1 = m+n-1$, meaning that $ind(X \cup Y) \leq ind(X) + ind(Y) - 1$.*

Theorem 3.11. ***(The separation theorem)*** *For every closed sets A, B of a separable metric space X $ind(X) \leq n$, There is a partition L, such that $ind(L) \leq n-1$.*

Proof. We can decompose X into two spaces $Y, ind(Y) \leq n-1; Z, ind(Z) = 0$. There is a partition L between A, B and $L \cap Z = \emptyset \Rightarrow L \subseteq Y$ and $ind(L) \leq n-1$ as a sub-space of Y. □

Lemma 3.12. *M is a sub-space of metric X, A, B are closed and disjoint, $A \subseteq U$, $B \subseteq W$ are open in X and $\overline{U} \cap \overline{W} = \emptyset$.*

For every partition $L'$of $M \cap U, M \cap W$ there is a partition L of A, B , such that $M \cap L \subseteq L'$.

The last lemma says, that you can take a partition on M and extend it to a partition on X. We can use it to prove the following:

Theorem 3.13. ***(The second separation theorem)*** *X is a metric space, M is a separable sub-space and $ind(M) \leq n$. Then For every disjoint closed sets A, B there is a partition L such that $ind(L \cap M) \leq n - 1$.*

Theorem 3.14. ***(Theorem on partitions)*** *X is a separable metric space, $ind(X) \leq n \Leftrightarrow$. Then For every sequence of $(A_{1,}, B_1), \ldots (A_{n+1}, B_{n+1})$ of closed disjoint sets, The partitions $L_1, \ldots, L_{n+1}$exists, such that they have an empty intersection.*

Proof. We shall only prove ($\Rightarrow$)
Let's look at A_1, B_1 by the separation theorem, we can find a L_1,$ind(L_1) \leq n - 1$. Lets look at A_2, B_2 and let $M = L_1$. We can find a partition L_2, such that$ind(M \cap L_2) \leq n - 2$.

Lets look at A_i, B_i and let $M = L_1 \cap L_2 \cap \cdots \cap L_{i-1}$, we can find a partition L_i, such that$ind(M \cap L_i) = n - i$.

When we get to toA_{n+1}, B_{n+1}, we have $ind(L_1 \cap L_2 \cap \cdots \cap L_{n+1}) \leq -1 \Leftrightarrow L_1 \cap L_2 \cap \cdots \cap L_{n+1} = \emptyset$. □

We are 2-3 theorems away from showing that $ind(\mathbb{R}^n) = n$

Theorem 3.15. ***(Brouwer fixed point theorem)*** *If S is a non-empty, compact, closed and convex sub set of $\mathbb{R}^n$, then $f : S \to S$ has a fixed point.*

Why do we need this theorem? you shall soon find out!

Theorem 3.16. *Let $A_1 B_1, \ldots, A_n, B_n$ be the opposite faces of the I^n cube (meaning: the i-th coordinate of A_i is 0 and in B_i is 1).*
Now, if L_i is a partition between A_i and B_i , then $L_1 \cap \cdots \cap L_n \neq \emptyset$.

Proof. As L_i is a partition between A_i and B_i, by definition exists open sets $U_i, W_i; A_i \subseteq U_i, B_i \subseteq W_i$. and $L_i = (U_i \cup W_i)^c$. The following function is well defined:

$$f_i = \begin{cases} \frac{1}{2} \frac{d(x, L_i)}{d(A_i, x) + d(x, L_i)} + \frac{1}{2} & x \in W_i^c \\ -\frac{1}{2} \frac{d(x, L_i)}{d(B_i, x) + d(x, L_i)} + \frac{1}{2} & x \in U_i^c \end{cases}$$

f_i is continuous. We also have: $f^{-1}(\frac{1}{2}) = L_i$ and $f_i(A_i) = \{1\}$ and $f_i(B_i) = \{0\}$ (Note that the i-th coordinate of $x \in A_i$ is 0, and of $x \in B_i$ is 1). Lets define $f(x) = (f_1, \ldots, f_n)$.
if $L_1 \cap \cdots \cap L_n = \emptyset$ then $\forall x\, f(x) \neq (\frac{1}{2}, \ldots, \frac{1}{2})$.
Let $p : I^n \setminus \{(\frac{1}{2}, \ldots, \frac{1}{2})\}$ be the projection from a point in the cube to its boundary (We stretch a line from the middle, to the point, until it reaches the boundary).

Define $g = p \circ f$. We have that $g(A_i) \subseteq B_i$ and $g(B_i) \subseteq A_i$and that $g(I^n)$is on one of the $B_i$or A_i.That means that $g(x) \neq x$ for every x. contradicting Brouwer fixed point theorem. □

Can you see where this is heading?

We have already seen that $I^n \leq n$ is true. let's show that $I^n \leq n-1$ is false. If $I^n \leq n-1$ is true, then it has we can find n partitions of the faces that has an empty intersection. (the faces of I^n are sequence of n disjoint closed sets, so we apply the Theorem on partitions)

But by the previous theorem showed that any selection of partitions of the faces will always have a non-empty intersection! and we get:

$$ind(I^n) = n$$

And due to the sub-space theorem:

$$ind(\mathbb{R}^n) = n$$

This also shows us, that $I^n \approx I^m \Leftrightarrow n = m$.

4 The Large inductive dimension

One can see that for every separable metric space X with $ind(X) = n$, for every closed subset F of every open subset U of X, there is an open V in between, such that the $ind(\partial V) \leq n-1$, this suggest a modification in the definition of the small inductive dimension consisting in replacing the point x by a closed set A.

Definition 4.1. The large inductive dimension of normal space X is notated $Ind(X)$, and is defined as follows:

1. We say that the dimension of a space X ($Ind(X)$) is -1 iff $X = \emptyset$
2. $Ind(X)$ $\leq$n if for every closed set $C \subseteq X$ and for every open set U exists an open V, $C \subseteq V$ such that $\overline{V} \subseteq U$, and $Ind(\partial V) \leq n-1$. Where ∂V is the boundary of V.
3. $Ind(X) = n$ if (2) is true for n, but false for $n-1$.
4. $Ind(X) = \infty$if for every n, $Ind(X) \leq n$ is false.

Remainder: X is a **normal space** if, given any disjoint closed sets E and F, there are a neighborhood U of E and a neighborhood V of F that are also disjoint.

Proposition 4.1. *A normal space X satisfies the inequality $Ind(X) \leq n$ iff for every pair A, B of disjoint closed subset of X exists a partition L between A and B such that $Ind(X) \leq n-1$*

We have seen this before...

Theorem 4.2. *For every separable space X we have $ind(X) = Ind(X)$*

Proof. For every normal space X we have $ind(X) \leq Ind(X)$ by definition.

To show that $Ind(X) \leq ind(X)$ we will use induction with respect to $ind(X)$, clearly one can suppose that $ind(X) < \infty$.

If $ind(X) = -1 \Rightarrow ind(X) \leq Ind(X)$. Assume that the inequality is proven for all separable metric space X of $ind(X) < n$ and consider a separable metric space X such that $ind(X) = n$.

Let A and B be a pair of disjoint closed subset of X, according to the first separation theorem there exists a partition L between A and B such that $ind(L) \leq n-1$, by the inductive assumption for every $k < n$ $Ind(L) \leq n-1$ and according to the previous proposition $Ind(X) \leq n$ and finally we got $Ind(X) \leq ind(X) \Rightarrow Ind(X) = ind(X)$. □

5 The Lebesgue covering dimension

Lebesgue covering dimension or topological dimension of a topological space is defined to be the minimum value of n, such that any open cover has a *refinement* in which no point is included in more than n+1 elements, if a space does not have Lebesgue covering dimension m for any m, it is said to be infinite dimensional.

In this context, a *refinement* is a second open cover such that every set of the second open cover is a subset of some set in the first open cover.

It is named after Henry Lebesgue, although it was independently arrived at by a number of mathematicians.

Example 5.1. consider some arbitrary open cover of the unit circle. This open cover will have a refinement consisting of a collection of open arcs. The circle has dimension 1, by this definition, because any such cover can be further refined to the stage where a given point x of the circle is contained in at most 2 arcs.

That is, whatever collection of arcs we begin with some can be discarded, such that in the remainder still covers the circle, but with simple overlaps.

Similarly, consider the unit disk in the two-dimensional plane. It is not hard to visualize that any open cover can be refined so that any point of the disk is contained in no more than three sets.

5.1 Properties

Property 5.1. Lebesgue covering dimension is a topological property

two homeomorphic spaces have the same dimension.

Property 5.2. $\mathbb{R}^n$ has dimension n

Theorem 5.1. *For every closed subspace M of a normal space X we have* $dimM \leq dimX$

Proof. The theorem is obvious if $dimX = \infty$, so that we can assume that $dimX = n < \infty$.

Consider a finite open cover $U = U_{i}{}_{i=1}^{k}$ of the space M. For i=1,2...,k let W_i be an open subset of X such that $U_i = M \bigcap W_i$. The family $X \backslash M \bigcup W_{i}{}_{i=1}^{k}$ is an open cover of the space X and since $dimX \leq n$ it has a finite open refinement γ which no point is included in more than n+1 elements of γ. one easily see that the family $\gamma \backslash M$ is a finite open cover of space M, refines U and has no point of M is included in more than n+1 elements of $\gamma \backslash M$, so that $dimM \leq n = dimX$. □

Theorem 5.2. *For X metrizable space* $IndX = dimX$.

Theorem 5.3. *For X normal space* $dimX \leq IndX$

5.2 Some topological constructions

The definition of the Lebesgue covering dimension can be used to build some topological sets, such as the Sierpinski carpet.

A construction can proceed as follows:
Consider, for example, a finite open covering for the two-dimensional unit disk. This covering can always be refined so that no point in the disk belongs to more than three sets. Now, we will remove all of the points in the disk that belong to three sets. Depending on the refinement, this will leave possibly one or more

holes in the disk. The remaining object is again two-dimensional, and again has a finite open cover.

The process of selecting a cover and refining, and then punching out holes can be repeated, ad infinitum. The resulting object is homeomorphic to the Sierpinski carpet.

What is curious about this construction is that the carpet has a Lebesgue covering dimension of one, and not two, although at any step of the creation the shape had dimension of two. The proof of this is essentially by contradiction: were there a covering which required membership to three sets, then the affected area would have been punched out during the construction phase. Similar constructions can be performed in higher dimensions; the three-dimensional analogue is called the Menger sponge. Curiously, the Lebesgue covering dimension of the Menger sponge is again one.

6 Fractals

The word "fractal" denotes a shape that is recursively constructed or self-similar, that is, a shape that appears similar at all scales of magnification and is therefore often referred to as "infinitely complex". There is also a mathematics fractal definition that we intrudes later.
Some examples that are:

1. *Cantor set*
2. *Sierpinski carpet*

   Which we already see, more known examples are:

3. *The Koch curve:* one can imagine that it was created by starting with a line segment, then recursively altering each line segment as follows:
   (a) divide the line segment into three segments of equal length.
   (b) draw an equilateral triangle that has the middle segment from step 1 as its base.
   (c) remove the line segment that is the base of the triangle from step 2.

   The Koch curve is in the limit approached as the above steps are followed over and over again

4. *The Mandelbrot set:* this set can be defined as the set of parameters c for which the critical point 0 of $f_c : \mathbb{C} \to \mathbb{C}; z \mapsto z^2 + c$ does not tend to ∞, That is: $f_c^n(0) \not\to \infty$ where f_c^n is the n-fold composition of f_c with itself. This definition lends itself immediately to the production of computer generated renderings.

   Example to approximate fractals are easily found in nature. These objects display self-similar structure over an extended, but finite, scale range. Examples include clouds, snow flakes, mountains, river networks, and systems of blood vessels. Famous example in this class is

5. *The coast line of Britain.*

6.1 Fractals dimension

We can observe the *Cantor set* as a key example to understanding fractals dimensions. The Cantor set has topological dimension of zero, but yet it has the same cardinality as the real line - in that sense we'd expect its dimension to be one. But the Cantor set has no interval in it - and in that sense we'd expect its dimension to be zero.

The answer then, lies somewhere in the middle. The Cantor set should have a dimension greater than zero, but smaller than one.

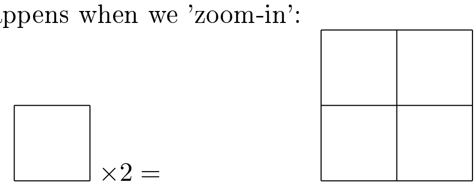
7 Hausdorff dimension

7.1 Introduction

The topological dimensions that we saw gives us a notion of geometrical dimension for topological space. And as we expect the topological dimension of $\mathbb{R}^n$ is indeed n. But there are some complex sets as we seen before that the topological dimensions seems too naive for.

For this purpose, the Hausdorff dimension (fractal dimension) was invented. The topological dimension were built on some topological notions, like the notion that boundary of the space has a dimension smaller by 1. This dimension has a different notion that is not topological.

We will now look at $\mathbb{R}^n$ and observe another feature that involves it's geometrical dimension, to get an intuition of the Hausdorff dimension.
Imagine a square in $\mathbb{R}^2$ with side length of 1. its area is also 1. let's see what happens when we 'zoom-in':



Comparing the two squares:

Zoom	Side	Area	Factor: $\frac{log(Area)}{log(Side)}$
1	1	1	-
2	2	4	2
3	3	9	2
4	4	16	2

Imagine a cube in $\mathbb{R}^3$ with side length of 1. its volume is also 1. let's see again what happens when we 'zoom-in'

Zoom	Side	Volume	Factor: $\frac{log(Area)}{log(Side)}$
1	1	1	-
2	2	8	3
3	3	27	3
4	4	64	3

More generally, if we take the cube I^n, and we 'zoom in' the space k times, the cube will grow by k^n.

As we can see this is another effect of the geometrical dimension. The Hausdorff dimension definition is based on this notion of the dimension, and that's why it requires a metric (so we can measure the growth of the space). This dimension gives nice results for fractals, as we will demonstrate, but first lets formalize the definition.

7.2 Formal definition

Note that the Hausdorff dimension is defined only for metric spaces - it uses concepts like length and volume, that require a metric.

Remainder: Let X be a metric set and C be a collection of sub-sets. The

mesh of C is $mesh(C) = sup\{diam(B) : B \in C\}$.

The definition of the Hausdorff dimension is done using a measure on the space, so we shall first define the Hausdorff measure:

Definition 7.1. Let A be a subset of X, we annotate $H_d(A, \epsilon) = inf\{\sum diam(B_i)^d : C = \{B_i\}, mesh(C) < \epsilon\}$ where C is a countable cover of A.

Remember that we are trying to use the geometrical notion to define a new dimension. The sum $\sum diam(B_i)$ can be thought of as the 'size' of a 1-d segment in the space. The sum $\sum diam(B_i)^d$ is the volume of the number of d-dimensional cubes needed to cover the space.

Now, to be able to use H_d comfortably, we define a measure:

Definition 7.2. The Hausdorff p-measure M_d is $M_d(A) = sup\{H_d(A, \epsilon)\}$

Note: $a < b \Leftrightarrow H_d(A, a) \geq H_d(A, b)$, so the following is also true:

$$M_d(A) = lim_{\epsilon \to 0} H_d(A, \epsilon)$$

. This measure 'tells' the 'volume' of A if it was in a d dimensional space.

Definition 7.3. The Hausdorff dimension of A, is $dim_H(A) = sup\{d : 0 < M_d(A) < \infty\}$.

This definition is quite natural - the dimension of A is the highest dimension that A has a finite 'volume'.

Note that if $p \leq d$, then $M_p(A) \geq M_d(A)$.

Theorem 7.1. *If* $0 < M_d(A) < \infty$ *then* $dim_H(A) = d$.

Proof. We shall first show that if $p < d$ then $M_p(A) = \infty$.
$M_d(A) > 0 \Rightarrow$ there is a $1 > \delta > 0$, $t > 0$ such that $H_d(A, \delta) = t$, This is due to the definition of H_d.

And by definition, for every $\epsilon < \delta$ we have $H_d(A, \epsilon) \leq t$. Let's choose a ϵ, so the following inequality holds: $\epsilon^{d-p} < t/N$; $\epsilon < \delta$, Where N is an arbitrary number. Let $C = \{B_i\}, mesh(C) < \epsilon$. Now, ($\heartsuit$) Remember that $x < y \Leftrightarrow x^{-1} > y^{-1}$.
$\sum diam(B_i)^p = \sum diam(B_i)^{p-d} \cdot diam(B_i)^d \geq \epsilon^{p-d} \sum diam(B_i)^d \geq \epsilon^{p-d} H_d(A, \epsilon) \geq (N/t)t = N$.

For every N, there is an ϵ, $\sum diam(B_i)^p$ does not converge, for any cover C with $mesh(C) \leq \epsilon$. This gives us that $M_p(A) = lim_{\epsilon \to 0} H_d(A, \epsilon) = \infty$.

Now if $d < p$, then by the above proof, it implies that $M_d(A) = \infty$, contradicting the assumption.
Therefore, the proof is complete. □

This makes sense - if we were to try to cover a square (2-d shape) with lines (1-d shape) we would need infinity of lines. As we said, the measure gives us the 'volume' of the set. If d is the dimension, Then if we try to measure a set A with a dimension $p < d$ then its p dimensional volume will be ∞.

The last theorem gives us quite a lot help on determining the dimension of a space. when we find one d that has a non-zero Hausdorff measure we found the dimension.

7.3 Examples and Computations

7.3.1 What is the Hausdorff dimension of a countable set?

Let $A = a_1, a_2, ..., a_n$ be a finite subset of a pseudometric space (X, d). Suppose that $d(a_j, a_i) > 0$ for $i \neq j$. Then, we can see that $M_0 = n \Rightarrow dim_H(A) = 0$.

7.3.2 What is the Hausdorff dimension of an interval ?

Let X be the space of the real numbers with the usual pseudometric, and let $A = [a, b]$. We will show that $M_1(A) = b-a$ which will leads us to $dim_H(A) = 1$.

We first show that $M_1(A) \geq b - a$ by showing that $M_1 \geq b - a + \eta$ for every $\eta > 0$. Let $\epsilon > 0$, and let N be an integer such that $h = \frac{(b-a)}{N} < \frac{\epsilon}{2}$. For $i = 1, 2, 3..., N$, let $x_i = a+ih$. We define an open cover $C = \{C_i : i = 1, 2..., N\}$ for A as follows (we may assume $\eta < b - a$):

$C_1 = [a, x_1 + \frac{\eta}{N})$

$C_i = (x_i - \frac{\eta}{(2N)}, x_{i+1} + \frac{\eta)}{(2N)})$

$C_N = (x_{N-1} - \frac{\eta}{N}, b]$

for each i we have $diam(C_i) = h + \frac{\eta}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $m(C) < \epsilon$ and $\sum diam(C_i) = (h + \frac{\eta}{N}) + (N - 2)(h + \frac{\eta}{N}) + (h + \frac{\eta}{N}) = Nh + \eta = b - a + \eta$.

Therefore, $M_1(A) \leq b - a + \eta$ for every $\eta > 0$, which implies that $M_1(A) \leq b - a$. Now we will show that $M_1(A) \geq b - a$.

Let $C = \{C_j : j \in Z_+\}$ be an open cover of A. Because A is compact there is a Lebesgue number $\epsilon > 0$ for C; that is, whenever $x, y \in A$ such that $|x - y| < \epsilon$, than $\exists C_i \in C$ that $x, y \in C_i$. Let N be a positive integer such that $h = \frac{(b-a)}{N} < \epsilon$, and define $x_i = a+ih$ for $i = 0, 1, 2..., N$. Then for each i there is a $C_{j(i)}$ so that x_{i-1} and x_i in $C_{j(i)}$; this implies that $x_i - x_{i-1} = h \leq diam(C_{j(i)}$.

Hence,

$b - a = \sum_{i=1}^{N}(x_i - x_{i-1}) < \sum diam(C_{j(i)})$.

were the summation is over distinct values $j(i)$. Therefore

$b - a \leq \sum_{i=1}^{\infty} diam(C_j)$ for every countable open cover C; consequently, $M_1(A) \geq b - a$.

Thus $M_1(A) = b - a$.

And in general the Hausdorff dimension of a n-dimension surface is n.

As we can see, The Hausdorff dimension of these sets is intuitive, and resembles the geometrical dimension and the topological dimension. These examples are important, because they give us the 'right' to call the Hausdorff dimension a dimension (and not 'The Hausdorff strange function')

7.4 Fractals

Now that we grasped the Hausdorff dimension, it's time to see some uses of it in the fractals field. The Hausdorff dimension used to formalize the definition of fractals.

Definition 7.4. The set A is a said to be a **Fractal** if its Hausdorff dimension is different from its topological dimension (dim). Some define a fractral as a set with non-integer Hausdorff dimension.

7.4.1 Examples

If we find a nice enough fractal, we can calculate its Hausdorff dimension with ease using the geometrical notion of the Hausdorff dimension.

The dimension of a fractal is intuitively, The factor between the zoom, and the number of self resembling parts after the zoom $\frac{log(Smaller\, self\, resmbling\, parts)}{log(Zoom)}$.

Lets give an example with a common fractal *The Cantor set*.

Zoom	Number of smaller self resembling parts	Factor: $\frac{log(Smaller\, self\, resmbling\, parts)}{log(Zoom)}$
1	1	-
3	2	0.630929754
9	4	0.630929754

$D = \frac{log(N)}{log(r)} \Rightarrow D = \frac{log(2)}{log(3)} = 0.63$.

We got an object with Hausdorff dimension different from his topology dimension.

We can also notice *The Cantor set* dimension is between a point (dimensionality 0) and a line (dimensionality 1) just like we would expected.

Another well known and loved fractal example is *The Koch curve*
We can see that the length of the curve increases with each iteration, so it has infinite length. This is a good example of a bound shape of infinite length. But we still can obtain the Hausdorff dimensions from the formula $D = \frac{log(N)}{log(r)} \Rightarrow D = \frac{log(4)}{log(3)} \Rightarrow D = 1.26$

References

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